SIMPLIFIED VARIATIONAL PRINCIPLES FOR NON-BAROTROPIC MAGNETOHYDRODYNAMICS FURTHER DETAILS

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ABSTRACT

Variational principles for magnetohydrodynamics (MHD) were introduced by previous authors both in Lagrangian and Eulerian form. In this paper we introduce simpler Eulerian variational principles from which all the relevant equations of non-barotropic magnetohydrodynamics can be derived. The variational principle is given in terms of five independent functions for non-stationary barotropic flows. This is less then the eight variables which appear in the standard equations of barotropic magnetohydrodynamics which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$, the entropy s and the density ρ.

Keywords: Magnetohydrodynamics, Variational principles

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INTRODUCTION

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [1] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [2] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$ and the pressure P. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats has shown that the variables he suggested can be utilized to describe the motion of arbitrary discontinuity surfaces [4, 5]. Sakurai [6] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [1]. A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos [7]. Yahalom & Lynden-Bell [8] combined the Lagrangian of Sturrock [1] with the Lagrangian of Sakurai [6] to obtain an Eulerian Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [11] (see also [12]). Yahalom [9] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [10] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [13] in terms of 15 functions and V.A. Kats [3] in terms of 20 functions. The authors of this paper suspect that this number can be somewhat reduced. Moreover, A. V. Kats in a remarkable paper [21] (section IV,E) has shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions, this implies that the number of degrees of freedom can be reduced. Here we will show that only five functions will suffice to describe non barotropic magnetohydrodynamics in the case that we enforce a Sakurai [6] representation for the magnetic field.

We anticipate applications of this study both to linear and non-linear stability analysis of known non barotropic magnetohydrodynamic configurations [14, 15] and for designing efficient numerical schemes for integrating the equations of fluid dynamics and magnetohydrodynamics [16, 17, 18, 19]. Another possible application is connected to obtaining new analytic solutions in terms of the variational variables [20].

The plan of this paper is as follows: First we introduce the standard notations and equations of non-barotropic magnetohydrodynamics. Next we introduce a generalization of the barotropic variational principle suitable for the non-barotropic case. Later we simplify the Eulerian variational principle and formulate it in terms of eight functions. We conclude by showing how three variational variables can be integrated algebraically thus reducing the variational principle to five functions.

STANDARD FORMULATION OF NON-BAROTROPIC MHD

The standard set of equations solved for non-barotropic magnetohydrodynamics are given below:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{3}$$

$$\rho \frac{d\vec{v}}{dt} = \rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}. \tag{4}$$

$$\frac{ds}{dt} = 0. \tag{5}$$

The following notations are utilized $\frac{\partial}{\partial t}$ the temporal derivative, I $\frac{d}{dt}$ the temporal material derivative anc $\vec{\nabla}$ has its standard meaning in vector calculus. $\vec{B}$ is the magnetic field vector, $\vec{v}$ is the velocity field vector, ρ is the fluid density and s is the specific entropy. Finally p(ρ, s) is the pressure which depends on the density and entropy (the non-barotropic case). The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [1]). Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (2) describes the fact that the magnetic field is solenoidal, equation (3) describes the conservation of mass and equation (4) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\vec{J} = \frac{\vec{\nabla} \times \vec{B}}{4\pi}, \tag{6}$$

is the electric current density which is not connected to any mass flow. Equation (5) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic magnetohydrodynamics and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight $(\vec{v}, \vec{B}, \rho, s)$ and the number of equations (1,3,4,5) is also eight. Notice that equation (2) is a condition on the initial $\vec{B}$ field and is satisfied automatically for any other time due to equation (1).

VARIATIONAL PRINCIPLE OF NON-BAROTROPIC MHD

In the following section we will generalize the approach of [8] for the non-barotropic case. Consider the action:

$$\begin{aligned}
A &\equiv \int \mathcal{L} d^3x dt, \\
\mathcal{L} &\equiv \mathcal{L}_1 + \mathcal{L}_2, \\
\mathcal{L}_1 &\equiv \rho(\frac{1}{2}\vec{v}^2 - \varepsilon(\rho, s)) + \frac{\vec{B}^2}{8\pi}, \\
\mathcal{L}_2 &\equiv \nu[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v})] - \rho\alpha\frac{d\chi}{dt} - \rho\beta\frac{d\eta}{dt} - \rho\sigma\frac{ds}{dt} - \frac{\vec{B}}{4\pi} \cdot \vec{\nabla}\chi \times \vec{\nabla}\eta.
\end{aligned} \tag{7}$$

Obviously ν, α, β, σ are Lagrange multipliers which were inserted in such a way that the variational principle will yield the following equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \\
&\rho\frac{d\chi}{dt} = 0, \\
&\rho\frac{d\eta}{dt} = 0. \\
&\rho\frac{ds}{dt} = 0.
\end{aligned} \tag{8}$$

It is not assumed that ν, α, β, σ are single valued. Provided ρ is not null those are just the continuity equation (3), entropy conservation and the conditions that Sakurai's functions (to be described below) are comoving. Taking the variational derivative with respect to $\vec{B}$ we see that:

$$\vec{B} = \hat{\vec{B}} \equiv \vec{\nabla}\chi \times \vec{\nabla}\eta. \tag{9}$$

Hence $\vec{B}$ is in Sakurai's form and satisfies equation (2). It can be easily shown that provided that $\vec{B}$ is in the form given in equation (9), and equations (8) are satisfied, then also equation (1) is satisfied.

For the time being we have showed that all the equations of non-barotropic magnetohydrodynamics can be obtained from the above variational principle except Euler's equations. We will now show that Euler's equations can be derived from the above variational principle as well. Let us take an arbitrary variational derivative of the above action with respect to $\vec{v}$, this will result in:

$$\delta_{\vec{v}} A = \int dt \{ \int d^3x dt \rho \delta\vec{v} \cdot [\vec{v} - \vec{\nabla}\nu - \alpha\vec{\nabla}\chi - \beta\vec{\nabla}\eta - \sigma\vec{\nabla}s] + \oint d\vec{S} \cdot \delta\vec{v}\rho\nu + \int d\vec{\Sigma} \cdot \delta\vec{v}\rho[\nu] \}. \tag{10}$$

The integral $\oint d\vec{S} \cdot \delta\vec{v}\rho\nu$ vanishes in many physical scenarios. In the case of astrophysical flows this integral will vanish since $\rho = 0$ on the flow boundary, in the case of a fluid contained in a vessel no flux boundary conditions $\delta\vec{v} \cdot \hat{n} = 0$ are induced ($\hat{n}$ is a unit vector normal to the boundary). The surface integral $\int d\vec{\Sigma}$ on the cut of ν vanishes in the case that ν is single valued and $[\nu] = 0$. In the case that that ν is not single valued only a Kutta type velocity perturbation [18] in which the velocity perturbation is parallel to the cut will cause the cut integral to vanish.

Provided that the surface integrals do vanish and that $\delta_{\vec{v}} A = 0$ for an arbitrary velocity perturbation we see that $\vec{v}$ must have the following form:

$$\vec{v} = \hat{\vec{v}} \equiv \vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s. \tag{11}$$

Let us now take the variational derivative with respect to the density ρ we obtain:

$$\begin{aligned} \delta_\rho A &= \int d^3x dt \delta\rho [\frac{1}{2}\vec{v}^2 - w - \frac{\partial\nu}{\partial t} - \vec{v} \cdot \vec{\nabla}\nu] \\ &+ \int dt \oint d\vec{S} \cdot \vec{v}\delta\rho\nu + \int dt \int d\vec{\Sigma} \cdot \vec{v}\delta\rho[\nu] + \int d^3x \nu\delta\rho|_{t_0}^{t_1}. \end{aligned} \tag{12}$$

In which $w = \frac{\partial(\varepsilon\rho)}{\partial\rho}$ is the specific enthalpy. Hence provided that $\oint d\vec{S} \cdot \vec{v}\delta\rho\nu$ vanishes on the boundary of the domain and $\int d\vec{\Sigma} \cdot \vec{v}\delta\rho[\nu]$ vanishes on the cut of ν in the case that ν is not single valued and in initial and final times the following equation must be satisfied:

$$\frac{d\nu}{dt} = \frac{1}{2}\vec{v}^2 - w, \tag{13}$$

Finally we have to calculate the variation with respect to both χ and η this will lead us to the following results:

$$\delta_\chi A = \int d^3x dt \delta\chi [\frac{\partial(\rho\alpha)}{\partial t} + \vec{\nabla}\cdot(\rho\alpha\vec{v}) - \vec{\nabla}\eta\cdot\vec{J}] + \int dt \oint d\vec{S}\cdot[\frac{\vec{B}}{4\pi}\times\vec{\nabla}\eta - \vec{v}\rho\alpha]\delta\chi$$
$$+ \quad \int dt \int d\vec{\Sigma}\cdot[\frac{\vec{B}}{4\pi}\times\vec{\nabla}\eta - \vec{v}\rho\alpha][\delta\chi] - \int d^3x\rho\alpha\delta\chi|_{t_0}^{t_1}, \qquad (14)$$

$$\delta_\eta A = \int d^3x dt \delta\eta [\frac{\partial(\rho\beta)}{\partial t} + \vec{\nabla}\cdot(\rho\beta\vec{v}) + \vec{\nabla}\chi\cdot\vec{J}] + \int dt \oint d\vec{S}\cdot[\vec{\nabla}\chi\times\frac{\vec{B}}{4\pi} - \vec{v}\rho\beta]\delta\eta$$
$$+ \quad \int dt \int d\vec{\Sigma}\cdot[\vec{\nabla}\chi\times\frac{\vec{B}}{4\pi} - \vec{v}\rho\beta][\delta\eta] - \int d^3x\rho\beta\delta\eta|_{t_0}^{t_1}. \qquad (15)$$

Provided that the correct temporal and boundary conditions are met with respect to the variations $\delta\chi$ and $\delta\eta$ on the domain boundary and on the cuts in the case that some (or all) of the relevant functions are non single valued. We obtain the following set of equations:

$$\frac{d\alpha}{dt} = \frac{\vec{\nabla}\eta\cdot\vec{J}}{\rho}, \qquad \frac{d\beta}{dt} = -\frac{\vec{\nabla}\chi\cdot\vec{J}}{\rho}, \qquad (16)$$

in which the continuity equation (3) was taken into account. By correct temporal conditions we mean that both $\delta\eta$ and $\delta\chi$ vanish at initial and final times. As for boundary conditions which are sufficient to make the boundary term vanish one can consider the case that the boundary is at infinity and both $\vec{B}$ and ρ vanish. Another possibility is that the boundary is impermeable and perfectly conducting. A sufficient condition for the integral over the "cuts" to vanish is to use variations $\delta\eta$ and $\delta\chi$ which are single valued. It can be shown that χ can always be taken to be single valued, hence taking $\delta\chi$ to be single valued is no restriction at all. In some topologies η is not single valued and in those cases a single valued restriction on $\delta\eta$ is sufficient to make the cut term null. Finally we take a variational derivative with respect to the entropy's:

$$\delta_s A = \int d^3x dt \delta s [\frac{\partial(\rho\sigma)}{\partial t} + \vec{\nabla}\cdot(\rho\sigma\vec{v}) - \rho T] + \int dt \oint d\vec{S}\cdot\rho\sigma\vec{v}\delta s$$
$$- \quad \int d^3x\rho\sigma\delta s|_{t_0}^{t_1}, \qquad (17)$$

in which the temperature is $T = \frac{\partial\varepsilon}{\partial s}$. We notice that according to equation (11) σ is single valued and hence no cuts are needed. Taking into account the continuity equation (3) we obtain for location in which the density ρ is not null the result:

$$\frac{d\sigma}{dt} = T, \qquad (18)$$

provided that $\delta_s A$ vanished for an arbitrary δs.

EULER'S EQUATIONS

We shall now show that a velocity field given by equation (11), such that the equations for α, β, χ, η, ν, σ, s satisfy the corresponding equations (8,13,16,18) must satisfy Euler's equations. Let us calculate the materi$\vec{v}$ derivative of:

$$\frac{d\vec{v}}{dt} = \frac{d\vec{\nabla}\nu}{dt} + \frac{d\alpha}{dt}\vec{\nabla}\chi + \alpha\frac{d\vec{\nabla}\chi}{dt} + \frac{d\beta}{dt}\vec{\nabla}\eta + \beta\frac{d\vec{\nabla}\eta}{dt} + \frac{d\sigma}{dt}\vec{\nabla}s + \sigma\frac{d\vec{\nabla}s}{dt}. \qquad (19)$$

It can be easily shown that:

$$\begin{aligned}
\frac{d\vec{\nabla}\nu}{dt} &= \vec{\nabla}\frac{d\nu}{dt} - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k} = \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k}, \\
\frac{d\vec{\nabla}\eta}{dt} &= \vec{\nabla}\frac{d\eta}{dt} - \vec{\nabla}v_k\frac{\partial\eta}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\eta}{\partial x_k}, \\
\frac{d\vec{\nabla}\chi}{dt} &= \vec{\nabla}\frac{d\chi}{dt} - \vec{\nabla}v_k\frac{\partial\chi}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\chi}{\partial x_k}, \\
\frac{d\vec{\nabla}s}{dt} &= \vec{\nabla}\frac{ds}{dt} - \vec{\nabla}v_k\frac{\partial s}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial s}{\partial x_k}. \qquad (20)
\end{aligned}$$

In which x_k is a Cartesian coordinate and a summation convention is assumed. Inserting the result from equations (20,8) into equation (19) yields:

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= -\vec{\nabla}v_k(\frac{\partial\nu}{\partial x_k} + \alpha\frac{\partial\chi}{\partial x_k} + \beta\frac{\partial\eta}{\partial x_k} + \sigma\frac{\partial s}{\partial x_k}) + \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) + T\vec{\nabla}s \\
&+ \frac{1}{\rho}((\vec{\nabla}\eta\cdot\vec{J})\vec{\nabla}\chi - (\vec{\nabla}\chi\cdot\vec{J})\vec{\nabla}\eta) \\
&= -\vec{\nabla}v_k v_k + \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) + T\vec{\nabla}s + \frac{1}{\rho}\vec{J}\times(\vec{\nabla}\chi\times\vec{\nabla}\eta) \\
&= -\frac{\vec{\nabla}p}{\rho} + \frac{1}{\rho}\vec{J}\times\vec{B}. \qquad (21)
\end{aligned}$$

In which we have used both equation (11) and equation (9) in the above derivation. This of course proves that the non-barotropic Euler equations can be derived from the action given in equation (7) and hence all the equations of non-barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts.

SIMPLIFIED ACTION

The reader of this paper might argue here that the paper is misleading. The author has declared that they are going to present a simplified action for non-barotropic magnetohydrodynamics instead he has added six more functions α, β, χ, η, ν, σ to the standard set $(\vec{v}, \vec{B}, \rho, s)$. In the following we will show that this is not so and the action given in equation (7) in a form suitable for a pedagogic presentation can indeed be simplified. It is easy to show that the Lagrangian density appearing in equation (7) can be written in the form:

$$\mathcal{L} = -\rho[\frac{\partial \nu}{\partial t} + \alpha\frac{\partial \chi}{\partial t} + \beta\frac{\partial \eta}{\partial t} + \sigma\frac{\partial s}{\partial t} + \varepsilon(\rho, s)] + \frac{1}{2}\rho[(\vec{v} - \hat{\vec{v}})^2 - (\hat{\vec{v}})^2]$$
$$+ \frac{1}{8\pi}[(\vec{B} - \hat{\vec{B}})^2 - (\hat{\vec{B}})^2] + \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla}\cdot(\nu\rho\vec{v}). \qquad (22)$$

In which $\hat{\vec{v}}$ is a shorthand notation for $\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s$ (see equation (11)) and $\hat{\vec{B}}$ is a shorthand notation for $\vec{\nabla}\chi \times \vec{\nabla}\eta$ (see equation (9)). Thus $\mathcal{L}$ has four contributions:

$$\mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}_{\vec{v}} + \mathcal{L}_{\vec{B}} + \mathcal{L}_{boundary},$$
$$\hat{\mathcal{L}} \equiv -\rho\left[\frac{\partial \nu}{\partial t} + \alpha\frac{\partial \chi}{\partial t} + \beta\frac{\partial \eta}{\partial t} + \sigma\frac{\partial s}{\partial t} + \varepsilon(\rho, s) + \frac{1}{2}(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)^2\right]$$
$$- \frac{1}{8\pi}(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2$$
$$\mathcal{L}_{\vec{v}} \equiv \frac{1}{2}\rho(\vec{v} - \hat{\vec{v}})^2,$$
$$\mathcal{L}_{\vec{B}} \equiv \frac{1}{8\pi}(\vec{B} - \hat{\vec{B}})^2,$$
$$\mathcal{L}_{boundary} \equiv \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla}\cdot(\nu\rho\vec{v}). \qquad (23)$$

The only term containing $\vec{v}$ is $\mathcal{L}_{\vec{v}}$, it can easily be seen that this term will lead, after we nullify the variational derivative with respect to $\vec{v}$, to equation (11) but will otherwise have no contribution to other variational derivatives. Similarly the only term containing $\vec{B}$ is $\mathcal{L}_{\vec{B}}$ and it can easily be seen that this term will lead, after we nullify the variational derivative, to equation (9) but will have no contribution to other variational derivatives. Also notice that the term $\mathcal{L}_{boundary}$ contains only complete partial derivatives and thus can not contribute to the equations although it can change the boundary conditions. Hence we see that equations (8), equation (13),equations (16) and equation (18) can be derived using the Lagrangian density:

$$\hat{\mathcal{L}}[\alpha, \beta, \chi, \eta, \nu, \rho, \sigma, s] = -\rho[\frac{\partial \nu}{\partial t} + \alpha\frac{\partial \chi}{\partial t} + \beta\frac{\partial \eta}{\partial t} + \sigma\frac{\partial s}{\partial t}$$
$$+ \varepsilon(\rho, s) + \frac{1}{2}(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)^2] - \frac{1}{8\pi}(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2 \qquad (24)$$

in which $\hat{\vec{v}}$ replaces $\vec{v}$ and $\hat{\vec{B}}$ replaces $\vec{B}$ in the relevant equations. Furthermore, after integrating the eight equations (8,13,16,18) we can insert the potentials $\alpha, \beta, \chi, \eta, \nu, \sigma, s$ into equations (11) and (9) to obtain the physical quantities $\vec{v}$ and $\vec{B}$. Hence, the general non-barotropic magnetohydrodynamic problem is reduced from eight equations (1,3,4,5) and the additional constraint (2) to a problem of eight first order (in the temporal derivative) unconstrained equations. Moreover, the entire set of equations can be derived from the Lagrangian density $\hat{\mathcal{L}}$.

FURTHER SIMPLIFICATION

Elimination of Variables

Let us now look at the three last three equations of (8). Those describe three comoving quantities which can be written in terms of the generalized Clebsch form given in equation (11) as follows:

$$\frac{\partial \chi}{\partial t} + (\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)\cdot\vec{\nabla}\chi = 0$$
$$\frac{\partial \eta}{\partial t} + (\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)\cdot\vec{\nabla}\eta = 0$$
$$\frac{\partial s}{\partial t} + (\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)\cdot\vec{\nabla}s = 0 \qquad (25)$$

Those are algebraic equations for α, β, σ. Which can be solved such that α, β, σ can be written as functionals of χ, η, ν, s, resulting eventually in the description of non-barotropic magnetohydrodynamics in terms of five functions: ν, ρ, χ, η, s. Let us introduce the notation:

$$\alpha_i \equiv (\alpha,\beta,\sigma), \quad \chi_i \equiv (\chi,\eta,s), \quad k_i \equiv -\frac{\partial \chi_i}{\partial t} - \vec{\nabla}\nu\cdot\vec{\nabla}\chi_i, \qquad i \in (1,2,3) \quad (26)$$

in which the Einstein summation convention is assumed. In terms of the above notation equation (25) takes the form:

$$k_i = \alpha_j \vec{\nabla}\chi_i \cdot \vec{\nabla}\chi_j, \qquad j \in (1,2,3) \qquad (27)$$

in which the Einstein summation convention is assumed. Let us define the matrix:

$$A_{ij} \equiv \vec{\nabla}\chi_i \cdot \vec{\nabla}\chi_j \qquad (28)$$

Obviously this matrix is symmetric since $A_{ij} = A_{ji}$. Hence equation (27) takes the form:

$$k_i = A_{ij}\alpha_j, \qquad j \in (1,2,3) \qquad (29)$$

Provided that the matrix A_{ij} is not singular it has an inverse A^{-1}_{ij} which can be written as:

$$A^{-1}_{ij} = |A|^{-1}\begin{pmatrix} A_{22}A_{33} - A_{23}^2 & A_{13}A_{23} - A_{12}A_{33} & A_{12}A_{23} - A_{13}A_{22} \\ A_{13}A_{23} - A_{12}A_{33} & A_{11}A_{33} - A_{13}^2 & A_{12}A_{13} - A_{11}A_{23} \\ A_{12}A_{23} - A_{13}A_{22} & A_{12}A_{13} - A_{11}A_{23} & A_{11}A_{22} - A_{12}^2 \end{pmatrix} \qquad (30)$$

In which the determinant |A| is given by the following equation:

$$|A| = A_{11}A_{22}A_{33} - A_{11}A^2_{23} - A_{22}A^2_{13} - A_{33}A^2_{12} + 2A_{12}A_{13}A_{23} \qquad (31)$$

In terms of the above equations the α_i's can be calculated as functionals of χ_i, ν as follows:

$$\alpha_i[\chi_i, \nu] = A_{ij}^{-1} k_j. \tag{32}$$

The velocity equation (11) can now be written as:

$$\vec{v} = \vec{\nabla}\nu + \alpha_i \vec{\nabla}\chi_i = \vec{\nabla}\nu + A_{ij}^{-1} k_j \vec{\nabla}\chi_i = \vec{\nabla}\nu - A_{ij}^{-1}\vec{\nabla}\chi_i(\frac{\partial \chi_j}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\chi_j). \tag{33}$$

Provided that the χ_i is a coordinate basis in three dimensions, we may write:

$$\vec{\nabla}\nu = \vec{\nabla}\chi_n \frac{\partial \nu}{\partial \chi_n}, \qquad n \in (1, 2, 3). \tag{34}$$

Inserting equation (34) into equation (33) we obtain:

$$\begin{aligned} \vec{v} &= -A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \chi_j}{\partial t} + \vec{\nabla}\nu - A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \nu}{\partial \chi_n}\vec{\nabla}\chi_n \cdot \vec{\nabla}\chi_j \\ &= -A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \chi_j}{\partial t} + \vec{\nabla}\nu - A_{ij}^{-1} A_{jn}\vec{\nabla}\chi_i \frac{\partial \nu}{\partial \chi_n} \\ &= -A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \chi_j}{\partial t} + \vec{\nabla}\nu - \delta_{in}\vec{\nabla}\chi_i \frac{\partial \nu}{\partial \chi_n} \\ &= -A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \chi_j}{\partial t} + \vec{\nabla}\nu - \vec{\nabla}\chi_n \frac{\partial \nu}{\partial \chi_n} \\ &= -A_{ij}^{-1}\vec{\nabla}\chi_i \frac{\partial \chi_j}{\partial t} \end{aligned} \tag{35}$$

in the above δ_{in} is a Kronecker delta. Thus the velocity $\vec{v}[\chi i]$ is a functional of χ_i only and is independent of ν.

Lagrangian Density and Variational Analysis

Let us now rewrite the Lagrangian density $\hat{\mathcal{L}}[\chi_i, \nu, \rho]$ given in equation (24) in terms of the new variables:

$$\hat{\mathcal{L}}[\chi_i, \nu, \rho] = -\rho[\frac{\partial \nu}{\partial t} + \alpha_k[\chi_i, nu]\frac{\partial \chi_k}{\partial t} + \varepsilon(\rho, \chi_3) + \frac{1}{2}\vec{v}[\chi_i]^2] - \frac{1}{8\pi}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2)^2 \tag{36}$$

Let us calculate the variational derivative of $\hat{\mathcal{L}}[\chi_i, \nu, \rho]$ with respect to χ_i this will result in:

$$\delta_{\chi_i}\hat{\mathcal{L}} = -\rho[\delta_{\chi_i}\alpha_k \frac{\partial \chi_k}{\partial t} + \alpha_{\underline{i}}\frac{\partial \delta \chi_{\underline{i}}}{\partial t} + \delta_{\chi_i}\varepsilon(\rho, \chi_3) + \delta_{\chi_i}\vec{v}\cdot\vec{v}] - \frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1 \times \vec{\nabla}\chi_2) \tag{37}$$

in which the summation convention is not applied if the index is underlined. However, due to equation (33) we may write:

$$\delta_{\chi_i}\vec{v} = \delta_{\chi_i}\alpha_k \vec{\nabla}\chi_k + \alpha_{\underline{i}}\vec{\nabla}\delta\chi_{\underline{i}}. \tag{38}$$

Inserting equation (38) into equation (37) and rearranging the terms we obtain:

$$\begin{aligned}\delta_{\chi_i}\hat{\mathcal{L}} &= -\rho[\delta_{\chi_i}\alpha_k(\frac{\partial\chi_k}{\partial t}+\vec{v}\cdot\vec{\nabla}\chi_k)+\alpha_{\underline{i}}(\frac{\partial\delta\chi_{\underline{i}}}{\partial t}+\vec{v}\cdot\vec{\nabla}\delta\chi_{\underline{i}})+\ \delta_{\chi_i}\varepsilon(\rho,\chi_3)] \\ &- \frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1\times\vec{\nabla}\chi_2). \end{aligned} \tag{39}$$

Now by construction $\vec{v}$ satisfies equation (25) and hence $\frac{\partial\chi_k}{\partial t}+\vec{v}\cdot\vec{\nabla}\chi_k=0$, this leads to:

$$\delta_{\chi_i}\hat{\mathcal{L}} = -\rho\left[\alpha_{\underline{i}}\frac{d\delta\chi_{\underline{i}}}{dt}+\delta_{\chi_i}\varepsilon(\rho,\chi_3)\right]-\frac{\vec{B}}{4\pi}\cdot\delta_{\chi_i}(\vec{\nabla}\chi_1\times\vec{\nabla}\chi_2). \tag{40}$$

From now on the derivation proceeds as in equations (14,15,17) resulting in equations (16,18) and will not be repeated. The difference is that now α, β and σ are not independent quantities, rather they depend through equation (32) on the derivatives of χ_i, ν. Thus, equations (14,15,17) are not first order equations in time but are second order equations. Now let us calculate the variational derivative with respect to ν this will result in the expression:

$$\delta_\nu\hat{\mathcal{L}} = -\rho[\frac{\partial\delta\nu}{\partial t}+\delta_\nu\alpha_n\frac{\partial\chi_n}{\partial t}] \tag{41}$$

However, $\delta_\nu\ \alpha_k$ can be calculated from equation (32):

$$\delta_\nu\alpha_n = A^{-1}_{nj}\delta_\nu k_j = -A^{-1}_{nj}\vec{\nabla}\delta\nu\cdot\vec{\nabla}\chi_j \tag{42}$$

Inserting the above equation into equation (41):

$$\delta_\nu\hat{\mathcal{L}} = -\rho[\frac{\partial\delta\nu}{\partial t}-A^{-1}_{nj}\vec{\nabla}\chi_j\frac{\partial\chi_n}{\partial t}\cdot\vec{\nabla}\delta\nu] = -\rho[\frac{\partial\delta\nu}{\partial t}+\vec{v}\cdot\vec{\nabla}\delta\nu] = -\rho\frac{d\delta\nu}{dt} \tag{43}$$

The above equation can be put to the form:

$$\delta_\nu\hat{\mathcal{L}} = \delta\nu[\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})]-\frac{\partial(\rho\delta\nu)}{\partial t}-\vec{\nabla}\cdot(\rho\vec{v}\delta\nu) \tag{44}$$

This obviously leads to the continuity equation (3) and some boundary terms in space and time. The variation derivative with respect to ρ is trivial and the analysis is identical to the one in equation (12) leading to equation (13). To conclude this subsection let us summarize the equations of non-barotropic magnetohydrodynamics:

$$
\begin{aligned}
\frac{d\nu}{dt} &= \frac{1}{2}\vec{v}^2 - w, \\
\frac{\partial\rho}{\partial t} &+ \vec{\nabla}\cdot(\rho\vec{v}) = 0 \\
\frac{d\sigma}{dt} &= T, \\
\frac{d\alpha}{dt} &= \frac{\vec{\nabla}\eta\cdot\vec{J}}{\rho}, \\
\frac{d\beta}{dt} &= -\frac{\vec{\nabla}\chi\cdot\vec{J}}{\rho},
\end{aligned}
\tag{45}
$$

in which α, β, σ, $\vec{v}$ are functionals of χ, η, s, ν as described above. It is easy to show as in equation (21) that those variational equations are equivalent to the physical equations.

CONCLUSION

It is shown that non-barotropic magnetohydrodynamics can be derived from a variational principle of five functions. Relation to a Hamiltonian formalism and possible application will be given in a future comprehensive paper.

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