Computing All Large Sums-of-Pairs in $\mathbb{R}^n$ and the Discrete Planar Two-Watchtower Problem

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Abstract

We observe that Matoušek’s algorithm for computing all dominances for a set $P$ of $n$ points in $\mathbb{R}^n$ can be employed for computing all pairs of points in such a set whose sum is greater or equal to a given point $a \in \mathbb{R}^n$. We apply this observation to the decision problem of the discrete planar two-watchtower problem and obtain an improved solution.

Keywords: Algorithms, computational geometry, facility location.

1 Introduction

Let $X$ be an $n \times n$ matrix of real numbers, and denote the $i$-th row of $X$ by $x_{i,*}$. In [3] Matoušek presents an algorithm for computing an $n \times n$ matrix $C$, such that $c_{i,j} = n$ if and only if the $i$’th row of $X$ is dominated by the $j$’th row of $X$, that is, $x_{i,k} \leq x_{j,k}$, for $k = 1, \ldots, n$. ($c_{i,j}$ is actually the number of coordinates $k$ for which $x_{i,k} \leq x_{j,k}$.) The running time of Matoušek’s algorithm is $O(n^3 M(n)^{1/2})$, where $M(n)$ is the time required for multiplying two $n \times n$ matrices. Currently $M(n) = O(n^{2.376})$ (see [2]), and therefore the running time of his algorithm is $O(n^{2.688})$.

In this short paper we observe that Matoušek’s algorithm can be employed to compute all pairs of rows in $X$, whose sum is greater or equal to a given $n$ vector $a = (a_1, \ldots, a_n)$, that is, all pairs of rows $x_{i,*}, x_{j,*}$, such that $x_{i,k} + x_{j,k} \geq a_k$, for $k = 1, \ldots, n$. The computation time remains $O(n^3 M(n)^{1/2})$ (which is currently $O(n^{2.688})$).

Next, we apply this observation to the discrete planar two-watchtower problem, to obtain an improved solution to the corresponding decision problem. The input to the discrete version of the planar two-watchtower problem

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is an $x$-monotone polygonal line $T$, i.e., a 2-dimensional (polygonal) terrain. The goal is to place two watchtowers at two of the vertices of $T$, such that (i) each point of the terrain is seen from at least one of the watchtowers, and (ii) the height of the higher watchtower is as small as possible; see Figure 1.

The corresponding decision problem is thus, given a height $h$, determine whether the entire terrain can be viewed from two watchtowers of height $h$ that are located at two of $T$’s vertices. Bespamyatnikh et al. [1] present an $O(n^3)$ solution to this decision problem, where $n$ is the number of vertices in $T$. We present an improved solution to this problem that is based on the algorithm for the large sums problem above. The running time of our solution is $O(n^2M(n^{3/2}))$ (which is currently $O(n^{2.688})$).

2 Computing all large sums-of-pairs in $\mathbb{R}^n$

Let $X$ be an $n \times n$ matrix of real numbers, and let $a = (a_1, \ldots, a_n)$ be an $n$ vector. We shall compute a matrix $C$, such that $c_{i,j} = n$ if and only if the sum of the $i$'th and $j$'th rows of $X$ is at least $a$, that is, $x_{i,k} + x_{j,k} \geq a_k$, for $k = 1, \ldots, n$. ($c_{i,j}$ is actually the number of coordinates $k$ for which $x_{i,k} + x_{j,k} \geq a_k$.)

Let $Y$ be the matrix obtained from $X$ and $a$ as follows. The $i$'th row of $Y$ is $a - x_{i,*}$. Notice that $x_{i,*} + x_{j,*}$ is greater or equal to $a$ if and only if $y_{j,*}$ is dominated by $x_{i,*}$ and $y_{i,*}$ is dominated by $x_{j,*}$.

We form four $n \times n$ matrices as follows:

1. $M^1$ contains the first $n/2$ rows of $X$ and the first $n/2$ rows of $Y$.
2. $M^2$ contains the first $n/2$ rows of $X$ and the last $n/2$ rows of $Y$.
3. \( M^3 \) contains the last \( n/2 \) rows of \( X \) and the first \( n/2 \) rows of \( Y \).

4. \( M^4 \) contains the last \( n/2 \) rows of \( X \) and the last \( n/2 \) rows of \( Y \).

We now apply Matoušek’s algorithm to each of the four matrices \( M^1, \ldots, M^4 \), obtaining matrices \( C^1, \ldots, C^4 \). The matrix \( C \) can now be easily computed from the matrices \( C^1, \ldots, C^4 \) as follows. Consider, e.g., the matrix \( C^3 \) (obtained from the matrix \( M^3 \)). Then \( c^3_{i,j} = n \), for \( n/2 + 1 \leq i \leq n \) and \( 1 \leq j \leq n/2 \), if and only if \( y_{i-n/2,*} \) is dominated by \( x_{j+n/2,*} \), or, in other words, \( x_{i-n/2,*} + x_{j+n/2,*} \) is greater or equal to \( a \). Thus, the bottom left quadrant of \( C^3 \) is the top right quadrant of \( C \). Similarly, the bottom left quadrant of \( C^1 \) is the top left quadrant of \( C \), and the bottom left quadrants of \( C_2 \) and \( C_4 \) are the bottom left and bottom right quadrants of \( C \), respectively.

**Theorem 2.1** Given a set of \( n \) points in \( \mathbb{R}^n \) and a point \( a \in \mathbb{R}^n \), one can find all pairs of points whose sum, in each of the coordinates \( k \), is at least \( a_k \) in \( O(n^{3/2} M(n)^{1/2}) \) time.

**Remark.** Matoušek’s algorithm can be adapted to solve the large sum-of-pairs problem directly.

### 3 The discrete planar two-watchtower problem

We solve the decision problem of the discrete planar two-watchtower problem. Given a 2-dimensional terrain \( T = (v_0, \ldots, v_n) \) and a height \( h \), determine whether the entire terrain can be viewed from two watchtowers of height \( h \) that are located at two of \( T \)'s vertices.

For each vertex \( v_i \) of \( T \), we shall place a watchtower of height \( h \) at \( v_i \) and compute the region \( R_i \) of \( T \) that is visible from this watchtower. We use the following easy and known observation (see Figure 2).

**Observation:** If \( e_k = [v_{k-1}, v_k] \) is an edge of \( T \) lying to the right of \( v_i \), then either (i) \( e_k \cap R_i = \emptyset \) (e.g., in Figure 2, \( e_8 \cap R_2 = \emptyset \)), or (ii) \( e_k \cap R_i = v_{k-1} \) (e.g., \( e_{10} \cap R_2 = v_9 \)), or (iii) \( e_k \cap R_i \) consists of a single line segment anchored at \( v_k \) (e.g., \( e_9 \cap R_3 \)).

Thus, \( R_i \) is the union of \( n \) (possibly empty or degenerate) line segments, and \( R_i \) can be computed in \( O(n) \) time.

Next we define an \( n \times n \) matrix \( X \). The \( i \)'th row of \( X \) is obtained from the region \( R_i \) as follows. Let \( e_k = [v_{k-1}, v_k] \) be the \( k \)'th edge of \( T \). Then

\[
x_{i,k} = \frac{|e_k \cap R_i|}{|e_k|}.
\]

Notice that if \( v_i \) and \( v_j \) are two vertices of \( T \) such that the entire terrain can be viewed from their watchtowers, then the sum of the \( i \)'th and \( j \)'th
Figure 2: The region of $T$ that is visible from the watchtower at $v_2$.

rows of $X$ is greater or equal to $(1, \ldots, 1)$. That is, $x_{i,k} + x_{j,k} \geq 1$, for $k = 1, \ldots, n$.

In the following lemma we prove that the opposite statement is also true, that is

Lemma 3.1 If the sum of the $i$'th and $j$'th rows of $X$ is greater or equal to $(1, \ldots, 1)$, then the entire terrain can be viewed from the watchtowers at $v_i$ and $v_j$.

Proof: Assume $i < j$ and let $e_k = [v_{k-1}, v_k]$ be an edge of $T$. If $e_k$ lies between $v_i$ and $v_j$, then the part of $e_k$ that is visible from the watchtower at $v_i$ (resp. $v_j$) is a single segment anchored at $v_k$ (resp. $v_{k-1}$) (see observation above). Therefore, since $x_{i,k} + x_{j,k} \geq 1$, the edge $e_k$ is entirely covered by the watchtowers at $v_i$ and $v_j$.

Assume now that $e_k$ lies, e.g., to the right of $v_j$. We show that $e_k$ is entirely covered by one of the two watchtowers. If $v_{k-1} = v_j$, then $e_k$ is entirely covered by the watchtower at $v_j$, so assume $v_{k-1} \neq v_j$. Notice that by the observation above, if $0 < x_{i,k} < 1$ (resp. $0 < x_{j,k} < 1$), then the vertex $v_{k-1}$ cannot be seen from the watchtower at $v_i$ (resp. $v_j$), and, therefore, $x_{i,k-1}$ (resp. $x_{j,k-1}$) must be 0. Now, if $e_k$ is not entirely covered by one of the two watchtowers, then, since we are assuming $x_{i,k} + x_{j,k} \geq 1$, we have $0 < x_{i,k} < 1$ and $0 < x_{j,k} < 1$, and therefore $x_{i,k-1} + x_{j,k-1} = 0$, contradicting our assumption. □

According to the lemma above and to the paragraph preceding it, there exists a solution to our decision problem if and only if there exist two rows in $X$ whose sum is greater or equal to $(1, \ldots, 1)$. In order to determine whether
two such rows exist, we simply apply the algorithm from the preceding
section with \( a = (1, \ldots, 1) \). Thus we obtain

**Theorem 3.2** The decision problem of the discrete planar two-watchtower
problem can be solved in \( O(n^{3/2} M(n)^{1/2}) \) time.

It is easy to verify that Megiddo’s parametric search technique [4] can
now be used to obtain an improved solution to the discrete planar two-
watchtower problem. (The \( \log^2 n \) factor in the bound below is the cost of
applying parametric search.)

**Theorem 3.3** The discrete planar two-watchtower problem can be solved
in \( O(n^{3/2} M(n)^{1/2} \log^2 n) \) time.

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