Unboundedness of Solutions and Nonstability of Differential Equations of the Second Order with Delayed Argument

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Abstract

It is well known that, for $\varepsilon = 0$, all solutions of the equations

$$
\begin{align*}
-x''(t) + p(t)x(t - \varepsilon) &= 0, \quad t \in [1, +\infty), \\
x''(t) + p(t)x(t) + \frac{1}{t^\alpha}x \left( t - \frac{\varepsilon}{t^\beta} \right) &= 0, \quad t \in [1, +\infty),
\end{align*}
$$

are bounded on $[1, +\infty)$ and even tend to zero, as $p(t) \xrightarrow{t \to +\infty} +\infty$. Here we obtain the following results:

1) for each positive $\varepsilon$ there exist unbounded solutions of the first equation;

2) for each positive $\varepsilon$ there exist unbounded solutions of the second equation in case when $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ and $p(t)$ is bounded;

3) all solutions of the equation

$$
x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [1, +\infty),
$$

with positive nondecreasing and bounded on $[1, +\infty)$ coefficient $p(t)$ are bounded if and only if

$$
\int_{1}^{+\infty} \tau(t) \, dt < \infty.
$$

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1 Introduction

It was proved by J. J. A. M. Brands [15] that for each bounded delay $\tau(t)$ the equation
\[ x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \]
is oscillatory if and only if the corresponding ordinary differential equation
\[ x''(t) + p(t)x(t) = 0, \quad t \in [0, +\infty), \]
is oscillatory. Thus, the following question arises: are the asymptotic properties of ordinary differential equation inherited by the delay equation? The answer is negative. A. D. Myshkis [3] proved that there exists unbounded solution of the equation
\[ x''(t) + p(t)x(t - \varepsilon) = 0, \quad t \in [0, +\infty), \]
where $p$ and $\varepsilon$ are positive constants. The problem of unboundedness of the solutions in case of nonconstant coefficients was formulated in [3] as an open one. The first results in this direction were obtained by A. Domoshnitsky in [16]. Unboundedness of solutions to system of second order equations with periodic coefficient $p(t)$ and constant delay were investigated in a recent paper by Yu. Dolgii and S.G. Nikolaev [17]. Results on boundedness of delay equations’ solution were obtained by D. V. Izumova [9]. The asymptotic formula of solutions to second order equation in case, when $\tau(t)$ is a summable function, was obtained by M. Pinto [18].

Our approach is based on the research of behavior of the amplitudes of oscillating solutions. The amplitudes’ behavior of solutions of ordinary differential equations was investigated by C. T. Taam [4], L. Lasota [5] and A. Elbert [10]. Results on boundedness of solutions of ordinary differential equations were obtained by I. T. Kiguradze [12,13] and D. V. Izumova [12] and on the unboundedness — by F. Hartman [11, p.413]. Results on both the boundedness and the unboundedness were also proposed in the monography of V. N. Shevelo [2].

Let us consider the following equation
\[ x''(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty), \quad (1.1) \]
\[ x(\xi) = \varphi(\xi) \text{ for } \xi < 0, \quad (1.2) \]

where \( f, \varphi : [0, +\infty) \to (-\infty, +\infty), \varphi : (-\infty, 0) \to (-\infty, +\infty) \) are measurable functions bounded in essential, \( p_i \) and \( \tau_i : [0, +\infty) \to [0, +\infty) \) are locally summable functions.

It is known [1] that general solution of equations (1.1), (1.2) has the following representation

\[ x(t) = \int_0^t C(t, s) \bar{f}(s) \, ds + x_1(t)x(0) + x_2(t)x'(0). \quad (1.3) \]

Here \( C(t, s) \) is the Cauchy function of equation

\[ x''(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, +\infty), \quad (1.4) \]

\[ x(\xi) = 0 \text{ for } \xi < 0. \quad (1.5) \]

Note that for every fixed \( s \in [0, +\infty) \) function \( C(\cdot, s) \) is the solutions of “s-truncated” equation

\[ (L_s x)(t) \equiv x''(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [s, +\infty), \quad (1.6) \]

\[ x(\xi) = 0 \text{ for } \xi < s. \quad (1.7) \]

The function \( \bar{f} \) is determined by the equality

\[ \bar{f}(t) = f(t) - \sum_{i=1}^n p_i(t)\varphi(t - \tau_i(t)) \left( 1 - \sigma(t - \tau_i(t), 0) \right), \quad (1.8) \]

where \( \sigma(t, s) = \begin{cases} 1 & \text{for } t \geq s, \\ 0 & \text{for } t < s, \end{cases} \)

Functions \( x_1 \) and \( x_2 \) are solutions of equation (1.4), (1.5), satisfying the conditions \( x_1(0) = 1, \ x_1'(0) = 0, \ x_2(0) = 0, \ x_2'(0) = 1. \)

Equation (1.1), (1.2) is said to be unstable if for each positive \( \varepsilon \) there exist two solutions \( x \) and \( \bar{x} \) so that

\[ |x(0) - \bar{x}(0)| < \varepsilon \text{ and } |x'(0) - \bar{x}'(0)| < \varepsilon, \]

but their difference \( x(t) - \bar{x}(t) \) is unbounded on \([0, +\infty)\).
From representation (1.3) of the solutions it is clear that existence of unbounded solution of equation (1.4), (1.5) is equivalent to instability of equation (1.1), (1.2).

In this paper several criteria for existence of unbounded solutions to equation (1.4), (1.5) have been obtained. The following examples illustrate some of them. If \( \varepsilon = 0 \), then all solutions of the equations

\[
x''(t) + e^t x(t - \varepsilon) = 0, \tag{1.9}
\]
\[
x''(t) + t^2 x(t) + t^{3/2} x \left( t - \frac{\varepsilon}{t} \right) = 0, \tag{1.10}
\]
\[
x''(t) + t^\alpha x \left( t - \frac{\varepsilon}{t^\beta} \right) = 0, \quad \alpha + 2 > 2 \beta, \tag{1.11}
\]
\[
x''(t) + x(t) + \frac{1}{\sqrt{t}} x \left( t - \varepsilon/\sqrt{t} \right) = 0, \tag{1.12}
\]
\[
x''(t) + x(t - \varepsilon) = 0, \tag{1.13}
\]
are bounded on \((1, +\infty)\), and for equations (1.9)–(1.11) they even tend to zero when \( t \to +\infty \) [4,5 and 2, p. 24]. If \( \varepsilon > 0 \), then there exist unbounded solutions to each of these equations.

We will obtain the following criteria of boundedness of all solutions of the equation

\[
x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty),
\]
\[x(\xi) = 0 \text{ for } \xi < 0, \tag{1.14}
\]

**Theorem 1.1.** All solutions of equation (1.14) with positive nondecreasing and bounded coefficient \( p(t) \) are bounded if and only if

\[
\int_0^\infty \tau(t) \, dt < \infty. \tag{1.15}
\]

Let us introduce the following operator \( K_{\nu\mu} : C_{[\nu, \mu]} \to C_{[\nu, \mu]} \) by the following equality

\[
(K_{\nu\mu} x)(t) = -\int_\nu^\mu G_{\nu\mu}(t, s) \sum_{j=1}^m p_j(s) x(s - \tau_j(s)) \, ds, \tag{1.16}
\]
where \( x(\xi) = 0 \) for \( \xi < \nu \),
\( G_{\nu\mu}(t, s) \) is the Green function of boundary value problem

\[
x''(t) = f(t), \quad x(\nu) = 0, \quad x(\mu) = 0.
\] (1.17)

Denote by \( \lambda_{\nu,\mu} \) the minimal positive characteristic number of operator \( K_{\nu\mu} \).

Denote by \( h_i(t) = t - \tau_i(t) \) and \( h(t) = \min_{1 \leq i \leq m} h_i(t) \). All the results of this paper are obtained under the condition

\[
\lambda_{h(t), t} > 1 \text{ for } t \in (0, +\infty),
\] (1.18)

known as H-condition [6]. Each of the following conditions a), b) and c) guarantees inequality (1.18) [6,7,14]:

a) \( (t - h(t)) \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \, ds \leq 4 \text{ for } t \in (0, +\infty), \)

b) \( (t - h(t))^2 \sup_{s \in [h(t), t]} \sum_{i=1}^{m} p_i(s) \leq 8 \text{ for } t \in (0, +\infty), \)

c) \( m = 1 \) and \( h \) is a nondecreasing function.

Note: H-condition ensures that there is no more than one zero of nontrivial solution \( x \) on \([h(t), t]\) for every \( t \in (0, +\infty)\).

\section{Main results}

Let us formulate results on unboundedness of solutions of equation (1.4), (1.5).

\textbf{Theorem 2.1.} Let it be that

\[
M \equiv \sup_{t \in [0, +\infty)} \sum_{j=1}^{m} p_j(t) < +\infty
\]

and there exists \( i \in \{1, \ldots, m\} \) so that

\[
\int_{0}^{\infty} p_i(t) \tau_i(t) \left( \frac{2\sqrt{2}}{\sqrt{M}} - \tau_i(t) \right) \, dt = +\infty.
\]
then there exists unbounded solution of equation (1.4), (1.5).

For equation
\[ x''(t) + p_1(t)x(t) + p_2(t)x(t - \tau_2(t)) = 0, \quad t \in [0, +\infty), \]
\[ x(\xi) = 0 \text{ for } \xi < 0, \quad (2.1) \]
we will obtain the following result.

**Theorem 2.2.** Let \( p_1 \) and \( p_2 \) be bounded on \([0, +\infty), \tau_2(t) \xrightarrow{t \to +\infty} 0 \) and
\[ \int_0^\infty p_2(t)\tau_2(t) \, dt = +\infty. \quad (2.2) \]
Then there exists unbounded solution of equation (2.1).

**Example 2.3.** The equation
\[ x''(t) + p(t)x(t) + \frac{1}{t^\alpha} x \left( t - \frac{\varepsilon}{t^\beta} \right) = 0, \quad t \in [1, +\infty), \quad (2.3) \]
has unbounded solution if \( \alpha + \beta \leq 1, \alpha \geq 0, \beta \geq 0. \)
Unboundedness of solution of equation (1.12) follows from the above assertion in case \( \alpha = \frac{1}{2}, \beta = \frac{1}{2} \) and \( p = 1. \)
Denote \( \tau(t) = \min_{1 \leq i \leq m} \tau_i(t). \)

**Theorem 2.4.** Let there be index \( i \) so that
\[ \int_0^\infty p_i(t)\tau(t) \, dt = \infty. \quad (2.4) \]
Assume that at least one of the following two conditions a) or b) is fulfilled:

a) there exists \( \varepsilon > 0 \) so that \( \tau(t) \geq \varepsilon \) for \( t \geq \nu \geq 0; \)

b) vrai sup \( t \in [\nu, +\infty) \sum_{i=1}^m p_i(t) < \infty. \)

Then there exists unbounded solution of equation (1.4), (1.5).
Existence of unbounded solutions of equations (1.9) and (1.13) comes from Theorem 2.4 (with condition a)).

**Example 2.5.** The equation

$$x''(t) + t^\alpha |\sin t| x(t - \varepsilon) = 0, \quad t \in (0, +\infty)$$

has unbounded solution if $\alpha \geq -1$.

### 3 Increase of Wronskian and Existence of Unbounded Solutions

Denote by

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x'_1(t) & x'_2(t) \end{vmatrix}$$

Wronskian of the fundamental system of equation (1.4) (1.5). For simplicity let us assume that $W(0) > 0$.

**Theorem 3.1.** If

$$\lim_{t \to +\infty} W(t) = +\infty$$

and there exists positive $\varepsilon$ so that $\tau_i(t) \geq \varepsilon$ for $i = 1, \ldots, m$ and almost all $t \geq \nu$, then there exist unbounded solutions of equation (1.4), (1.5).

Introduce function $\theta: [0, +\infty) \mapsto [0, +\infty)$ so that the minimal positive characteristic number $\lambda_{\nu, \theta(\nu)}$ of the operator $K_{\nu, \theta(\nu)}$ satisfies the inequality $\lambda_{\nu, \theta(\nu)} \leq 1$ for each $\nu \in [0, +\infty)$.

Denote

$$R = \text{vraisup}_{s \in [t, \theta(t)]} \sum_{i=1}^{m} p_i(s).$$

**Theorem 3.2.** Let it be that

$$\text{vraillim}_{t \to +\infty} \frac{W(t)}{\sqrt{R(t)}} = \infty.$$
Then there exists unbounded solution of equation (1.4), (1.5).

**Corollary 3.3.** If

$$\operatorname{vraisup}_{t \in [0, +\infty)} \sum_{i=1}^{m} p_i(t) < \infty \text{ and } \lim_{t \to +\infty} W(t) = \infty,$$

then there exists unbounded solutions of equation (1.4), (1.5).

**Remark 3.4.** In many cases it is possible to replace (3.2) by the following condition

$$\operatorname{vrai\lim}_{t \to +\infty} \frac{W(t)}{\sqrt{\sum_{i=1}^{m} p_i(s)}} = \infty. \quad (3.3)$$

This replacement is interesting only in such cases, that conditions of Theorem 3.1 and Corollary 3.3 are not fulfilled, i.e. there exist an index $i$ so that

$$\operatorname{vraisup}_{t \in [0, +\infty)} p_i(t) = \infty$$

and an index $j$ so that

$$\operatorname{vrai\inf}_{t \in [0, +\infty)} \tau_j(t) = 0.$$

More typical case is the following: functions $p(t) = \sum_{i=1}^{m} p_i(t)$, $h_i(t) = t - \tau_i(t)$ are nondecreasing on $[\nu, +\infty)$. In this case the function $\theta$ can be assigned, for example, by the following formula

$$\theta(t) = t + \frac{\pi}{2 \sum_{i=1}^{m} p_i(t)} + \bar{g}(t), \quad (3.4)$$

where $\bar{g}(t) \geq \max_{1 \leq i \leq m} \tau_i(t)$, $\bar{g}(t) \sum_{i=1}^{m} p_i(t) \geq 1$ (see Lemma 3.6 below).

It is clear that the replacement is possible if

$$\operatorname{vrai\lim}_{t \to +\infty} \frac{\sum_{i=1}^{m} p_i \left( t + \frac{\pi}{2 \sqrt{p(t)}} + \bar{g}(t) \right)}{p(t)} = K < \infty, \text{ where } p(t) = \sum_{i=1}^{m} p_i(t). \quad (3.5)$$

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For example, this condition is fulfilled for polynomial coefficients $p_i, i = 1, \ldots, m$.

**Proof of Theorem 3.2.** Let $x_1$ and $x_2$ be a fundamental system of equation (1.4), (1.5). Let us assume that $x_1$ is bounded. Without loss of generality suppose that $\max_{t \in [0, +\infty]} |x_1(t)| \leq 1$ and

$$W(0) \equiv \begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix} = A > 0.$$  

Let us start with the option of oscillating solution $x_2$, i.e. there exists a sequence $\{t_j\}$ such that $x_2(t_j) = 0$ for $j = 1, 2, 3, \ldots, 0 \leq t_1 < t_2 < \ldots < t_j < t_{j+1} < \ldots$. At these points $W(t_j) = x_1(t_j)x_2'(t_j)$. We assume that $x_2'(t_j) > 0$. It is clear that $x_2'(t_j) \geq W(t_j)$.

Solution $x_2$ satisfies the following equality

$$x''_2(t) + \sum_{i=1}^m p_i(t)x_2(h_i(t))\sigma(h_i(t), t_j) =$$

$$= -\sum_{i=1}^m p_i(t)x_2(h_i(t))[1 - \sigma(h_i(t), t_j)].$$

on the segment $[t_j, t_{j+1}]$. H-condition [6] implies that $h_i(t) \geq t_{j-1}$ for $i = 1, \ldots, m$ and almost all $t \geq t_j$. Now it is evident that $\varphi(t) \equiv -\sum_{i=1}^m p_i(t)x_2(h_i(t))[1 - \sigma(h_i(t), t_j)] \geq 0$ for $i = 1, \ldots, m$ and almost all $t \in [t_j, t_{j+1}]$.

Estimate $x_2$ from below on the segment $[t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}]$.

Set

$$v(t) = \frac{W(t_j)}{\sqrt{R(t_j)}} \sin[\sqrt{R(t_j)}(t - t_j)].$$

(3.6)

It is clear that $v(t_j) = 0$, $v'(t_j) = W(t_j)$ and

$$\psi(t) \equiv v''(t) + \sum_{i=1}^m p_i(t)v(h_i(t))\sigma(h_i(t), t_j) \leq 0$$

for almost all $t \in \left[t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}\right]$. Now, by the known result [6], the Cauchy function $C(t, s)$ is positive in triangle $s, t \in \left(t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}\right), s < t$.  

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Since $\psi(t) \leq 0 \leq \varphi(t)$ for almost all $t \in \left[ t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}} \right]$, then $v(t) \leq x_2(t)$ for $t \in \left[ t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}} \right]$. If $j \to \infty$, then $t_j \to \infty$ (see [6]). We obtain that

$$\text{vrai lim}_{t \to +\infty} \frac{W(t)}{\sqrt{R(t)}} \leq \lim_{t \to +\infty} \max_{s \in [0,t]} |x_2(s)|.$$

Now let us consider another option, i.e. nonoscillating solutions. Let $t_0$ be the last zero of $x_2(t)$ on $[0, +\infty)$. Without loss of generality we assume that $x_2'(t_0) > 0$. Let $\tilde{t}_0$ be the last zero of solution $x_1(t)$ on $[0, +\infty)$. Without loss of generality we suppose that $x_1'(\tilde{t}_0) > 0$. According to the known result [6] Wronskian $W(t)$ does not decrease:

$$W(t) = x_1(t)x'_2(t) - x_1'(t)x_2(t) \geq W(0) > 0.$$

Since $x_2(t)x_1'(t) \geq 0$ for sufficiently large $t$, then $W(t) \leq x_1(t)x'_2(t)$ and $W(0) \leq W(t) \leq x'_2(t)$. It means that $x_2$ is not bounded on $(0, +\infty)$. Theorem 3.2 has been completely proved. □

Nondecreasing of $W(t)$ implies the following:

**Corollary 3.5.** If

$$\text{vrai lim}_{t \to +\infty} \sum_{i=1}^{m} p_i(t) = 0,$$

then there exists unbounded solution of equation (1.4), (1.5).

**Proof of Theorem 3.1.** Let $x_1$ and $x_2$ be a fundamental system of (1.4), (1.5),

$$\max_{t \in [0, +\infty)} |x_1(t)| \leq 1 \text{ and } W(0) > 0.$$

We will prove that $x_2$ is unbounded solution. In the nonoscillating option unboundedness of $x_2$ is obvious from the proof of Theorem 3.2. In the oscillating option let us consider sequence $\{t_j\}$ so that $x_2(t_j) = 0$, $j = 1, 2, 3, \ldots$, $t_1 < t_2 \ldots < t_j < t_{j+1} < \ldots$. At these points $W(t_j) = x_1(t_j)x'_2(t_j)$. Under the assumption that $\max_{t \in [0, +\infty)} |x_1(t)| \leq 1$ one obtains that $W(t_j) \leq x_2'(t_j)$. Let us observe the proof of Theorem 3.2 for $v(t) = W(t_j)(t - t_j)$.  

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It is evident that $v(t_j) = 0$, $v(t_j + \epsilon) = W(t_j)\epsilon$ and

$$
\psi(t) \equiv v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\sigma(h_i(t), t_j) \leq 0
$$

for almost all $t \in [t_j, t_j + \epsilon]$.

Since $v(t) \leq x_2(t)$ for $t \in [t_j, t_j + \epsilon]$, (see the proof of Theorem 3.2), then $x_2(t_j + \epsilon) \geq W(t_j)\epsilon$ and we conclude that $x_2$ is unbounded.

**Lemma 3.6.** Let $p$ and $h$ be nondecreasing functions and

$$
\mu = \nu + \frac{\pi}{2\sum_{i=1}^{m} p_i(\nu)} + \bar{g}(\nu) \text{ for } \nu \in [0, +\infty).
$$

Then $\lambda_{\nu, \mu} \leq 1$.

**Proof.** Let us set

$$
v(t) = \begin{cases} 
\sin \left[ \sqrt{\sum_{i=1}^{m} p_i(\nu)(t - \nu - \bar{g}(\nu))} \right] + \sqrt{\sum_{i=1}^{m} p_i(\nu)\bar{g}(\nu)}, & t \geq \bar{g}(\nu) + \nu, \\
\sqrt{\sum_{i=1}^{m} p_i(\nu)(t - \nu)}, & t < \bar{g}(\nu) + \nu.
\end{cases}
$$

for each fixed $\nu$. It is evident that

$$
v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\sigma(h_i(t), t_j) \geq 0
$$

for almost all $t \in [\nu, \mu]$. It implies the inequality $v(t) \leq (K_{\nu, \mu}v)(t)$ for $t \in [\nu, \mu]$. According to the known result [8, p. 81] the following estimate follows: $\lambda_{\nu, \mu} \leq 1$.

## 4 Estimates of Wronskian

In order to use the results of Part 3 we have to obtain estimates of Wronskian.
Theorem 4.1. Wronskian $W(t)$ of the fundamental system satisfies the following differential inequality

$$W'(t) \geq \sum_{i=1}^{m} p_i(t) C(t, h_i(t)) W(h_i(t)), \quad t \in [0, +\infty),$$

(4.1)

where $W(s) = 0$ for $s < 0$, $C(\cdot, s) = 0$ for $s < 0$.

Proof. Let us introduce the following function of two variables:

$$Q(t, s) = \begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix}.$$

For a fixed $s$ function $q_s(t) \equiv Q(t, s)$, as a function of argument $t$ only, is a solution of equation (1.4), (1.5) and moreover, $q_s(s) = 0$, $q'_s(s) = W(s)$.

It is clear that

$$W'(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = -\begin{vmatrix} x_1(t) & x_2(t) \\ \sum_{i=1}^{m} p_i(t)x_1(h_i(t)) & \sum_{i=1}^{m} p_i(t)x_2(h_i(t)) \end{vmatrix} =$$

$$= -\sum_{i=1}^{m} p_i(t)[x_1(t)x_2(h_i(t)) - x_2(t)x_1(h_i(t))] =$$

$$= -\sum_{i=1}^{m} p_i(t)Q(h_i(t), t) = \sum_{i=1}^{m} p_i(t)Q(t, h_i(t)).$$

In order to continue the proof let us obtain the following:

Lemma 4.2. If $\lambda_{s,t} > 1$ $(0 \leq s < t < \infty)$, then

$$Q(t, s) \geq W(s)C(t, s).$$

Proof. The solution $q_s$ of equation (1.4), (1.5) satisfies the equality

$$(L_s q_s)(\xi) = -\sum_{i=1}^{m} p_i(\xi)q_s(h_i(\xi))[1 - \sigma(h_i(\xi), s)] \text{ for } \xi \in [s, t].$$
Denote zero of \( q_s \) nearest to \( s \) from below by \( \nu \).

It is apparent that

\[
\varphi(\xi) \equiv -\sum_{i=1}^{m} p_i(\xi) q_s(h_i(\xi))[1 - \sigma(h_i(\xi), s)] \geq 0, \quad \xi \in [s, t].
\]

From the condition \( \lambda_{s,t} > 1 \) positivity of the Cauchy function \( C(\xi, \eta) \) for \( \xi, \eta \in (s, t) \), \( \xi > \eta \), follows. Inequality \( \varphi \geq 0 \) implies that \( Q(\xi, s) \geq W(s)C(\xi, s) \) for \( s \leq \xi \leq t \).

Lemma 4.2 has been completely proved.

Continue the proof of Theorem 4.1.

From Lemma 4.2 it follows that

\[
\sum_{i=1}^{m} p_i(t)Q(t, h_i(t)) \geq \sum_{i=1}^{m} p_i(t)W(h_i(t))C(t, h_i(t)).
\]

The equality

\[
W'(t) = \sum_{i=1}^{m} p_i(t)Q(t, h_i(t))
\]

finishes the proof of Theorem 4.1. Theorem 4.1 makes it possible to obtain estimates of Wronskian \( W(t) \). Let us use the following estimate

\[
W(t) \geq W(0)(1 + \int_{0}^{t} \sum_{i=1}^{m} p_i(s)C(s, h_i(s)) \, ds), \quad (4.2)
\]

where \( C(t, h_i(t)) = 0 \) if \( h_i(t) < 0 \), in order to obtain the following result.

**Theorem 4.3.** If there exists a function \( v(\cdot, \cdot) : [\nu, +\infty) \mapsto [0, +\infty) \) so that

1) \( v(\cdot, s) \) for each fixed \( s \) has an absolutely continuous derivative on each segment \([s, b]\);

2) \( v(\cdot, h_i(\cdot)) : [\nu, +\infty) \mapsto [0, +\infty) \) is measurable for \( i = 1, \ldots, m \);

3) 

\[
v(t, h_i(s)) = \begin{cases} 
> 0 & t \in (h_i(s), s), \quad h_i(s) \in [\nu, s), \\
= 0 & t = h_i(s), \\
= 0 & t \in [\nu, +\infty), \quad h_i(s) \notin [\nu, s), 
\end{cases}
\]

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\end{cases}
\]
\[ v'(h_i(s), h_i(s)) = \begin{cases} 
1, & h_i(s) \in [\nu, s), \\
0, & h_i(s) \notin [\nu, s), 
\end{cases} \]

and

\[ \psi(t) = v''(t, h_i(s)) + \sum_{j=1}^{m} p_j(t)v(h_j(t), h_i(s)) \leq 0 \]

for \( i = 1, \ldots, m \) and almost all \( t \in [h_i(s), s] \).

Then

\[ W(t) \geq W(\nu)(1 + \int_{\nu}^{t} \sum_{i=1}^{m} p_i(s)v(s, h_i(s)) \, ds), \quad t \in [\nu, +\infty). \quad (4.2) \]

**Proof.** In order to prove Theorem 4.3 let us show that \( v(s, h_i(s)) \leq C(s, h_i(s)) \) for \( i = 1, \ldots, m \) and almost all \( s \in [\nu, +\infty) \). De la Vallée-Poussin’s theorem [6] implies that \( C(t, \xi) \geq 0 \) for \( t, \xi \in [h_i(s), s], \, t \geq \xi \). Function \( C(\cdot, h_i(s)) \) for almost all fixed \( h_i(s) \in [\nu, +\infty) \) is a solution of equation

\[ (\mathcal{L}_{h_i(s)}x)(t) = 0, \quad t \in [h_i(s), s], \]

and function \( v(\cdot, h_i(s)) \) satisfies equation

\[ (\mathcal{L}_{h_i(s)}x)(t) = \psi(t), \quad t \in [h_i(s), s], \]

where \( \psi \leq 0 \).

Positivity of the Cauchy function \( C(t, \xi) \) implies inequality \( v(s, h_i(s)) \leq C(s, h_i(s)), \quad s \in [\nu, +\infty) \).

5 Proofs

In order to prove Theorem 2.1, let us substitute

\[ v(t, s) = (t - s) \left( s + 2\sqrt{2}/\sqrt{M} - t \right) \]

in conditions of Theorem 4.3. Theorem 2.2 is a corollary of Theorem 2.1.
**Proof of Theorem 2.4.** Let us substitute \( v(t, s) = t - s \) in conditions of Theorem 4.3. Condition (2.4) implies that \( \lim_{t \to +\infty} W(t) = +\infty \). Theorem 3.1 implies sufficiency of condition a) and Theorem 3.2 implies sufficiency of condition b) for existence of unbounded solution.

**Corollary 5.1.** If there exists index \( i \in \{1, \ldots, m\} \) so that
\[
\int_{-\infty}^{\infty} \frac{p_i(t)}{R(t - \tau_i(t))} \sin \left( \sqrt{R(t - \tau_i(t))} \tau_i(t) \right) dt = \infty,
\]
then \( \lim_{t \to +\infty} W(t) = +\infty \).

If also
\[
\varlimsup_{t \to +\infty} \frac{\int_{-\infty}^{\infty} \frac{p_i(s)}{R(s - \tau_i(s))} \sin \left( \sqrt{R(s - \tau_i(s))} \tau_i(s) \right) ds}{\sqrt{R(t)}} = \infty,
\]
then there exists unbounded solution of equation (1.4), (1.5).

In order to prove Corollary 5.1, let us set
\[
v(t, s) = \frac{1}{\sqrt{R(s)}} \sin[\sqrt{R(s)}(t - s)].
\]
Note that existence of unbounded solution of equation (1.10) follows from Corollary 5.1.

If conditions a) and b) of Theorem 2.4 are not fulfilled, the following assertion is proposed.

**Corollary 5.2.** If there exists index \( i \in \{1, \ldots, m\} \) so that at least one of the following conditions is fulfilled
\[
\varlimsup_{t \to +\infty} \tau(t) \int_{-\infty}^{t} p_i(s) \tau(s) ds = \infty,
\]
(5.1)
or
\[
\varlimsup_{t \to +\infty} \frac{\int_{-\infty}^{t} p_i(s) \tau(s) ds}{\sqrt{R(t)}} = \infty,
\]
(5.2)
then there exists unbounded solution of equation (1.4), (1.5).

In order to prove Corollary 5.2 let us set \( v(t,s) = t - s \). Sufficiency of condition (5.1) results from Proof of Theorem 3.1. Sufficiency of condition (5.2) results from Theorem 3.2.

Existence of unbounded solution of equation (1.11) follows from Corollary 5.2.

**Proof of Theorem 1.1.** Whereas we have proved that necessity follows from Theorem 2.4, sufficiency was proved by Izumova D.V. in the paper [9].

**Remark 5.3.** Nondecreasing of the coefficient \( p(t) \) in Theorem 1.1 is essential, for example: equation

\[
x''(t) + \frac{1}{t^2} x \left( t - \frac{1}{t^2} \right) = 0, \quad t \in [1, +\infty),
\]

has an unbounded solution by Corollary 3.5, but

\[
\int_{\tau(t)}^{\infty} t \, dt = \int_{\frac{1}{t^2}}^{\infty} \tau(t) \, dt < \infty.
\]

**Remark 5.4.** In all assertions of this paper it is said about existence of unbounded solution. Can we say about unboundedness of all solutions? No, what is given in the following example: function \( x = \sin t \) is one of solutions of equation

\[
x''(t) + x(t - \tau(t)) = 0, \quad t \in [0, +\infty),
\]

where

\[
\tau(t) = \begin{cases} 
0, & 0 \leq t \leq \frac{\pi}{2}, \\
2t - \pi, & \frac{\pi}{2} < t < \pi,
\end{cases}
\]

\( \tau(t + \pi) = \tau(t). \)

Other solutions are unbounded by Theorem 1.1.
References


