Question 1

In East Prussia, each of the two million citizens owns a "prussak". It is also known that exactly half of the citizens own 12 "prussaks" each, and the other half own 24 "prussaks" each. Two "prussaks" are called "comrades" if they are owned by the same master (every "prussak" is also his own "comrade"). Find the difference between the average number of "comrades" each "prussak" has and the average number of "prussaks" each citizen of Prussia owns.

Answer: 2

Solution:

The average number of "prussaks" that each citizen of the country owns equals $\frac{1000000 \cdot 12 + 1000000 \cdot 24}{2000000} = \frac{12+24}{2} = 18$.

The total number of "prussacks" in east Prussia is $1000000 \cdot 12 + 1000000 \cdot 24 = 36000000$. Among them, 12000000 have 12 "comrades", and 24000000 have 24 "comrades". Therefore, the total number of "comrades" is $12000000 \cdot 12 + 24000000 \cdot 24 = 720000000$. The average number of "comrades" each "prussack" has equals $\frac{720000000}{36000000} = 20$. Therefore, the number that we need to find is 20-18=2.

Comment: In the general case (when each citizen owns a random number of "prussacks") the average number of "prussacks" that each citizen owns does not exceed the average number of a "prussack's" "comrades". Equality is attained only if the "prussacks" are distributed equally. A similar problem: the percentage of people who ride in crowded buses is higher than the percentage of crowded buses (and nobody rides in empty buses). This explains "Murphy's Law': when we need a bus, on average we have to wait for a long time, and when we don't, empty buses pass by us.

Question 2

A pedestrian ran one third of his way at the speed of $v_1 = 12{,}5$ km/h, walked one third of the time at the speed $v_2 = 4{,}5$ km/h, and walked the rest of the way at a speed equal to the average speed of the entire journey. What was the average speed of the pedestrian?

Answer: 7.5 km/h

Solution: Let the total length of the pedestrian's path equal *S*, his mean velocity equal v_3 km/h, the time it took him to cover the entire distance equal *t* h, the time it took him to cover the the first third of the way equal t_1, and the time it took him to cover the the last third of the way equal t_3.

It follows from the equality $S = v_1 t_1 + v_2 \cdot \frac{t}{3} + v_3 t_3 = v_3 (t_1 + \frac{t}{3} + t_3)$ that

$$v_1 t_1 + v_2 \cdot \frac{t}{3} = v_3 t_1 + v_3 \frac{t}{3}. \qquad (1)$$

In addition,

$$v_1 t_1 = v_3 \frac{t}{3}. \qquad (2)$$

It follows from (1) and (2) that

$$v_3 t_1 = v_2 \cdot \frac{t}{3}. \qquad (3)$$

Diving (2) by (3), we obtain

$v_3{}^2 = v_1 \cdot \mathrm{v}_2 = 12{,}5 \cdot 4{,}5 = 56{,}25 (v_{cp} > 0)$. Therefore, $v_{cp} = 7{,}5$ km/h.

Question 3

Let $f_1(x) = x - x^2$, $f_2(x) = x + x^2$, $g_1(x) = x - x^3$, $g_2(x) = x + x^3$.

Find $\lim\limits_{x \to 0} \dfrac{x - f_1(g_1(f_2(g_2(x))))}{x^3}$.

Answer: 2

Solution: Let us consider the expansion of functions with $x \to 0$ up to third-order terms.

We obtain:

$$f_2(g_2(x)) = \left(x + x^3\right) + \left(x + x^3\right)^2 = x + x^3 + x^2 + o\left(x^3\right),$$

$$g_1(f_2(g_2(x))) = (f_2(g_2(x))) - (f_2(g_2(x)))^3 =$$

$$= \left(x + x^3 + x^2 + o\left(x^3\right)\right) - \left(x + x^3 + x^2 + o\left(x^3\right)\right)^3 = x + x^2 + o\left(x^3\right),$$

$$f_1(g_1(f_2(g_2(x)))) = (g_1(f_2(g_2(x)))) - (g_1(f_2(g_2(x))))^2 =$$

$$= \left(x + x^2 + o\left(x^3\right)\right) - \left(x + x^2 + o\left(x^3\right)\right)^2 = x - 2x^3 + o\left(x^3\right).$$

Hence,

$$\lim_{x\to 0}\frac{x-f_1(g_1(f_2(g_2(x))))}{x^3}=\lim_{x\to 0}\frac{x-\left(x-2x^3+o\left(x^3\right)\right)}{x^3}=\lim_{x\to 0}\frac{2x^3+o\left(x^3\right)}{x^3}=2.$$

Comment: If we took the composition $f_1g_1f_1^{-1}g_1^{-1}$ it would differ from x by members of the fourth order, while f_1^{-1} and g_1^{-1} differ from f_2 and g_2, respectively, by members of a higher order.

Question 4

All participants of a certain contest receive cards at the end of the contest. The number of participants is n. The contestant who takes first place is given one card and one tenth of all remaining cards, the contestant who takes second place is given two cards and one tenth of all remaining cards, etc., and the contestant who finishes in the last, n-th, place is given n cards and one tenth of the remaining cards. If after all of the contestants have received their cards no extra cards remain, how many contestants were there?

Answer: 9 contestants (and 81 cards)

Solution: Let us consider the last participant. If all the cards have been divided among the contestants, then he got exactly *n* cards (since 0 cards were left). The previous, (*n*-1)-th participant, got (*n*-1) cards plus 1/10 of the remaining cards, after which *n* cards were left. Hence, 9/10 of the remaining cards equal *n*, i.e. the (*n*-1)-th participant got $n-1+n/9$ cards. Therefore, *n* is divisible by 9 and $n\ge 9$.

Thus, *n* cards remain for the *n*-th participant,

$n-1+\frac{10}{9}n$ cards for the (*n*-1)-th,

$n-2+\frac{10}{9}\left(n-1+\frac{10}{9}n\right)=(n-2)+\frac{10}{9}(n-1)+\left(\frac{10}{9}\right)^2 n$ for the (*n*-2)-th,

$$n-3+\frac{10}{9}\left((n-2)+\frac{10}{9}(n-1)+\left(\frac{10}{9}\right)^2 n\right)=$$

$$=(n-3)+\frac{10}{9}(n-2)+\left(\frac{10}{9}\right)^2(n-1)+\left(\frac{10}{9}\right)^3 n$$

for the (*n*-3)-th, etc.

Therefore, the number of cards that remain before the (*n*-*k*)-th participant receives his cards equals

$$(n-k)+\frac{10}{9}(n-k+1)+\ldots+\left(\frac{10}{9}\right)^{k-1}(n-1)+\left(\frac{10}{9}\right)^{k}n=\sum_{i=0}^{k}(n-k+i)\left(\frac{10}{9}\right)^{i}$$

And in the particular case when $k=n-1$ (the initial situation) we have

$$\sum_{i=0}^{n-1}(n-(n-1)+i)\left(\frac{10}{9}\right)^{i}=\sum_{i=0}^{n-1}(i+1)\left(\frac{10}{9}\right)^{i}.$$

However,

$$\sum_{i=0}^{n-1}(i+1)x^{i}=\left(\sum_{i=0}^{n-1}x^{i+1}\right)'=\left(\frac{x(x^{n}-1)}{x-1}\right)'=\frac{nx^{n+1}-(n+1)x^{n}+1}{(x-1)^{2}}=$$

$$=\frac{x^{n}}{(x-1)^{2}}\cdot(nx-n-1)+\frac{1}{(x-1)^{2}}.$$

Therefore,

$$\sum_{i=0}^{n-1}(i+1)\left(\frac{10}{9}\right)^{i}=\frac{\left(\frac{10}{9}\right)^{n}}{\left(\frac{10}{9}-1\right)^{2}}\cdot\left(n\cdot\frac{10}{9}-n-1\right)+\frac{1}{\left(\frac{10}{9}-1\right)^{2}}=$$

$$=81\cdot\left(\frac{10}{9}\right)^{n}\cdot\left(\frac{n}{9}-1\right)+81.$$

Since $n\geq 9$, the obtained expression can only be an integer if $\frac{n}{9}-1=0$. Consequently, $n=9$, and the total number of cards that the participants received equals 81.

Question 5

Find the integral $\int_{0}^{2014}\frac{f(x)}{f(x)+f(2014-x)}dx$.

Answer: 1007

Solution:

$$\int_0^{2014}\frac{f(x)}{f(x)+f(2014-x)}dx=\left|\begin{array}{l}t=2014-x,\ dx=-dt,\\ x\to 2014\Rightarrow t\to 0,\\ x\to 0\Rightarrow t\to 2014\end{array}\right|=-\int_{2014}^{0}\frac{f(2014-t)}{f(t)+f(2014-t)}dt=$$

$$=\int_0^{2014}\frac{f(2014-x)}{f(x)+f(2014-x)}dx.$$

Then $2\cdot\int_0^{2014}\frac{f(x)}{f(x)+f(2014-x)}dx=\int_0^{2014}\frac{f(x)}{f(x)+f(2014-x)}dx+$

$+\int_0^{2014}\frac{f(2014-x)}{f(x)+f(2014-x)}dx=\int_0^{2014}\frac{f(x)+f(2014-x)}{f(x)+f(2014-x)}dx=\int_0^{2014}dx=2014.$

And therefore, $\int_0^{2014}\frac{f(x)}{f(x)+f(2014-x)}dx=1007.$

Question 6

A flea is jumping inside a unit square. Initially it can be anywhere in this square. Every second it chooses a vertex, and gets four times closer to it by jumping towards it. Find the area of the set of points where the flea can be after the fifth jump. (For example, the area of the set of points where the flea can be after the first jump is equal to 1).

Answer: 1/1024.

Solution: Let M_n be the set of all possible positions of the flea after its *n*-th jump. M_o is the initial unit square *K*. Since its area equals 1, it suffices to show that the area M_{n+1} is 4 times smaller than the area of M_n.

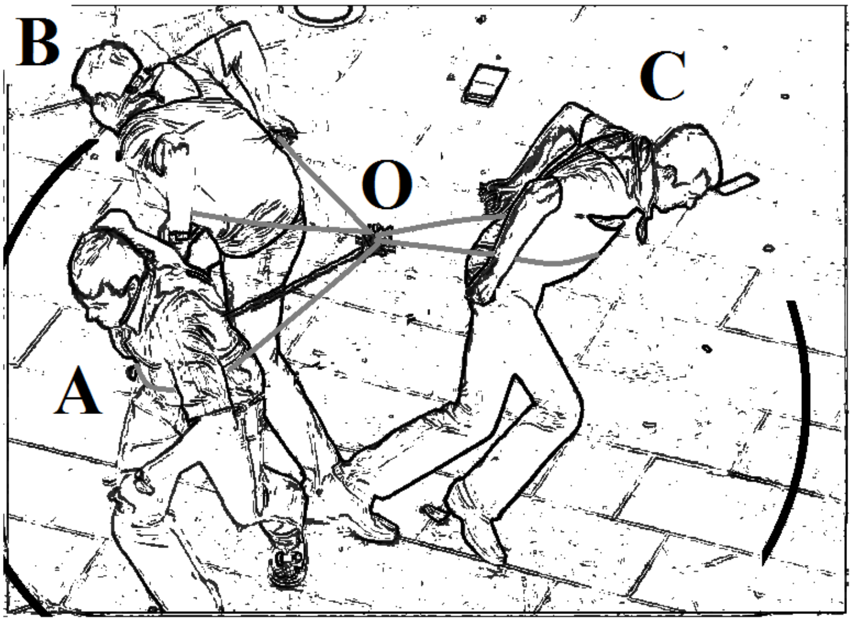
The set M_{n+1} is the union of four homothetic images of M_n with a factor of 1/4, while the centers of the homotheties are the vertices of square *K*. The area of each such set is 1/16 of the area of M_n, and these sets are disjoint. M_n is contained in K, and the images of K under the homothety with center at its vertices do not intersect if the factor of dilation is a positive number less than ½.

Thus, the area of the union of these four sets is equal to the sum of their areas or quadruple the area of any one of them. In other words, it equals the area of M_n divided by 4.

Comment: In the limit the set that arises is a fractal.

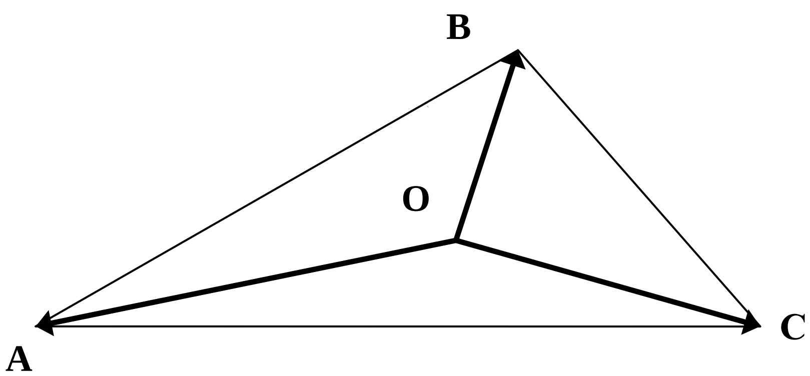
Question 7

Three people are taking part in a tug of war competition (see figure).



The goal of each player is to cross the line drawn in front of him.

Player C is planning to pull his rope with force $\overrightarrow{F_C}$, which is equal in magnitude to the product of the distance from the knot O, which ties the three players' ropes together, to player C, multiplied by the area of the triangle formed by the vectors leading from the knot to his rivals. ($\overrightarrow{F_C} = S_C \cdot \overrightarrow{OC}$, where S_C - is the area of triangle AOB, see figure).



Prove that if the other two players follow the same strategy, none will be able to move ($\overrightarrow{F_C}+\overrightarrow{F_A}+\overrightarrow{F_B}=\vec{0}$).

Solution: According to the question

$\overrightarrow{F_A}=S_A\cdot\overrightarrow{OA}, \overrightarrow{F_B}=S_B\cdot\overrightarrow{OB}, \overrightarrow{F_C}=S_C\cdot\overrightarrow{OC}$, where S_A, S_B, S_C are the areas of the triangles BCO, CAO, ABO, respectively.

Thus, we must prove that the following inequality holds:

$S_A\cdot\overrightarrow{OA}+S_B\cdot\overrightarrow{OB}+S_C\cdot\overrightarrow{OC}=\vec{0}$.

Solution 1. Consider the vector product $\left[\overline{OB},\overline{OC}\right]$. Its length equals $2S_A$, and its direction is perpendicular to the plane of the figure. Hence, $\left[\overline{OA},\left[\overline{OB},\overline{OC}\right]\right]$is a vector whose length equals $2S_A\cdot\left|\overline{OA}\right|$, and whose direction is obtained from the direction of the vector $\overline{OA}$ by rotating it clockwise 90^{o}. When we consider the other pairs of vectors we see that the vector

$\left[\overline{OA},\left[\overline{OB},\overline{OC}\right]\right]+\left[\overline{OB},\left[\overline{OC},\overline{OA}\right]\right]+\left[\overline{OC},\left[\overline{OA},\overline{OB}\right]\right]$ is obtained from the vector $S_A\cdot\overrightarrow{OA}+S_B\cdot\overrightarrow{OB}+S_C\cdot\overrightarrow{OC}$ by doubling it and rotating 90^{o} clockwise.

However, $\left[\bar{a},\left[\bar{b},\bar{c}\right]\right]=\bar{b}\left(\bar{a},\bar{c}\right)-\bar{c}\left(\bar{a},\bar{b}\right)$.

Therefore,

$$\left[\overline{OA},\left[\overline{OB},\overline{OC}\right]\right]+\left[\overline{OB},\left[\overline{OC},\overline{OA}\right]\right]+\left[\overline{OC},\left[\overline{OA},\overline{OB}\right]\right]=\overline{OB}\left(\overline{OA},\overline{OC}\right)-\overline{OC}\left(\overline{OA},\overline{OB}\right)+$$
$$+\overline{OC}\left(\overline{OB},\overline{OA}\right)-\overline{OA}\left(\overline{OB},\overline{OC}\right)+\overline{OA}\left(\overline{OC},\overline{OB}\right)-\overline{OB}\left(\overline{OC},\overline{OA}\right)=\bar{0}.$$

And consequently, $S_A\cdot\overrightarrow{OA}+S_B\cdot\overrightarrow{OB}+S_C\cdot\overrightarrow{OC}=\bar{0}$.

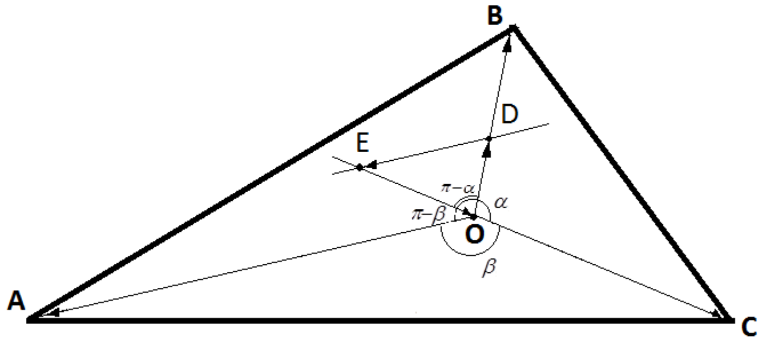
Solution 2.

Let $\vec{e_1}=\frac{1}{\left|\overrightarrow{OA}\right|}\cdot\overrightarrow{OA}, \vec{e_2}=\frac{1}{\left|\overrightarrow{OB}\right|}\cdot\overrightarrow{OB}, \vec{e_3}=\frac{1}{\left|\overrightarrow{OC}\right|}\cdot\overrightarrow{OC}$ be unit vectors of $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ and $\angle BOC=\alpha, \angle COA=\beta, \angle AOB=\gamma$.

Then,

$$S_A\cdot\overrightarrow{OA}+S_B\cdot\overrightarrow{OB}+S_C\cdot\overrightarrow{OC}=S_A\cdot\overrightarrow{OA}+S_B\cdot\overrightarrow{OB}+S_C\cdot\overrightarrow{OC}=$$
$$=\frac{1}{2}\cdot\left|\overrightarrow{OB}\right|\cdot\left|\overrightarrow{OC}\right|\cdot\sin\alpha\cdot\left|\overrightarrow{OA}\right|\cdot\vec{e_1}+\frac{1}{2}\cdot\left|\overrightarrow{OC}\right|\cdot\left|\overrightarrow{OA}\right|\cdot\sin\beta\cdot\left|\overrightarrow{OB}\right|\cdot\vec{e_2}+$$
$$+\frac{1}{2}\cdot\left|\overrightarrow{OA}\right|\cdot\left|\overrightarrow{OB}\right|\cdot\sin\gamma\cdot\left|\overrightarrow{OC}\right|\cdot\vec{e_3}=\frac{1}{2}\cdot\left|\overrightarrow{OA}\right|\cdot\left|\overrightarrow{OB}\right|\cdot\left|\overrightarrow{OC}\right|\cdot\left(\vec{e_1}\sin\alpha+\vec{e_2}\sin\beta+\vec{e_3}\sin\gamma\right).$$

From point O, let us draw vector $\overline{OD} = \overrightarrow{e_2} \sin\beta$, and from point D – vector $\overrightarrow{DE}$ (see figure below), collinear with the vector $\overline{OA}$ ($\overrightarrow{DE} = m \cdot \overrightarrow{e_1}, m > 0$), so that the point E would lie on the line OC ($\overrightarrow{EO} = n \cdot \overrightarrow{e_3}, n > 0$).



Then, $\overrightarrow{OD} + \overrightarrow{DE} + \overrightarrow{EO} = \vec{0}$. Let us find the numbers m and n.

Since $ED \| OA$, $\angle AOE = \angle OED = \pi - \beta$, $\angle ODE = \pi - \gamma$. In addition, $\angle EOD = \pi - \alpha$. According to the law of sines,

$$\frac{\left|\overrightarrow{OD}\right|}{\sin\angle OED} = \frac{\left|\overrightarrow{DE}\right|}{\sin\angle EOD} = \frac{\left|\overrightarrow{EO}\right|}{\sin\angle ODE} \Rightarrow \frac{\left|\overrightarrow{e_2}\sin\beta\right|}{\sin(\pi-\beta)} = \frac{\left|m\cdot\overrightarrow{e_1}\right|}{\sin(\pi-\alpha)} = \frac{\left|n\cdot\overrightarrow{e_3}\right|}{\sin(\pi-\gamma)}.$$

However, $\left|\overrightarrow{e_1}\right| = \left|\overrightarrow{e_2}\right| = \left|\overrightarrow{e_3}\right| = 1$ and $m, n > 0$.

We find $1 = \frac{m}{\sin\alpha} = \frac{n}{\sin\gamma} \Rightarrow m = \sin\alpha, n = \sin\gamma$.

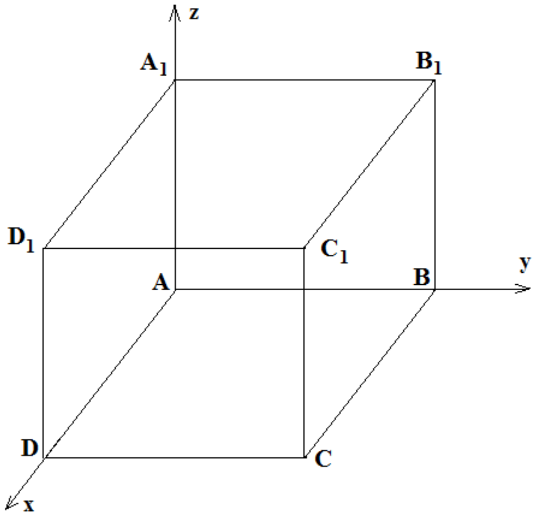
Consequently, $\overrightarrow{e_1}\sin\alpha + \overrightarrow{e_2}\sin\beta + \overrightarrow{e_3}\sin\gamma = \vec{0}$ and $S_A \cdot \overrightarrow{OA} + S_B \cdot \overrightarrow{OB} + S_C \cdot \overrightarrow{OC} = \vec{0}$.

Question 8

How many different non-singular matrices of order 3 exist, the elements of which are "0" or "1"?

Answer: 174.

Solution: Let us denote the first, second and third row of the original matrix A by vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively. Then $|A| = \vec{a}\vec{b}\vec{c}$ is a mixed product of these vectors. Each vector-row of the matrix A is then assigned a radius-vector in the basis $\left(\vec{i}; \vec{j}; \vec{k}\right)$, with the end at the vetex of the cube $ABCDA_1B_1C_1D_1$ (see figure below).



The total number of such non-zero radius-vectors is 7 (the cube has 8 vertices minus the zero vector $\overline{AA}$). The number of distinct ordered sets of three vectors $\left(\vec{a};\vec{b};\vec{c}\right)$ (the number of different matrices) is equal to $A_7^3 = 210$. Let us subtract from this number the number of coplanar sets of three vectors (for which $\vec{a}\vec{b}\vec{c} = 0$) lying in planes passing through the origin and the three vertices of the cube (these are three faces of the cube AA_1DD_1, ABB_1A_1, $ABCD$ and the planes AA_1CC_1, ABC_1D_1, ADB_1C_1). Each set of three coplanar vectors allows $3!$ permutations. Consequently, the total number of ordered sets of three coplanar vectors equals $6 \cdot 3! = 36$.

The number of non-degenerate matrices equals $210 - 36 = 174$.

Question 9

Find the integer part of the sum of 4028 terms

$$\sqrt{2014^2+1} + \sqrt{2014^2+2} + \ldots \sqrt{2014^2 + 2\cdot 2014} = \sum_{k=1}^{2\cdot 2014} \sqrt{2014^2 + k}.$$

Answer: 8114406.

Solution: Let us solve the problem in general terms. First, we will find the integer part of the sum of $2n$ terms

$$\sqrt{n^2+1} + \sqrt{n^2+2} + \ldots \sqrt{n^2+2n} = \sum_{k=1}^{2n} \sqrt{n^2+k},$$ where n is a positive integer.

In order to do this, we will transform the sum of the roots:

$$S = \sum_{k=1}^{2n} \sqrt{n^2+k} = \sum_{k=1}^{2n} \left(n + \left(\sqrt{n^2+k} - n \right) \right) = 2n^2 + \sum_{k=1}^{2n} \frac{k}{\sqrt{n^2+k}+n}.$$

Let us denote $\Delta = \sum_{k=1}^{2n} \frac{k}{\sqrt{n^2+k}+n}$ and estimate this value.

On the one hand, $\frac{k}{\sqrt{n^2+k}+n} < \frac{k}{2n}$. Hence,

$$\Delta < \sum_{k=1}^{2n} \frac{k}{2n} = \frac{1}{2n} \cdot (1+2+\ldots+2n) = \frac{1}{2n} \cdot \frac{1+2n}{2} \cdot 2n = n + \frac{1}{2}.$$

On the other hand, $\frac{k}{\sqrt{n^2+k}+n} > \frac{k}{\sqrt{n^2+2n+1}+n} = \frac{k}{2n+1}$ and

$$\Delta > \sum_{k=1}^{2n} \frac{k}{2n+1} = \frac{1}{2n+1} \cdot (1+2+\ldots+2n) = \frac{1}{2n+1} \cdot \frac{1+2n}{2} \cdot 2n = n.$$

This means that $n < \Delta < n + \frac{1}{2}$. Therefore, the integer part of the number Δ equals n.

Consequently, the integer part of S equals $2n^2 + n$. For $n = 2014$ we obtain $2n^2 + n = 2 \cdot 2014^2 + 2014 = 8114406$.

Question 10

Let $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n-1}, a_{2n}$ all be different positive divisors of the number $N = 2^{2014} + 1$, in ascending order. Find $(a_{n+1} - a_n)$.

Answer: 2^{505}.

Solution: Let x and y be integer divisors of the number N, and $xy = N$. Let $x = 2^{1007} - a$ (a is an integer and $a > 0$), then $y = \frac{2^{2014}+1}{2^{1007}-a} = 2^{1007} + a + \frac{a^2+1}{2^{1007}-a}$. Obviously, x and y are adjacent divisors of N if their difference $y - x = 2a + \frac{a^2+1}{2^{1007}-a}$ takes the smallest possible integer value. Note that $2a + \frac{a^2+1}{2^{1007}-a}$ is an increasing function with $a \in [1; 2^{1007} - 1]$. It can be easily shown that $\frac{a^2+1}{2^{1007}-a} \neq 1$ for any integer a (if a is an even number, then a^2+1 is odd, and $2^{1007}-a$ is even, and if a is an odd number, then a^2+1 is even, and $2^{1007}-a$ is odd).

If $\frac{a^2+1}{2^{1007}-a}=2$, then $a=2^{504}-1$ and

$a_{n+1}-a_n=(2^{1007}+2^{504}+1)-(2^{1007}-2^{504}+1)=2^{505}$.

Question 11

Victor and Michael are playing a game. Victor chooses a number from 1 to *n*. Michael has to deduce which number Victor chose by asking questions, such as: “Is the number even?”, “Is the number greater than 5?” etc. If the answer to the question is yes, Michael has to pay Victor 1$, and if the answer is no – 10$. If Michael has 32$, what is the maximal n for which he may be able to deduce the number?

Answer: 125.

Solution: Let $T(k)$ be the maximal *n* for which it is possible for Michael to guess which number from 1 to *n* Victor has chosen if Michael has *k* dollars. Then, $T(k)=1$ for $k=1,\dots,9$, $T(10)=2$. For $k=11,\dots,19$ the number $T(k)$ will be 3,4,5,6,7,8,9,10,11, respectively (for the above *k*, the following algorithm can be used: 1-st question: Is it true that the chosen number is smaller than *n*? If the answer is no, then the chosen number is *n*, and if the answer is yes then the 2nd question will be: Is it true that the chosen number is smaller than $n-1$? If the answer is no, then the chosen number is $n-1$, and if the answer is yes, then we move on to the next question, and so on).

For *k*>19 the following recurrent formula is true

$T(k+1)=T(k)+T(k-9)$.

After we ask the first question, the set of possibilities is divided into two subsets M_1 and M_2 - one for which the answer is “yes”, and another for which the answer is ‘no”. In the first case, Michael will have *k* dollars left, and therefore, $|M_1|=T(k)$. In the second case, Michael will have $k-9$ dollars left, and therefore, $|M_2|=T(k-9)$. The total number of possibilities equals

$n=|M_1|+|M_2|=T(k)+T(k-9)$.

Therefore, $T(k+1)=T(k)+T(k-9)$.

On the other hand, let us divide the set of numbers from 1 to $T(k)+T(k-9)$ into two subsets – from 1 to $T(k)$, and from $T(k+1)$ to $T(k)+T(k-9)$, and ask the question: is the chosen number smaller than $T(k+1)$? If the answer is “yes”, we have to guess a number from 1 to $T(k)$ with k dollars, which is possible, and if the answer is "no", then we have to guess a number from $T(k-9)$ with $k-9=k+1-10$ dollars, which is also possible.

According to this formula,

$T(20)=T(19)+T(10)=11+2=13,$

$T(21)=T(20)+T(11)=13+3=16,$

$T(22)=T(21)+T(12)=16+4=20,.$

For $k=13,\ldots,19$ (a total of 7 numbers) $T(k)$ will equal 5,6,7,8,9,10,11, respectively.

Since $T(k+1)=T(k)+T(k-9)$,

$$T(32)=T(22)+\left(T(13)+T(14)+\ldots+T(19)+T(20)+T(21)+T(22)\right)=$$

$$=20+(5+6+\ldots+11)+13+16+20=20+\frac{5+11}{2}\cdot 7+49=125\ .$$

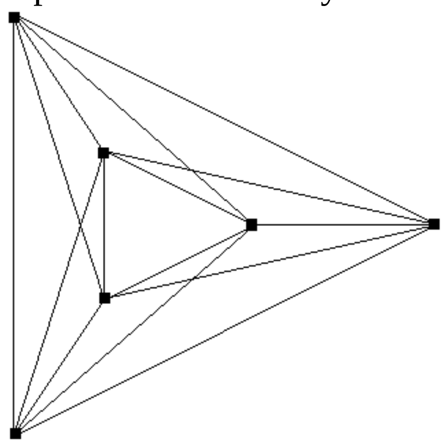
Comment: In problems regarding value of information, linear recurrence arises, while in our case $T(k+1)=T(k)+T(k-9)$.

Question 12

What is the maximal number N of points which can be positioned on a plane in such a way that every two points will be connected by segments, some of which are red and some –blue, and segments of the same color will not intersect with each other, while segments of different colors will intersect with each other no more than once?

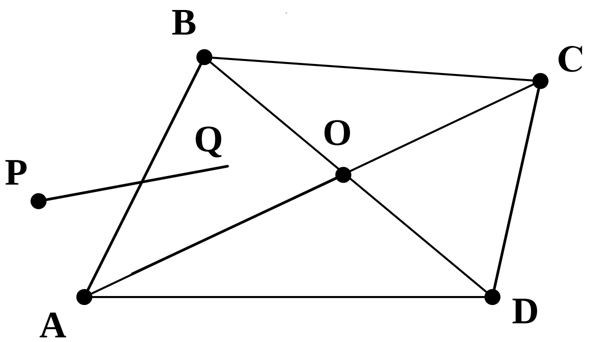
Answer: $N=6$.

Solution: For $N=6$ this is possible (see figure below). For smaller *N*, a similar picture can be easily obtained by removing points.



Consider a complete graph *G* with 7 vertices which satisfies the conditions of the question. Let us show that it cannot be placed on a plane in such a way that each edge would have no more than one point of intersection with the other edges. (If we prove this, then we will also have proven that a graph with a larger number of vertices cannot be placed on a plane in such a way either). The graph *G* has $\frac{n(n-1)}{2}=21$ edges, and *X* intersections between the edges. We can assume that the number *X* is the minimal possible number.

Let *O* be the point of intersection of edges AC and BD (see figure). Consider the quadrilateral ABCD. Its opposite sides do not intersect. Moreover, none of its sides intersect with anything else. For instance, if the edge PQ intersected AB, the path from Q to the outer region of the quadrilateral ABCD would be impossible, since all the arcs that form the triangle AOB have already been intersected.



Thus, each pair of intersecting edges of *G* corresponds to four that do not intersect pairwise. Each of the four edges is counted no more than twice. Consequently, the number of edges without intersection points is no less than $2X$. On the other hand, the number of edges with intersection points equals $2X$. Therefore, $4X<21$, meaning that $X\le 5$.

Now let us consider another graph *GG*, whose vertices are the vertices of graph *G* and the points of intersection of the edges of graph *G*, and whose edges are "halves" of intersecting edges, as well as the original edges of graph *G* which have no points of intersection. This is a planar graph (in contrast to the original graph G) and it satisfies the Euler formula. The graph G has $V=7+X$ vertices, $E=21+2X$ edges, and $F\le\frac{2}{3}E$ faces. According to Euler's formula,

$$2=V-E+F\le 7+X-(21+2X)+\frac{2}{3}(21+2X)=\frac{1}{3}X\ .$$

Therefore, $6\le X$, which contradicts the previously proven inequality $X\le 5$.

Comment: It would be interesting to generalize this problem for an arbitrary limit of the number of intersections of the edges of a graph. We invite you to send us your ideas on this subject.