Optimal LQR Control of Structures using Linear Modal Model

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Abstract
According to the seismic modal approach, optimal modal control design should derive a control signal affecting only the dominant modal coordinates. Additionally, story drifts are the main issue for many seismic design problems where the design goal is to reduce drifts in excited structures. This study formulates optimal modal feedback design based on the infinite horizon LQR method. New state-space similarity transformations are introduced. Efficiency of the method is demonstrated in a numerical example, presenting a modal LQR design of a 20 story building constructed in a seismic area.

keywords: modal control, LQR, optimal structural control.

1 Introduction
In the last decades, there is an increasing interest in the field of structural control. It suggests the use of approaches and tools of control theory for analysis and manipulation of structures dynamic behavior. A dynamic response of a structure can be described by the displacements in a selected set of degrees of freedom, also known as nodal coordinates [1]. These displacements are calculated by solving the corresponding equations of motion.

In civil engineering, the difference between the displacements of two adjacent floors is usually denoted as story drift [2, 3]. Story drifts are the main issue for many seismic design problems, where the design goal is to reduce drifts in the excited structure. In most cases, the control forces entering the structure are inter-story forces. Therefore, from engineering viewpoint, a drift model is more attractive than a nodal one as it highlights the design scope and allows a more intuitive choice of quadratic optimization index coefficients, used for optimal seismic designs based on LQR criteria. The use of story drifts instead of nodal displacements was suggested in the design of supplemental dampers for structural control [4].

The equations of motion in the nodal coordinates and the drifts coordinates are internally coupled through the structure inertia, damping and stiffness properties. Moreover, structural models are quite large, compared to other controlled systems. It creates computational difficulties in finding an efficient control design for large structures. A common solution to this problem is known as modal decomposition [5]. It leads to a
simpler and convenient design process [1] and it is also physically interpreted and adequate from structural engineering viewpoint.

The contribution of each modal coordinate to the overall dynamic response is different, therefore, coordinates with small contribution may be neglected [5, 6, 7, 8]. For structures, subjected to earthquake excitation, it is known that a seismic design, taking in account merely the dominant low frequency modal coordinates set is mostly adequate.

An optimal control design that relies on the structure modal properties is denoted as optimal modal control. According to the seismic modal approach, optimal modal control design should derive a control signal that affects only the dominant modal coordinates [9]. The problem is that in general case, when a control design is done, each single control signal generates multiple modal control signals and therefore couples the structural modal coordinates [10]. In other words, the control signals affect multiple modes simultaneously. This coupling also happens for feedback control signals. It is unwanted since the modal control design tries to increase the control efficiency by manipulating only the dominant modes.

In independent modal space control (IMSC) strategy a feedback control signal is designed in such a way that only the desired modes are affected. According to this method, modal control signals are designed in the modal space for each modal coordinate separately. Next, the implemented control is derived from the modal control signals [11].

Though in the general case the feedback couples the modes of the system, in the case of independent modal space control, where a modal control design deals merely the designed modal coordinates, the closed loop remains decoupled [12]. A problem that can be troublesome is that a feedback, designed by the IMSC approach might require estimation of the modal coordinates. Another problem is that the use of IMSC might affect the uncontrolled modes such that their contribution to the overall response might increase [8].

An approach, requiring no modal state estimation is direct feedback control, whereby the sensors are collocated with the actuators and a given actuator force is a linear function of the sensor output at the same point [13].

A Modified IMSC was recently proposed [8]. However, the method is questionable. It requires pseudo-inversion of matrices which provide the designer with an approximated solution, because the pseudo-inverse is a solution to the least squares problem. Additionally, for the case of MIMSC design which controls less than half of the modes, the proposed pseudo-inverse is singular.

Design of dampers, based on modal LQR was also studied [6]. The dominance of the first set of modes from energy viewpoint was emphasized and control design approach of the dominant modes by LQR method was suggested.
2 Aims and Scope

This study aims to find a verified optimal modal control based on the well known LQR control design method for Rayleigh damped models. Additionally it is expected that the modal approach will provide a more efficient and intuitive optimal control design of structures.

2 Equations of Motion in Nodal Model Form

2.1 Second Order ODE

The equation of motion for a controlled structure, subjected to seismic excitation is:

\[ M \ddot{z}(t) + C \dot{z}(t) + K z(t) = -M \gamma \ddot{\xi}_g(t) + W u(t) \]  

(1)

where \( z(t) \) is a vector function of the nodal displacements; \( M, C, K \) are the structural mass, damping and stiffness matrices, respectively; \( \ddot{\xi}_g(t) \) represents the earthquake ground acceleration signal; \( u(t) \) is a vector of the forces produced by the actuators; \( W \) is the dampers distribution matrix that relates the input forces in the dampers to the forces in the nodal coordinates; \( \gamma \) is the ground acceleration distribution vector. It should be noted that in this study, the damping of the structure is modeled as Rayleigh damping [5].

2.2 State-Space Form and LQR Control

In order to use the infinite horizon LQR method, a state-space formulation for Eq. (1) is used. As LQR design does not take in account disturbance signals, the ground acceleration is omitted. In the state space Eq. (1) takes the following form [1]:

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \]  

(2)

where \( x(t) \) is the states vector of the model and \( A, B \) are the state matrices:

\[ A = \begin{bmatrix} 0 & I \\ M^{-1}K & M^{-1}C \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ M^{-1}W \end{bmatrix} \]

An infinite horizon LQ cost function [20], for the model described by Eq. (2), is:

\[ J(x,u) = \frac{1}{2} \int_0^\infty x^T Q x + u^T R u \, dt \]  

(3)

The time dependency notation in this equation was removed for simplicity. According the infinite horizon LQR design method, the optimal closed loop control signals vector is [14]:

\[ u(t) = -G x(t) \]

where \( G \) is derived by

\[ G = R^{-1} B^T P \]

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and $P$ is the solution of the algebraic matrix Riccati equation [14]:

$$-PA - A^TP - Q + P(BR^{-1}B^T)P = O$$

2.3 Modal Form of the Undamped Model

It is known that a real and nonsingular matrix $\Phi$ that diagonalizes $M$ and $K$, such that $\Phi^TM\Phi$ and $\Phi^TK\Phi$ are diagonal matrices, can be found by solving the undamped structure eigenproblem [5]:

$$(M^{-1}K - \lambda_i I)\phi_i = 0.$$  

The matrix $\Phi$ is defined by

$$\Phi = [\phi_1 \ \phi_2 \ \cdots \ \phi_l]$$

2.4 State Space Modal Decomposition

The nodal state space model can be transformed from Eq. (2) into a modal state space model, composed from $l$ internally independent modal subsystems, by using a state space similarity transformation. Consider the block diagonal matrix

$$\Phi_0 = \text{diag}(\Phi, \Phi)$$  \hspace{0.5cm} (4)

Denote $e_k$ as the $k^{th}$ column in the identity matrix $I \in R^{nn}$. A permutation matrix $T_0$ is:

$$T_0 = [e_1 \ e_2 \ \cdots \ e_{n-1} \ e_2 \ e_4 \ \cdots \ e_l]^T$$

The similarity transformation matrix is

$$T = \Phi_0 T_0$$

The transformation into the modal model is given by

$$x(t) = Tq(t); \quad A^* = T^{-1}AT = \begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ 0 & A_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & A_l^* \end{bmatrix}; \quad B^* = T^{-1}B = \begin{bmatrix} B_1^* \\ B_2^* \\ \vdots \\ B_l^* \end{bmatrix}$$

where $l$ is the number of modal coordinates. The block diagonal form of $A^*$ implies that it is composed of $l$ internally independent subsystems whose dynamics are governed by the blocks $A_i^*$.

2.4 State Space Drift Transformation

Structural control design, based on story drifts instead of ground related displacements, was suggested in various studies.

The transformation of the system from nodal to drift space can be done in the state space
by using a non-singular similarity transformation matrix

\[ T_i = \begin{bmatrix} T_d & 0 \\ 0 & T_d \end{bmatrix} \]

where \( T_d \) elements are defined by

\[ T_{d,ij} = \begin{cases} 1 & i \leq j \\ 0 & \text{otherwise} \end{cases} \]

The similarity transformation from the nodal to the drift space is:

\[ x(t) = T_i x_d(t); \quad A_d = T_i^{-1} A_T; \quad B_d = T_i^{-1} B \]

The drift model can be decomposed into corresponding modal model by using \( \Phi_d = T_d^{-1} \Phi \) instead of \( \Phi \) in Eq. (4).

A closed loop system, designed in the drift space, has the form

\[ \dot{x}_d(t) = (A_d - B_d G_d)x_d(t); \quad G_d = GT_i \]

3 Reduced Modal LQR Design

Since seismic control design can be carried out only for the dominant modes, it can be applied to reduced model by taking in account only the dominant modes. Consider

\[ A^*_{11} = \begin{bmatrix} A^*_1 & 0 & \cdots & 0 \\ 0 & A^*_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A^*_k \end{bmatrix}; \quad 1 \leq k \leq l \]

which is the leading block from \( A^* \). It describes the dynamics of the modes to be controlled. Here \( k \) is the number of modes considered by the control design. Denote \( B^*_{11} \in \mathbb{R}^{2k \times n_u} \) as the corresponding block from \( B^* \). It describes how the control inputs affect the selected modes.

Therefore, the algebraic Ricatti's equation for the reduced model is

\[ -P^*_{11} A^*_{11} - (A^*_{11})^T P^*_{11} - Q^*_{11} + P^*_{11} (B^*_{11} R^{-1} B^*_{11}^T) P^*_{11} = 0 \]  

(5)

Here \( Q^*_{11} \in \mathbb{R}^{2k \times 2k} \) is the modes weighting and \( R \) is the same as in Eq. (3). By solving Eq. (5), the full model gain matrix can be derived

\[ G = [R^{-1} B^*_{11}^T P^*_{11} \quad 0] T^{-1} \]

And the control signals are

\[ u(t) = -G x(t) = [-G_{11} \quad G_{12}] \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = -G_{11} z(t) - G_{12} \dot{z}(t) \]

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Note that in the general case, \( G \) is a full matrix. It implies that a single control signal depends on both the displacements and the velocities throughout the model. In order to realize such feedback an active control should be used.

4 Numerical Example- 20 story model

In order to verify the suggested approach, structural seismic response was simulated numerically. An active control was designed for a dynamic model according to two design approaches. A drift space modal LQR approach and classical drift space LQR design. Seismic response to the following four earthquakes was obtained: El-Centro, Hachinoche, Kobe, and Northridge. The earthquakes acceleration records were scaled to PGA = 0.3g. Scaling was used in order to obtain better comparison basis of the different responses.

The 20 story model is based on the evaluation model described by Spencer et. al [15], with slight modifications. The model originates from a planned north-south moment resisting frame (MRF) of a 20 story, steel building which is also a typical medium to high-rise building in the Los Angeles, California region.

The modeled MRF is 30.48 m in plan and 80.77 m in elevation. It consists of 20 story and 2 basement levels with five bays of 6.10 m. The floors are composite made of concrete and steel and the columns are made of 345 [MPa] steel. Typical story heights are 3.96 m, except for the heights of the two basement levels and the ground floor which are 3.65 m and 5.49 m respectively. The column bases are modeled as pinned and secured to the ground. Concrete foundation walls and surrounding soil are assumed to restrain the structure at the first floor from horizontal displacement. The floors are composite and consist of 248 [MPa] steel wide-flange beams acting compositely with the floor slab. In accordance with common practice, the floor system, which provides diaphragm action, is assumed to be rigid in the horizontal plane. The basement floor beams are simply connected to the columns.

The seismic mass of the structure first level is \( 5.32 \times 10^5 \) kg, for the second level is \( 5.65 \times 10^4 \) kg, for the third level to 20th level is \( 5.51 \times 10^5 \) kg and for the roof is \( 5.83 \times 10^5 \) kg. The inertial effects of each level are assumed to be carried evenly by the floor diaphragm to each perimeter MRF, hence each frame resists one half of the seismic mass associated with the entire structure. Floor inertial loads are uniformly distributed at the nodes of each respective floor assuming a consistent mass formulation. A more detailed
description of the model can be found in [15].

Guyan reduction is used to reduce the number of DOF to a manageable size. In each above ground floor one horizontal DOF (at the 6th column) and two vertical DOF (at the 2nd and 5th columns), were defined as active DOF. Namely, the tested model consists of 60 active DOF. The damping matrix correspond Rayleigh damping.

The nodal model was transformed into the drift space and then decomposed into corresponding modal model. The control design considered 16 modes out of 60 modes and the modes weights are illustrated in Figure 2. The weights were chosen so that the magnitude of the force in the actuator would not exceed 220 kN for El-Centro response.

![Figure 2: Model B, Modal Weighting.](image)

The affect of the modal control on the structure frequency response is demonstrated by a Bode diagram in Figure 3. As it follows from the figure, the gains in the resonance frequencies of the selected modes were reduced. The change in the model eigenvalues is illustrated in Figure 4. It can be seen that only the controlled eigenvalues moved due to the feedback.

![Figure 3. Model B, 1st story drift earthquake gains for the uncontrolled and the closed loop models.](image)
The peak response and time history analysis of the model to El-Centro earthquake are presented in Figures 5 and 6. The response to the other earthquakes is similar. Analysis of the control energy shows that the overall energy consumed by the actuators is of the same order for both modal and classical design approaches. The form of $G$ shows that each control force depends on all the nodal velocities. It means that active control approach is required for accurate implementation of such feedback. However, this control design might provide a platform for approximation of active devices by passive or semi-active control approaches.

Comparison of the time required for the design process in MATLAB shows that the calculation time of the reduced modal design was 25 times smaller than the calculation time of the classical design.

5 Conclusions

A method for modal LQR control of lumped mass structures was formulated and verified numerically. The method offers a design algorithm that reduces the numerical calculation effort required for the design by controlling only the desired mode shapes.

The method reduces the order of Riccati’s equation that should be solved. It was verified numerically that LQR modal control does exist and that the closed loop eigenvalues differs from the uncontrolled eigenvalues only for the selected mode shapes.

A flexible 20 story building was analyzed. Response of to 4 scaled natural earthquake records was simulated in order to verify the proposed method.

It was found that for the 60 DOF model, the modal design approach performs 25 times faster than the classical approach while the design results are similar.
Figure 5. Peak response: (a) - Nodal Displacements [cm], (b) - Drift Displacements [cm], (c) - Control Forces [$10^3$ kN].

Figure 6. Time history
References


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