VARIATIONAL PRINCIPLES FOR NON - BAROTROPIC MAGNETOHYDRODYNAMICS A TOOL FOR EVALUATION OF PLASMA PROCESSES

A. Yahalom

Ariel University, Ariel 40700, Israel
asya@ariel.ac.il

Variational principles for magnetohydrodynamics (MHD) were introduced by previous authors both in Lagrangian and Eulerian form. In this paper we introduce simpler Eulerian variational principles from which all the relevant equations of non-barotropic MHD can be derived for certain field topologies. The variational principle is given in terms of five independent functions for non-stationary non-barotropic flows. This is less then the eight variables which appear in the standard equations of barotropic MHD which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$, the entropy s and the density ρ.

The case of non-barotropic MHD in which the internal energy is a function of both entropy and density was not discussed in previous works which were concerned with the simplistic barotropic case. It is important to understand the rule of entropy and temperature for the variational analysis of MHD. Thus we introduce a variational principle of non-barotropic MHD and show that five functions will suffice to describe this physical system.

We will also discuss the implications of the above analysis for topological constants. It will be shown that while cross helicity is not conserved for non-barotropic MHD a variant of this quantity is.

Introduction

Cross Helicity was first described by Woltjer [1, 2] and is given by:

$$H_C \equiv \int \vec{B} \cdot \vec{v} d^3 x, \tag{1}$$

in which $\vec{B}$ is the magnetic field, $\vec{v}$ is the velocity field and the integral is taken over the entire flow domain. H_C is conserved for barotropic or incompressible MHD and is given a topological interpretation in terms of the knottiness of magnetic and flow field lines. An analogous conserved helicity for fluid dynamics was obtained by Moffatt [3]. Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [4-7]. In the non-barotropic case cross helicity is not conserved, hence one may ask if this quantity can be amended in such a way that we obtain a conserved quantity. We will give a positive answer to this question in this paper and will use variational analysis to generate a conserved cross helicity for non-barotropic MHD.

Variational principles for MHD were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [8] has discussed in his book a Lagrangian variational formalism for MHD. Vladimirov and Moffatt [9] in a series of papers have discussed an Eulerian variational principle for incompressible MHD. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible MHD which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$ and the pressure P. Kats [10] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats has shown that the variables he suggested can be utilized to describe the motion of arbitrary discontinuity surfaces [11, 12]. Sakurai [13] has introduced a two function Eulerian variational principle for force-free MHD and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [8]. A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos [15]. Yahalom & Lynden-Bell [7] combined the Lagrangian of Sturrock [8] with the Lagrangian of Sakurai [13] to obtain an **Eulerian** Lagrangian principle for barotropic MHD which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic MHD without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [18] (see also [19]). Yahalom [16] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [17] are topological local conserved quantities.

Previous work was concerned only with barotropic MHD. Variational principles of non barotropic MHD can be found in the work of Bekenstein & Oron [20] in terms of 15 functions and V.A. Kats [10] in terms of 20 functions. The author of this paper suspect that this number can be somewhat reduced. Moreover, A. V. Kats in a remarkable paper [21] (section IV, E) has shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions, this implies that the number of degrees of freedom can be reduced. Here we will show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [13] representation for the magnetic field. Morrison [14] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [14]).

The plan of this paper is as follows: First we introduce the standard notations and equations of non-barotropic MHD. Next we introduce a generalization of the barotropic variational principle suitable for the non-barotropic case. Later we simplify the Eulerian variational principle and formulate it in terms of eight functions. Next we show how three variational variables can be integrated algebraically thus reducing the variational principle to five functions. We conclude by discussing cross helicity conservation for non-barotropic MHD.

Standard formulation of non-barotropic MHD

The standard set of equations solved for non-barotropic MHD are given below:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\mathrm{B}}), \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{4}$$

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}) = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}. \tag{5}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0. \tag{6}$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{\mathrm{d}}{\mathrm{d}t}$ is the temporal material derivative and $\vec{\nabla}$ has its standard meaning in vector calculus. $\vec{B}$ is the magnetic field vector, $\vec{v}$ is the velocity field vector, ρ is the fluid density and s is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case).

The justification for those equations and the conditions under which they apply can be found in standard books on MHD (see for example [8]). The above applies to a collision - dominated plasma in local thermodynamic equilibrium. Such conditions are seldom satisfied by physical plasmas, certainly not in astrophysics or in fusion-relevant magnetic confinement experiments. Never the less it is believed that the fastest macroscopic instabilities in those systems obey the above equations [17], while instabilities associated with viscous or finite conductivity terms are slower. It should be noted that due to a theorem by Bateman [23] every physical system can be described by a variational principle (including viscous plasma) the trick is to find an elegant variational principle usually depending on a small amount of variational variables. The current work will discuss only ideal MHD while viscous MHD will be left for future endeavors.

Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (3) describes the fact that the magnetic field is solenoidal, equation (4) describes the conservation of mass and equation (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\vec{J} = \frac{\vec{\nabla} \times \vec{B}}{4\pi}, \tag{7}$$

is the electric current density which is not connected to any mass flow. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight ($\vec{v}$, $\vec{B}$, ρ, s) and the number of equations (2, 4, 5, 6) is also eight. Notice that equation (3) is a condition on the initial $\vec{B}$ field and is satisfied automatically for any other time due to equation (2).

Variational principle of non-barotropic MHD

In the following section we will generalize the approach of [7] for the non-barotropic case. Consider the action:

$$A \equiv \int L d^3x dt,$$

$$L \equiv L_1 + L_2, \tag{8}$$

$$L_1 \equiv \rho(\frac{1}{2}\vec{v}^2 - \varepsilon(\rho, s)) + \frac{\vec{B}^2}{8\pi},$$

$$L_2 \equiv \nu[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v})] - \rho\alpha \frac{d\chi}{dt} - \rho\beta \frac{d\eta}{dt} - \rho\sigma \frac{ds}{dt} - \frac{\vec{B}}{4\pi} \cdot \vec{\nabla}\chi \times \vec{\nabla}\eta.$$

In the above ε is the specific internal energy (internal energy per unit of mass). The reader is reminded of the following thermodynamic relations which will become useful later:

$$d\varepsilon = Tds - Pd\frac{1}{\rho} = Tds + \frac{P}{\rho^2} d\rho$$

$$\frac{\partial \varepsilon}{\partial s}=T,\ \frac{\partial \varepsilon}{\partial \rho}=\frac{P}{\rho^2}$$
$$w=\varepsilon+\frac{P}{\rho}=\varepsilon+\frac{\partial \varepsilon}{\partial \rho}\rho=\frac{\partial(\rho\varepsilon)}{\partial \rho} \tag{9}$$
$$dw=d\varepsilon+d(\frac{P}{\rho})=Tds+\frac{1}{\rho}dP$$

in the above T is the temperature and w is the specific enthalpy. Obviously ν, α, β, σ are Lagrange multipliers which were inserted in such a way that the variational principle will yield the following equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0,\\
&\rho\frac{d\chi}{dt}=0,\\
&\rho\frac{d\eta}{dt}=0.\\
&\rho\frac{ds}{dt}=0.
\end{aligned} \tag{10}$$

It **is not** assumed that ν, α, β, σ are single valued. Provided ρ is not null those are just the continuity equation (4), entropy conservation and the conditions that Sakurai's functions are comoving. Taking the variational derivative with respect to $\vec{B}$ we see that

$$\vec{B}=\hat{\vec{B}}\equiv\vec{\nabla}\chi\times\vec{\nabla}\eta. \tag{11}$$

Hence $\vec{B}$ is in Sakurai's form and satisfies equation (3). It can be easily shown that provided that $\vec{B}$ is in the form given in equation (11), and equations (10) are satisfied, then also equation (2) is satisfied.

For the time being we have showed that all the equations of non-barotropic MHD can be obtained from the above variational principle except Euler's equations. We will now show that Euler's equations can be derived from the above variational principle as well. Let us take an arbitrary variational derivative of the above action with respect to $\vec{v}$, this will result in:

$$\delta_{\vec{v}} A = \int dt \{ \int d^3 x dt \rho \delta \vec{v} \cdot [\vec{v} - \vec{\nabla} \nu - \alpha \vec{\nabla} \chi - \\ - \beta \vec{\nabla} \eta - \sigma \vec{\nabla} s] + \oint d\vec{S} \cdot \delta \vec{v} \rho \delta + \int d\vec{\Sigma} \cdot \delta \vec{v} \rho [\nu] \}. \tag{12}$$

The integral $\oint d\vec{S} \cdot \delta \vec{v} \rho \nu$ vanishes in many physical scenarios. In the case of astrophysical flows this integral will vanish since $\rho = 0$ on the flow boundary, in the case of a fluid contained in a vessel no flux boundary conditions $\delta \vec{v} \cdot \hat{n} = 0$ are induced ($\hat{n}$ is a unit vector normal to the boundary). The surface integral $\int d\vec{\Sigma}$ on the cut of ν vanishes in the case that ν is single valued and [ν] = 0 as is the case for some flow topologies. In the case that ν is not single valued only a Kutta type velocity perturbation [22] in which the velocity perturbation is parallel to the cut will cause the cut integral to vanish. An arbitrary velocity perturbation on the cut will indicate that ρ = 0 on this surface which is contradictory to the fact that a cut surface is to some degree arbitrary as is the case for the zero line of an azimuthal angle. We will show later that the "cut" surface is co-moving with the flow hence it may become quite complicated. This uneasy situation may be somewhat be less restrictive when the flow has some symmetry properties.

Provided that the surface integrals do vanish and that $\delta_{\vec{v}} A = 0$ for an arbitrary velocity perturbation we see that $\vec{v}$ must have the following form:

$$\vec{v} = \hat{\vec{v}} \equiv \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta + \sigma \vec{\nabla} s. \tag{13}$$

The above equation is reminiscent of Clebsch representation in non magnetic fluids [24, 25]. Let us now take the variational derivative with respect to the density ρ we obtain:

$$\delta_\rho A = \int d^3 x dt \delta \rho [\frac{1}{2} \vec{v}^2 - w - \frac{\partial \nu}{\partial t} - \vec{v} \cdot \vec{\nabla} \nu] + \\ + \int dt \oint d\vec{S} \cdot \vec{v} \delta \rho \nu + \int dt \int d\vec{\Sigma} \cdot \vec{v} \delta \rho [\nu] + \int d^3 x \nu \delta \rho |_{t_0}^{t_1} . \tag{14}$$

In which $w = \frac{\partial (\varepsilon \rho)}{\partial \rho}$ is the specific enthalpy. Hence provided that $\oint d\vec{S} \cdot \vec{v} \delta \rho \nu$ vanishes on the boundary of the domain and $\int d\vec{\Sigma} \cdot \vec{v} \delta \rho [\nu]$

on the cut of ν in the case that ν is not single valued[1] and in initial and final times the following equation must be satisfied:

$$\frac{d\nu}{dt} = \frac{1}{2}\vec{v}^2 - w. \tag{15}$$

Since the right hand side of the above equation is single valued as it is made of physical quantities, we conclude that:

$$\frac{d[\nu]}{dt} = 0. \tag{16}$$

Hence the cut value is co-moving with the flow and thus the cut surface may become arbitrary complicated. This uneasy situation may be somewhat be less restrictive when the flow has some symmetry properties.

Finally we have to calculate the variation with respect to both χ and η this will lead us to the following results:

$$\begin{aligned} \delta_\chi A = \int d^3x dt \delta\chi[\frac{\partial(\rho\alpha)}{\partial t} + \vec{\nabla}\cdot(\rho\alpha\vec{v}) - \vec{\nabla}\eta\cdot\vec{J}] + \int dt \oint d\vec{S}\cdot[\frac{\vec{B}}{4\pi}\times\vec{\nabla}\eta - \\ -\vec{v}\rho\alpha]\delta\chi + \int dt \int d\vec{\Sigma}\cdot[\frac{\vec{B}}{4\pi}\times\vec{\nabla}\eta - \vec{v}\rho\alpha][\delta\chi] - \int d^3x \rho\alpha\delta\chi |_{t_0}^{t_1}, \end{aligned} \tag{17}$$

$$\begin{aligned} \delta_\eta A = \int d^3x dt \delta\eta[\frac{\partial(\rho\beta)}{\partial t} + \vec{\nabla}\cdot(\rho\beta\vec{v}) + \vec{\nabla}\chi\cdot\vec{J}] + \int dt \oint d\vec{S}\cdot[\vec{\nabla}\chi\times\frac{\vec{B}}{4\pi} - \\ -\vec{v}\rho\beta]\delta\eta + \int dt \int d\vec{\Sigma}\cdot[\vec{\nabla}\chi\times\frac{\vec{B}}{4\pi} - \vec{v}\rho\beta][\delta\eta] - \int d^3x \rho\beta\delta\eta |_{t_0}^{t_1}. \end{aligned} \tag{18}$$

Provided that the correct temporal and boundary conditions are met with respect to the variations $\delta\chi$ and $\delta\eta$ on the domain boundary and on the cuts in the case that some (or all) of the relevant functions are non single valued. We obtain the following set of equations:

$$\frac{d\alpha}{dt} = \frac{\vec{\nabla}\eta\cdot\vec{J}}{\rho}, \quad \frac{d\beta}{dt} = -\frac{\vec{\nabla}\chi\cdot\vec{J}}{\rho}, \tag{19}$$

[1] Which entails either a Kutta type condition for the velocity in contradiction to the "cut" being an arbitrary surface, or a vanishing density perturbation on the cut.

in which the continuity equation (4) was taken into account. By correct temporal conditions we mean that both $\delta\eta$ and $\delta\chi$ vanish at initial and final times. As for boundary conditions which are sufficient to make the boundary term vanish on can consider the case that the boundary is at infinity and both $\vec{B}$ and ρ vanish. Another possibility is that the boundary is impermeable and perfectly conducting. A sufficient condition for the integral over the "cuts" to vanish is to use variations $\delta\eta$ and $\delta\chi$ which are single valued. It can be shown that χ can always be taken to be single valued, hence taking $\delta\chi$ to be single valued is no restriction at all. In some topologies η is not single valued and in those cases a single valued restriction on $\delta\eta$ is sufficient to make the cut term null.

Finally we take a variational derivative with respect to the entropy s:

$$\delta_s A = \int d^3x dt \delta s[\frac{\partial(\rho\sigma)}{\partial t} + \vec{\nabla}\cdot(\rho\sigma\vec{v}) - \rho T] + \\ + \int dt \oint d\vec{S}\cdot\rho\sigma\vec{v}\delta s - \int d^3x \rho\sigma\delta s |_{t_0}^{t_1}, \tag{20}$$

in which the temperature is $T = \frac{\partial\varepsilon}{\partial s}$. We notice that according to equation (13) σ is single valued and hence no cuts are needed. Taking into account the continuity equation (4) we obtain for locations in which the density ρ is not null the result:

$$\frac{d\sigma}{dt} = T, \tag{21}$$

provided that $\delta_s A$ vanished for an arbitrary δs.

Euler's equations

We shall now show that a velocity field given by equation (13), such that the equations for α, β, χ, η, ν, σ, s satisfy the corresponding equations (10, 15, 19, 21) must satisfy Euler's equations. Let us calculate the material derivative of $\vec{v}$:

$$\frac{d\vec{v}}{dt} = \frac{d\vec{\nabla}\nu}{dt} + \frac{d\alpha}{dt}\vec{\nabla}\chi + \alpha\frac{d\vec{\nabla}\chi}{dt} + \frac{d\beta}{dt}\vec{\nabla}\eta + \beta\frac{d\vec{\nabla}\eta}{dt} + \frac{d\sigma}{dt}\vec{\nabla}s + \sigma\frac{d\vec{\nabla}s}{dt}. \tag{22}$$

It can be easily shown that:

$$\frac{d\vec{\nabla}\nu}{dt} = \vec{\nabla}\frac{d\nu}{dt} - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k} = \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k},$$

$$\frac{d\vec{\nabla}\eta}{dt} = \vec{\nabla}\frac{d\eta}{dt} - \vec{\nabla}v_k\frac{\partial\eta}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\eta}{\partial x_k},$$
$$\frac{d\vec{\nabla}\chi}{dt} = \vec{\nabla}\frac{d\chi}{dt} - \vec{\nabla}v_k\frac{\partial\chi}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\chi}{\partial x_k}, \tag{23}$$
$$\frac{d\vec{\nabla}s}{dt} = \vec{\nabla}\frac{ds}{dt} - \vec{\nabla}v_k\frac{\partial s}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial s}{\partial x_k}.$$

In which x_k is a Cartesian coordinate and a summation convention is assumed. Inserting the result from equations (23, 10) into equation (22) yields:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}v_k\left(\frac{\partial\nu}{\partial x_k} + \alpha\frac{\partial\chi}{\partial x_k} + \beta\frac{\partial\eta}{\partial x_k} + \sigma\frac{\partial s}{\partial x_k}\right) + \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 - w\right) + T\vec{\nabla}s +$$
$$+\frac{1}{\rho}((\vec{\nabla}\eta\cdot\vec{J})\vec{\nabla}\chi - (\vec{\nabla}\chi\cdot\vec{J})\vec{\nabla}\eta) = -\vec{\nabla}v_k v_k + \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 - w\right) + T\vec{\nabla}s + \tag{24}$$
$$+\frac{1}{\rho}\vec{J}\times(\vec{\nabla}\chi\times\vec{\nabla}\eta) = -\frac{\vec{\nabla}p}{\rho} + \frac{1}{\rho}\vec{J}\times\vec{B}.$$

In which we have used both equation (13) and equation (11) in the above derivation. This of course proves that the non-barotropic Euler equations can be derived from the action given in equation (8) and hence all the equations of non-barotropic MHD can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts.

Simplified action

The reader of this paper might argue here that the paper is misleading. The author has declared that he is going to present a simplified action for non-barotropic MHD instead he added six more functions α, β, χ, η, ν, σ to the standard set $\vec{B}$, $\vec{v}$, ρ, s. In the following I will show that this is not so and the action given in equation (8) in a form suitable for a pedagogic presentation can indeed be simplified. It is easy to show that the Lagrangian density appearing in equation (8) can be written in the form:

$$L = -\rho\left[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \sigma\frac{\partial s}{\partial t} + \varepsilon(\rho,s)\right] + \frac{1}{2}\rho[(\vec{v}-\hat{\vec{v}})^2 - (\hat{\vec{v}})^2] +$$
$$+\frac{1}{8\pi}[(\vec{B}-\hat{\vec{B}})^2 - (\hat{\vec{B}})^2] + \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla}\cdot(\nu\rho\vec{v}). \tag{25}$$

In which is a shorthand notation for notation for $\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s$ (see equation (13)) $\vec{v}$ and $\vec{B}$ is a shorthand notation for $\vec{\nabla}\chi\times\vec{\nabla}\eta$ (see equation (11)). Thus L has four contributions:

$$
\begin{aligned}
L &= L + L_{\vec{v}} + L_{\vec{B}} + L_{\text{boundary}}, \\
L &\equiv -\rho\left[\frac{\partial\nu}{\partial t}+\alpha\frac{\partial\chi}{\partial t}+\beta\frac{\partial\eta}{\partial t}+\sigma\frac{\partial s}{\partial t}+\varepsilon(\rho,s)+\frac{1}{2}(\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s)^2\right]- \\
&\quad -\frac{1}{8\pi}(\vec{\nabla}\chi\times\vec{\nabla}\eta)^2, \\
L_{\vec{v}} &\equiv \frac{1}{2}\rho(\vec{v}-\hat{\vec{v}})^2, \\
L_{\vec{B}} &\equiv \frac{1}{8\pi}(\vec{B}-\hat{\vec{B}})^2, \\
L_{\text{boundary}} &\equiv \frac{\partial(\nu\rho)}{\partial t}+\vec{\nabla}\cdot(\nu\rho\vec{v}).
\end{aligned}
\tag{26}
$$

The only term containing $\vec{v}$ is[2] $L_{\vec{v}}$, it can easily be seen that this term will lead, after we nullify the variational derivative with respect to $\vec{v}$, to equation (13) but will otherwise have no contribution to other variational derivatives. Similarly the only term containing $\vec{B}$ is $L_{\vec{B}}$ and it can easily be seen that this term will lead, after we nullify the variational derivative, to equation (11) but will have no contribution to other variational derivatives. Also notice that the term L_{boundary} contains only complete partial derivatives and thus can not contribute to the equations although it can change the boundary conditions. Hence we see that equations (10), equation (15) , equations (19) and equation (21) can be derived using the Lagrangian density:

$$
\begin{aligned}
L[\alpha,\beta,\chi,\eta,\nu,\rho,\sigma,s] &= -\rho[\frac{\partial\nu}{\partial t}+\alpha\frac{\partial\chi}{\partial t}+\beta\frac{\partial\eta}{\partial t}+\sigma\frac{\partial s}{\partial t}+ \\
&+\varepsilon(\rho,s)+\frac{1}{2}(\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s)^2]-\frac{1}{8\pi}(\vec{\nabla}\chi\times\vec{\nabla}\eta)^2
\end{aligned}
\tag{27}
$$

---

[2] L_{boundary} also depends on $\vec{v}$ but being a boundary term is space and time it does not contribute to the derived equations.

in which $\hat{\vec{v}}$ replaces $\vec{v}$ and $\hat{\vec{B}}$ replaces $\vec{B}$ in the relevant equations. Furthermore, after integrating the eight equations (10, 15, 19, 21) we can insert the potentials α, β, χ, η, ν, σ, s into equations (13) and (11) to obtain the physical quantities $\vec{v}$ and $\vec{B}$. Hence, the general non-barotropic MHD problem is reduced from eight equations (2, 4, 5, 6) and the additional constraint (3) to a problem of eight first order (in the temporal derivative) unconstrained equations. Moreover, the entire set of equations can be derived from the Lagrangian density L.

Further simplification

Elimination of Variables

Let us now look at the three last three equations of (10). Those describe three comoving quantities which can be written in terms of the generalized Clebsch form given in equation (13) as follows:

$$\begin{aligned}
&\frac{\partial\chi}{\partial t}+(\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s)\cdot\vec{\nabla}\chi=0\\
&\frac{\partial\eta}{\partial t}+(\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s)\cdot\vec{\nabla}\eta=0\\
&\frac{\partial s}{\partial t}+(\vec{\nabla}\nu+\alpha\vec{\nabla}\chi+\beta\vec{\nabla}\eta+\sigma\vec{\nabla}s)\cdot\vec{\nabla}s=0
\end{aligned}\qquad(28)$$

Those are algebraic equations for α, β, σ. Which can be solved such that α, β, σ can be written as functionals of χ, η, ν, s, resulting eventually in the description of non-barotropic MHD in terms of five functions: ν, ρ, χ, η, s. Let us introduce the notation:

$$\alpha_i\equiv(\alpha,\beta,\sigma),\quad \chi_i\equiv(\chi,\eta,s),\quad k_i\equiv-\frac{\partial\chi_i}{\partial t}-\vec{\nabla}\nu\cdot\vec{\nabla}\chi_i,\quad i\in(1,2,3)\qquad(29)$$

In terms of the above notation equation (28) takes the form:

$$k_i=\alpha_j\vec{\nabla}\chi_i\cdot\vec{\nabla}\chi_j,\quad j\in(1,2,3)\qquad(30)$$

in which the Einstein summation convention is assumed. Let us define the matrix:

$$A_{ij}\equiv\vec{\nabla}\chi_i\cdot\vec{\nabla}\chi_j\qquad(31)$$

obviously this matrix is symmetric since $A_{ij}=A_{ji}$. Hence equation (30) takes the form:

$$k_i = A_{ij}\alpha_j,\ j \in (1,2,3). \tag{32}$$

Provided that the matrix A_{ij} is not singular it has an inverse A^{-1}_{ij} which can be written as:

$$A^{-1}_{ij} = |A|^{-1}\begin{pmatrix} A_{22}A_{33} - A_{23}^2 & A_{13}A_{23} - A_{12}A_{33} & A_{12}A_{23} - A_{13}A_{22} \\ A_{13}A_{23} - A_{12}A_{33} & A_{11}A_{33} - A_{13}^2 & A_{12}A_{13} - A_{11}A_{23} \\ A_{12}A_{23} - A_{13}A_{22} & A_{12}A_{13} - A_{11}A_{23} & A_{11}A_{22} - A_{12}^2 \end{pmatrix} \tag{33}$$

In which the determinant $|A|$ is given by the following equation:

$$|A| = A_{11}A_{22}A_{33} - A_{11}A_{23}^2 - A_{22}A_{13}^2 - A_{33}A_{12}^2 + 2A_{12}A_{13}A_{23}. \tag{34}$$

In terms of the above equations the $\alpha_i's$ can be calculated as functionals of χ_i, ν as follows:

$$\alpha_i[\chi_i,\ \nu] = A_j^{-1}k_j. \tag{35}$$

The velocity equation (13) can now be written as:

$$\vec{v} = \vec{\nabla}\nu + \alpha_i\vec{\nabla}\chi_i = \vec{\nabla}\nu + A^{-1}_{ij}k_j\vec{\nabla}\chi_i = \vec{\nabla}\nu - A^{-1}_{ij}\vec{\nabla}\chi_i\left(\frac{\partial\chi_j}{\partial t} + \vec{\nabla}\nu\cdot\vec{\nabla}\chi_j\right). \tag{36}$$

Provided that the χ_i is a coordinate basis in three dimensions, we may write:

$$\vec{\nabla}\nu = \vec{\nabla}\chi_n\frac{\partial\nu}{\partial\chi_n},\ n \in (1,\ 2,\ 3). \tag{37}$$

Inserting equation (37) into equation (36) we obtain:

$$\begin{aligned} \vec{v} &= -A^{-1}_{ij}\vec{\nabla}\chi_i\frac{\partial\chi_j}{\partial \mathrm{t}} + \vec{\nabla}\nu - A^{-1}_{ij}\vec{\nabla}\chi_i\frac{\partial\nu}{\partial\chi_n}\vec{\nabla}\chi_n\cdot\vec{\nabla}\chi_j = -A^{-1}_{ij}\vec{\nabla}\chi_i\frac{\partial\chi_\mathrm{j}}{\partial t} + \\ &+\vec{\nabla}\nu - A^{-1}_{ij}\mathrm{A}_{jn}\vec{\nabla}\chi_i\frac{\partial\nu}{\partial\chi_n} = -A^{-1}_{ij}\vec{\nabla}\chi_\mathrm{i}\frac{\partial\chi_j}{\partial t} + \vec{\nabla}\nu - \delta_{in}\vec{\nabla}\chi_i\frac{\partial\nu}{\partial\chi_n} = \\ &= -A^{-1}_{ij}\vec{\nabla}\chi_i\frac{\partial\chi_j}{\partial t} + \vec{\nabla}\nu - \vec{\nabla}\chi_n\frac{\partial\nu}{\partial\chi_n} = -A^{-1}_{ij}\vec{\nabla}\chi_i\frac{\partial\chi_j}{\partial t} \end{aligned} \tag{38}$$

in the above δ_{in} is a Kronecker delta. Thus the velocity $\vec{v}\,[\chi_i]$ is a functional of χ_i only and is independent of ν.

Lagrangian Density and Variational Analysis

Let us now rewrite the Lagrangian density $L[\chi_i, \nu, \rho]$ given in equation (27) in terms of the new variables:

$$L[\chi_i,\nu,\rho] = -\rho[\frac{\partial \nu}{\partial t} + \alpha_k[\chi_i, nu]\frac{\partial \chi_k}{\partial t} + \varepsilon(\rho,\chi_3) + \\ + \frac{1}{2}\vec{v}[\chi_i]^2] - \frac{1}{8\pi}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2)^2. \tag{39}$$

Let us calculate the variational derivative of $L[\chi_i, \nu, \rho]$ with respect to χ_i this will result in:

$$\delta_{\chi_i} L = -\rho[\delta_{\chi_i}\alpha_k \frac{\partial \chi_k}{\partial t} + \alpha_i \frac{\partial \delta\chi_i}{\partial \mathrm{t}} + \delta_{\chi_i}\varepsilon(\rho,\chi_3) + \delta_{\chi_i}\vec{v}\cdot\vec{v}] - \\ - \frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2) \tag{40}$$

in which the summation convention is not applied if the index is underlined. However, due to equation (36) we may write:

$$\delta_{\chi_i}\vec{v} = \delta_{\chi_i}\alpha_k \vec{\nabla}\chi_k + \alpha_i \vec{\nabla}\delta\chi_i. \tag{41}$$

Inserting equation (41) into equation (40) and rearranging the terms we obtain:

$$\delta_{\chi_i} L = -\rho[\delta_{\chi_i}\alpha_k(\frac{\partial \chi_k}{\partial t} + \vec{v}\cdot\vec{\nabla}\chi_k) + \alpha_i(\frac{\partial \delta\chi_i}{\partial t} + \vec{v}\cdot\vec{\nabla}\delta\chi_i) + \\ + \delta_{\chi_i}\varepsilon(\rho,\chi_3)] - \frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2). \tag{42}$$

Now by construction $\vec{v}$ satisfies equation (28) and hence $\frac{\partial \chi_k}{\partial t} + \vec{v}\cdot\vec{\nabla}\chi_k = 0$, this leads to:

$$\delta_{\chi_i} L = -\rho\left[\alpha_i \frac{d\delta\chi_i}{dt} + \delta_{\chi_i}\varepsilon(\rho,\chi_3)\right] - \frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2). \tag{43}$$

From now on the derivation proceeds as in equations (17, 18, 20) resulting in equations (19, 21) and will not be repeated. The difference is that now α, β and σ are not independent quantities, rather they depend through equation (35) on the derivatives of χ_i, ν. Thus, equations (17, 18, 20) are not first order equations in time but are second order equations. Now let us calculate the variational derivative with respect to ν this will result in the expression:

$$\delta_\nu L = -\rho[\frac{\partial \delta\nu}{\partial t} + \delta_\nu \alpha_n \frac{\partial \chi_n}{\partial t}]. \tag{44}$$

However, $\delta_\nu \alpha_k$ can be calculated from equation (35) :

$$\delta_\nu \alpha_n = \mathrm{A}^{-1}_{nj} \delta_\nu \mathrm{k}_j = -A^{-1}_{nj} \vec\nabla \delta\nu \cdot \vec\nabla \chi_\mathrm{j}. \tag{45}$$

Inserting the above equation into equation (44):

$$\delta_\nu L = -\rho[\frac{\partial \delta\nu}{\partial t} - A^{-1}_{nj} \vec\nabla \chi_j \frac{\partial \chi_n}{\partial t} \cdot \vec\nabla \delta\nu] = -\rho[\frac{\partial \delta\nu}{\partial t} + \vec v \cdot \vec\nabla \delta\nu] = -\rho \frac{d\delta\nu}{\mathrm{d}t}. \tag{46}$$

The above equation can be put to the form:

$$\delta_\nu L = \delta\nu[\frac{\partial \rho}{\partial t} + \vec\nabla \cdot (\rho \vec v)] - \frac{\partial(\rho\delta\nu)}{\partial t} - \vec\nabla \cdot (\rho \vec v \delta\nu). \tag{47}$$

This obviously leads to the continuity equation (4) and some boundary terms in space and time. The variational derivative with respect to *ρ* is trivial and the analysis is identical to the one in equation (14) leading to equation (15). To conclude this subsection let us summarize the equations of non-barotropic MHD:

$$\begin{aligned} \frac{d\nu}{d\mathrm{t}} &= \frac{1}{2}\vec v^2 - w, \\ \frac{\partial \rho}{\partial t} + \vec\nabla \cdot (\rho \vec v) &= 0, \\ \frac{d\sigma}{dt} &= T, \\ \frac{d\alpha}{dt} &= \frac{\vec\nabla \eta \cdot \vec J}{\rho}, \\ \frac{d\beta}{dt} &= -\frac{\vec\nabla \chi \cdot \vec J}{\rho}, \end{aligned} \tag{48}$$

in which α, β, σ, $\vec{v}$ are functionals of χ, η, s, ν as described above. It is easy to show as in equation (24) that those variational equations are equivalent to the physical equations.

It is shown in [26] that the Lagrangian density can be written standard quadratic form:

$$\begin{aligned} L[\chi_i, \nu, \rho] = \rho[\frac{1}{2} A^{-1}_{jn} \frac{\partial \chi_j}{\partial t} \frac{\partial \chi_n}{\partial t} + \frac{\partial \nu}{\partial \chi_m} \frac{\partial \chi_m}{\partial t} - \\ - \frac{\partial \nu}{\partial t} - \varepsilon(\rho, \chi_3)] - \frac{1}{8\pi} (\vec{\nabla} \chi_1 \times \vec{\nabla} \chi_2)^2. \end{aligned} \tag{49}$$

In which A^{-1}_{jn} plays the rule of a "metric". The Lagrangian is thus composed of a kinetic terms which is quadratic in the temporal derivatives, a "gyroscopic" terms which is linear in the temporal derivative and a potential term which is independent of the temporal derivative.

Cross helicity

In non-barotropic MHD one can calculate the temporal derivative of the cross helicity (1) using the above equations and obtain:

$$\frac{dH_C}{dt} = \int T \vec{\nabla} s \cdot \vec{B} d^3 x. \tag{50}$$

Hence, generally speaking cross helicity is not conserved. A clue on how to define cross helicity for non-barotropic MHD can be obtained from the variational analysis described in the previous sections.

Let us now write the cross helicity given in equation (1) in terms of equation (11) and equation (13), this will take the form:

$$H_C = \int d\Phi[\nu] + \int d\Phi \oint \sigma ds \tag{51}$$

in which: $d\Phi = \vec{B} \cdot d\vec{S} = \vec{\nabla}\chi \times \vec{\nabla}\eta \cdot d\vec{S} = d\chi d\eta$ and the closed line integral is taken along a magnetic field line. $d\Phi$ is a magnetic flux element which is comoving according to equation (2) and is an infinitesimal area element. Although the cross helicity is not conserved for non-barotropic flows, looking at the right hand side we see that it is made of a sum of two terms. One which is conserved as both $d\Phi$ and $[\nu]$ are comoving (see equation (??)) and one which is not. This suggests the following definition for the non barotropic cross helicity H_{CNB}:

$$H_{CNB} \equiv \int d\Phi[\nu] = H_C - \int d\Phi \oint \sigma ds. \tag{52}$$

Which can be written in a more conventional form:

$$H_{CNB} = \int \vec{B} \cdot \vec{v}_t d^3 x \tag{53}$$

where the topological velocity field is defined as follows:

$$\vec{v}_t \equiv \vec{v} - \sigma \vec{\nabla} s. \tag{54}$$

It should be noticed that H_{CNB} is conserved even for an MHD not satisfying the Sakurai topological constraint given in equation (11), provided that we have a field σ satisfying the equation $\frac{d\sigma}{dt} = T$. Thus the non barotropic cross helicity conservation law:

$$\frac{dH_{CNB}}{dt} = 0, \tag{55}$$

is more general than the variational principle described by equation (49) as follows from a direct computation using equations (2, 4, 5, 6). Also notice that for a constant specific entropy s we obtain $H_{CNB} = H_C$ and the non-barotropic cross helicity reduces to the standard barotropic cross helicity. To conclude we introduce also a local topological conservation law in the spirit of [17] which is the non barotropic cross helicity per unit of magnetic flux. This quantity which is equal to the discontinuity of ν is conserved and can be written as a sum of the barotropic cross helicity per unit flux and the closed line integral of *s*dσ along a magnetic field line:

$$[\nu] = \frac{dH_{CNB}}{d\Phi} = \frac{dH_C}{d\Phi} + \oint s d\sigma. \tag{56}$$

Conclusion

Topological invariants have always been informative for the analysis of plasma processes and there are such invariants in MHD flows. For example magnetic helicity and cross helicity have long been useful in research into the problem of hydrogen fusion and in various astrophysical scenarios although magnetic helicity appeared to be more useful. This may be connected to the fact that cross helicity is not conserved in non barotropic flow. In previous works [4, 7, 17] connections between helicities with symmetries of the barotropic fluid equations were made. The variables of the current variational principles were useful in identifying and characterizinga cross helicity invariant for **non-barotropic** MHD but the work done here is in no way exhausting and thus more invariants may be found.

References

1. *Woltjer L. 1958 a Proc. Nat. Acad. Sci. U.S.A. 44. 489-491.*
2. *Woltjer L. 1958 b Proc. Nat. Acad. Sci. U.S.A. 44. 833-841.*
3. *Moffatt H. K. J. Fluid Mech. 1969. 35. 117.*
4. *Yahalom A. Helicity Conservation via the Noether Theorem // Math. Phys. 1995. 36. P.1324-1327 [Los-Alamos Archives solv-int/9407001].*
5. *Padhye N. and Morrison P.J. Phys. Lett. A. 1996. 219, 287.*
6. *Padhye N. and Morrison P.J. Plasma Phys. Rep. 1996. 22. 869.*
7. *Yahalom A. and Lynden-Bell D. Simplified Variational Principles for Barotropic Magnetohydrodynamics (Los-Alamos Archives- physics/0603128) // Journal of Fluid Mechanics. 2008. V.607. P.235–265.*
8. *Sturrock P.A. Plasma Physics (Cambridge University Press, Cambridge). 1994.*
9. *Vladimirov V.A. and Moffatt H.K. // J. Fluid. Mech. 1995. 283. P.125-139.*
10. *Kats A.V. Los Alamos Archives physics – 0212023. 2002. JETP Lett.77. 657. 2003.*
11. *Kats A.V. and Kontorovich V. M. Low Temp. Phys. 1997. 23. 89.*
12. *Kats A.V. Physica D. 2001. 459. P.152-153.*
13. *Sakurai T. Pub. Ast. Soc. Japan. 1979. 31. 209.*
14. *Morrison P.J. Poisson Brackets for Fluids and Plasmas, AIP Conference proceedings. V.88. Table 2.*
15. *Yang W.H., Sturrock P.A. and Antiochos S. Ap. J. 1986. 309. 383.*
16. *Yahalom A.A. Four Function Variational Principle for Barotropic Magnetohydrodynamics. EPL. 89. 2010. 34005, doi: 10.1209/0295-5075/89/34005 [Los - Alamos Archives - arXiv: 0811.2309].*
17. *Yahalom A. Aharonov - Bohm Effects in Magnetohydrodynamics Physics Letters A. V. 377. Issues 31-33. 2013. P.1898-1904.*
18. *Frenkel A., Levich E. and Stilman L. Phys. Lett. A. 1982. 88. 461 p.*
19. *Zakharov V. E. and Kuznetsov E. A. Usp. Fiz. Nauk. 1997. 40. 1087.*
20. *Bekenstein J.D. and Oron A. Physical Review E. 2000. V.62, N.4, P.5594-5602.*
21. *Kats A.V. Phys. Rev E. 2004. 69. 046303.*
22. *Yahalom A., Pinhasi G.A. and Kopylenko M. A Numerical Model Based on Variational Principle for Airfoil and Wing Aerodynamics, proceedings of the AIAA Conference, Reno, USA. 2005.*
23. *Bateman H. On Dissipative Systems and Related Variational Principles. Phys. Rev. 1931. 38. 815.*
24. *Clebsch A. Uber eine allgemeine Transformation der hydro-dynamischen Gleichungen. J. reine angew. Math. 1857. 54. P.293–312.*
25. *Clebsch A. Uber die Integration der hydrodynamischen Gleichungen. J. reine angew. Math. 1859. 56. P.1–10.*
26. *Yahalom A. Simplified Variational Principles for non-Barotropic Magnetohydrodynamics. (arXiv: 1510.00637 [Plasma Physics]) J. Plasma Phys. 2016. V.82. 905820204. doi:10.1017/S0022377816000222.*