CRITICAL WAVE LENGTHS AND INSTABILITIES IN
GRADIENT-ENRICHED CONTINUUM THEORIES

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ABSTRACT: Continuum material models can be enriched with additional gradients in
order to model phenomena that are driven by processes at lower levels of observation.
For instance, higher-order gradients of strains or inertia can be included in the equations
of elasticity in order to model wave dispersion that occurs in heterogeneous media.
These higher-order gradients may be obtained by homogenisation or continualisation
procedures applied to heterogeneous micro-structured media. For a systematic
comparison of various gradient-enriched continua, dispersion analysis may be used. In
this contribution, we will explore the occurrence of critical wave lengths and its
implications for the material stability. In particular, we will present a unifying theorem
that permits to assess the stability of elastic, hardening and softening gradient-enriched
continua by means of a critical wave length analysis, whereby the upper or lower bound
nature of the critical wave length indicates whether the model is stable or unstable.

Introduction

Gradient theories are continuum theories with additional spatial derivatives of
the state variables. Gradient theories can be used as an alternative to standard continuum
theories (in which these additional gradient are absent) in cases where the latter fail. Examples of such applications include
1. wave dispersion in heterogeneous media. The standard equations of
elasticity cannot predict any dispersive effects, whereas gradient-enriched
elasticity formulations can be used to simulate the attenuation of high-
frequency waves, for instance [1,2];
2. stress and strain fields at sharp crack tips. Whereas standard elasticity
formulations predict singularities in stress and strain, these can be removed when
gradient elasticity is used [3];
3. the modelling of post-peak phenomena. It is well documented that the
standard formulations of damage and/or plasticity lead to physically
meaningless results in case strain softening is included in the description [4,5];
4. the modelling of size effects. The relative strength of a structure depends on
its actual size, whereby in general smaller structures are relatively stronger.
This cannot be modelled with a standard continuum theory, whereas size
effects can be simulated when using a gradient-enriched theory [6, 7].

In recent decades, many different gradient theories have been suggested and
formulated. Apart from the various motivations to use gradient theories as outlined
above, a host of methodologies have been used to incorporate gradients in the
continuum description. The two main categories are a phenomenological postulation
of the gradients in the stress-strain relation or the evolution equations of the internal
variables on the one hand, and the transition from micro structural models to enriched
continuum theories by means of homogenisation and continualisation on the other hand.

This has led to a wide range of available gradient theories, of which the
similarities and differences are often difficult to assess. In this paper, dispersion analysis will be developed as a tool for a systematic comparison between different gradient theories. Dispersion is the phenomenon that the different harmonic components of a propagating wave travel with different velocities — this property is shared between gradient theories, whereas standard continuum theories are normally not dispersive. Of particular interest is the formation of stationary zones, the width of which is determined through the wave number that corresponds to zero propagation velocity. The appearance of a stationary zone denotes a transition from wave numbers with real phase velocities to wave numbers with imaginary phase velocities or vice versa. This occurrence of a stationary zone is appreciated remarkably differently in the various research communities that employ gradient theories. In particular,

- in the strain-softening analysis community, a stationary zone indicates that strain localisation and, thus, energy dissipation takes place in a zone with finite width — unlike standard continua in which this zone unrealistically has zero width. In other words, this stationary zone is a key ingredient for the objective modeling of material instabilities [4,5,7,8];
- in the homogenisation community, the focus is often on capturing the propagation characteristics of various regimes of wave numbers [1,2]. The emergence of a stationary zone also indicates a transition point beyond which certain wave numbers may be accompanied by imaginary phase velocities that indicate loss of uniqueness or loss of stability [9].

Hence, stationary zones are seen either as a crucial and desirable ingredient to model material instabilities or as an undesirable artefact that introduces instabilities. These two concepts may seem contradicting, but in this paper we will set out to present a unified treatment based on the notions of Askes and Sluys [10,11]. In particular, we will argue that the upper or lower bound nature must be established of the wave length associated with the stationary zone.

Another property that is sometimes used to assess stability of a higher-order theory is positive-definiteness of the underlying energy potentials. This aids in identifying which of the contributing terms are stabilising and which are destabilising. However, in a boundary value problem normally all the contributing terms of the model interact and it could be that the effects of certain destabilising terms are balanced by the effects of certain other stabilising terms. In this respect, dispersion analysis is more appropriate as it assesses the stability of all terms together, rather than of each term individually. After revisiting certain basic notions on wave propagation and dispersion analysis, a theorem is formulated that relates stability of the model to the upper or lower bound nature of the emerging critical wave length. This is then illustrated by means of an example.

**DISPERSION ANALYSIS**

In a one-dimensional analysis of dispersive waves the equation of motion for an infinite medium is considered. For nonlinear governing equations a uniform reference state is normally assumed around which the equation of motion is linearised. A general harmonic perturbation $\delta u$ of the form

$$\delta u = A \exp(ik(x - ct))$$

is taken, whereby $A$ is the amplitude of the perturbation, $k$ is the wave number and $c$ is the phase velocity. The resulting expression relates the phase velocity of each harmonic component to its wave number, and in gradient theories each wave number has in general a different phase velocity.
Stationary zones can be retrieved by solving the wave number \( k \) for the case that \( c = 0 \). In this work, the resulting wave number is denoted as \( k_{\text{crit}} \) and the corresponding wave length \( \lambda_{\text{crit}} = 2\pi/k_{\text{crit}} \). Obviuously, \( \lambda_{\text{crit}} \) is the width of the stationary zone. For nonlinear material models, the value of the critical wave number and critical wave length often depends on the level of (inelastic) loading, hence their values must be evaluated at various strain levels. As such, in many previous works the evolution of the critical wave number or critical wave length as a function of the strain history is studied [5,7,8,10,11].

**STABLE AND UNSTABLE WAVE PROPAGATION**

In this Section, two fundamental properties of wave propagation through a dispersive medium will be treated. Firstly, it is argued that wave numbers accompanied with imaginary phase velocities can be destabilising (this concerns the time component of the general harmonic wave). Secondly, it is shown that wave numbers with imaginary phase velocity can only destabilise the solution in a zone that is larger than the wave length of the destabilising wave number (this concerns the space component of the general harmonic wave).

If it assumed that the phase velocity in Eq. (1) is imaginary, i.e. \( c = ic_r \) with \( c_r \) a real number, then Eq. (1) particularises as

\[
\delta u = A \exp(i k x + k c_r t) 
\]

where \( c_r \) can adopt positive as well as negative values. Since Eq. (2) only constitutes one of the components of the total solution, normally both positive and negative \( c_r \) are present. As is obvious from Eq. (2), a positive \( c_r \) leads to an unlimited growth in time of the associated harmonic component. If this unbounded growth is not triggered by the flux of external energy into the system, it is an indication of an instability. Next, an instantaneous consideration of the stationary zone is made, whereby this zone is isolated from the surrounding material. Building on the assumption of a uniform reference state, it can be stated that the strain profile corresponding to Eq. (2) inside the stationary zone is symmetric: its values at either end of the stationary zone must be equal, and it consists of a number of sine and cosine components as indicated by Eq. (2). Furthermore, its derivatives at either end of the stationary zone must be zero to ensure continuity with the remainder of the problem domain that obeys the uniform reference state. It then follows that the wave length governing the harmonic excitation must be smaller than the width of the stationary zone.

**UPPER / LOWER BOUND THEOREM**

The two key ingredients for formulating an upper/lower bound theorem on stability are outlined in the previous section, and in reverse order they can be stated as

1. Upon formation of a stationary zone, only waves with a wave length *smaller* than the width of the stationary zone can exist in this zone;

2. If these waves have imaginary phase velocities they destabilise the solution.

This then leads to the aforementioned theorem:

If the critical wave length serves as a *lower bound* (that is, if smaller wave lengths have imaginary phase velocities and larger wave lengths have real
phase velocities), then the model can become unstable. If the critical wave length serves as an upper bound, then the model is stable.

Two particular cases need further explanation:

**Reduction to statics:** The static or quasi-static case can be obtained from the general dynamic case by asymptotically ignoring the inertia effects. In dispersion analysis the same procedure is followed: the static case is considered to be the limit case of the general dynamic case. Thus, the theorem can be used for static loading cases as well, as is confirmed by various numerical tests [10,11].

**Multiple critical wave lengths:** In certain gradient theories not only second gradients but also fourth gradients of for instance the strain are added, as illustrated in [10,11]. These models may lead to the simultaneous occurrence of two critical wave lengths. If this is the case, one of them will serve as an upper bound and the other will serve as a lower bound. Irrespective of which of the two is the larger, the upper bound critical wave length indicates instability (compare for instance the various cases presented in [10] and [11]).

An example will serve to illustrate these concepts.

**EXAMPLE**

As an example, a one-dimensional higher-order gradient theory of the damage type is studied, in particular the so-called explicit gradient model as suggested in [5,11], whereby the stress \( \sigma \) relates to the strain \( \varepsilon \) as

\[
\sigma = (1 - \omega)E\varepsilon
\]  

with the damage \( \omega \) given as a function of the strain history \( \kappa \) as

\[
\omega = \frac{\kappa_u(\kappa - \kappa_i)}{\kappa(\kappa_u - \kappa_i)}; \quad \kappa = \max(\kappa, \kappa_i, \varepsilon)
\]  

Upon loading beyond the elastic limit, it holds that \( \kappa = \varepsilon \). The gradient-enriched strain \( \varepsilon \) is expressed as

\[
\varepsilon = \varepsilon + \frac{1}{2} \ell^2 \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{1}{8} \ell^4 \frac{\partial^4 \varepsilon}{\partial x^4}
\]  

![Figure 1: Bilinear stress-strain relation in the absence of higher-order gradients](image)

In the above, \( \kappa_i \) is the strain level that denotes the end of the elastic regime and \( \kappa_u \) is the ultimate strain level beyond which all load-carrying capacity is exhausted. The
resulting bilinear stress-strain curve for the case $\ell = 0$ is plotted in Fig. 1. The above equations can be substituted into the equation of motion given by

$$\rho \ddot{u} = \frac{\partial \sigma}{\partial x} \tag{6}$$

together with the kinematic relation

$$\varepsilon = \frac{\partial u}{\partial x} \tag{7}$$

while furthermore the assumption of a uniform reference state is used. Next, linearisation around the reference state (denoted by $\varepsilon_0$ and $\omega_0$) is carried out where it has been used that derivatives of the reference strain $\varepsilon_0$ vanish. Substituting Eq. (1) leads after some algebra to

$$\frac{c^2}{c_o^2} = (1 - \omega_0) - \varepsilon_0 \frac{\partial \omega}{\partial \varepsilon} \left( 1 - \frac{1}{2} \ell^2 k^2 + \frac{1}{8} \ell^4 k^4 \right) \tag{8}$$

The behaviour of Eq. (8) is plotted in Fig. 2 for various levels of the reference strain $\varepsilon_0$. Furthermore, distinction is made between the so-called second-order (whereby the $\ell^2$-term is kept but the $\ell^4$-term is ignored) and the so-called fourth-order model (that includes both the $\ell^2$-term and the $\ell^4$-term). The material parameters are taken as $\kappa_1 = 10^{-4}$ and $\kappa_2 = 10^{-2}$. It can be seen that in both cases a range of wave numbers have imaginary phase velocities. In the second-order model, this concerns only wave numbers smaller than the intercept wave number (the critical wave number mentioned above), whereas in the fourth-order model imaginary phase velocities are found for the small wave numbers as well as the large wave numbers.

The critical wave number is found numerically by setting $c = 0$ and solving Eq. (8) for $k$ which is then denoted $k_{\text{crit}}$. The corresponding critical wave length $\lambda_{\text{crit}}$ reads [11]

second-order model: $\lambda_{\text{crit}} = \pi \ell \sqrt{\frac{2 \varepsilon_0 \omega'}{\varepsilon_0 \omega' + \omega_0 - 1}} \tag{9}$

fourth-order model: $\lambda_{\text{crit}} = \pi \ell \sqrt{\frac{\varepsilon_0 \omega' \pm \sqrt{\varepsilon_0 \omega' (2 - 2 \omega_0 - \varepsilon_0 \omega')}}{\varepsilon_0 \omega' + \omega_0 - 1}} \tag{10}$

Figure 2: Normalised phase velocity $c/c_o$ versus normalised wave number $\ell k$ for different strain levels — second-order model (left) and fourth-order model (right)
Figure 3: Normalised critical wave length $\lambda_{\text{crit}}/l$ versus strain level — second-order model (left) and fourth-order model (right), areas that refer to complex phase velocities are shaded.

where $\omega' = \partial \omega / \partial \epsilon$. Figure 3 shows the critical wave length as a function of the reference strain level. An additional feature of this Figure is that areas with imaginary phase velocities are shaded, thus enabling to identify whether the critical wave length serves as an upper and/or lower bound. As can be seen, in the second-order model the critical wave length only serves as an upper bound, and therefore stability of this model is ensured. On the other hand, in the fourth-order model two critical wave lengths are found. One of them serves as a lower bound and is thus destabilising.

A theorem has been founded and formulated that captures this fundamental characteristic of critical wave lengths. It has been illustrated by an example that shows how dispersion analysis can be an easy tool to assess the stability of a gradient theory. This can now be extended to assess and compare a wide range of gradient theories such as for instance presented in [7,8], while including the upper bound or lower bound nature of the critical wave length.

**DISCUSSION**

In this paper, the stability of gradient theories is assessed via an analysis of dispersive waves. Wave dispersion can occur in various waves and often it is found that a stationary zone forms associated with a critical value of the wave number and the wave length. In different communities this occurrence of a stationary zone is viewed differently, but in this paper a unifying assessment of this zone is given. It is based on whether the critical wave number acts as a lower bound or as an upper bound — in the former case, the model can become unstable while it is stable in the latter case.

**REFERENCES**


