DYNAMICS OF STRUCTURED ELASTIC HALFSPACE UNDER ACTION OF TRAVELING LOADING

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Abstract. Wave propagation processes are explored in an elastic halfspace covered by elastic layer and (or) thin elastic plate. The structure is subjected by an external load travelling along the free surface. In the steady-state formulation, waveguide properties of considered systems are analyzed. Multiple roots of dispersion equations are revealed with the aim to determine critical velocities of travelling loads caused resonant processes developing with time. Together with the well-known resonance existing in the homogeneous halfspace in the case when the loading travels with the Rayleigh's velocity, a set of additional resonances were discovered in the structured halfspase. Obtained asymptotes of solutions of dispersion equations show that the Rayleigh resonance exists only in the longwave case. In the middle spectrum of wave lengths, solutions of dispersion equations are numerically obtained, and a set of resonant velocities are established. In the non-steady-state formulation, computer solutions are completed determining the initiation and development of resonant perturbations in the considered systems. Simplified models of structures are proposed possessing the similar spectral features in the middle length wave zone as the mentioned structured halfspase: (*a*) thin plate upon an elastic foundation, and (*b*) the model (*a*) with inertial media continuously connected to the plate by elastic springs. These models allow the qualitative asymptotical solutions for resonant processes to be established and analyzed. Performed in parallel computer simulations enable quantitative features of the process to be explored. Numerical and asymptotic results are compared to reveal the range of the asymptotic solution acceptability.

1. Introduction

Studying the dynamics properties of elastic solids and structures subjected by travelling loads has a long history. Together with the understandable significance scientific interest in the topic 'transient waves and vibrations in structured solids', this problem has obvious practical applications related to actual problems of geophysics, geodynamics, aerodrome runways, bridge construction, etc. Particular attention in this research should be given to the study of resonance processes excited by loads moving with critical velocities. As far as to the author's knowledge, development of resonant waves in plane models of structurally inhomogeneous solids were initially considered in the works [1, 2] while dispersion properties of such structures and critical velocities of the moving loading were previously explored in [3]. In work [4], the more complicated problem was studied in the case of cylindrical symmetry.

Studies of the problems under discussion in the case of homogeneous bodies were carried out much earlier. First of all note the pioneer work [5] related to the plane problem for free half-space and the loading moving with the Rayleigh velocity along the surface. There was shown, first, a steady-state limit of the solution is absent; secondly, particle velocities and stresses near the surface rise linearly. Then diverse aspects of resonant waves were studied for elongated structures, in general, for cylindrical shells interacting with acoustic media (see [6 - 10]). Special points in the case of a hollow cylinder were examined in [6 - 9], asymptotic and computer solutions of flexural resonant waves are obtained in [11, 12] for various types of a moving loads.

Note, the main contribution to the theory of resonant waves in solid and structures has come from Slepyan [9, 10]. In monograph [9], basic aspects of non-steady-state wave propagation have been considered and the principal problem of excitation of resonant waves in elastic waveguides by moving load has been explored. The main tool discovered by Slepyan to the analytical exploring travelling resonant processes is an approach to reversing the double Laplace-Fourier transforms in the vicinity of a moving wave ($x = ct$, c is the wave speed, t is time). This tool enables asymptotic solutions (for large values of time) to be designed. Notably on this basis have been obtained analytical solutions in [3, 4, 6–9, 11, 12]. Different aspects of the considered problems can be found e.g. in [13 – 16], note work [15] where a wide set of related publications can be found.

Nevertheless the many questions of the problem under consideration have been solved; we note that surface resonances in a structured have not yet been examined so far in spite of their evident theoretical and practical importance. Besides an important question is not revealed up to present related to applicability of asymptotical solutions in actual problems.

In this work we have based on the well known point of the considered process. So in the case of the step loading, a wave resonance is formed in the vicinity of the front, if the load velocity equals critical one, $V_{\text{er.}}$, which coincides with phase, c, and group, c_g, velocities of a free wave propagating in the systems ($V = V_{\text{cr}} = c = c_g$): a load moving along the waveguide axis enables permanent pumping up of external energy into the energy of the oscillating waves arising with time. The spectrum of such waves (or a set of waves) possessing the same phase and group velocities can be determined by special solutions of the dispersion equation representing dependence $c(q)$, where q is the wavenumber, $q = 2\pi/\lambda$, while λ is the wavelength. Taking into account that the group velocity is $c_g = c + q\partial c/\partial q$, we show that special solutions (they noted below as special points in dispersion curves), q_*, c_*, where $c_* = c(q_*) = c_g(q_*)$, appear (*i*) in the longwave spectrum or more precisely – in the limiting case of the infinitely long waves: $q_* = 0,\ c_* = c(0) = c_g(0)$ and (*ii*) in the middle-spectrum of q where extreme points of the solution $c = c(q)$ exist. Besides, the asymptotic values of $c = c(q)$ exist in the case of infinitely short waves $q \to \infty$, but only limiting values of q are considered here because the used Bernoulli model of the plate bending is not relevant for short waves.

In the case of a free halfspace (it is 'the dispersionless surface waveguide' in the terminology used here), the critical velocity is equal to the Rayleigh velocity c_R, which does not depend on the wavelength: $c(0) = c_g(0) = c_R$. The main result related to wave formation in the case $V = c_R$ is that a surface resonant process is formed, in which the growth rate of perturbations is linear, proportional to t (t is time) [6].

In this work, critical velocities are obtained in the considered above structure (dispersion waveguides). It is shown below that this structure possesses, *first*, a special point in the longwave spectrum ($q = 0$, $c = c_g = c_R$): a long surface wave overlooks finite thickness structures covering the halfspace, and *second*, it possess a set of special points (maximum, minimum, inflection) in a medium-wave spectrum, which appear due to the structured surface. The growth rate of middle spectrum resonances turns out less than linear: due to dispersion, some part of the energy pumped by the external load is diverted with time for non-growing perturbations with parameters near point $(q_*,\ c_*)$.

* In [10] there was shown that resonant waves can be excited by an oscillating load travelling with constant velocity *V* and frequency $\omega = vs$.

2. PROBLEM FORMULATION

Consider a dynamic problem for the plane system: thin plate - thick layer - halfspace. On the external surface (y = 0) beginning at time $t = 0$, normal surface stresses of a given magnitude *Q* move with constant velocity *V* along the right and left directions of axes *x*.

In Fig. 1, the system geometry is shown and the following significations are used: *H* is the Heaviside step function, numbers 0, 1, 2 refer to the plate, layer and halfspace, respectively; c_0 is the velocity of longitudinal waves in the plate, ρ_j (j = 0,1,2) are densities, h_j (j = 0,1) are thicknesses, subindices *l* and *s* related to velocities of longitudinal and shear waves, respectively: c_{jl}, and c_{js} (j = 1,2).

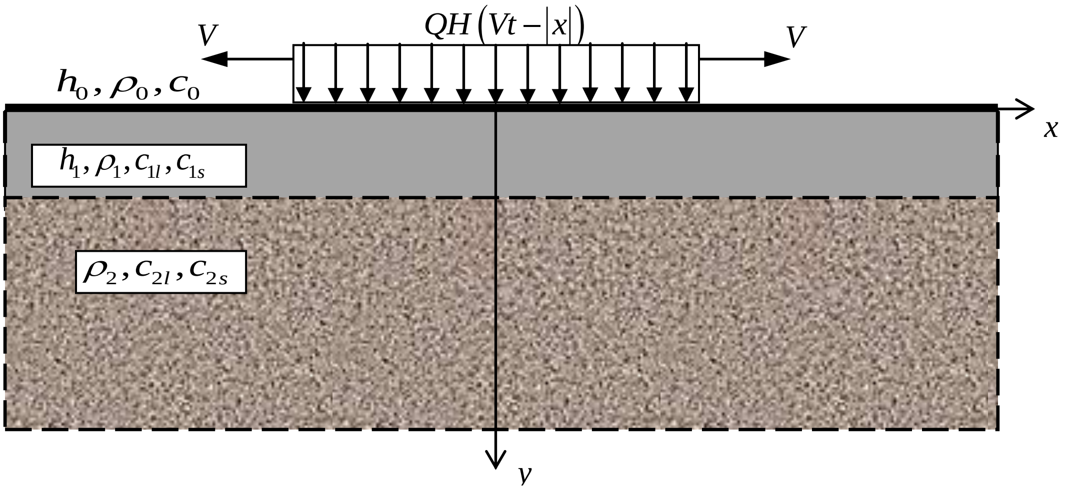


Fig. 1: The considered system: thin plate-layer-halfspace

The theory of dynamic elasticity describes the motion of the layer ($0 < y < h_1$: $j = 1$) and halfspace ($y > h_1$: $j = 2$) as follows:

$$\begin{aligned} \ddot{u}_j &= c_{jl}^2 u''_{j,xx} + c_{js}^2 u''_{j,yy} + \left(c_{jl}^2 - c_{js}^2\right) w''_{j,xy}, \\ \ddot{w}_j &= c_{jl}^2 w''_{j,yy} + c_{js}^2 w''_{j,xx} + \left(c_{jl}^2 - c_{js}^2\right) u''_{j,xy} \quad , \end{aligned} \tag{1}$$

while the classic Bernoulli equation is used for plate dynamic bending:

$$y = 0: \quad \ddot{w} + c_0^2 \left(h_0^2/12\right) w_x^{(IV)} = \left(P - R\right)/\rho_0 h_0 , \tag{2}$$

where $P = QH(Vt - |x|)$ is a moving step load, *R* is the normal reaction of the layer to the plate motion:

$$R(x,t) = \sigma_{yy}^{(1)}(x,0,t) = \rho_1 \left[c_{1l}^2 w'_{1,y} + \left(c_{1l}^2 - 2c_{1s}^2\right) u'_{1,x}\right] \tag{3}$$

All the components of the composition are connected by a rigid contact excluding longitudinal connection between the plate and layer (or the halfspace in the reduced system: plate-halfspace) which are assumed to be absent. So the following relations are proved:

$$\begin{aligned} &y = 0: \quad w_1\left(x,0,t\right) = \sigma_{xy}^{(1)} = 0; \\ &y = h_1: \quad u_1 = u_2,\ w_1 = w_2,\ \sigma_{yy}^{(1)} = \sigma_{yy}^{(2)},\ \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}; \\ &y \to \infty: \quad u_2 \to 0,\ \ w_2 \to 0. \end{aligned} \tag{4}$$

Note that the load source is manifested here somewhat conditionally, it is convenient for analytical and numerical procedures, there is no problem to use the sources appeared in practical problems. For example external pressure waves caused by the air blast in a far field; the action of a plane landing onto a strip; internal sources related to earthquake or underground explosions (then the superposition method of wave theory can be used to calculate the parameters of the moving surface loading).

In the steady-state formulation, the solution of the problem (1) – (4) is found as a superposition of Fourier harmonics, $\exp\left[iq(x \pm ct) - \varsigma y\right]$, propagating along the *x*-axis and exponentially decaying at $y \to \infty$. Factor ζ is calculated from Fourier-transforms of original equations for the halfspace. Then, by using the boundary conditions, the dispersion equation connecting the phase velocity, *c*, and the wave number, *q*, is obtained. This equation is transcendental, it has a cumbersome structure, and its formal expression only is presented below:

$$L\left(q, c; h_0, c_0, h_1, c_{1l}, c_{2l}, \rho_1, c_{2l}, c_{2s}, \rho_2\right) = 0 \qquad (5)$$

In the general case, the analytical solution of Eq. (5) it is not possible to obtain, but there is no problem its computer solving. The some complexity, however, is to conduct (more or less) analysis of its solutions – dependences $c = c\left(q;\Pi\right)$ – where Π is the set of nine parameters signed in (5). Using some of them as measurement units, the set of free parameters can be significantly reduced. Below calculation results are presented in the way to describe common dispersion features with the minimal varying of parameters.

3. DISPESION ANALYSIS

The aim of analysis of Eqn. (5) is to find special points and examine the behavior of dispersion curves in their vicinities.

3.1 Thin plate-halfspace

First, consider a system: thin plate-halfspace, which is the simplest (single-mode) special case of the considered system. Formally, to obtain the mathematical formulation of the dynamics of this reduced system, there is enough to equating parameters of the layer and the halfspace. We introduce the following notation: $c_l = c_{1l} = c_{2l}$, $c_s = c_{1s} = c_{2s}$, $\rho = \rho_1 = \rho_2$. Then Eqn. (5) is written as follows:

$$\rho_0 q\sqrt{1-c^2}\left(c^2 - c_0^2 q^2/12\right)c^2 + c_s^4 L_R = 0,\ L_R = \left(2 - c^2/c_s^2\right)^2 - 4\sqrt{1-c^2}\sqrt{1-c^2/c_s^2} \qquad (6)$$

where c_l, ρ and h_0 are taken as measurement units. Here $L_R = 0$ is the Rayleigh equation for a free halfspace (its single real root is $c = c_R$), and. Eqn. (6) has a single mode $c = c(q)$, which is real if $c < c_s$.

If $q \to 0$, then $c \to c_R$: a plate of finite rigidity and mass does not influence the long wave dispersion (more accurately: the infinitely long wave dispersion). The asymptotic behavior of phase velocity *c* obtained from (6) is

$$\begin{aligned} c &= c_R\left[1 - \alpha\rho_0 q + O\left(q^2\right)\right], \quad \alpha = \tfrac{1}{4}\sqrt{1-c_R^2}\left(c_R/c_s\right)^2 L_1^{-1} > 0, \\ L_1 &= \left(1 + c_s^2 - 2c_R^2\right)\left[\left(1-c_R^2\right)\left(1 - c_R^2/c_s^2\right)\right]^2 - 2 + c_R^2/c_s^2. \end{aligned} \qquad (7)$$

If q is relatively small, c linearly decreases with q increasing. The decreasing rate in c strongly depends on the plate mass and, vice versa, is independent of the plate rigidity. With further increase in q, the decay of $c(q)$ within a middle spectrum stops, the dispersion curve reaches minimum $\left(q_m = q_*,\ c_m = c_*\right)$, and after that it monotonically rises up to $c = c_s$ (remind that real values of c are only if $c \le c_s$).

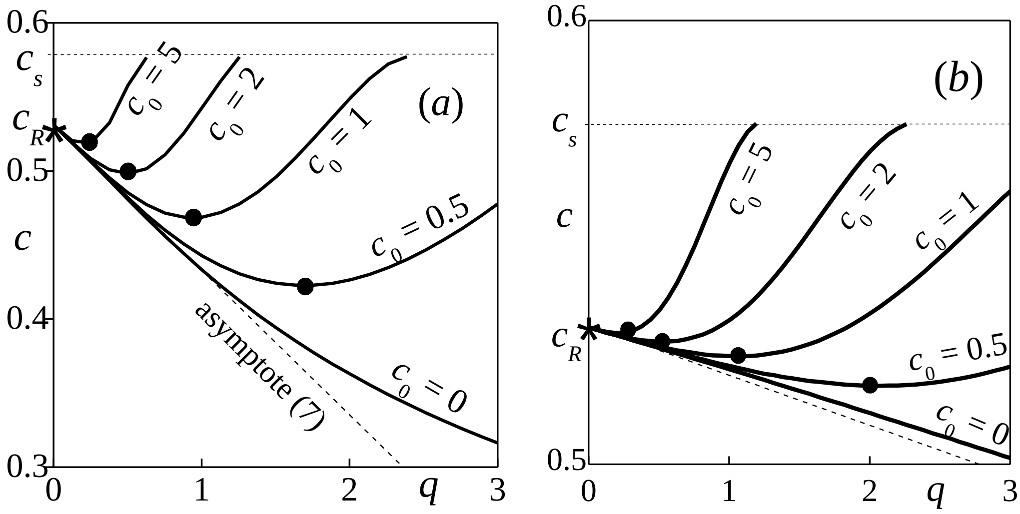


Fig. 2: Dispersion curves for system: plate-halfspace,
(*a*) "heavy" plate, $\rho_0 = 2$; (*b*) "light" plate, $\rho_0 = 0.5$.

Black circles in Fig. 2 are special points (minima) in the middle wave spectrum, while the asterisk is the special point of the Rayleigh surface resonance. Asymptote (7) coincides with a linear part of all curves. Note that here and in the subsequent examples, the Poisson ratios are the same: $\nu_1 = \nu_2 = 0.25$, then $c_{ks} = c_{kl}/\sqrt{3} \approx 0.5774 \ \ (k = 1, 2)$.

It can be shown that points of the minimum are inside the upper domains between two straight lines: asymptote (7) and the plate bending mode $c = c_0 hq/\sqrt{12}$. Coordinates q_m, c_m can be obtained from the following formulas:

$$q_m = 2c_m/c_0\ , \quad (\rho_0/c_0)c_m^5\sqrt{1-c_m^2} + \sqrt{3/8}L_R(c_m) = 0 \qquad (8)$$

So, in the considered case, two critical velocities exist: $V_{cr} = c_R$ in a long wave spectrum $(q \to 0)$ with a low frequency $(\omega \to 0)$, and $V_{cr} = c_m$, within a medium wave spectrum ($q = q_m$).

If it is possible to approximate the dispersion curve in the vicinity q_m by dependence $c \approx c_m + \beta(q - q_m)^n$, β is constant, then the larger index n, the less the dispersion in the vicinity q_m, the wider becomes the wavelength spectrum that shapes the resonance disturbances and the more intense is their growth in time. As it is shown in [10], with the existing the above-mentioned approximation, the resonance growth rate is asymptotically proportional to $t^{(n-1)/n}$ $(t \to \infty)$, while number n ($n \ge 2$) is the first natural number for which $\partial^n c/\partial q^n \ne 0$. Therefore, in the case of a light and pliable plate (i. e. small ρ_0 and c_0) resonance regimes in the medium wave spectrum are to be suspected as more intensive. On the other hand, with increase in c_0 the value of c_m approaches c_R, while q_m is removed into the long wave domain. This fact shows the possibility of a strong superposition of surface waves in the halfspace ($V_{cr} = c_R$, $q \to 0$) and bending waves in the plate ($V_{cr} = c_m$, $q = q_m$), which will considerably strengthen disturbances if V is within interval (c_m, c_R,).

3.2 The system thin plate-thick layer-halfspace

Here as distinct from the previous case infinite number exist of the dispersion equation roots (modes) corresponding free wave modes propagating in a layer of finite thickness. If $q \to 0$, then as in the previous case $c \to c_{R2}$, the Rayleigh velocity in the halfspace. The first (lower) mode has real roots, whereas the higher modes can be real or complex depending on the relation between system parameters. A set of special points can exist in these modes.

Longwave ($q \to 0$) asymptote of the dispersion equation (5) is obtained as the following:

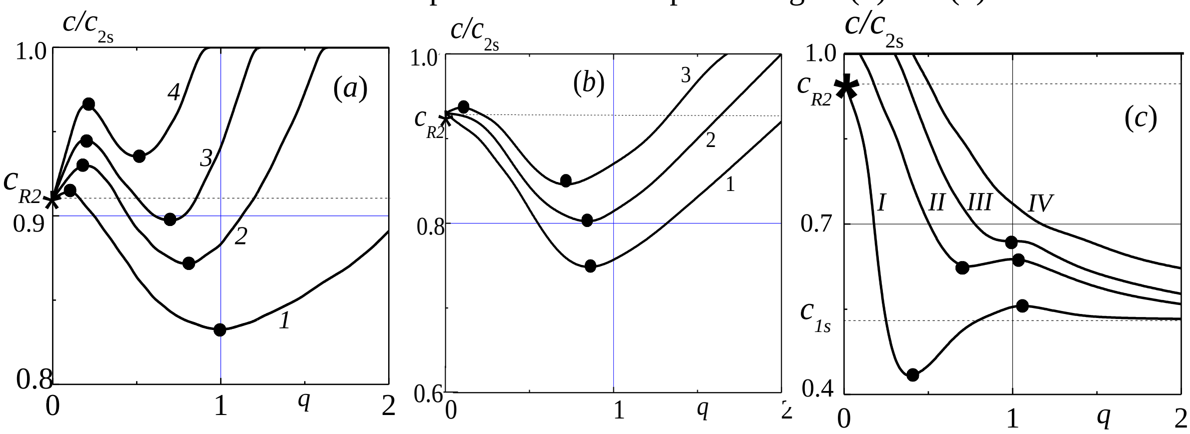
$$L \sim c_{2s}^4 L_{R2} + q\left[\rho_0 c^4 \sqrt{1-c^2} + \rho_1 h_1 \Phi\left(c_{1l}; c_{1s}, c_{2s}\right)\right] \tag{9}$$

where Φ is a finite function. The measurement units are: ρ_2, c_{2l} and h_0. Remind, we assume $\nu_0 = \nu_1 = \nu_2 = 0.25$, then $c_s / c_l = 0.577$ and $c_R = 0.92$. Then only five free parameters are remain in the problem: ρ_0, c_0, ρ_1, h_1, c_{1l}.

As in the previous case, a linear asymptote $c(q)$ is proved by Eq. (9). Here the Rayleigh velocity is also critical for relatively long waves and the plate stiffness does not influence the longwave asymptote.

The main distinction that brings the layer existing is that velocity $c(q)$ can change drastically (decrease or increase) with growth in q. Obtaining analytical estimations of special points in the middle spectrum turns out problematic, but in this case, there is not problematic for the numerical solutions of the original dispersion equation shown as the formal expression in (5).

Below, in Fig. 3 (*a*), (*b*) first modes are depicted calculated for a set of structure parameters, while four first modes can be observed in Fig. 3 (c). For a relatively rigid and heavy layer ($\rho_1 > \rho_2$, $c_{1l} > c_{2l}$), the first mode can receive points of maximum and minimum in the medium wave spectrum: see examples in Figs 4 (*a*) and (*b*).



$\rho_0 = 2.5$, $c_0 = 1.05 c_{1l}$, $\rho_1 = 2$, $h_1 = 10$
1, 2, 3, 4: $c_{1l}/c_{2l} = 1.2, 1.6, 2.0, 2.4$

$\rho_0 = 2.5$, $c_0 = 1.92$, $\rho_1 = 2$, $c_{1l} = 1.6$
curves *1, 2, 3*: $h_1 = 2, 3, 5$

$\rho_0 = 2$, $c_0 = 1.92$, $\rho_1 = 0.6$, $h_1 = 5$,
$c_{1l} = 1.6$; *I - IV* – mode numbers

Fig. 3: Dispersion curves in the system: plate-layer-halfspace

In the example for a lighter ($\rho_1 < \rho_2$) and more pliable layer ($c_{1l} < c_{2l}$) – Figure 3,c, curves of the first and second modes have maximum and minimum, while the third mode they has an inflection point with a tangent parallel to the axis of q. The presence of diverse special points in the narrow spectrum of different modes testifies to the possibility superposing of different oscillation forms with close wavelengths in a narrow range of critical velocities.

4. NON-STEADY-STATE PROBLEM. DEVELOPMENT OF RESONANT WAVES

The diversity of special points in a wide spectrum revealed above proves the development of a set of resonant disturbances if the surface load moves with critical velocities $V_{CR} = c_m$ and $V_{CR} = c_R$. Such processes were examined on the basis of direct numerical modeling of the problem considered. With this aim, an explicit finite-difference scheme is applied with using the special method of the mesh dispersion minimization initially introduced in [17] and developed in [18]. The method enables long- and short-wave components to be calculated with the same accuracy with a static difference mesh.

In Figure 4, an initial stage of resonant wave formation in the system plate-halfspace is shown for two critical velocities of the moving step load: $V = c_R$ and $V = c_m$ and $V_{CR} = c_R$. The depicted curves are rising values of normalized normal reaction R/Q in the mentioned cross-sections on the surface (y = 0). Measurement units are h_0, c_0, ρ_0. One can see a clear distinction between these two cases: (a) a strong rise and a fixed frequency of the flexural resonance at $V = c_m$, while a weak rise and decreasing frequency are detected with time if $V = c_R$. Such peculiarities can partially be forecasted by the analysis of dispersion roots in vicinities of special points, where noticeable flattening of the dispersion curve in the vicinity of the minimum point (q_m, c_m) is detected, which means to a decrease in the level of dispersion and, consequently, the increase in the growth rate of resonant perturbations. On the other hand, a relatively strong dispersion is revealed in the point corresponding to the Rayleigh resonance (q = 0, $c = c_R$). Such estimations can have a local character without claiming to be a kind of the generality: in the beginning of the wave process, it is problematically to establish the common characteristics of propagation of non-steady-state perturbations. In the next Section, we have tried to build analytical solutions allowing highlighting the physical consequences of interest.

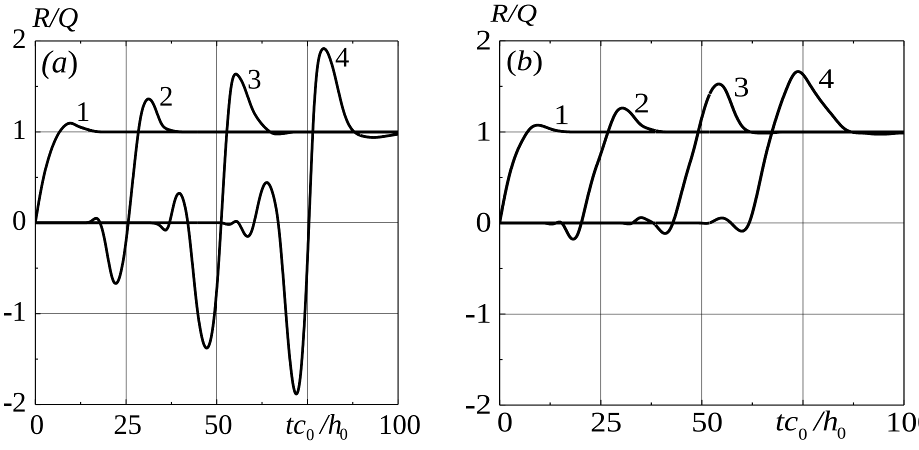


Fig. 4: Formation of resonant process in the system plate-halfspace: ρ_2 = 0.4, c_{2l} =1. Curves 1, 2, 3, 4 correspond to cross-sections x= 0, 10, 20, 30. (a): $V = c_m$ = 0.48, (b): $V = c_R$= 0.53.

The similar process is detected in the system plate-layer-halfspace. In Fig.5 the results are shown calculated in the case of a relatively rigid and heavy halfspace. It can be seen, that in the case of the bending resonance ($V = c_m$), the growth of perturbations is more pronounced.

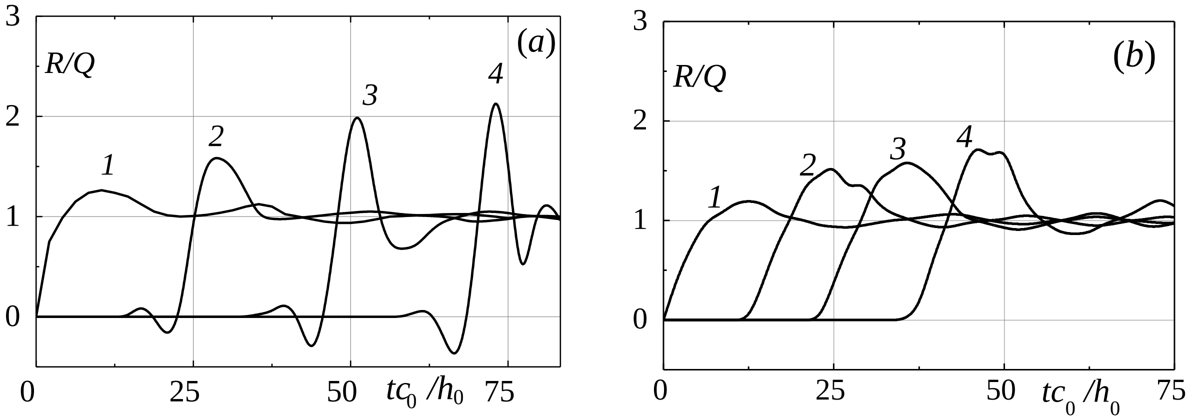


Fig. 5: Formation of resonant process in the system plate-plate-halfspace:
$h_1 = 5$, $\rho_1 = 1.25$, $c_{1l} = 1$, $\rho_2 = 0.4$, $c_{2l} = 0.4$.
Curves 1, 2, 3, 4 correspond to cross-sections $x = 0, 10, 20, 30$.
(a): $V = c_m = 0.45$, (b): $V = c_R = 0.58$.

5. SIMPLIFIED MODELS

Unfortunately, obtaining analytical solutions for resonant waves in the considered systems is problematical. However, such solutions can be successively found on the basis of the well-known approach in which the original complex structure is changed by a simplified structure possessing the closest spectral properties in a detailed spectral band. This approach is notably applicable in the considered above case of flexural resonances where a narrow spectrum surrounding a special point is of interest. With this aim, we have used below two simple models: (a) thin plate upon an elastic foundation and (b) the same system supplemented with distributed inertial masses connected with the plate by inertionless elastic springs (it is clear that there are no Rayleigh points in these models).

The measurement units for these two models are parameters of the plate: h, c, ρ, other parameters of systems are indicated in Fig. 6, the foundation rigidity g - model (a) (the single free parameter), and three free parameters in the model (b): g; G - the rigidity of links connected the plate with the amortized medium, and its mass M.

Fig. 6: Considered simplified systems

The dimensionless equations describing dynamics of simplified models subjected to action of the travelling load are

$$
\begin{aligned}
&(a) \quad \ddot{w} + (1/12)\, w_x^{(IV)} = P - R, \quad R = g_0 w; \\
&(b) \quad \ddot{w} + (1/12)\, w_x^{(IV)} = P - R, \quad R = gw + G(w - W), \\
&\qquad M\ddot{W} - G(w - W) = 0.
\end{aligned}
\tag{10}
$$

where $P = QH(Vt - |x|)$ is the external travelling load, R – is (*a*) the elastic reaction of the foundation, (*b*) the total reaction of the foundation and of the amortized medium. Dispersion equations (5) for these models are disclosed as follows:

$$
\begin{aligned}
&(a) \quad L(q,c;g) = q^4/12 - q^2c^2 + g = 0, \\
&(b) \quad L(q,c;g,G,M) = \left[q^4/12 - q^2c^2 + g + G\right]\left[G - Mq^2c^2\right] - G^2 = 0.
\end{aligned}
\tag{11}
$$

In the case of system (*a*), a single special point - minimum is found:

$$
(a) \quad q_m = (12g)^{1/4}, \;\; c_m = (g/3)^{1/4}, \tag{12}
$$

while in model (*b*), special numerical procedures to solving Eq. (11, *b*) are required to find the coordinates of special points. As it was numerically obtained, the model (*b*) can have from one (minimum) to three (minimum, maximum and inflection) such points, depending on structure parameters.

By variations of free parameters in Eq. (11) we choose such of them that ensure the closest proximity of the dispersion curves of the original and the simplified model within a given spectral band. Such a procedure can be completed, for example, by the least square method. It is clear that the model (*b*) has substantially more possibilities than the model (a) to coincide the dispersion patterns in the original model and simplified those. Nevertheless, we consider just the simplest model (*a*) below with the aim to build an analytic solution for the explored problem.

In Fig. 7, dispersion curves are depicted calculated from (11, *a*) for some values of rigidity *g*. The special points of minimums can be indicated.

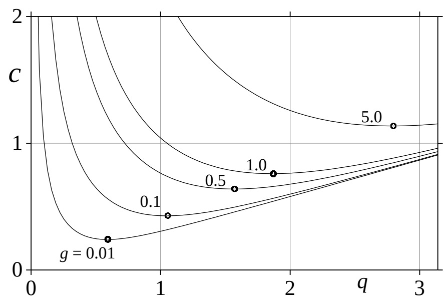


Fig. 7: Dispersion curves for the model (*a*)

6. ASYMPTOTE OF THE FLEXURAL RESONANT PROCESS

Asymptotic solutions we have built basing on the Slepyan approach [9], in which a double Laplace-Fourier transform is reduced to the Laplace transform at the ray $x = V_{cr}t$ with further asymptotic expression of the double Laplace-Fourier images in the vicinity $q = q_m$ (the wave number q is the Fourier transform parameter) and for a great values of time: $t \to \infty$ (that correspond to asymptotic condition $s \to 0$ in the Laplace-Fourier images, where s is the Laplace transform parameter). After these procedures, the precise reversion of the joint Laplace-Fourier transform is completed.

Below we omit rather cumbersome mathematical calculations and show the final formulas of the asymptotic solution to the problem (10, *a*):

$$w(x,t) \sim \frac{\sqrt{t}}{\pi q_m^2 c_m \left(1-c_m^2\right)\sqrt{\varphi}}\left[F_1(\kappa)\cos\eta q_m + F_2(\kappa)\sin\eta q_m\right],$$

$$F_1(\kappa) = \int_0^\infty \frac{\sin z^2 \cos\kappa z dz}{z^2}, \quad F_2(\kappa) = \int_0^\infty \frac{(1-\cos z^2)\cos\kappa z dz}{z^2}, \tag{13}$$

$$\varphi = \frac{1}{2}\left(q\frac{\partial^2 c}{\partial q^2}\right)_{q=q_m} \neq 0, \quad \kappa = \frac{\eta}{\sqrt{\varphi t}}, \quad \eta = c_m t - x.$$

Here η is the stationary phase in the travelling wave. In the vicinity of this phase, wave pattern is marked by the growth of resonant perturbations proportional to $\sqrt{t}$; the oscillating process is described by the sum of envelopes F_1 and F_2 with the sinusoidal saturation with the carrier resonant frequency $\omega_m = c_m q_m$. Integrals F_1 and F_2 have the following analytical expressions:

$$F_1(\kappa) = \sqrt{\frac{\pi}{2}}\left[\cos\left(\frac{\kappa^2}{4}\right)+\sin\left(\frac{\kappa^2}{4}\right)\right] - \frac{\pi|\kappa|}{2}\left[C\left(\frac{\kappa^2}{4}\right)-S\left(\frac{\kappa^2}{4}\right)\right],$$

$$F_2(\kappa) = \sqrt{\frac{\pi}{2}}\left[\cos\left(\frac{\kappa^2}{4}\right)-\sin\left(\frac{\kappa^2}{4}\right)\right] - \frac{\pi|\kappa|}{2}\left[1-C\left(\frac{\kappa^2}{4}\right)-S\left(\frac{\kappa^2}{4}\right)\right],$$

where C(…) and S(…) are the Fresnel integrals. The graph expression of F_1 and F_2 can be seen in Fig. 8.

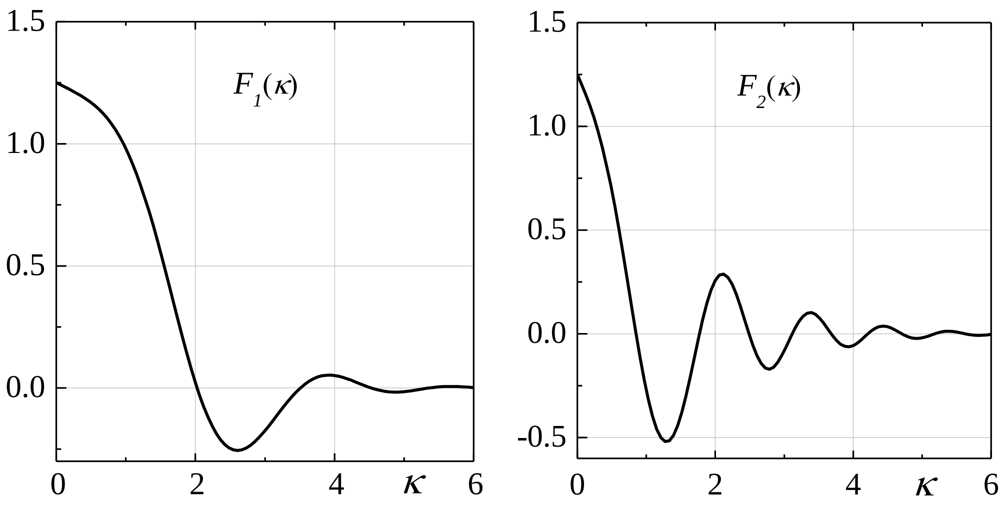


Fig. 8: Envelopes of resonant waves

The envelopes were named as quasi-stationary those [9], and the obtained solution is proved within the vicinity of the stationary phase $\eta = c_m t - x$. This vicinity extends with time proportionally $\sqrt{t}$.

Below we have presented the computer solution of Eq. (10,*a*) in the case g =0.01 (then, q_m = 0.5771, c_m = 0.2295). Finite-difference mesh together with the mesh dispersion minimization approach has been applied with this aim. In Fig.9, the comparison is presented of wave patterns at the resonance velocity of the moving load: (*a*) $V = V_{cr} = c_m$, and velocities closed to it: (*a*) $V = 0.9V_{cr}$ and (*c*) $V = 1.1V_{cr}$.

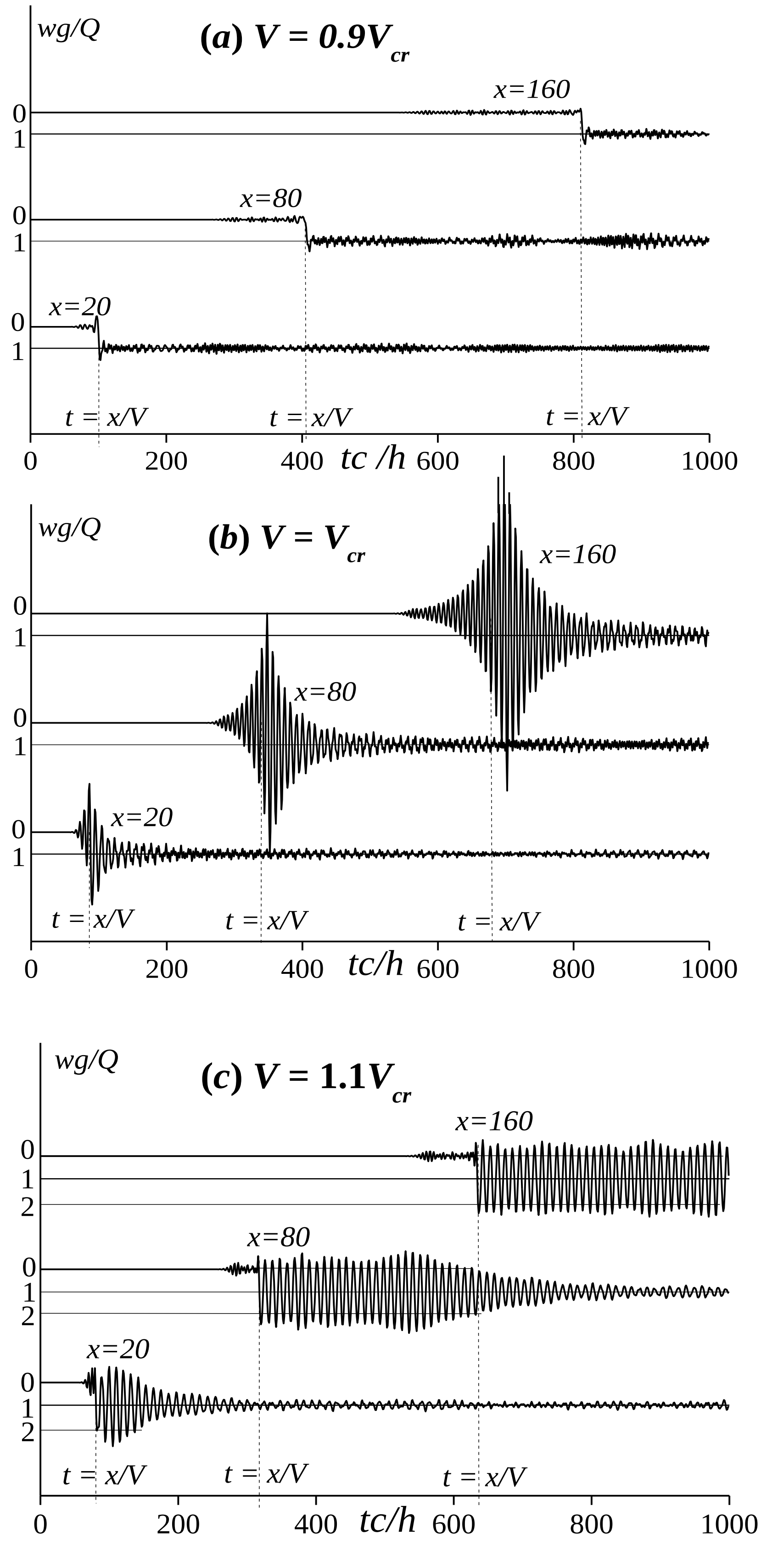


Fig. 9: Wave patterns at different velocities of the travelling load

Note that positive direction of axis y is trended downward. The quasi-static solution of the problem is w=Q/g $w = (Q/g) H(Vt - x)$. It can be seen strong distinction for resonance and non-resonance processes in spite of a small difference of the velocities *V*. The comparison of computation results and the asymptotic solution (13) show that the asymptotic solution turns out practically exact if the cross-sections in which recording conducted of the observed process is located to the right of the point x =25h.

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