# Game Theoretic Modeling and Computational Analysis of N-Player Conflicts over Resources


Noam Hazon
*School of Computer Science*
*Carnegie Mellon University*
*Pittsburgh, USA*
*noamh@cs.cmu.edu*

Nilanjan Chakraborty
*School of Computer Science*
*Carnegie Mellon University*
*Pittsburgh, USA*
*nilanjan@cs.cmu.edu*

Katia Sycara
*School of Computer Science*
*Carnegie Mellon University*
*Pittsburgh, USA*
*katia@cs.cmu.edu*



***Abstract*—This paper considers the problem of $n$-player conflict modeling, arising due to competition over resources. Each player represents a distinct group of people and has some resource and power. A player may either attack other players (i.e., groups) to obtain their resources or do nothing. We present a game-theoretical model for interaction between the players, and show that key questions of interest to policy makers can be answered efficiently, i.e., in polynomial time in the number of players. They are: (1) Given the resources and the power of each group, is no-war a stable situation? and (2) Assuming there are some conflicts already in the society, is there a danger of other groups not involved in the conflict joining the conflict and further degrading the current situation? We show that the pure strategy Nash equilibrium is not an appropriate solution concept for our problem and introduce a refinement of the Nash equilibrium called the asymmetric equilibrium. We also provide an algorithm (that is exponential in the number of players) to compute all the asymmetric equilibria and propose heuristics to improve the performance of the algorithm.**



***Keywords*-conflict modeling; game theory;**


## I. Introduction

Empirical research in the social sciences [1] identified both material and non-material causes for conflicts in multicultural societies. Material causes include uneven distribution of resources (sometimes caused by population explosion or environmental degradation) and uneven economic development. Non-material causes include factors like history of violence, fear of losing ethno-religious identity or ethnic identity based politics. Although both material and non-material factors influence conflicts to different degrees, in certain cases, scarcity of natural and economic resources is a key factor in precipitating conflicts. For example, in the Darfur area in Sudan, although ethnicity and Sudanese politics play a major role, the shrinkage of pastoral land for nomads due to severe drought was a key factor in starting the conflict [2]. Within the social sciences literature there has been extensive use of game theory for international or inter-state conflict modeling (see [3], [4] for a review). However, armed conflicts in regions with heterogeneous (in terms of ethnicity) population, has not been studied adequately from a game-theoretic perspective. In particular, there is a lack of a $n$-player game-theoretic model that is rich enough to model important socio-economic and political factors (e.g., resources, political power). In this paper, we present a $n$-player game for modeling conflicts due to competition over resources, and analyze its computational aspects.

In our game model, we consider each ethnic group to be a player. We assume that each group (or player) has a certain amount of resources and a certain amount of power. Each player has the possibility to attack any other player. There is a battle between the two players if any one of the players decides to attack the other. There is a cost of the battle and the probability of a player winning in a battle is assumed to be proportional to the relative power of the player with respect to its opponent. The winner of a battle takes all the resources of the opponent (in case of multiple players winning against the same opponent, the resources are shared equally). We assume the players to be rational in the sense that a player will have an incentive to attack another player if its total expected resources (from the battles it wins) is more than its current resources. With the suggested model, we can provide answers to key questions of interest to analysts and policy makers, efficiently.

More precisely, we show that we can efficiently determine (in $O(n^2)$ time, where $n$ is the number of players) if, in a multi-cultural society, (1) no war is a stable situation, and (2) assuming there are already some conflicts in the society, if there is a danger that other players not involved in the conflict may initiate new wars and further degrade the situation. We demonstrate that the pure strategy Nash equilibrium (NE) is not an appropriate solution concept for our problem. Hence, we introduce a *refinement* of the NE called the *asymmetric equilibrium*. We also present an algorithm for computing all possible asymmetric equilibria (that takes exponential time and space in the number of players). We further present heuristics that use the structure of our problem to improve the performance of the algorithm.

This paper is organized as follows. In Section II we define our game-theoretic model. In Section III, we show why NE is not completely appropriate for our game, and define the new refinement, the asymmetric equilibrium. In Section IV we show that whether the no-war situation is stable can be determined in polynomial time, and in Section V we show that whether a scenario with wars is stable can be determined in polynomial time. In Section VI we discuss the algorithm to find all the equilibriums, and some heuristics to improve its actual running time. Section VII discusses the related work, and Section VIII concludes the paper by summarizing our results and proposing several directions for future research.

## II. Terminology and Definition of Our Model

A *game* is given by a tuple $\langle \mathcal{P}, \{\Sigma_i\}_{i=1}^n, \{u_i\}_{i=1}^n \rangle$, where $\mathcal{P}$ is a set of $n$ *players*, $\Sigma_i$ is a finite *strategy* space for player $i \in \mathcal{P}$, and $u_i(\cdot)$ is a *utility* or *payoff* function of player $i$. A *strategy profile* or a *state* is an association of strategies to players, that is an n-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for each $i \in \mathcal{P}$, $\sigma_i \in \Sigma_i$. We denote by $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$ the strategy profiles space of the $n$ players. For a strategy profile $\sigma \in \Sigma$ we denote by $\sigma_{-i}$ the strategies of player $j \neq i$, i.e., $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$. We assume that the players have complete information, i.e, every player knows the payoffs and strategies available to other players.

In our game, the strategy for each player is which players he decides to attack. Therefore, for each $i \in \mathcal{P}$, $|\Sigma_i| = 2^{n-1}$. We assume that the game has an initial or a *status-quo* state, denoted by $\sigma^0$, where there are no attacks. Each player $i$ has an associated *power* or *size*, $s_i$, and *resources*, $r_i$. We normalize them such that $\sum_{i=1}^n s_i = 1$ and $\sum_{i=1}^n r_i = 1$. In every strategy profile $\sigma \neq \sigma^0$, if either player $i$ attacks player $j$ or player $j$ attacks player $i$ we say that there is a *battle* between $i$ and $j$ in $\sigma$, and its outcome is a Bernoulli random variable - either $i$ wins or loses. Let $X_{i,j}$ be the Bernoulli random variable which represents the outcome of the battle between $i$ and $j$ such that,

$$X_{i,j} = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, every battle can be represented from both sides perspectives and we get $X_{i,j} = 1 - X_{j,i}$. We make the following assumptions on the possible outcome of the battles:

1) Independence: a player may lose in some battles and win in others at the same time.
2) Offence-defence tradeoff: the outcome of a battle does not depend on which side initiated the attack.

We can represent a strategy profile $\sigma$ by a directed graph $\vec{G_\sigma} = (V, \vec{E_\sigma})$, where the set of nodes $V = \{v_1, v_2, \ldots, v_n\}$ corresponds to the set of players, and the set of directed edges $\vec{E_\sigma} \subseteq V \times V$ represents the attacks in $\sigma$ such that if player $i$ attacks player $j$, $(v_i, v_j) \in \vec{E_\sigma}$. We call $\vec{E_\sigma}$ the *directed battles scenario* associated with $\sigma$. However, since we make the offence-defence tradeoff assumption, it is more convenient to work with the undirected graph $G_\sigma = (V, E_\sigma)$, where $E_\sigma$ is the undirected version of $\vec{E_\sigma}$. We call $E_\sigma$ the *battles scenario* associated with $\sigma$. Since a battles scenario omits the information of who initiated the attack, it corresponds to more than one strategy profile (unless $E_\sigma$ is the empty set). For each player $i$, let $N_i$ be the set of immediate neighbors of $v_i$ in $G_\sigma$. $\vec{N_i}$ is defined similarly in $\vec{G_\sigma}$. The total number of battles that player $i$ faces is therefore $|N_i|$, while the number of battles that this player initiated is $|\vec{N_i}|$. If $|N_i| > 0$, we assume that player $i$ divides its power equally among its battles, allocating $\hat{s_i} = s_i / |N_i|$ for each one of them. The (probabilistic) outcome of each battle is thus determined by the ratio between the allocated powers of the players, $P(X_{i,j} = 1) = \frac{1}{1 + \hat{s}_j / \hat{s}_i}$. Note that this definition preserves the probability constraints, i.e., $P(X_{i,j} = 1) = 1 - P(X_{j,i} = 1) = P(X_{j,i} = 0)$. Every battle also entails a cost. We denote by $C_{i,j}$ the cost of a battle between players $i$ and $j$.

Now, if $|N_i| = 0$ then the player's utility in $\sigma$ is simply $r_i$. Otherwise, since we deal with probabilities, we assume that each player tries to maximize its expected utility. The expected utility of every player $i \in \mathcal{P}$ is composed of the expected gain, which is the expected amount of resources that $i$ wins in its battles, minus the cost of all the battles that $i$ is involved in. We assume that in the case of multiple players winning against the same opponent the resources are shared equally. In addition, $i$ keeps its current resources $r_i$ only if he does not lose in any battle. Formally, given a strategy profile $\sigma$ and a player $i$, a *relevant event* e is an assignment of values (0 or 1) to all the random variables in the set $\mathcal{X} = \{X_{i,j} : v_j \in N_i\} \cup \{X_{j,k} : v_j \in N_i, v_k \in N_j\}$. Let $\mathrm{e}(X_{i,j})$ be the value assigned by e to $X_{i,j}$. The probability of e occurring is $P(\mathrm{e}) = \prod_{X_{i,j} \in \mathcal{X}} P(X_{i,j} = \mathrm{e}(X_{i,j}))$, and the gain for $i$ if this event happens is,

$$Gain(\mathrm{e}) = \begin{cases} 0 & \text{if } \forall j, \mathrm{e}(X_{i,j}) = 0 \\ r_i + \hat{G} & \text{if } \forall j, \mathrm{e}(X_{i,j}) = 1 \\ \hat{G} & \text{otherwise} \end{cases}$$

where $\hat{G} = \sum_{j : \mathrm{e}(X_{i,j}) = 1} \frac{r_j}{|\{k : \mathrm{e}(X_{k,j}) = 1\}| + 1}$.

The *expected gain*, $EG$, is thus $\sum_{\mathrm{e}} (P(\mathrm{e}) \cdot Gain(\mathrm{e}))$. Therefore, the (expected) utility of player $i$ in $\sigma$ is,

$$u_i(\sigma) = EG - \sum_{j : v_j \in N_i} C_{i,j}$$

Actually, this is also the expected utility of player $i$ in the corresponding battles scenario $E$. We will denote it by $u_i(E_\sigma)$.

## III. New Solution Concept: Asymmetric Equilibrium

After defining our model, we need to decide on a *solution concept*, which is a formal rule for predicting how the game will be played. In our case we have several players that are making decisions at the same time, and the outcome depends on the decisions of the others. For that case, the *Nash Equilibrium (NE)* is a well-known solution concept, that is used to analyze the outcome of the strategic interaction of several decision makers.

*Definition 1:* A strategy profile $\sigma^* \in \Sigma$ is a *Nash Equilibrium (NE)* if no player $i \in \mathcal{P}$ can benefit by unilaterally deviating from its strategy to another strategy, i.e., for every $i \in \mathcal{P}, \sigma_i \in \Sigma_i$, it holds that $u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*)$.

Informally, a strategy profile is a NE if no player can do better by unilaterally changing its strategy. A game can have either a pure strategy or a mixed strategy NE (i.e., a probability distribution over pure strategies), but in our case we consider only pure strategies. The reason is that seldom do people make their choices following a lottery. Furthermore, in our case the game is played only once, and the decisions of a player have a significant effect- whether there will be a war or not.

One of the main characteristics in our model is the offence-defence tradeoff. According to Jervis [5], there are cases where the attacker is more likely to win, and there may be cases where the defender has the advantage. However, there are many factors involved, like the available technology of war, territorial features, etc. The exact characterization is too coarse for quantitative models, thus we assumed in our model that the outcome of a battle does not depend on the side that initiated the attack. Our model assigns the same utility for each strategy profile that is represented by the same battles scenario. We are thus interested in predicting which battles scenario will emerge according to the decisions of the players. Since NE is defined over strategies (which corresponds to directed battles scenarios) we adjust it to our case.

*Definition 2:* A battles scenario $E_\sigma$ is a NE if there is at least one strategy profile $\sigma$ which is a NE, and $\sigma$ is represented by $E_\sigma$.

The use of NE in our case to predict what will happen is a little problematic. We illustrate it using the following example.

*Example 1:* Suppose that we have 3 players with sizes and resources as described in Table I, and the cost of war for every pair of players is $0.1$. Table II shows the expected utility for each player, in each possible battles scenario.

Table I
EXAMPLE 1: 3-PLAYER GAME

| Player | Size | Resource |
|---|---|---|
| 1 | 0.55 | 0.6 |
| 2 | 0.25 | 0.2 |
| 3 | 0.20 | 0.2 |

Table II
EXAMPLE 1: THE UTILITIES FOR THE 3-PLAYER GAME

| Battles Scenario | | | Utilities | | |
|---|---|---|---|---|---|
| 1 Vs. 2 | 1 Vs. 3 | 2 Vs. 3 | Player 1 | Player 2 | Player 3 |
| | | | 0.6 | 0.2 | 0.2 |
| ✓ | | | 0.45 | 0.15 | 0.2 |
| | ✓ | | 0.49 | 0.2 | 0.12 |
| | | ✓ | 0.6 | 0.12 | 0.08 |
| ✓ | ✓ | | 0.2 | 0.22 | 0.18 |
| ✓ | | ✓ | 0.5 | 0.01 | 0.1 |
| | ✓ | ✓ | 0.52 | 0.12 | -0.04 |
| ✓ | ✓ | ✓ | 0.31 | 0.07 | 0.02 |

Now, let's examine the strategy profile $\sigma^*$ in which player 1 attacks player 2, player 2 attacks player 1, and player 3 does nothing. Clearly, this strategy profile corresponds to the second battles scenario in Table II. In $\sigma^*$, player 1 can not unilaterally cancel the battle with player 2. As seen from the table, player 1 does not gain a higher utility by, in addition, initiating a battle with player 3. The same argument holds for player 2. The highest utility for player 3 is when it is not involved in any battle, thus player 3 does not have an incentive to deviate from the current state. Therefore, $\sigma^*$ is a NE. However, it is clear that both players 1 and 2 will prefer not to initiate the battle among them in advance. Example 1 demonstrates that a strategy profile could be technically a NE, but it will still not happen. Instead, the players will deviate from the strategy profile together, even without explicit coordination. The flaw in using the NE concept in our case is that it avoids the presence of some actions that can not be canceled by a single player (like the battle between players 1 and 2). We therefore suggest a refinement to the NE solution concept, that we call *asymmetric equilibrium*. We start by defining the notion of symmetric strategies. For a strategy profile $\sigma \in \Sigma$ let $\sigma_{-i,j}$ be the strategies of player $k \neq i, j$, i.e., $\sigma_{-i,j} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_n)$.

*Definition 3:* In a given game, two strategies $\sigma_i$ and $\sigma_j$ for players $i$ and $j$, respectively, are *symmetric strategies* if for every strategy profile $\sigma \in \Sigma$ and for every player $x \in \mathcal{P}$, $u_x(\sigma_{-i,j}, \sigma_i, \sigma_j) = u_x(\sigma_{-i}, \sigma_i) = u_x(\sigma_{-j}, \sigma_j)$

In our game, for every two players $i$ and $j$, if $i$ attacks $j$ and $j$ attacks $i$, $\sigma_i$ and $\sigma_j$ are symmetric strategies. Symmetric strategies could be also found in any other game where players interact with each other, and the outcome of the interaction does not depend on who initiated it. Actually, Definition 3 is far more general, since it is not restricted to a specific game type. If the NE contains symmetric strategies, we might get the unwanted situation that appeared in Example 1. We therefore define the asymmetric equilibrium.

*Definition 4:* A strategy profile is an *asymmetric equilibrium* if it is a NE, and it does not contain any symmetric strategies.

The asymmetric equilibrium resembles another solution concept, the Coalition-Proof Nash Equilibrium (CPNE) [6]. The CPNE is a refinement of the NE, that requires stability against deviations of coalitions. It assumes that players may jointly deviate in a way that is mutually beneficial and self-enforcing (i.e., immune to deviations by sub-coalitions). Our asymmetric equilibrium goes in the same vein, as can be seen in Example 1: $\sigma_*$ is not asymmetric equilibrium since the symmetric attack actions of players 1 and 2 can not be canceled by only one of them, and they both will prefer to deviate to the status-quo state. However, there is a key difference between the underlying assumptions of the two solution concepts. The CPNE assumes that there is a round before the games starts, when the players have unlimited (but non-binding) communication. The players can use this round to coordinate and decide which coalitions should be formed. In our setting, we do not assume that the players can communicate. Therefore, the CPNE eliminates any NE which is susceptible to deviation of coalitions of any size. Our asymmetric equilibrium eliminates any NE which is susceptible to deviation of two players with symmetric strategies. The asymmetric equilibrium tries to fix a technical problem with the NE, that arises when there are symmetric strategies. In that case, two players may form an implicit coalition to deviate, without the use of communication.

As noted before, we are interested in predicting which battles scenario will emerge. We therefore adjust Definition 2 to use asymmetric equilibrium.

*Definition 5:* A battles scenario $E_\sigma$ is an *asymmetric equilibrium* if there is at least one strategy profile $\sigma$ which is an asymmetric equilibrium, and $\sigma$ is represented by $E_\sigma$.

This definition means that when checking if a battle scenario is an asymmetric equilibrium, one needs to check the incentive of players to add and remove battles. Checking if a battle scenario is a NE is simpler: one can assume that every battle is initiated by both sides, and thus check only the incentive to add battles (the exact procedure for finding all the asymmetric equilibria will be described in Section VI). Nevertheless, since we demonstrated the problems with the NE in our case, we will adopt the asymmetric equilibrium as our solution concept, and for the rest of the paper we will refer to asymmetric equilibrium as "equilibrium".

## IV. The Stability of the Status-Quo

A key question in analyzing our strategic multi-player environment is to determine whether the current status-quo is stable, i.e, whether no-war is an equilibrium or not. In this section we show that we can answer this question in polynomial time, with regard to the number of players.

We recall that the status-quo state in our case is the strategy profile $\sigma^0$ where all the actions of every player is to do nothing. This is the only strategy profile which is represented by exactly one battles scenario (which is a graph with no edges) and this strategy also maximizes the social welfare. Furthermore, in every other strategy profile there is at least one battle. In practice, players will not always choose the action with the highest expected utility if they can avoid being involved in a battle. Even though we modeled the cost of war, we believe that the natural tendency of a player is not to be involved in a war. Therefore, if the status-quo is in equilibrium, it is very reasonable to assume that this will be the actual outcome of the game even if there are other profiles that are also in equilibrium.

Since every player can attack any subset of players, the number of possible deviations from $\sigma^0$ is $n \cdot (2^{n-1}-1)$. Nevertheless, we can decide if $\sigma^0$ is an equilibrium in polynomial time using the following lemma.

*Lemma 6:* Given player $\hat{a}$, let $B \subseteq \mathcal{P} \setminus \hat{a}$ be a subset of players not containing $\hat{a}$, and suppose that the current strategy profile is $\sigma^0$. If $\hat{a}$ does not have an incentive to attack any player $i \in B$ *separately*, then it does not have an incentive to attack all the players in $B$ *simultaneously*.

*Proof:* Without loss of generality, let $B = \{i\}_{i=1}^{t}$. We are given that for each $i \in B$,

$$\frac{1}{1+\hat{s}_i/\hat{s}_{\hat{a}}} \cdot (r_{\hat{a}} + r_i) - C_{\hat{a},i} \leq r_{\hat{a}}$$

Since players $\hat{a}$ and $i$ face only one battle, $\hat{s}_{\hat{a}} = s_{\hat{a}}$ and $\hat{s}_i = s_i$. We get that,

$$r_i \leq \frac{C_{\hat{a},i} \cdot (s_{\hat{a}} + s_i) + s_i r_{\hat{a}}}{s_{\hat{a}}} \tag{1}$$

We need to prove that $\hat{a}$ does not have any incentive to attack all the players in $B$. In that case, $\hat{s}_{\hat{a}} = s_{\hat{a}}/t$ and for each $i \in B$, $\hat{s}_i = s_i$. Thus, we can write the probability $P(X_{\hat{a},i}=1)$ as $\frac{s_{\hat{a}}}{s_{\hat{a}}+ts_i}$. To calculate the expected utility, we need to sum over all the $2^t-1$ relevant events. We do so by dividing all the events to $t$ disjoint classes, according to the number of battles that $\hat{a}$ loses. For every $k \in \{0,1,\ldots,t-1\}$, let $S^k$ be the set of events where $\hat{a}$ loses $k$ times, and let $E^k$ be the expected gain from the events in $S^k$, i.e., $E^k = \sum_{\mathrm{e} \in S^k}(P(\mathrm{e}) \cdot Gain(\mathrm{e}))$. We write explicitly,

$$E^0 = \frac{s_{\hat{a}}^t (r_{\hat{a}} + \sum_{k=1}^t r_k)}{\prod_{j=1}^t (s_{\hat{a}} + ts_j)}$$

$$E^1 = \frac{\sum_{j_1=1}^t (s_{\hat{a}}^{t-1} \cdot ts_{j_1} \sum_{k=1, k\neq j_1}^t r_k)}{\prod_{j=1}^t (s_{\hat{a}} + ts_j)}$$

$$E^2 = \frac{\sum_{j_1=1}^{t-1} \sum_{j_2=j_1+1}^t (s_{\hat{a}}^{t-2} \cdot t^2 s_{j_1} s_{j_2} \sum_{k=1, k\neq j_1, j_2}^t r_k)}{\prod_{j=1}^t (s_{\hat{a}} + ts_j)}$$

and so on. Therefore, we need to prove that

$$\sum_{j=0}^{t-1} E^j - \sum_{j=1}^{t} C_{\hat{a},j} \overset{?}{\leq} r_{\hat{a}} \tag{2}$$

Note that the denominator of all the $E^i$'s is the same. Let $Num(E)$ be the sum of numerators of all the $E^i$'s. Thus,

$$Num(E) = s_{\hat{a}}^t r_{\hat{a}} + \sum_{k=1}^{t} [r_k (s_{\hat{a}}^t + \sum_{\substack{j_1=1 \\ j_1 \neq k}}^{t} (s_{\hat{a}}^{t-1} ts_{j_1}) +$$

$$\sum_{\substack{j_1=1 \\ j_1 \neq k}}^{t-1} \sum_{\substack{j_2=j_1+1 \\ j_2 \neq k}}^{t} (s_{\hat{a}}^{t-2} t^2 s_{j_1} s_{j_2}) + \ldots + s_{\hat{a}} t^{t-1} \prod_{\substack{j_1=1 \\ j_1 \neq k}}^{t} s_{j_1})] =$$

$$s_{\hat{a}}^t r_{\hat{a}} + \sum_{k=1}^{t}[r_k s_{\hat{a}} \prod_{\substack{j=1\\j\neq k}}^{t}(s_{\hat{a}} + ts_j)]$$

Since we are given (1),

$$Num(E) \leq s_{\hat{a}}^t r_{\hat{a}} + \sum_{k=1}^{t}[(C_{\hat{a},k}(s_{\hat{a}}+s_k)+s_k r_{\hat{a}}) \prod_{\substack{j=1\\j\neq k}}^{t}(s_{\hat{a}}+ts_j)]$$

The left side of the inequality in (2) is thus,

$$\frac{Num(E)}{\prod_{j=1}^{t}(s_{\hat{a}} + ts_j)} - \sum_{j=1}^{t} C_{\hat{a},j} \leq \frac{s_{\hat{a}}^t r_{\hat{a}}}{\prod_{j=1}^{t}(s_{\hat{a}} + ts_j)} +$$

$$\frac{\sum_{k=1}^{t}[(C_{\hat{a},k}(s_{\hat{a}} + s_k) + s_k r_{\hat{a}}) \prod_{j=1, j\neq k}^{t}(s_{\hat{a}} + ts_j)]}{\prod_{j=1}^{t}(s_{\hat{a}} + ts_j)} -$$

$$\sum_{j=1}^{t} C_{\hat{a},j} = \sum_{j=1}^{t} C_{\hat{a},j}(\frac{s_{\hat{a}} + s_k}{s_{\hat{a}} + ts_k} - 1) +$$

$$\frac{s_{\hat{a}}^t r_{\hat{a}} + \sum_{k=1}^{t}[s_k r_{\hat{a}} \prod_{j=1, j\neq k}^{t}(s_{\hat{a}} + ts_j)]}{\prod_{j=1}^{t}(s_{\hat{a}} + ts_j)}$$

Since $\frac{s_{\hat{a}}+s_k}{s_{\hat{a}}+ts_k} < 1$, it suffices to show that,

$$\frac{s_{\hat{a}}^t r_{\hat{a}} + \sum_{k=1}^{t}[s_k r_{\hat{a}} \prod_{j=1, j\neq k}^{t}(s_{\hat{a}} + ts_j)]}{\prod_{j=1}^{t}(s_{\hat{a}} + ts_j)} \overset{?}{\leq} r_{\hat{a}}$$

That is,

$$s_{\hat{a}}^t + \sum_{k=1}^{t}[s_k \prod_{\substack{j=1\\j\neq k}}^{t}(s_{\hat{a}} + ts_j)] \overset{?}{\leq} \prod_{j=1}^{t}(s_{\hat{a}} + ts_j) \qquad (3)$$

However, the left term in (3) is equivalent to

$$s_{\hat{a}}^t + \sum_{k=1}^{t}[\frac{ts_k + s_{\hat{a}}}{t} \prod_{\substack{j=1\\j\neq k}}^{t}(s_{\hat{a}} + ts_j)] - \sum_{k=1}^{t}[\frac{s_{\hat{a}}}{t} \prod_{\substack{j=1\\j\neq k}}^{t}(s_{\hat{a}} + ts_j)] =$$

$$s_{\hat{a}}^t + \prod_{j=1}^{t}(s_{\hat{a}} + ts_j) - (s_{\hat{a}}^t + R)$$

where $R > 0$. Therefore, the inequality in (3) holds. ■

We are now ready to prove our theorem.

*Theorem 7:* Deciding if $\sigma^0$ is an equilibrium can be done in $O(n^2)$ steps.

*Proof:* For each $i, j \in \mathcal{P}, i \neq j$, let $\sigma^{i\to j}$ be the strategy profile where $i$ deviates from $\sigma^0$ by attacking $j$. If there exists an $i$ such that $u_i(\sigma^{i\to j}) > r_i$, then $i$ has an incentive to deviate from $\sigma^0$, and therefore $\sigma^0$ is not an equilibrium. Otherwise, every player has no incentive to deviate by attacking any subset of players, according to Lemma 6, and $\sigma^0$ is an equilibrium. Checking if $u_i(\sigma^{i\to j}) > r_i$ for every $i \neq j$ takes only $O(n^2)$ operations. ■

Apart from being the basis for Theorem 7, Lemma 6 leads to a better understanding of the conditions that are required for a conflict to arise. The next proposition is an example to one of the consequences of Lemma 6.

*Proposition 8:* Suppose $\forall i \in \mathcal{P}$, $s_i = r_i$, and $\forall j \neq i$, $C_{i,j} > 0$ in a given game. Then, $\sigma^0$ is an equilibrium.

This result corresponds to the finding in Powell [7, ch. 3], that war is less likely when the existing distribution of resources among the players reflects the underlying distribution of power. Although he uses a model of bargaining, our model leads to the same result: if every player has a power which is proportional to its size, and the cost of every war is greater than zero, the no-war state is in equilibrium.

## V. Stability of Status-Quo with Wars

In the previous section we analyzed the stability of the no-war status-quo state. In some cases, the assumption that the status-quo state does not contain any battle does not hold. Players may initiate a battle even if it is not worth it for several reasons. They may lack knowledge about the power or resources of other players, so they estimate it wrongly. Players may also have ethnographic, religious or emotional factors which affects their decision to fight, and these factors are not modeled in our utility function. We can also think of a situation where the no-war is not in equilibrium, thus players may initiate some battles. Then, new players arrive, and the status-quo state for the new game will contain the battles that are already there.

We assume that the players that are already involved in battles are committed to their actions. They can not cancel them unilaterally, or initiate new battles against other players. However, the battles are still active, so the resources and powers of the players that are involved in the battle are still the same. We now show how to generalize Lemma 6 and Theorem 7, so that given any initial strategy profile, we can decide if it is an equilibrium in polynomial time.

*Lemma 9:* Given player $\hat{a}$, let $B \subseteq \mathcal{P} \setminus \hat{a}$ be a subset of players not containing $\hat{a}$, and suppose that in the current strategy profile $\hat{a}$ is not involved in any battle. If $\hat{a}$ does not have an incentive to attack any player $i \in B$ *separately*, then it does not have an incentive to attack all the players in $B$ *simultaneously*.

*Proof:* (sketch) The proof's idea is to show an alternative way to represent the expected payoff of a player. Given a player $\hat{a}$, we show that from $\hat{a}$'s point of view, a battles scenario $E_{\hat{a}}$ is equivalent to a battles scenario which contains only edges from the set $\{(v_{\hat{a}}, v_j) : v_j \in N_i\}$, but with different resources for the players. In this "dual" representation there are no battles, except from the battles of $\hat{a}$ with other players. Therefore, we can plug-in this representation for the use with Lemma 6. We omit the details due to space constraints. ■

Since only players that are not currently involved in a battle can initiate one, the proof of the following theorem is essentially similar to the proof of Theorem 7, but uses Lemma 9 instead of Lemma 6. We get:

*Theorem 10:* Given a strategy profile $\sigma$ as an initial state, we can decide if $\sigma$ is an equilibrium in $O(n^2)$ steps.

## VI. Finding all the Equilibria

We now return to the setting where the statu-quo is $\sigma^0$ and present the basic algorithm that finds all the equilibria. We then suggest several heuristics, to improve the algorithm's actual run-time. To understand the motivation for finding all the equilibria, consider the following example.

*Example 2:* Suppose that we have 3 players with sizes and resources as described in Table III; the cost of war for every pair of players is $0.05$. Table IV shows the expected utility for each player, in each possible battles scenario.

Table III
Example 2: 3-player game

| Player | Size | Resource |
|---|---|---|
| 1 | 0.8 | 0.8 |
| 2 | 0.1 | 0.1 |
| 3 | 0.1 | 0.1 |

Table IV
Example 2: the utilities for the 3-player game

| Battles Scenario | | | Utilities | | |
|---|---|---|---|---|---|
| 1 Vs. 2 | 1 Vs. 3 | 2 Vs. 3 | Player 1 | Player 2 | Player 3 |
| | | | 0.8 | 0.1 | 0.1 |
| ✓ | | | 0.75 | 0.05 | 0.1 |
| | ✓ | | 0.75 | 0.1 | 0.05 |
| | | ✓ | 0.8 | 0.05 | 0.05 |
| ✓ | ✓ | | 0.57 | 0.14 | 0.14 |
| ✓ | | ✓ | 0.76 | -0.02 | 0.05 |
| | ✓ | ✓ | 0.76 | 0.05 | -0.02 |
| ✓ | ✓ | ✓ | 0.66 | 0.02 | 0.02 |

In this example, the no-war status-quo is in equilibrium, since no player can gain a higher utility by deviating unilaterally. However, there is another equilibrium when players 2 and 3 attack player 1 together. In this battles scenario they both get an expected payoff of $0.14$, which is higher than their payoff in the status-quo state. If players 2 and 3 are risk-seeking, each one of them will initiate an attack on player 1, with the hope that the other player will join. If the players are risk-averse, the status-quo state will be likely kept. Our model does not define how the players will behave if we have more than one equilibrium (or no equilibrium at all). However, to have a better understanding of the possible outcomes of the game, it is important to identify all the battles scenarios that are in equilibrium.

### A. The Basic Algorithm

In general games, finding all the equilibria is not an easy task. The trivial approach would be to do an exhaustive search over the space of all possible strategy profiles. In our case, for each $i \in \mathcal{P}$, $|\Sigma_i| = 2^{n-1}$. Therefore, the complexity of finding all the equilibria by searching over the strategies space is $O(2^{n(n-1)} n 2^{n-1})$. Fortunately, we are interested in predicting which battles scenarios might happen. We thus get an improved complexity of $O(3^{n(n-1)/2} n 2^{n-1})$ with our algorithm. The details follow. To simplify notation, we will abbreviate $\{N_i \cup v_i\}$ as $\hat{N}_i$.

**Algorithm 1** FindAllEq(players $\mathcal{P}$)

1: Init EqList $\leftarrow \{\}$
2: **for all** $E \subseteq V \times V$ and $i \in \mathcal{P}$ **do**
3: $\quad T[E,i] \leftarrow u_i(E)$
4: **for all** $E \subseteq V \times V$ **do**
5: $\quad$ **for** $i \leftarrow 1$ to $n$ **do**
6: $\quad\quad$ **for all** $V' \subseteq V \setminus \hat{N}_i, V' \neq \emptyset$ **do**
7: $\quad\quad\quad E' \leftarrow E \cup \{(v_i, v')\}_{v' \in V'}$
8: $\quad\quad\quad$ **if** $T[E',i] > T[E,i]$ **then**
9: $\quad\quad\quad\quad$ **goto** line 4 and continue with the next battles scenario
10: $\quad$ **if** $E = \sigma^0$ **then**
11: $\quad\quad$ Add $E$ to EqList
12: $\quad\quad$ **goto** line 4 and continue with the next battles scenario
13: $\quad$ **for all** possible conversion of $E$ to $\vec{E}$ **do**
14: $\quad\quad$ **for** $i \leftarrow 1$ to $n$ **do**
15: $\quad\quad\quad$ **for all** $V_1' \subseteq V \setminus \hat{N}_i$ and $V_2' \subseteq \vec{N}_i, V_2' \neq \emptyset$ **do**
16: $\quad\quad\quad\quad E' \leftarrow E \cup \{(v_i, v_1')\}_{v_1' \in V_1'} \setminus \{(v_i, v_2')\}_{v_2' \in V_2'}$
17: $\quad\quad\quad\quad$ **if** $T[E',i] > T[E,i]$ **then**
18: $\quad\quad\quad\quad\quad$ **goto** line 13 and continue with the next possible conversion
19: $\quad\quad$ Add $E$ to EqList
20: $\quad\quad$ **goto** line 4 and continue with the next battles scenario
21: **return** EqList

The algorithm first calculates the utilities of each player in every battles scenario, and stores them in a table (lines 2-3). It then iterates over all the battles scenarios, to check if each one of them is an equilibrium. The first phase checks whether any player has an incentive to initiate additional attacks (lines 5-12). Thus, the algorithm iterates over the players (line 5), and for each player it tries all the possible options to add battles (lines 6-7). If the utility of a player in this modified battles scenario is higher than its current utility then the battles scenario is not an equilibrium, and the search proceeds to the next possible battles scenario (lines 8-9). Otherwise, the algorithm proceeds to the second phase, to check whether any player has an incentive to initiate additional attacks while also canceling some of its current battles (lines 13-20). However, if the current battles scenario is the status-quo state, no battle can be canceled. Thus, the status-quo state is added to the equilibriums list, and the search proceeds to the next possible battles scenario (lines 10-12). In the second phase, the algorithm first converts the battles scenario to a possible directed battles scenario, by fixing a direction for each edge (line 13). This step is necessary to ensure that a player may be able to deviate by removing only the battles that it initiated. The algorithm iterates over the players (line 14), and for each player tries all the possible options to add and delete battles (lines 15-16). If the utility of a player in this modified directed battles scenario is higher than its current utility then the directed battles scenario is not an equilibrium, and the algorithm tries a different conversion of the battles scenario (lines 17-18). If in any conversion there is some player with an incentive to deviate, then this battles scenario is not an equilibrium, and the algorithm proceeds to check the next possible battles scenario. Otherwise, the battles scenario is added to the equilibriums list, and the search proceeds (lines 19-20). Since the algorithm does an exhaustive search over all the options, it finds all the battles scenarios that are in equilibrium. We now show the algorithm's complexity.

*Theorem 11:* Algorithm 1 uses $O(n2^{n(n-1)/2})$ space and requires $O(3^{n(n-1)/2}n2^{n-1})$ time.

*Proof:* In a graph with $n$ vertices there are at most $n(n-1)/2$ edges. Therefore, the number of possible battles scenarios is $2^{n(n-1)/2}$ and the algorithm needs a space of $O(n2^{n(n-1)/2})$ to store its table. To build the table, the algorithm calculates the expected utility in each battles scenario. There are $\binom{n(n-1)/2}{i}$ battles scenarios with $i$ battles and each battle has exactly two possible outcomes. Thus the total time required to calculate the expected utilities is,

$$\sum_{i=0}^{n(n-1)/2} \binom{n(n-1)/2}{i} 2^i \cdot n = n3^{n(n-1)/2}$$

In line 4 the algorithm iterates over $2^{n(n-1)/2}$ options. In line 5 it iterates over at most $n$ options and in line 6 it iterates over at most $2^{n-1}$ options. Therefore, the first phase of computations (lines 4-9) requires $O(2^{n(n-1)/2}n2^{n-1})$ time. In the second phase (lines 13-20), the algorithm first considers all the possible conversions of $E$ to $\vec{E}$. Every undirected edge can be converted to two directed edges. Thus the number of iterations in line 13 is $2^i$, where $i = |E|$. In line 14, the algorithm iterates over at most $n$ options and in line 15, it iterates over at most $2^{n-1}$ options. Since

$$\sum_{i=0}^{n(n-1)/2} \binom{n(n-1)/2}{i} 2^i \cdot n \cdot 2^{n-1} = 3^{n(n-1)/2}n2^{n-1}$$

the time complexity of the second phase, which dominates the overall time complexity, is $O(3^{n(n-1)/2}n2^{n-1})$. ■

### B. Heuristics

The worst case running time of Algorithm 1 is exponential in the number of players. Therefore, we propose heuristics for reducing the algorithm's actual running time.

*1) On-demand Table Building:* Our first suggestion is to reduce the time that is spent on building the battles scenario table. Instead of building the whole table (lines 2-3), the algorithm can just allocate the required space without actually calculating the utilities. Then, whenever there is a need for a value from the table (lines 8 and 17), the algorithm should first check if this value was already calculated. If not, it should be calculated and stored for future use. In this way, the algorithm can save the expensive calculation of the utilities, for battles scenario that are not being checked at all. Note however, that whenever there is a need for the utility of a specific player in a battles scenario, the algorithm should calculate the utilities of all the players in this battles scenario, in order not to increase the overall running time.

*2) Extended Equilibrium:* Our next suggestion is to reduce the number of iterations of the big loop, which starts on line 4. In fact, it is quite straightforward if at least one equilibrium is found. Clearly, if a battles scenario is in equilibrium, then any other battles scenario with an addition of exactly one edge is not in equilibrium. Since the algorithm already holds a table with all the battles scenarios, it can just add a flag bit for each battles scenario, to indicate if it has to be checked or not. Initially, all the battles scenario should be marked as "needs to be checked". If an equilibrium is found (line 19), the algorithm should mark all the battles scenarios with an addition of exactly one edge as "does not need to be checked" and continue the loop in line 4. The additional time to check the flag bit at the beginning of the big loop (line 4) and to mark it if an equilibrium is found does not increase the overall time complexity, but can reduce the actual running time significantly since it enables to skip on several iterations as a whole.

*3) Not Attacked:* This heuristic uses Lemma 9 to reduce the number of checks when a player is not involved in any battle, or it cancels all its battles. For each battles scenario, the algorithm can keep a list of all the players not involved in any battle. For these players, the loop in line 6 should iterate only on single nodes, $v' \in V \setminus N_i$, rather than on all the possible set of nodes. Similar modification can be done in line 15 when an agent cancels all of its battles, i.e., $V_2' = N_i$. Although the heuristic helps only when an agent is not involved in any battle, whenever applicable, it reduces the number of checks from $O(2^{n-1})$ to $O(n)$.

*4) Passive Agent:* Given a battles scenario, the algorithm iterates over all possible conversions to a directed battles scenario. For each conversion, it iterates over the number of players. However, if a player does not have any incentive to deviate where it can cancel all of its battles, there is no need to check this player again in any other conversion. Therefore, in line 14, the algorithm should first check for players with zero incoming edges. Each such player that does not have any incentive to deviate should be kept in a list. When moving to the next conversion, the algorithm should skip all the agents in this list in the loop in line 14.

## VII. Related Work

There has been extensive use of game theory in inter-state conflict modeling in the political science literature during the last five decades. Here we give a very brief overview of the work (please see [3], [4] for more details). Most of the games have been stylized 2-player (often, 2-action) single shot or sequential games. Some examples include a 2-player, 2-action, zero sum game for guerrilla warfare, Prisoner's Dilemma game for arms race [3], and a 2-player sequential game for mutual deterrence [8]. In general, there are more than two players with conflicting interests and the number of actions available to a player is more than two. Moreover, the causes for conflicts, like competition over resources, is usually not modeled.

Competition over resources has been one of the primary causes of inter-state or intra-state ethno-religious conflict [9], [10]. Historically, many wars have been fought over cultivable land, water, precious minerals and oil. Over the last two decades, control over natural resources have driven conflicts in African countries like Angola, Congo, Somalia, Sudan. Powell [7] has used a 3-player game to study conflicts as a result of competition for resources. His focus was on inter-state conflicts. However, in many situations (e.g., Sudan), there are a large number of different ethnic groups that may be occupying the same territory and in such cases the interaction is a general $n$-player game. Thus, the model presented in our paper is for a $n$-player game and it can be thought of as an extension of Powell's model.

Apart from game theoretic modeling, agent-based models have also been used for modeling conflicts [10]–[12]. Although, the models presented in [10], [12] consider scarcity of resources, the agents are not strategic in nature. In the model presented in [11], the agents are strategic. However, they have a population level model of the society and they do not model resource constraints. In this paper, we assume that each group (or state) is one player and they have to take a strategic decision of whether to attack other players in order to obtain their resources.

## VIII. Conclusion and Future Work

In this paper, we presented a game-theoretic model for multi-player conflict driven by competition over resources. We proposed a rich model, which takes into account heterogeneous players with different sizes and resources. In order to predict the outcome, we suggest a new refinement to the pure-strategy Nash equilibrium, called asymmetric equilibrium. We showed that our model can be used to answer questions of policy interest efficiently. In particular, we prove that we can check whether no war is an equilibrium in polynomial time in the number of players. We also prove that whether a situation where some players are engaged in a conflict can further degrade due to other players joining the conflict can be checked in polynomial time. We also proposed an algorithm to find all the asymmetric equilibria, and suggest several heuristics to improve the algorithm's performance.

For future work, there are a number of interesting directions. From a game-theory perspective, it is interesting to investigate the extension of our model to a repeated game. In this game, the first stage will be the simultaneous game that we defined. The next stage will use the same game's model, but without all the players that lost in the previous stage and with different sizes and resources of the remaining players. The game will end when the no-war state is in equilibrium. In this game, one should also consider the use of mixed strategies, and their effect on the model (e.g., is there a need for the asymmetric equilibrium refinement with mixed strategies?).

From the political science perspective, it will be interesting to run our algorithm to find all the equilibria in different scenarios, in order to learn the effect of the different parameters. We intend to use real-world data, and to learn from our model about the possible conditions on the creation of future multilateral conflict situations.


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