RESONANT WAVES AND LOCALIZATION PHENOMENA IN LATTICE STRUCTURES

M. Ayzenberg-Stepanenko\textsuperscript{a} and G. Mishuris\textsuperscript{b}

\textsuperscript{a}Ben-Gurion University of the Negev, Beer-Sheva, Israel
\textsuperscript{b}Institute of Mathematics and Physics, Aberystwyth University, Aberystwyth, UK

ABSTRACT

Transient wave processes are numerically simulated in 2D/3D square-cell lattice structures modeling nanomaterials with deterministic/random inhomogeneities. Considered structures are subjected by local excitations. The molecular mechanics approach is used for description the lattice dynamics. The aim of simulations is (a) to reveal the sensibility of resonant phenomena and the stellar-beaming localization appeared in inhomogeneous lattices, and (b) to discover special features inherent them. As it shown, a localization picture can be formed and remain relatively stable for a long time. Some unexpected features of wave localizations are detected; one of them shows that in the unbounded lattice with defects, a standing resonance similar that in a 1DOF system can be developed in a group of particles at the source vicinity.

1 INTRODUCTION

Mathematical description of mechanical properties of nanostructures is based, as a rule, on relatively simple but appropriate models of periodic lattices. Such models are suitable for both analytical and numerical methods and, at the same time, they will allow us to reveal the main mechanical phenomena, to provide physical insight and to theoretically predict the stress state and fracture toughness of nanomaterials and periodic-like composites [1-3].

The well-studied wave phenomenon inherent to periodic solids is that free wave propagation takes place only within certain discrete bands of frequencies known also as pass-bands alternated with stop-bands, where the steady-state wave propagation is forbidden. In the field of waves in lattices or, more generally, waves in periodic media, the monograph by Brillouin [4] has been basic for subsequent investigations of various theoretical and engineering aspects (see, e.g., [5-7]).

During recent decades, this topic got a second wind, when so-called ‘artificial crystals’ used as band-gap materials were discovered. Artificial phononic (or zonic) crystals are periodic lattices or composite structures designed to control sound and vibration waves. Some important results related to band-gaps in phononic crystals can be found, e.g., in [8-9]. In the frequency spectrum of band-gap materials, there exist resonant frequencies, which usually demarcate pass- and stop-bands. In the 1D case, the group velocity of the wave equals zero at these frequencies: there is no steady-state solution corresponding to an external non-self-equilibrated excitation, and the wave energy flows from a source decelerating with time, like heat, and not as a wave [10].

It was shown in [11] that in 2D/3D uniform lattices, resonant frequencies can also exist in the interior of pass bands. Such frequencies differ from those in the 1D case, since the group velocity is zero only for some special wave orientations. A local harmonic source functioning with one of these frequencies excites a resonance process and the pronounced localization effect – the stellar-beaming pattern of propagated waves – is developed.

The line-localization phenomenon reported in [12] is inherent for a 2D square-cell lattice with an included spring layer that can be mentioned as an infinite linear defect. There, a waveguide-like pattern is characterized by wave propagation along the layer
direction, while an exponential attenuation occurs along the layer normal. Roughly speaking, the 2D problem transits into a quasi 1D one, analysis of which is essentially simplified.

A set of steady-state localization phenomena in lattices with defects and in random lattices has been discovered beginning with the well-known in solid-state physics work by Anderson [13] who predicts that an electron may become immobile when placed in a disordered atomic lattice (see also [14,15]). A rapid progress in the area of ‘localization and fracture propagation in lattices’ is associated, for example, with works [16-19].

This paper is aimed to further studying resonant waves in uniform lattices including 3D those and to conduct a comparative description of wave patterns in nonuniform lattice structures possessing various deterministic and random defects. Note that dispersion relations in such structures do not exist (that prevents the spectral analysis), analytical solutions aren’t developed, while computer simulations allows us to analyze as transient dynamics, as well as steady-state patterns. The molecular mechanics approach is used in numerical calculations. Some results for uniform rectangular-cell lattices obtained in [20,21] serve as reference those for analysis of nonuniform lattices. Although computer simulations of waves in lattices (with corresponding advanced methods and tools) can be found in many works, the purposeful analysis of resonant waves and localizations has not been performed.

2 RESONANT WAVES IN UNIFORM SQUARE-CELL LATTICE

2.1 The 1D Lattice

To explain the usage term ‘resonant wave’, we compare frequencies in both 1D finite and infinite simplest uniform mass-spring lattices (MSLs) where point particles of mass $M$ are linked by massless elastic bonds of unit length and stiffness $g$. In Fig. 1(a) and (b), such two systems are schematically depicted.

![Fig. 1: (a) - finite, (b) - infinite MSLs and (c) - dispersion pattern in the infinite MSL](image)

In the case of a system of $N$ oscillators, a finite spectrum of $N$ eigenfrequencies: is proved: $\omega_1 < \omega_m < \omega_N$, $(m = 2, \ldots, N-1)$ wherein the minimal frequency (all particles perform in-phase motion only two boundary springs are subjected to deformation) is $\omega_{\text{min}} = \sqrt{2g/M}$. On other hand, $\lim_{N \to \infty} \omega_N = \sqrt{4g/M}$ – all particles tend to oscillate in contra-phase motion.

For an infinite MSL we have a continuous frequency spectrum which can be found using the Floquet approach. The system of homogeneous equations of free waves propagating in the infinite MSL is:

$$
\ddot{u}_m - c_0^2 L_2^1 (u_m) = 0, \quad c_0 = \sqrt{g/M}, \quad L_2^1 (u_m) = u_{m+1} - 2u_m + u_{m-1}
$$

where $\dot{z} = \partial z/\partial t$, $c_0$ is the sound velocity in an effective solid homogeneous spring, and $L_2^1 (u_m)$ is the difference operator of the second order.
We seek a solution of the homogeneous system (1) in the form of a traveling wave:
\[ u_m(t) = U e^{i(\omega t + k x)} \]  
(2)
where \( \omega, k \) and \( \lambda = 2\pi/k \) are, respectively, the temporal frequency, the space frequency and the space wavelength.

Substituting (2) into (1), we obtain the frequency and the group velocity spectra
\[ \omega = \pm 2c_0 \sin(k/2), \quad c_g = d\omega/dk = \pm c_0 \cos(k/2). \]  
(3)

This result has a simple physical sense: the longer the waves \( k \) decreases, the lesser the influence of the waveguide discreteness on wave propagation that is disappeared in the case of long waves: \( c_g \rightarrow \pm c_0 \quad (k \rightarrow 0) \).

Below, natural parameters \( g \) and \( M \) are taken as measurement units, and due to symmetry, only the Brillouin zone (interval \([0, \pi]\)) for \( k \) is considered. The frequency \( \omega = 2 \) demarcates pass/stop bands and determines zero group velocity (i.e. zero energy flux) and, for this reason, the steady-state solution is absent. In the stop band \((\omega > 2)\) free waves do not propagate. In Fig. 1 (c), dispersion dependencies \( \omega(k) \) and \( c_g(k) \) are depicted.

Spatial forms of the MSL motion depending on the wave length are determined by the ratio of displacements of neighboring particles. One can see from (2) that in the case of the minimal wavelength, \( \lambda = 2(k = \pi) \), this ratio equals \(-1\), i.e. the shortest waves perform anti-phase oscillations.

Differences in resonant patterns for finite and infinite 1D mass-spring lattices are distinct. In the \( N \)-DOF MSL, each \( n^{th} \) eigenfrequency is the resonant one. An external excitation of the system with such a frequency results in a linear growth of the solution with time. In the case of an infinite MSL, we have for problem (1) with added zero initial conditions and with a harmonic force applied in the zero cross-section of the waveguide:
\[ Q = H(t)\delta(x)\sin\omega_0 t \quad (H(t) \text{ and } \delta(x) \text{ are Heaviside and Dirac functions }) \]
resonant frequency \( \omega_0 = \omega_{res} = 2 \) the following asymptotic solution \([20]\):
\[ u_m(t) \sim \sqrt{t} \left[ F_2(\lambda)\sin(2t - \pi m) - F_1(\lambda)\cos(2t - \pi m) \right], \quad \lambda = 2|m|/\sqrt{t} \quad (t \rightarrow \infty, \quad |m| > t). \]  
(4)

Here oscillating and spreading with the growth of \( \lambda \) functions \( F_1(\lambda) \) and \( F_2(\lambda) \) are the well-known in the theory of transient processes, (see, e.g., \([11, 20]\)).

Thus, the resonant wave (4) propagated along the lattice has a speed value decrease with time as \( t^{-1/2} \) that, generally speaking, corresponds to the heat propagation law. Within this process, amplitudes \( u_m(t) \) increase with time as \( t^{1/2} \).

![Fig. 2: Resonant process in a MSL. (a) Envelopes of displacement oscillations in nodes \( m = 0, 10, 25, 50, 100 \) vs. time; (b) Snapshots of \( u_m \) at \( t=300, 600, 900 \).](image-url)
contrary to the linear growth in a finite MSL.

In Fig.2, calculation examples are presented of resonant waves in an infinite MSL. Envelopes $U_m(t)$ of displacements $u_{m}(t)$ for a set of nodes can be seen in Fig.2 (a), while distributions of $u_{m}(t)$ along $n$-axis at $t = 300, 600$ and $900$ are presented in Fig.2 (b). In accordance with (4), the main perturbations move at the velocity decreasing with time as $t^{-3/2}$, while their amplitudes increase with time as $t^{1/2}$. The larger the distance from the considered node to the loading point, the longer the time period needed to reach a good of correspondence of the compared solutions.

2.2 Uniform Square-Cell Lattice

In a uniform square-cell lattice, in addition to resonances excited by the frequency demarcating pass/stop bands, there resonances exist with the frequency located in the interior of a pass band.

Here and below we will consider transversal oscillations of a square-cell lattice shown in Fig. 3(a). In mechanical terms, this structure represents a plane net of bonds possessing stiffness $g$ and point particles (of mass $M$ ) at nodes $(m,n)$, $m,n = 0, \pm 1, \pm 2, \ldots$. In the case of massless bonds, in the linear approximation, a homogeneous system of dynamic equations of a square-cell lattice is

$$M\ddot{u}_{m,n} = g \left( u_{m,n+1} + u_{m,n-1} + u_{m+1,n} + u_{m-1,n} - 4u_{m,n} \right) \quad \text{for} \quad (m,n) = (0,-1) \ldots$$

(5)

where $u_{m,n}$ is a transversal displacement of node $(m,n)$. Below, along with the integers $m$ and $n$ we use continuous coordinates $x$ and $y$, respectively. The bond stiffness, $g$, the particle mass, $M$, and the cell size, $l$, are assumed to be measurement units (so that particle velocities are normalized by $c_0 = \sqrt{g/M} = 1$, the sound velocity in the lattice). A general solution of homogeneous systems (2) is represented by superposition of sinusoidal waves of type

$$u_{m,n} = U_{m,n}(\omega) e^{i \omega t}, \quad U_{m,n}(\omega) = \exp \left[ -i(k_x m + k_y n) \right]$$

(6)

where $k_x$ and $k_y$ are wave numbers (projections of the wave vector $k$), $U_{m,n}$ is the envelope. Substituting (6) into (5) we obtain the dispersion relation

$$\omega = \sqrt{4 - 2 \cos k_x - 2 \cos k_y}$$

(7)

which defines, in particular, two resonant frequencies: $\omega_1 = 2 \left( q_x = \pm \pi \pm q_x \right)$ located inside the pass band, and $\omega_2 = \sqrt{8} \left( k_y = k_x = \pi \right)$ demarcated pass- and stop bands. Their positions are shown in Fig. 3(b) in the Brillouin diagram that usually built to describe the wave dispersion in band-gap materials (see, e.g., [8,9]). The equifrequency contour $\omega = 2$ possesses a square perimeter, $k_x = \pm \pi \pm k_x$, – Fig. 3(c). Remind that the energy flux of the sinusoidal wave and its group velocity vector, $c_g = \{\omega_{g,x}, \omega_{g,y}\}$, for an arbitrary frequency are oriented as the external normal to the corresponded contour.

In the considered case ( $\omega = 2$) we have: $2c_{g,x} = 2c_{g,y} = \sin q_x = \sin q_y$, while the energy flux is absent along axes $x$ and $y$: $c_{g,x} = 0(k_x = 0)$, $c_{g,y} = 0(k_y = 0)$.

The direction of the group velocity is independent of the wave orientation within each quadrant in the $xy$-plane, and it coincides with the orientation of the discovered in [11] so-called localized primitive waveforms (LPWs). The LPW is a “selfequilibrated” standing anti-symmetric wave strictly localized on a line of a certain orientation. Any
sinusoidal wave, and hence any wave at this frequency, can be represented as a set of the LPWs, and it bears evidence of the features of the LPW. In particular, the sinusoidal-wave group velocity orientation coincides with the LPW orientation nearest to the wave propagation direction, and this is why it is piece-wise constant at this frequency.

As shown in [11], the modulus of the group wave, $|c_g|$, for $\omega = 2$ and its orientation, $\beta$ (the angle between the $x$-axis and $c_g$ that determines the direction of energy flux), are $|c_g| = \sqrt{2}/2$ and $\beta = \pm \pi/4$, i.e. the group velocity is maximal in these directions. It equals to zero at those and only at those four points where its direction changes: $q_x = \pm \pi$, $q_y = 0$ and $q_y = \pm \pi$, $q_x = 0$. If $\omega = \sqrt{8}$ (the resonant frequency at the pass/stop band interface) group velocities are also equal to zero: $q_x = q_y = \pi \Rightarrow (c_g)_x = (c_g)_y = 0$. In spite of the fact that the group velocity is non-zero almost everywhere, the resonance (in the above sense) does exist, although the oscillation amplitudes grow very slowly with time: a logarithmic law was proved by the asymptotic representation obtained in [11]. The latter is used below to compare analytical and calculation results.

2.2.1 A local source. In the case of transient harmonic loading,

$$Q(0,0,t) = Q_0 \sin \omega t \cdot H(t),$$

with $a_0 = 2$ applied to the node $(0,0)$ at $t = 0$, the asymptotical solutions ($t \to \infty$) for envelopes, $U_{m,n}(t)$, can be rewritten by the following manner:

(a) if the sum $m+n$ is even, then

$$|U_{0,0}(t)| \sim \alpha \left[ \ln t + \ln 4 + E \right], \quad \alpha = \frac{Q_0}{2\pi}; \quad |U_{m,n}(t)| \sim \left[ \ln t + \ln 4 + E - \ln |m| - 2 \right] (m \neq 0); \quad (9)$$

$$|U_{m,n}(t)| \sim \alpha \left[ \ln t + \ln 16 + E - \ln |m^2 - n^2| - 4 \right] (m \neq n), \quad E \approx 0.577 - \text{the Euler constant}; \quad (10)$$

(b) $\lim_{t \to \infty} |U_{m,n}| = 0.25Q_0$, the sum $m+n$ is odd.

At the kinematical sinusoidal excitation in the case (a), the same logarithmic growth is proved, while some differences for coefficients are detected. The main distinction is that displacements in the case (b) tend asymptotically to zero with time.

Presented here and below calculation results were obtained with use the explicit finite difference scheme applied to Eqn. (5) (or to its modifications in the case of a nonuniform lattice structure) with zero initial conditions and transient excitations: the dynamic source like to (8) or a kinematical source located at a node. In used explicit finite-difference

Fig. 3: (a) The lattice; (b) The Brillouin diagram: frequency $\omega$ is plotted against $k_x$, $k_y$, and the diagonal $k_x = k_y$; (c) The equifrequency contour $\omega = 2$. Arrows show the direction of energy flux in the stationary wave.
algorithms, the time step and boundaries of the calculated domain are chosen so that their influence on the obtained results are not detected.

Consider results related to the resonant excitation ($\omega_0 = 2$). Curves in Fig. 4 (a) show values of $\max |U_{m,n}|$ vs. time. Results in the upper window correspond to the main diagonal ($m = n$) – bold curves and to the diagonals for which $m+n$ ($m \neq n$) is even. In the former case, the coincidence with asymptote (9) is detected practically soon after the wave incoming to the current node, while in the latter case, significantly greater time values are needed to reach the asymptote (10): a non-monotonic process is detected becoming more pronounced with increasing in the distance from the source. In the lower window, the calculation results are taken in diagonals for which the sum $m+n$ is odd. The asymptote (11) possessing a qualitative sense shows a good adequacy to results. The closer a node to the source, the faster the limit (11) is reached.

In general, the above-described patterns indicate a strong localization of the propagated wave near the main diagonals of the coordinate system. To show this pattern in more details the following type of the result representation was chosen: maxima $\bar{U}_{m,n} = \max |U_{m,n}|$ are fixed in all lattice nodes at a certain time (more exactly, in a small vicinity of this time, where the considered maximum is detected), then boundaries in the calculation domain are depicted separating its into subdomains, in which magnitudes of $\bar{U}_{m,n}$ (normalized by $\bar{U}_{0,0}$) remain less than chosen values.

In the case $\omega_0 = 2$, such subdomains are shown in Fig. 4 (b) at $t = 400$. In outer areas of four-pointed stellar contours depicted by thick, median, thin solid and thin dashed curves, we have, correspondingly, $\bar{U}_{m,n}/U < 0.20, 0.10, 0.05$ and 0.01, while at this time

![Graphs showing the results](image_url)
\( U = \max |U_{0,0}| = 1.21 \) if \( Q_0 = 1 \). Numbers at the axis \( m \) (here and below) are coordinates of beam vertexes \((n = m)\). Below these numbers serve for the comparative quantitative estimate of the localization level. The wave spatial dispersion results in the 'beam swelling' for the relatively small magnitude limit, essential difference of the latter inner area from the three former those is decreased with time.

The used representation of result allows the star-like localization of the resonant wave to be simply visualized.

2.2.2 Groups of sources. Similar calculations were conducted in the case of action of grouped sources – dipoles and quadrupoles. Each of sources within the group is described by the form (8). Although the asymptotic representation of the wave pattern can be obtained by the simple superposition of formulas (9) – (11), the analysis of simulations results is more preferable to describe the wave pattern in all the calculating area at fixed time values starting from the beginning of loading (remind, that the asymptotic solution can be used for finite time values with the condition that \( t >> \sqrt{m^2 + n^2} \), where \( m \) and \( n \) are coordinates of the considered node).

Shown in Fig. 5 calculation results are related to problems with dipole sources, functioning with resonant frequency \( \omega_0 = 2 \) in (a) co-phase and (b) contra-phase modes (see all details of loading within the figure where sources marked by crosses).

\[(a) \text{ dipole}\]
\[
\begin{cases}
  m^* = 0 \\
  n^* = \pm 1
\end{cases}
\]

\[(b) \text{ co-phase loading}\]
\[u_{0,1} = u_{0,-1} = 0.5 \sin 2t \cdot H(t)\]

\[(c) \text{ contra-phase loading}\]
\[u_{0,1} = -u_{0,-1} = \sin 2t \cdot H(t)\]

\[U = 0.83\]
\[U = 0.60\]

**Fig. 5:** Action of dipoles. (a) Scheme of loading. Envelope snapshots (at \( t = 200 \)) in cases of action of (b) the co-phase dipole and (c) the contra-phase dipole.

Magnitude of the co-phase dipole is the same that in the considered monopole case, while the maximal magnitude of the each source in the self-equilibrated contra-phase dipole is equal to 1. For deterministic problem, here and below, only the first coordinate quarter of the lattice is depicted. Due to the double symmetry of the problem at the loading located in the coordinate origin, this quarter is enough to imagine and perceive the whole picture of the spatial distribution of the wave field.

The visual analysis shows the same diagonal localization of the wave pattern that in the referent monopole case, while a qualitative difference is detected in star shapes in the two mentioned dipole cases. If in the case of the co-phase dipole, the specific dwelling is adjacent to the axis, then in the contra-phase case – to the diagonal. Besides, the oscillation
level in the case of dipole excitations ($U_{\text{co-phase}} = 0.83$ and $U_{\text{contra-phase}} = 0.60$) is significantly lesser than in the monopole case ($U = 1.21$).

The same conclusions are valid in the cases of quadrupole sources – see Fig. 6.

\[ u_{1,1} = u_{-1,1} = u_{1,-1} = u_{-1,-1} = 0.25 \sin 2t \cdot H(t) \]

**Fig. 6:** Action of quadrupoles. (a) Scheme of loading. Wave patterns produced by (b) the co-phase quadrupole and (b) the contra-phase quadrupole at $t = 200$

### 3 RESONANT WAVES IN 3D LATTICE STRUCTURES

Consider a multi-layered lattice structures consisting on $S$ uniform square-cell lattices (see Fig. 3 (a)) of unite parameters connected each other serially in corresponded nodes by the same elastic transversal springs of stiffness $\gamma$.

The system of $S$ differential equations describing the time-dependent transversal motion of the mentioned lattice system is

\[
\begin{align*}
\dot{u}_{m,n,1} &= V_{m,n,1} + \gamma (u_{m,n,2} - u_{m,n,1}) + Q_{m,n,1} \\
\dot{u}_{m,n,2} &= V_{m,n,2} + \gamma (u_{m,n,3} - 2u_{m,n,2} + u_{m,n,1}) + Q_{m,n,2}
\end{align*}
\]

\[ 2 < s < S - 1: \quad \dot{u}_{m,n,s} = V_{m,n,s} + \gamma (u_{m,n,s+1} - 2u_{m,n,s} + u_{m,n,s-1}) + Q_{m,n,s} \]

\[ s = S - 1: \quad \dot{u}_{m,n,S-1} = V_{m,n,S-1} + \gamma (u_{m,n,S} - 2u_{m,n,S-1} + u_{m,n,S-2}) + Q_{m,n,S-1} \]

\[ s = S: \quad \dot{u}_{m,n,S} = V_{m,n,S} + \gamma (u_{m,n,S} - u_{m,n,S-1}) + Q_{m,n,S} \]

where $V_{m,n,s} = u_{m,n+1,s} + u_{m+1,n,s} + u_{m-1,n,s} + u_{m,n-1,s} - 4u_{m,n,s}$ is the difference operator for $s^{th}$ lattice (see Eqn. (5)), and $Q_{m,n,s} = \sin \omega_q t \cdot H(t)$ ($m,n = 0, \pm 1, \pm 2, \ldots; s = 1, S$) are the transversal transient force applied to node $(m,n,s)$. 

191
The dispersion operator of homogeneous system (12) \((Q_{m,n,s} = 0)\) is the following:

\[
\begin{bmatrix}
\omega^2 - \Omega^2 - \gamma & \omega^2 - \Omega^2 - 2\gamma & \cdots & \gamma \\
\gamma & \omega^2 - \Omega^2 - 2\gamma & \cdots & \gamma \\
0 & \cdots & 0 & \omega^2 - \Omega^2 - 2\gamma \\
0 & \cdots & 0 & \omega^2 - \Omega^2 - \gamma
\end{bmatrix}
= 0, \quad (13)
\]

where \(\Omega = \sqrt{4 - 2\cos k_x - 2\cos k_y}\).

If \(S = 2\) or \(S = 3\), we have analytically found equifrequency contours from (14), as for a single lattice above. They are turned out the same squares: \(k_i = \pm \pi \pm k_i\) (see Fig. 3(c)) that determine the following resonant frequencies:

\[
\omega_{1,\text{res}}, \omega_{2,\text{res}} = 2, \sqrt{4 + 2\gamma} \quad (S = 2); \quad \omega_{1,\text{res}}, \omega_{2,\text{res}}, \omega_{3,\text{res}} = 2, \sqrt{4 + \gamma}, \sqrt{4 + 3\gamma} \quad (S = 3) \quad (14)
\]

Note that the lowest frequency, \(\omega_{1,\text{res}} = 2\), does not depend on the number of lattices in the package. It corresponds to the zero mode detrined the in-phase mutual motion of corresponded nodes of lattices is realized without interaction, while in-lattice wave processes remain the same as in the single lattice described above. The modal analysis of the next frequencies results in the following conclusions:

- with resonant frequency \(\omega_{2,\text{res}} = \sqrt{4 + 2\gamma}\) \((S = 2)\), the identically numbered nodes of lattices are in free opposite motion so that the system of two lattices is equivalent to the single lattice upon the elastic foundation of stiffness \(2g\);
- the second resonant frequency, \(\omega_{2,\text{res}} = \sqrt{4 + \gamma}\) \((S = 3)\), is related to the free oscillation mode, where corresponded nodes of two outer lattices have the same (in absolute value) magnitudes, but are in the contra-phase motion, while nodes of the inner lattice remain immobile;
- the third resonant frequency, \(\omega_{3,\text{res}} = \sqrt{4 + 3\gamma}\) \((S = 3)\), determines the following free oscillation form: \(u_{m,n,1} : u_{m,n,2} : u_{m,n,3} = -0.5 : 1 : -0.5\).

The aim of computer simulations below is to reveal localization patterns excited by harmonic sources with resonant frequencies predicted above. In presented examples we consider the three-layered lattice structure \((S = 3)\), besides we set \(\gamma = 1\).

Let us discuss results of two wave processes excited by sinusoidal forces located in node \((0,0)\) of one of lattices and possessing resonant frequencies, \(\omega_0 = \omega_{2,\text{res}} = \sqrt{5}\) or \(\omega_0 = \omega_{3,\text{res}} = \sqrt{7}\). Vibrations of nodes \((20,20)\) are depicted in Fig. 7 in the case \(Q_1 = 3\sin \omega_0 t \quad (\omega_0 = \sqrt{5}), \quad Q_2 = Q_3 = 0\) (the factor 3 is chosen to compare results for single and three layered lattices). The specific snapshots of the localization patterns in this case can be seen in Fig. 8.

The resonant growth is detected of contra-phase oscillations in outer lattices ('s = 1' and 's = 3'), while the level of displacements in the corresponded node of the inner lattice does not (practically) depend on time and remains small. This is in qualitative accordance with the above-presented modal analysis at \(\omega_0 = \omega_{2,\text{res}} = \sqrt{5}\): the immobility of the inner lattice in
the case of free waves corresponds to its relatively small response in the nonstationary process (as compared with that in outer lattices).

The analysis of obtained results shows some specific features of wave localization patterns. For example, the inner lattice (s = 2) transmits oscillations from the loaded outer lattice (s = 1) to the second outer lattice (s = 3) node amplitudes of which just after the excitation receives relatively great values, and at the same time the along-diagonal localization is realized as in the loaded lattice. In the latter, however, this process is detectably stronger. The localization in the inner lattice is not developed.

![Fig. 7: Oscillations of displacements in the three-layered lattice system at the resonant excitation of the outer (s=1') lattice](image)

![Fig. 8: Snapshots of max|U_m,n| in three-layered lattice structure excited by the resonant frequency ω_0 = ω_{2,res} = \sqrt{5} (t = 200). Curves separate outer areas where max|U_m,n|/U < 0.2, 0.01, and 0.05 as in Fig. 4 (b) and the following similar figures.](image)
Fig. 9: Snapshots of max|Um,n| in three-layered lattice structure excited by the resonant frequency (t = 200).

In the case $Q_2 = 3\sin \alpha_0 t$, $\alpha_0 = \sqrt{7}$ and $Q_1 = Q_3 = 0$, similar snapshots are shown in Fig.9 at $t = 200$. The inner lattice is loaded now. Due to the identity of lattices, outer those are in co-phase motion at the excitation of inner (as it was indicated above, the amplitude ratio in the stationary regime is $u_1/u_2/u_3 = -0.5/1/-0.5$). Calculations show that the solution of the transient problem is not drastically differed from the stationary prediction: we have obtained the same sign phase interrelation and the similar ratio of amplitudes. At the same time, the localization level in outer lattices is detectably lower than in the loaded inner lattice.

4 WAVE PATTERNS IN NONUNIFORM LATTICE STRUCTURES

In this, final stage of the work, transient wave patterns under local sinusoidal loadings are numerically simulated in square-cell lattices possessing deterministic or random massive inclusions (below, defects). Our aim is to discover similarities and differences of resonant and localization processes as compared with those described above for a uniform lattice.

4.1 Deterministic Problems

Among of many lattices with defects, we consider specimens containing some 'defective' nodes possessing masses different from those in conventional nodes. Such specimens consist of groups ('crosses') – see Fig. 10: (a) nodes of the first group, $(0,n^*),(0,-n^*),(m^*,0),(-m^*,0)$, are placed in axes, and (b) nodes of the second group, $(m^*,m^*),(m^*,-m^*),(-m^*,m^*),(-m^*,-m^*)$, are placed in diagonals. Such configurations are motivated with features of the resonant localization in mind: the vast majority of the wave energy is captured by the near-diagonal areas, while in near-axes areas disturbances have relatively weak level. Therefore, we expect that group (a) will show a nonsignificant effect on the process of localization, and vice versa, group (b) can be essentially influenced. Besides, such groups preserve the problem symmetry that very convenient in the calculation analysis.

![Fig. 10](image)

Fig. 10: (a) and (b) – configurations of defects (bold squares), crosses indicate the source location; (c) schematic picture of a motion mode in the case of the standing resonance. Big squares are initially immobile nodes, small squares are the nodes that become ‘immobile’ at the transient process. Hollow circles are resonant nodes.

If in the case (a) we set $m^* = n^*$, then two varied parameters, $m^*$ and $M$, remain in the numerical experiments.

Solutions of the kind of travelling waves not exist here, and we are unable to obtain analytical predictions of dynamic behavior of considered structures in steady state/transient
regimes. We have conducted a set of purposeful simulations of the transient dynamics in described lattice structures loaded by the sinusoidal source \( Q_{0,0} = \sin \omega_0 t \cdot H(t) \) with frequency \( \omega_0 = 2 \) resonant for the uniform lattice.

4.1.1 The group of defects (a). Calculations show that independently on \( M \) the wave pattern is practically the same as in the uniform lattice (see above) if \( m' \geq 10 \). This effect was predicted above, and this boundary has a quantitative sense. But all the more surprising turned out the unexpected wave/oscillation patterns at some special values of \( m' \) and \( M \): the formation in an unbounded lattice of standing resonances.

\[
|U_{m,n}|_{m,n=0,0} = 2
\]

\[
|U_{m,n}|_{m,n=0,0} = 5
\]

\[
u_{3,0,0,3} = 0
\]

![Fig. 11: Envelopes of oscillations in nodes of inhomogeneous lattice with various masses of inclusions located in the axes of the lattice (the group (a)).](image)

Such an event can be associated with results presented in Fig. 11. In this example, inclusions are located in four nodes \((0,3),(3,0),(0,-3)\) and \((-3,0)\). Depicted curves correspond to envelopes of oscillations of nodes \((m;n) = (-4,\ldots,4;-4,\ldots,4)\) surrounding the source node, while only 15 nodes, \((m;n) = (-4,\ldots,4;0,\ldots,m)\), in the eighth of the plane \(m,n\) are shown (due to the double symmetry, they are enough to imagine the whole picture). Remind, that in Fig. 4 (a) such curves show the growth of the resonant wave in the uniform lattice. In the considered case, the pattern of perturbations is absolutely different: although their amplitudes in corresponding nodes are higher than in uniform lattice, the wave resonance is stopped, and instead it standing resonance is formed. In the third case where defected nodes are initially fixed (that corresponds to infinitely large masses \( M_{1,3,0,0,3} \sim \infty \)), in all nodes perturbations reach their stationary limits expect those shown in Fig. 10 (c) by 9 circles in which a rapid, practically linear, growth is detected (exactly such a growth proves the standing resonance).

Thus, in the unbounded lattices possessing the mentioned inclusions, local harmonic excitation with frequency \( \omega_0 = 2 \) results in the formation of the standing resonance. The described wave picture appears as the result of mutual influence of interferential, reflection and refraction waves so that nodes surrounding the resonant those and having significantly lower oscillation level play role of immobile those. There is easy to see that in the case of immobile nodes surrounding the mentioned 9 those shown in Fig. 10 (c), each of them oscillates independently with resonant frequency \( \omega_0 = 2 \) as 1DOF system with unite mass and the total stiffness equal to 4.

The first column in the table below, \( n^* \) determines the locations of the group of inclusions. Other columns are maximal values of disturbances (envelopes) for several values
of $M$ taken at $t = 200$ in three diagonal nodes – (0,0), (2,2) and (5,5). Bold characters indicate resonance and quasi-resonance amplitudes, a line above the number means that this amplitude corresponds to the steady state regime.

<table>
<thead>
<tr>
<th>$n/m,n$</th>
<th>$M = 2$</th>
<th>5</th>
<th>10</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.81/0.44/0.42</td>
<td>0.8/0.43/0.41</td>
<td>0.8/0.42/0.42</td>
<td>0.8/0.40/0.43</td>
</tr>
<tr>
<td>3</td>
<td>\textbf{1.48}/0.75/0.25</td>
<td>\textbf{4.81}/0.86/0.26</td>
<td>\textbf{8.0}/0.69/0.24</td>
<td>\textbf{10.6}/0.20/0.21</td>
</tr>
<tr>
<td>4</td>
<td>1.56/1.24/0.63</td>
<td>1.04/0.53/0.54</td>
<td>1.03/0.51/0.54</td>
<td>1.03/0.51/0.54</td>
</tr>
<tr>
<td>6</td>
<td>1.15/0.71/0.59</td>
<td>1.15/0.70/0.59</td>
<td>1.14/0.70/0.59</td>
<td>1.14/0.70/0.59</td>
</tr>
</tbody>
</table>

There is interesting to note that although wave resonances in lattices with the above considered defects are absent, the specific mixed pattern exists consisting of the along-diagonal localization in far field and standing resonance in near field.

4.1.2 The group of defects (b). The most intriguing result obtained in this case relates to the splitting of the main-diagonal-type localization at the resonant excitation in the uniform lattice to the three-diagonal localization pattern. In Fig. 12 examples of such a splitting are shown. The considered system of inclusions is located in four diagonals nodes $(m,n^*) = (5,5), (-5,5), (5,-5), (-5,-5)$.

**Fig. 12:** Envelopes of oscillations in nodes of inhomogeneous lattice with diagonal inclusions (the group (b)).

In Fig. 12 (a), as in the previous figures related to the localization pattern, bold, median and thin curves separate outer areas in which peak magnitudes of envelopes, $\max|U_{m,n}|$, are lesser than, correspondingly, 20%, 10% and 5% of the maximal value ($U$) in the loaded node. We have clearly indicated the main qualitative influence of the existing ‘diagonal’ system of inclusions: along-the-main-diagonal localization is transformed in the along-three-diagonal pattern with the inner main diagonal, while numbers and length of two outer symmetric diagonals depend on the inclusion mass and its distance from the source node. In this example, numbers of outer diagonals are 10, −10 (the main diagonal has number 0).

In Fig. 12 (b), (c) and (d), related to set of systems with fixed nodes, the above-mentioned curves separate outer domain in which $\max|U_{m,n}| < 0.1U$ (others curves of the level have the similar configuration). Numbers of outer diagonals are 14 (b), 21 (c) and 42 (d). The questions, why and how the three-beam configuration is formed, how to predict the distance between the mentioned beams we leave for future researches.

196
4.2 Random Problems

Consider lattices of randomly distributed masses in nodes. For more simple review of calculation results, the kinematical sinusoidal excitation

$$u_{00} = \sin \omega_0 t \cdot H(t)$$  \hspace{1cm} (16)

is applied with frequency $\omega_0 = 2$, resonant for the homogeneous case. So, for each numerical test we have the same maximal value: $U = \max u_{00} = 1$.

The pseudo Gaussian distribution is used to set mass values in lattice nodes:

$$M(\Delta M, \sigma) = 1 + \Phi^{-1}\left(\Phi(-\Delta M/\sigma) + N\left(\Phi(\Delta M/\sigma) - \Phi(-\Delta M/\sigma)\right)\right)\sigma,$$  \hspace{1cm} (17)

where $\sigma$ is standard deviation, $[-\Delta M, \Delta M]$ is the range, $\Phi(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{x} e^{-t^2/2} dt$ is the distribution function, $N$ are uniformly distributed numbers in interval $(0,1)$.

In examples of wave patterns presented in Fig. 13, the following parameters are used: $(a) \sigma = 0.02$, $\Delta M = 0.1$ and $(b) \sigma = 0.05$, $\Delta M = 0.2$. Because we have no symmetric state, the results below are presented for the whole plane occupied by the lattice, while instead of envelope images, the following presentation of results is used: coordinates of nodes are printed (by dots) in which perturbations to the present time are greater than or equal to the value 0.1.

(a) $\sigma = 0.02$, $\Delta M = 0.1$

(b) $\sigma = 0.05$, $\Delta M = 0.2$

Fig. 13: Configurations of the beam localization in lattices with random distributions of masses in nodes.

Calculation show that in the first case of relatively small deviation and range, the resonant stellar beaming is absolutely stable as in the homogeneous lattice, while in the second case, such a state remains for a some time and then begins to break down, although it traces can be detected for a long time.

Acknowledgement. GM gratefully acknowledges the support of the European Community’s Seven Framework Programme under the contract number PIAPP-GA-284544-PARM-2

REFERENCES