THE EFFECTIVE CONDUCTING PROPERTIES OF 2D CELLULAR MULTIPHASE MATERIALS

E. Pesetskaya\textsuperscript{1}, A. Öchsner\textsuperscript{1,2}, J. Grácio\textsuperscript{1,2}

\textsuperscript{1}Centre for Mechanical Technology and Automation
\textsuperscript{2}Department of Mechanical Engineering University of Aveiro, Portugal

ABSTRACT: The effective conductivity of 2D doubly periodic multiphase materials with temperature dependent and independent material properties is analytically investigated. On the base of an analytical approach, complex analytic methods are used. An arbitrary number of disjoint cylindrical parallel identical reinforcements in a representative cell are considered. The geometrical description of such kind of multiphase materials is a multiply connected unbounded domain in the complex plane. The mathematical model is a boundary value problem for the Laplace equation on the multiply connected domain. This boundary value problem is analytically solved by the method of functional equations. The effective conductivity is found in an explicit formula which contains the solution of the problem and the basic models’ parameters.

Introduction.

2D doubly periodic multiphase materials with a finite number of cylindrical parallel identical reinforcements in a representative cell are considered. The effective conductivity is analytically investigated in the case of arbitrary location of the disjoint reinforcements in the representative cell. A multiply connected unbounded domain in the complex plane can serve as a geometrical description of such kind of multiphase materials.

Steady heat-state heat conduction is considered. It is described by the following equations

\[ \nabla(\lambda(T)\nabla T) = 0, \quad q = -\lambda(T)\nabla T, \] \hspace{1cm} (1)

where \( \nabla := \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \), \( T = T(x, y) \) is the temperature on the plane \( (x, y) \), \( q = q(x, y) \) is the heat flux, \( \lambda(T) \) is the thermal conductivity. Two different situations are studied. First, the conductivity of the matrix is a constant. It means that the first equality in Eq.(1) is the Laplace equation with respect to \( T \). Second, the conductivity of the matrix is a piecewise continuous increasing function. The second situation is considered under the assumption that the conductivity of reinforcements is equal to zero, i.e. the multiphase material is a porous material. In this case, the first equality in Eq.(1) is not the Laplace equation, but with a special transformation, it can be reduced to the Laplace equation. Different boundary conditions in both situations produce different boundary value problems for the Laplace equation (linear and nonlinear) which are analytically solved by complex analytic methods, precisely, by the method of functional equations (1).

The effective conductivity of the multiphase materials is found in explicit forms in both cases. These explicit forms contain the solutions of the boundary value problems and basic characteristics of the model such as number, centres and radius of the reinforcements.

1-96
STATEMENT AND SOLUTION OF THE PROBLEMS

Let \( L \) be a lattice generated by two fundamental vectors \( 1 \) and \( i \) \((i^2 = -1)\) on the complex plane \( C \cong \mathbb{R}^2 \) of the variable \( z = x + iy \). Let
\[
Q_{(0,0)} := \{z = t_1 + it_2 \in C : -1/2 < t_p < 1/2, \ p = 1,2\}
\]
be the representative unit cell, and the lattice \( L \) consists of the cells
\[
Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + im_2 := \{z \in C : z - m_1 - im_2 \in Q_{(0,0)}\},
\]
where \( m_1, m_2 \in \mathbb{Z} \). It is assumed that mutually disjoint identical \( N \) cylindrical reinforcements
\[
D_k := \{z \in C : |z - a_k| < r\}, \ k = 1,...,N,
\]
with the boundary
\[
\partial D_k := \{z \in C : |z - a_k| = r\}
\]
are situated in the unit cell \( Q_{(0,0)} \) and periodically repeated in all cells \( Q_{(m_1,m_2)} \). Let
\[
D_0 = Q_{(0,0)} \bigcup_{k=1}^{N} (D_k \cup \partial D_k).
\]

![Fig. 1. Geometrical description of the model.](image)

The problem is to define the effective conductivity of the doubly periodic multiphase material with the matrix
\[
D_{\text{matr}} = \bigcup_{m_1, m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + im_2)
\]
and the reinforcements
\[
D_{\text{rein}} = \bigcup_{m_1, m_2, k=1}^{N} (D_k + m_1 + im_2)
\]
occupied by materials of conductivities \( \lambda_m \) and \( \lambda_r \), respectively.
First, the situation is considered when the conductivity of the matrix $\lambda_m$ is constant. In this case, the problem is equivalent to the determination of the potential of the corresponding field, i.e. to find a function $T(z)$ satisfying the Laplace equation in each component of the multiphase material:

$$\nabla^2 T(z) = 0, \quad z \in D_{\text{mat}} \cup D_{\text{res}},$$

and ideal contact conditions on the boundary between the matrix and reinforcements:

$$T^+(t) = T^-(t), \quad \lambda_m \frac{\partial T^+}{\partial n}(t) = \lambda_r \frac{\partial T^-}{\partial n}(t), \quad t \in \partial D_k, \quad k = 1, \ldots, N.$$  \hspace{1cm} (2)

where $\frac{\partial}{\partial n}$ is the outward normal derivative and

$$T^+(t) := \lim_{z \to t, z \in D_h} T(z), \quad T^-(t) := \lim_{z \to t, z \in D_h} T(z), \quad t \in \partial D_k, \quad k = 1, \ldots, N.$$

The first equality in Eq.(2) describes the temperature balance. The second one describes the flux balance. The quasi periodicity conditions are also imposed on $T(z)$, where $T(z)$ has a constant jump in the $x$-direction that corresponds to the direction of the external field:

$$T(z + 1) = T(z) + 1, \quad T(z + i) = T(z).$$  \hspace{1cm} (3)

The complex potentials $\varphi(z)$, $\varphi_k(z)$ which are analytic in $D_0$ and $D_k$, respectively, and continuously differentiable in the closures of $D_0$ and $D_k$, respectively, are introduced. The harmonic potential $T(z)$ and complex analytic potentials $\varphi(z)$, $\varphi_k(z)$ are related by the equalities

$$T(z) = \begin{cases} \text{Re}(\varphi(z) + z), & z \in D_0, \\ \frac{2\lambda_m}{\lambda_m + \lambda_r} \text{Re} \varphi_k(z), & z \in D_k. \end{cases}$$  \hspace{1cm} (4)

The conditions (2) can be rewritten in terms of the complex potentials (see (2))

$$\varphi(t) = \varphi_k(t) - \rho \overline{\varphi_k}(t) - t, \quad \|t - a_k\| = r, \quad k = 1, \ldots, N.$$  \hspace{1cm} (5)

Here, $\rho = \frac{\lambda_r - \lambda_m}{\lambda_r + \lambda_m}$ is the contrast parameter.

The problem given in Eq.(5) is a particular case of so-called R-linear conjugation problem (see (1)).

Second, let us consider the situation when the conductivity of the matrix is a piecewise continuous increasing function: $\lambda_m(T)$. It is assumed that the conductivity of the reinforcements is equal to zero. In this case, the temperature distribution $T(x, y)$ satisfies the quasi-linear equation

$$\nabla (\lambda(T) \nabla T) = 0 \quad \text{in} \quad \bigcup_{m_1, m_2} (D_0 + m_1 + im_2).$$  \hspace{1cm} (6)

It is assumed that on the boundary of the representative cell the following conditions hold

$$T(x, \frac{1}{2}) = T(x, -\frac{1}{2}), \quad \lambda(T(x, \frac{1}{2})) T_y (x, \frac{1}{2}) = \lambda(T(x, -\frac{1}{2})) T_y (x, -\frac{1}{2}),$$

$$\lambda(T(\pm \frac{1}{2}, y)) T_y (\pm \frac{1}{2}, y) = 0,$$

$$\lambda(T(\pm \frac{1}{2}, y)) T_y (\pm \frac{1}{2}, y) = \lambda(T(\frac{1}{2}, y)) T_y (\frac{1}{2}, y).$$  \hspace{1cm} (7)
where $T_x = \frac{\partial T}{\partial x}$, $T_y = \frac{\partial T}{\partial y}$. The condition on the boundary between the reinforcements and the matrix is
\[ \lambda(T) \frac{\partial T}{\partial n} = 0 \quad \text{on } \partial D_k. \tag{8} \]

Due to periodicity of the material, these conditions do satisfy on each cell $D_0 + m_1 + im_2$ and are compatible with the structure of the material if the external field is oriented to the $x$-direction. Transformation of this nonlinear problem (6)-(8) to a linear one can be done by the introduction of the function $u(x, y) = f(T(x, y))$, where $f(t) = \int_0^t \lambda(\xi) d\xi$. Equation (6) becomes the Laplace equation
\[ \nabla^2 u(x, y) = 0, \quad \text{in } \bigcup_{m_1, m_2} (D_0 + m_1 + im_2), \]
and the boundary conditions have the form
\[ u(x, \frac{1}{2}) = u(x, -\frac{1}{2}), u_y (x, \frac{1}{2}) = u_y (x, -\frac{1}{2}), u_y \left( \frac{1}{2}, y \right) = 0, u_y \left( -\frac{1}{2}, y \right) = u_y \left( -\frac{1}{2}, y \right), \frac{\partial u}{\partial n} = 0. \tag{9} \]

Let $\Phi(z) = u(x, y) + iv(x, y)$ be the complex potential, where $z = x + iy$. According to Cauchy-Riemann equation, $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}$, where $s$ is the natural parameter of $\partial D_k$. Cauchy-Riemann equation and integration of the last equality of Eq.(9) on $s$ give
\[ \text{Im } \Phi(t) = b \quad \text{on } \partial D_k, \]
where $b$ is an undetermined constant. Let us introduce in $D_0$ a new analytic function $\varphi(z) := \Phi(z) - z$ and in $D_k$ analytic functions $\varphi_k(z)$ which are continuously differentiable in the closures of $D_k$ with the boundary condition
\[ 2 \text{Re } \varphi_k(t) = \text{Re}(\varphi(t) + t), \quad t \in \partial D_k. \]

Boundary conditions (9) can be rewritten in the form of the R-linear conjugation problem
\[ \varphi(t) = \varphi_k(t) - \rho \bar{\varphi}_k(t) + ib - t, \quad t \in \partial D_k, \tag{10} \]
where $\rho = -1$ holds. Let the derivatives being denoted as
\[ \psi(z) := \frac{\partial \varphi}{\partial z}, \quad \psi_k(z) := \frac{\partial \varphi_k}{\partial z}. \]

Differentiating the problems (5) and (10), the following R-linear conjugation problem is obtained
\[ \psi(t) = \psi_k(t) + \rho \left( \frac{r}{t - a_k} \right)^2 \bar{\psi}_k(t) - 1, \quad t \in \partial D_k. \]

The solution of this problem is found by the method of functional equations (see (1)) and represented in the form of a series in $r^2$:
\[ \psi_k(z) = \psi_k^{(0)}(z) + r^2 \psi_k^{(1)}(z) + r^4 \psi_k^{(2)}(z) + \ldots. \]

Here, the coefficients of this series $\psi_m^{(p)}(z)$, $p = 0, 1, 2, \ldots$, are defined by the following recurrence relations:
\[ \psi_m^{(0)}(z) = 1, \]
$$\psi_m^{(1)}(z) = \rho \left[ \sum_{k,m} \widetilde{\psi}_{0k} E_2(z-a_k) + \widetilde{\psi}_{0m} \sigma_2(z-a_m) \right],$$
$$\psi_m^{(2)}(z) = \rho \left[ \sum_{k,m} \widetilde{\psi}_{1k} E_3(z-a_k) + \widetilde{\psi}_{1m} \sigma_3(z-a_m) + \sum_{k,m} \widetilde{\psi}_{0k} E_2(z-a_k) + \widetilde{\psi}_{0m} \sigma_2(z-a_m) \right],$$
$$...$$
$$\psi_m^{(p+1)}(z) = \rho \left[ \sum_{k,m} \widetilde{\psi}_{pk} E_{p+2}(z-a_k) + \widetilde{\psi}_{pm} \sigma_{p+2}(z-a_m) + \sum_{k,m} \widetilde{\psi}_{p-1k} E_{p+1}(z-a_k) + \widetilde{\psi}_{0m} \sigma_2(z-a_m) \right],$$

where $E_p(z) = \sum_{m_1,m_2} (z-m_1-i m_2)^p$ is the Eisenstein function, $\sigma_p(z) = E_p(z) - z^{-p}$ is a modification of the Eisenstein function (see (4)).

**EFFECTIVE CONDUCTIVITY**

In the case of the temperature independent conductivity of the matrix, the effective conductivity $\lambda_e$ of an isotropic multiphase material with $N$ identical reinforcements is found in the form of a series in the concentration $\nu = N\pi r^2$ (see (3)):

$$\lambda_e = \lambda_m + 2\lambda_m \rho \nu \sum_{p=0}^{6} A_p \nu^p + O(\nu^8),$$

where

$$A_p = \frac{1}{\pi^p N^{p+1}} \sum_{m=1}^{N} \psi_{m}^{(p)}(a_m), \quad p = 0, 1, 2, ...$$

Here, $\psi_m^{(p)}(z)$ is defined by (11), $a_m$ is a centre of a reinforcement, $r$ is the radius of the reinforcements. Obtained values for the effective conductivity as a function of the concentration $\nu$ are compared with independently obtained values by the numerical finite element method. In the scope of the finite element method, three geometries are considered (square arrangement of reinforcements for $\nu = 0.1, 0.3, 0.5$). A good correlation is shown in Fig. 2.
Fig. 2. Effective conductivity $\lambda_e$ in dependence on the concentration $\nu$.

The influence of small random perturbations of the reinforcements in the cell on the effective conductivity is investigated on the base of the representation (1) (see (3)). A small perturbation means that each disk arbitrarily moves in the prescribed part of the original cell and does not cross and touch the boundary of corresponding part (see Fig. 3).

![Figure 3: A small perturbation of the reinforcements.](image)

It is analytically shown that a symmetric structure with cylindrical reinforcements has in average a lower effective conductivity than any nonsymmetric structure when the conductivity of the matrix is less than the conductivity of the reinforcements.

In the case of the temperature dependent conductivity of the matrix, the effective conductivity in $x$-direction $\lambda_e^x$ of an anisotropic multiphase material with $N$ reinforcements is found in the following form (see (6))

$$\lambda_e^x(T) = \mu^x \cdot f''(T),$$

where

$$\mu^x = 1 + 2\rho \nu \sum_{p=0}^{6} B_p^x \nu^p + o(\nu^8),$$

$$B_p^x = \frac{1}{\pi^p N^{p+1}} \sum_{m=1}^{N} \text{Re} \psi_m^{(p)}(a_m), \quad p = 0, 1, 2, \ldots.$$ 

The effective conductivity in $y$-direction $\lambda_e^y$ is obtained in the form of the following equality (see (6))

$$\lambda_e^y(T) = \mu^y \cdot f''(T),$$

where

$$\mu^y = -2\rho \nu \sum_{p=0}^{6} B_p^y \nu^p + o(\nu^8),$$

1-101
\[ B_p = \frac{1}{\pi p N} \sum_{m=1}^{N} \text{Im} \psi_m^{(p)}(a_m), \quad p = 0, 1, 2, \ldots. \]

The effective conductivity in \( x \)-direction of 2D porous materials with disjoint parallel cylindrical pores is numerically investigated by the finite element method. The results obtained by both methods are presented in Table 1. A good correlation is discovered.

Table 1. Comparison of the effective conductivity \( \lambda_{\text{eff}} \) obtained by two different approaches \( (T_1 = 298, \nu = 0.5) \).

<table>
<thead>
<tr>
<th>( T_2 ) K</th>
<th>( T_{\text{mid}} ) K</th>
<th>( \lambda_{\text{eff}} ) (finite element method) W / mK</th>
<th>( \lambda_{\text{eff}} ) (analytical method) W / mK</th>
<th>deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>398</td>
<td>350.2556</td>
<td>43.9605</td>
<td>44.12746575</td>
<td>0.4</td>
</tr>
<tr>
<td>378</td>
<td>339.4711</td>
<td>43.1662</td>
<td>43.26932249</td>
<td>0.2</td>
</tr>
<tr>
<td>358</td>
<td>328.8435</td>
<td>42.3719</td>
<td>42.42365650</td>
<td>0.1</td>
</tr>
<tr>
<td>338</td>
<td>318.3822</td>
<td>41.5775</td>
<td>41.59122726</td>
<td>0.03</td>
</tr>
<tr>
<td>318</td>
<td>308.0974</td>
<td>40.7831</td>
<td>40.77284370</td>
<td>0.02</td>
</tr>
</tbody>
</table>

The temperature \( T_1 \) is constant 298K and the temperature \( T_2 \) is varied according to Table 1. Furthermore, the temperature \( T_{\text{mid}} \) at the centre of the representative cell is listed.

REFERENCES: