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Solutions

1. Let A and B denote square matrices of the same order, and let $ABA=BAB$. Prove that one of the following conditions is satisfied: one of the matrices is degenerate, or A and B have equal determinants.

Solution: We have $|ABA| = |BAB|$ ($|A|$ is the determinant of matrix A).

We use the multiplicative properties of the determinant:

$|A| \cdot |B| \cdot |A| = |B| \cdot |A| \cdot |B|$, hence $|A| \cdot |B| \cdot (|A| - |B|) = 0$.

If $|A| \neq |B|$, then either $\cdot |A| = 0$ or $|B| = 0$. If neither A nor B are degenerate, then $|A| - |B| = 0$ and $|A| = |B|$.

2. A pharmaceutical company is advertising a new product. Its representatives claim that using this product will result in a daily decrease in body weight or in cholesterol levels, or, during most days, both together. A customer used the new product regularly for a month, yet at the end of the month both his weight and his cholesterol levels were the same as before he started. Is it possible that the company's claims are true? Explain your answer.

3. Find all the solutions x_1, x_2, x_3, x_4, x_5 of the system

$$\begin{cases} x_5 + x_2 = yx_1 (1) \\ x_1 + x_3 = yx_2 (2) \\ x_2 + x_4 = yx_3 (3) \\ x_3 + x_5 = yx_4 (4) \\ x_4 + x_1 = yx_5 (5) \end{cases}$$, where y is a parameter.

Solution: This is a system of homogeneous linear equations. Therefore, the system of numbers $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ is a solution for any y. Next, we find "non-trivial" solutions where at least one x_i does not equal 0.

(1) and (5) can be written in equivalent form as

$$x_2 = yx_1 - x_5 \quad (6)$$

$$x_4 = yx_5 - x_1 \quad (7),$$

(2) and (4) can respectively be written as

$$x_3 = yx_2 - x_1$$
$$x_3 = yx_4 - x_5$$

We can exclude x_2 and x_4 by multiplying (6) and (7) by *y*, and subtracting from the expression for x_3:

$$x_3 = (y^2 - 1)x_1 - yx_5 \qquad (8)$$
$$x_3 = (y^2 - 1)x_5 - yx_1 \qquad (9).$$

Thus the system of the four relations (1), (2), (4), (5) is equivalent to the system of relations (6), (7), (8), (9).

By equating the right sides of the relations (8) and (9) we obtain:

$$(y^2 - 1)x_1 - yx_5 = (y^2 - 1)x_5 - yx_1$$

or

$$(y^2 + y - 1)x_1 = (y^2 + y - 1)x_5 \qquad (10).$$

Now, let us examine two cases:

1) $y^2 + y - 1 \neq 0$
2) $y^2 + y - 1 = 0$

In the first case, by dividing both sides of equation (10) by $y^2 + y - 1 \neq 0$, we obtain $x_1 = x_5$. Similarly, by cyclically changing the numeration of equations and variables we reach the conclusion that all x_i with neighboring indices are equal, i.e. $x_1 = x_2 = x_3 = x_4 = x_5$. Therefore, $y = 2$ for any x_i that does not equal 0. This can be verified by substitution.

Now let $y^2 + y - 1 = 0$, i.e. $y^2 - 1 = -y$. In this case (8) and (9) are the same equation written in two different ways:

$$x_3 = -y(x_1 + x_5) \qquad (11).$$

We can easily verify that equation (3) follows from (6), (7) and (11). Indeed,

$$x_2 + x_4 = (y - 1)(x_1 + x_5);$$
$$x_2 + x_4 = -x_3 \cdot \frac{y-1}{y}.$$

Let us compare with (3):

$$x_2 + x_4 = yx_3$$

In order to prove that they are equivalent, we subtract:

$$(x_2 + x_4) - (x_2 - x_4) = -x_3 \cdot \frac{y-1}{y} - yx_3$$

$$0 = x_3\left(y + \frac{y-1}{y}\right).$$

Therefore, if y is any one root of the equation $y^2 + y - 1 = 0$, then

$$D = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

$y_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, thus we can, for instance, choose x_1 and x_5 arbitrarily, define x_2 using formula (6), x_4 using formula (7), and x_3 using formula (11). The equations (1)-(5) are then automatically satisfied, since for these values of y they follow from the equations (6), (7) and (11).

Answer: a) $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ for any y;

b) $x_1 = x_2 = x_3 = x_4 = x_5 \neq 0$ for $y = 2$;

c) $y_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$; any x_1 and x_5; $x_2 = y_{1,2} \cdot x_1 - x_5$;

$x_4 = y_{1,2} \cdot x_5 - x_1$; $x_3 = -y_{1,2} \cdot (x_1 + x_5)$.

4. Let $0 < a < b$. Find the area of the union of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Answer: $4ab \cdot arctg(b/a)$.

Solution: The lines x = y, x = -y divide this figure into four equal segments. Since the areas of all of the segments are equal, we can simply calculate the area of one segment and then multiply it by 4.

We will use linear coordinate transformation: $x' = \frac{x}{b}$ and $y' = \frac{y}{a}$.

This transforms the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ into a circle the radius of which equals 1, while the lines $x = \pm y$ are transformed into lines $bx' = \pm ay'$. The sector of the circle is based on the arc $2arctg(b/a)$, and its area equals $arctg(b/a)$.

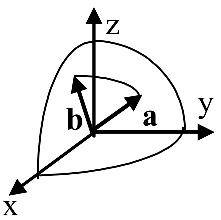
The transformation has reduced the area $a \cdot b$ times. Therefore, the area of the original segment equals $ab \cdot arctg(b/a)$, and the area of the entire figure equals 4 times that.

5. Is it possible to find 5 vectors in space such that all the angles between them are greater than 90^0?

Answer: No.

Solution: Let us place the system of coordinates in such a way that the direction of the "first" vector will be opposite to the direction of the axis Oz, and the origin of the system will be the beginning of all the vectors. Since the angles between all of the vectors have to be greater than 90^0 the ends of the four remaining vectors have to be above the coordinate plane Oxy. Let us choose the direction of axis Oy in such a way that the direction of the "second" vector will be perpendicular to axis Oy and opposite to the direction of axis Ox. In this case the remaining three vectors must lie in the part of space $x \geq 0$, $z \geq 0$. Therefore, two of them lie in the same octant and their scalar product is nonnegative, and the angle between them does not exceed 90^0.

In the figure below the ends of two vectors which lie in the same octant are connected by an arc of the circle of a unit radius. It is clearly seen that such an arc cannot exceed 90^0.



6. Calculate $\int_1^e \sqrt{\ln x}\; dx + \int_0^1 e^{x^2} dx$.

Answer: e

Solution: Let I_1 denote the first integral, and I_2 the second. We then transform I_1 as follows:

$$\int_1^e \sqrt{\ln x}\,dx = \left| \begin{array}{l} \sqrt{\ln x} = y,\ x = e^{y^2},\ dx = 2ye^{y^2}dy \\ x = 1 \Rightarrow y = 0, \quad x = e \Rightarrow y = 1 \end{array} \right| = \int_0^1 2y^2 e^{y^2} dy = \left| \begin{array}{ll} u = y, & du = dy, \\ dv = 2ye^{y^2}dy, & v = e^{y^2} \end{array} \right| =$$

$$= y \cdot e^{y^2} \Big|_0^1 - \int_0^1 e^{y^2} dy = e - I_2.$$

7. The function $F(x)$ is continuous for all $x > 0$. It is known that for any fixed $x > 0$ the sequence $F(x+n) \to 0$, as $n \to \infty$. Does this mean that $\lim\limits_{x\to\infty} F(x) = 0$?

Answer: No, it does not.

Solution: Let $f_o(x) = 1 - 2|x - 1/2|$. Let $f_k(x) = f_o(kx)$ for $x < 1/k$, and $f_k(x) = 0$ for $x > 1/k$. (We recommend that you draw a diagram of $f_k(x)$ in segment $[0;1]$). Let $g(x) = f_k(x)$ for $x \in [k; k+1]$. The function $g(x)$ is continuous, and $\lim\limits_{x\to\infty} g(x) \neq 0$ since $g(k + 1/k) = 1$ for any natural *k*.

On the other hand, let $x = n + \varepsilon$, where $n = [x]$ is a natural number and $\varepsilon < 1$. If $\varepsilon = 0$, then for any natural *m* $g(x+m) = 0$, so that $\lim\limits_{m\to\infty} g(x) = 0$. And if $\varepsilon \neq 0$, then for any sufficiently great $k > K_o$ $\varepsilon > 1/k$. Therefore, $g(x+k) = 0$, and once again $\lim\limits_{m\to\infty} g(x) = 0$.

8. A segment, the length of which is 1, is moving in such a way that both of its edges remain on the coordinate axis (in coordinate angle 1). As it moves along the plane, the segment changes the color of the part of the coordinate angle to its left. Find the equation of the line that separates the part of the plane the color of which has been changed from the part the color of which remains the same.

Answer: $x^{2/3} + y^{2/3} = 1$ (part of the asteroid).

Solution: Obviously, for any x $\in [0;1]$, the ordinate of the corresponding point of the line will be equal to the greatest value of ordinates of the points of the bar (for given x), for all the possible positions of the bar. Let us examine a position in which one of its ends is at A(t; 0), and the other at B(0; $\sqrt{1-t^2}$), t $\in$ $[0;1]$. Let y(x;t) denote the ordinate of the point of the bar, where the abscissa equals x, in the position where the ends of the bar are at points A and B. y = y(x) = max $\{y(x;t)\}$ for t $\in [x;1]$. (For obvious reasons, the position of the bar in which t < x does not need to be examined). Thus, $y(x;t) = (1 - \frac{x}{t})\sqrt{1-t^2}$ and

$y' = \frac{x}{t^2}\sqrt{1-t^2} - \left(1-\frac{x}{t}\right)\frac{t}{\sqrt{1-t^2}} = \frac{x-t^3}{t^2\sqrt{1-t^2}}$. The critical point $t = \sqrt[3]{x}$ is the point of maximum, therefore $y(x,\sqrt[3]{x}) = \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ or $x^{2/3} + y^{2/3} = 1$.

9. A ring has been placed on the end of a rod the length of which is 1 meter. At a certain point in time the rod begins to elongate uniformly (any two segments of equal length will be elongated equally during any given period of time). The ends of the rod are moving away from each other at the speed of 10cm/sec. At the same point in time the ring begins to move towards the opposite end of the rod at the speed of 1cm/sec. Will the ring ever reach the opposite end of the rod? If so, how much time will it take? (The width of the ring can be ignored).

Solution: Let l_0 denote the original length of the rod, v_1 - the velocity of the ring, v_2 - the velocity with which the ends of the rod are moving away from each other, *x* – the distance between the ring and the opposite end of the rod after *t* seconds from the time the ring started moving, i.e. $x = x(t)$.

The distance between the two ends of the rod at the time *t* equals $l(t) = l_o + v_2 \cdot t$. It is clear from the formulation of the problem, that the velocity with which the ends of any arbitrary segment of the rod move away from each other is proportional to the length of the segment. Therefore, the velocity with which the ends of a segment the length of which is *x* are moving away from each other at the time *t* is $\frac{x(t)}{l(t)} \cdot v_2$.

The overall velocity at which *x* is changing equals:

$$\frac{dx}{dt} = \frac{x(t)}{l(t)} \cdot v_2 - v_1 = \frac{x}{l_o + v_2 \cdot t} \cdot v_2 - v_1 .$$

This is a linear differential equation, the general solution of which is:

$$x = (l_o + v_2 \cdot t)\left(C - \frac{v_1}{v_2} \ln(l_o + v_2 \cdot t) \right).$$

The particular solution which satisfies the original condition $x(0) = l_o$ is:

$$x = (l_o + v_2 \cdot t)\left(1 + \frac{v_1}{v_2} \ln \frac{l_o}{l_o + v_2 \cdot t} \right).$$

The equation $(l_o + v_2 \cdot t)\left(1 + \frac{v_1}{v_2} \ln \frac{l_o}{l_o + v_2 \cdot t} \right) = 0$

has a positive solution $t = \frac{l_o}{v_2}(e^{\frac{v_2}{v_1}} - 1)$.

Therefore, the ring will reach the opposite end of the rod in finite time

$$t = \frac{1}{0.1}(e^{\frac{10}{1}} - 1) = 10 \cdot (e^{10} - 1) \text{ sec.}$$

Answer: The ring will reach the opposite end of the rod in $10 \cdot (e^{10} - 1)$ sec.

10. Suppose that 2009 guests are present at a banquet hall. Each person knows at least 45 of the other guests. Prove that it is possible to find at least four guests who can be seated at a round table in such a way that each will be sitting next to a person whom he or she knows.

Solution: Let us demonstrate that four such guests do exist. The number of ordered pairs of people who know each other must obviously be an integer. However, if each person knew only 45 others, the number – 2009*45/2 – would not be an integer. Therefore, we can conclude that at least one of the guests knows 46 others. We will choose this person, and prove that among the 46 guests he knows, at least two have a common acquaintance who is not one of these 46. If this weren't true, the total number of guests at the banquet would have to be at least 46*44=2024. This number is greater than 2009, which means that two such guests can indeed be found. Thus, we can choose the person who knows at least 46 other guests, then place those two who have a common acquaintance on either side of him, and the common acquaintance can be seated between them.

11. An infinite number of points is given. All of the distances between the points are integers.

a. Prove that all of the points are situated on the same straight line.
b. Can an infinite number of points which are not all on the same line, such that the distance between any two of the points is a rational number, exist on a plane?

Solution: a) Let A and B be 2 given points belonging to our set of points M, and X be an arbitrary point belonging to M. Note that due to the triangle inequality $\left\| XA \right| - \left| XB \right\| \le \left| AB \right|$. Since all of the distances between points are integers, the number of the possible values of $\left\| XA \right| - \left| XB \right\|$ cannot be greater than $\left| AB \right|$. Thus, all of the points lie on a finite number of hyperbolae $\left\| XA \right| - \left| XB \right\| = c; c = 0, \ldots, \left| AB \right|$ with focuses A and B.

For $c = 0$ and $c = \left| AB \right|$ these hyperbolae degenerate into lines (the middle perpendicular to segment $[AB]$ and line (AB), respectively). An infinite number of points which belong to set M (since set M is infinite) lie on one of these hyperbolae, H.

Now let us consider two points $C \ne D$ from set M, which lie on H, and build a family of hyperbolae H_i on which these two points lie. Since the number of points on H is infinite, an infinite subset of them lie on H_i, i.e. on the intersection $H \cap H_i$.

On the other hand, any two hyperbolae can either overlie each other, or intersect at no more than 4 points. Therefore, $H = H_i$. The focuses of hyperbola H_i lie on H; this can only be possible if both are a straight line. Let l denote that line.

Since not all of the points of set M lie on the same line, there is a marked point $O \in M$ which does not belong to l. Let $Q \in M \cap l$.

All that remains now is to examine the finite family of hyperbolae with focuses Q and O (to which points from M belong) and their intersection with line l. Since neither one of them can coincide with l, each intersect with l at no more than 2 points, i.e. a finite number of times. Therefore, the number of points from M on l is finite.

Thus, we have reached the desired contradiction.

b) Let us demonstrate that on a unit circle it is possible to choose a set of points, such that the distances between them are rational. If points A and B correspond to angles φ and ψ, then the distance between them is $2\sin((\varphi-\psi)/2)$. Therefore, if $\sin(n\varphi)$ is rational for any n, then all pair distances between points $A_k=(\sin(2k/\varphi),\cos(2k/\varphi))$ are rational.

Using the equalities $\sin(k+1)\varphi=\sin(k\varphi)\cos(\varphi)+\cos(k\varphi)\sin(\varphi)$ we can easily verify through induction that if $\sin(\varphi)$ and $\cos(\varphi)$ are rational, then $\sin(n\varphi)$ is rational for any n. If the degree measure of angle φ is irrational, then the points A_k are pair distinct and they form the original set.

Let us consider $\varphi=\arcsin 3/5$. In this case, $\sin(\varphi)=3/5, \cos(\varphi)=4/5$ so that $\sin(n\varphi)$ is rational for any n. All that remains now is to verify that the degree measure of angle φ is irrational.

In order to do this it is enough to verify that $\cos(2^n\varphi)\neq\cos(2^m\varphi)$ for $m\neq n$. And indeed, if $\varphi=p/q\cdot 2\pi,\ p, q$ are integers, $n\neq m$ for which $2^n \bmod q=2^m \bmod q$ can be found. In this case, the same point on a unit circle corresponds to both angles $2^n\varphi$ and $2^m\varphi$, and the cosines coincide.

Let us demonstrate that $\cos(2^n\varphi)$ has the form $p_n/3^{2^n}$, where p_n is not divisible by 3. Therefore, $\cos(2^n\varphi)\neq\cos(2^m\varphi)$ for $m\neq n$.

By virtue of the equality $\cos(2\alpha)=2\cos^2(\alpha)-1$ we have $p_{n+1}=2p_n^2-3^{2^n}$ (which is not divisible by 3), and the denominator of $\cos(2^{n+1}\varphi)$ equals $3^{2^{n+1}}$. Thus we achieve the solution of the problem through induction.

12. A robot is searching for a certain tool. It moves along a plane, and each of its steps equals 1. It can move in any direction it chooses. When the distance between it and the tool is less than one step it can pick the tool up. After each step the robot is informed whether the distance between him and the tool has decreased or increased. Suppose that at a given point in time the distance between the robot and the tool equals N steps and the robot knows the number of steps that separate him from the tool.

a. Prove that the robot can reach the tool in $[N+10\log_2 N]$ steps (where $[x]$ denotes the integer part of x).
b. Prove that no algorithm exists which can enable the robot to reach the tool in less than $N+\dfrac{\log_2 N}{10}$ steps.

<u>Solution</u>: a) Let us describe the algorithm, which consists of two phases.

1. Determining the necessary direction.
2. Moving in the chosen direction.

Let us define the *cone of possible directions, C*, as the set of possible directions from the starting point to the location of the tool. This is an angle, the value of which (the *span)* equals φ. In the beginning $\varphi = 2\pi$. An *action* consists of two steps – a step away from the starting point and a step back towards it.

Lemma: There exists an action that can reduce the span by half.

Proof: The case $\varphi = 2\pi$ can be verified directly. Let $\vec{l}$ denote a vector of bisector *K*, and $\vec{m}$ – a perpendicular vector. Obviously, an action associated with the direction of m will result in the above mentioned span reduction. The half of *K* which lies in the angle between $\vec{l}$, $\vec{m}$ corresponds to the case when the robot gets closer to the tool, while the other half, which lies in the angle between $\vec{l}$, $-\vec{m}$ corresponds to the case when the robot is moving away from the tool. Lemma is proved.

After $\log_2(n) + 2$ actions (i.e. $2[\log_2(n) + 3]$ steps) the direction of the location of the tool can be determined with accuracy up to angle $\varphi = 2\pi/(4n)$. In this case the deviation of the sides of the angle from the bisector will be $\pi/(4n) < \arcsin(1/n)$, and therefore from then on movement along the bisector will lead to success in *n*+1 steps.

b) Let us call the robot's step a *trial step* if the angle between the direction in which he moves and the direction of the tool is greater than $\pi/3$. *C* denotes the *arc of possible locations of the tool*. This is an arc of a circle the center of which is the starting point and the radius of which is *n*, such that all of the points on the arc correspond to the answers the robot receives. In the beginning, arc *C* is the whole circle.

Lemma 1. The number of trial steps *l* does not exceed $\log_2(n)/5$.

Proof. Trial steps bring the robot closer to the tool by no more than $l\cos\pi/3$. Therefore, the total number of steps is no less than $l + (n - l\cos\pi/3) = n + l(1 - \cos\pi/3) = n + l/2$ and if $l > \log_2(n)$ than the value $n+l/2$ will be greater than $n + \log_2(n)/10$ which is impossible given the conditions of the problem. Lemma proven.

Let us define the *initial area* as the first *n*/3 steps.

Lemma 2. After the robot makes the first step, only another trial step will lead to a change in the angle span. Following this first step, C is divided into two arcs, for one of which the answer to the robot's question is that it will bring him closer to the tool, and for the other that it will take it farther away (one of these arcs may be empty). The length $|C|$ of the larger one of them, C', does not exceed $|C|/2$.

Corollary. For any algorithm with l trial steps, there exists a location of the tool such that after the initial area the angle span of arc C will be no less than $\pi n / 2^l$.

The following lemma can be verified directly.

Lemma 3. If A denotes an arbitrary point, the distance from A to the most distant point of arc C can be no less than $d + |C| / 2\sqrt{n}$.

Therefore, it follows from lemma 1 that $|C| < \pi \cdot n / 2^{\log_2 n/5} = \pi \cdot n^{4/5}$, and after the first $n/3$ steps, the distance from the robot's position to the point on arc C which is farthest from it will be no less than $2n/3 + |C|/\sqrt{n} = 2n/3 + n^{0.3}/2$. The total number of steps in the worst case will therefore be no less than $n/3 + 2n/3 + n^{0.3}/2 + \log_2(n)/10$.

Problem solved.