Polychromatic Colorings of Bounded Degree Plane Graphs

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Abstract

A *polychromatic k-coloring* of a plane graph G is an assignment of k colors to the vertices of G such that every face of G has *all* k colors on its boundary. For a given plane graph G, we seek the *maximum* number k such that G admits a polychromatic k-coloring. We call a k-coloring in the classical sense (i.e., no monochromatic edges) that is also a polychromatic k-coloring a *strong polychromatic k-coloring*. In this paper, it is proven that every plane graph of maximum degree three, other than K_4 (the complete graph on four vertices), admits a strong polychromatic 3-coloring. This initiates the study of strong polychromatic colorings of plane graphs. Moreover, our proof is constructive and implies a polynomial-time algorithm.

Keywords: Plane graphs, vertex coloring with constraints on the faces, bounded degree graphs.

1 Introduction

A *polychromatic k-coloring* (abbreviated *PkC*) of a plane graph G is an assignment of k colors to the vertices of G such that each face of G has *all* k colors on its boundary. More formally, a polychromatic k-coloring of a plane graph G is a function $\varphi : V(G) \to \{1, \ldots, k\}$, such that for each face of G there exist k vertices $\{u_1, \ldots, u_k\}$ on its boundary with $\varphi(u_i) = i$, for $i = 1, \ldots, k$. Note that a polychromatic k-coloring allows monochromatic edges. For a plane graph G, the *polychromatic number* of G, denoted $\chi_f(G)$, is the maximum number k such that G admits a polychromatic k-coloring. A k-coloring (in the classical sense) of a plane graph G that is also a polychromatic k-coloring of G is called a *strong polychormatic k-coloring* (abbreviated *SPkC*) of G.

Polychromatic colorings of plane graphs have several applications in combinatorial and computational geometry. A first example of the importance and usefulness of the concept of polychromatic coloring is Fisk's proof [6] to the classical art gallery theorem, first proved by Chvátal [3]. Given a polygon P with n vertices, a subset of its vertices is a *guarding set* for P if for any point in P there exists a guard (i.e., vertex) in the set that sees P. Chvátal proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to guard a simple n-gon. Fisk presented an elegant proof to this theorem based on a graph coloring argument (and the fact that every simple polygon admits a triangulation). He showed that any triangulated simple polygon is 3-colorable in the classical sense. So that given such a polygon and coloring, each of the triangular faces has three colors on its boundary. By picking the vertices associated with the least frequent color, one obtains a guarding set of size at most $\lfloor \frac{n}{3} \rfloor$. Following Fisk's proof, results for other guarding problems using similar arguments have been obtained (see, e.g., [10, 7]). Observe that the crucial property enabling the final step in Fisk's proof is that each triangle of the triangulated polygon has all three colors on its boundary. In general, the more colors one can use while still ensuring this property, the better is the upper bound on the sufficient size of a guarding set. It follows that the concept of polychromatic coloring is an important tool for guarding and covering problems.

Another application is the following. Consider a plane graph G whose faces represent the rooms of a museum. On each vertex of G, a single battery operated motion detector is placed. The battery has a limited life span, say one time unit. Consequently, once a detector is switched on, it can operate for a single time unit, during which it covers all the rooms that are adjacent to it. We would like to partition the detectors into as many disjoint subsets as possible, such that the detectors in each subset provide full coverage of the museum. Suppose that G admits a polychromatic k-coloring. By switching on the detectors associated with color i at time $i - 1$, we can ensure full coverage of the museum for k time units.

Yet another application of polychromatic coloring of plane graphs in the context of facility location is the following. Consider a plane graph G whose faces represent neighborhoods or regions that require access to several types of facilities. In addition, suppose that facilities can only be placed on vertices of the graph, and a region has access to a facility only if the latter is placed on its boundary. Given k types of facilities, one may want to verify if there exists an assignment of facilities to the vertices of G, such that each region has access to all types of facilities. A polychromatic k-coloring of G, if exists, would provide such an assignment.

In the context of facility location problems one may find applications for strong polychromatic colorings of plane graphs. For instance, consider the former facility location problem. Suppose that at least one of the k types of facilities is hazardous, i.e., a high radiation transmitter of cellular networks. Still, one would like every region to have access to this type of facility, yet due to safety considerations, it is important to ensure that every region would be exposed to a limited number of installations of this type of hazardous facility. A strong polychromatic k-coloring of the graph G modeling the regions, ensures that every region R of size r would have at most $\frac{r-k+1}{2}$ installations of every type on its boundary. This would also provide a limit on the number of appearances of the hazardous facility on the boundary of every region. In addition, one can think of reasons why facilities of the same type should not be placed adjacent to one another. For instance, to avoid air pollution or over consumption of natural resources that are generated by a the same type of facility.

Related Work. Mohar and Škrekovski [9] proved that every simple plane graph admits a $P2C$. Their proof is short and relies upon the Four-Color Theorem for planar graphs. Bose *et al.* [2] provided an alternative proof of this result that does not relay on the Four-Color Theorem and yields a linear time algorithm for computing such a coloring.

Hoffmann and Kriegel [7] proved that all 2-connected plane bipartite graphs can be triangulated (by adding edges only) into a plane Eulerian triangulation. Since a plane *Eulerian triangulation* is 3-colorable in the classical sense [11] it follows that all 2-connected plane bipartite graphs admit a $SP3C$. Moreover, since the triangulation offered by Hoffmann and Kriegel [7] and 3-colorings of plane Eulerian triangulations can both be found efficiently [11], it follows that $SP3C$s of 2-connected bipartite graphs can be found efficiently.

Recently, a general and elegant result concerning polychromatic colorings of plane graphs was obtained by Alon *et al.* [1]. Let g be the length of a shortest face of a plane graph G. Clearly, $\chi_f(G) \leq g$. Alon *et al.* proved that for any plane graph G admits $\chi_f(G) \geq \lfloor (3g-5)/4 \rfloor$. They show that this bound is sufficiently tight by presenting plane graphs for which $\chi_f(G) \leq \lfloor (3g+1)/4 \rfloor$.

The lower bound of Alon *et al.* [1] on $\chi_f(G)$ is nontrivial as long as $g \geq 6$. One may inquire regarding what can be said about $\chi_f(G)$ if G is a plane graph with $3 \leq g \leq 5$? In this paper we consider such graphs.

Our motivation for the study of polychromatic colorings originated from the following problem. A *rectangular partition* is a partition of a plane rectangle into an arbitrary number of non-overlapping rectangles, such that no four rectangles meet at a common point. One may view a rectangular partition as a plane graph whose vertices are the corners of the rectangles and edges are the line segments connecting these corners. Dinitz *et al.* [5] proved that every rectangular partition admits a polychromatic 3-coloring that may miss the outer face, and conjectured that every rectangular partition admits a polychromatic 4-coloring.

While working on this conjecture, Horev *et al.* [8] proved that every 2-connected cubic bipartite plane graph admits a polychromatic 4-coloring that may miss the outer face. This result is tight, since by Euler's formula any such graph contains at least six faces of size four. Recently, Dimitrov and the authors [4] presented a simple proof of the conjecture of Dinitz *et al.* [5]. The proof of this conjuncture is based on the Four-Color Theorem for planar graphs.

Our Result. Let $\Delta(G)$ denote the maximum degree of a graph G. Our main result is as follows.

Theorem 1.1 *Let G be a simple plane graph with $\Delta(G) \leq 3$, other than K_4. Then G admits a strong polychromatic 3-coloring.*

Not all plane graphs admit a $P3C$. The complete graph on four vertices, namely K_4, and the wheel graph on k vertices, namely W_k, with k even does not admit a $P3C$. The graphs of Theorem 1.1 may contain faces of size three, four and five. Consequently, the result of Alon *et al.* [1] cannot be used on these graphs to obtain our result. Moreover, since the presence of faces of size three is allowed, our result is best possible in terms of the number of colors that can be used in a polychromatic coloring of such graphs.

Introducing vertices of higher degree to the graph generates interesting counter examples. For instance, allowing $\Delta(G) \leq 4$ will require one to forbid K_4 as a subgraph of G. Allowing $\Delta(G) \leq 5$ introduces counter examples like W_6, and so on. Thus allowing higher vertex degrees introduces more complicated graph structures.

By Brooks' theorem (see Theorem 2.1) the graphs considered in Theorem 1.1 are 3-colorable in the classical sense. Theorem 1.1 joins the well-known notion of classical vertex coloring and polychromatic coloring in planar graphs. The connection between these two types of colorings is not trivial and thus this paper introduces the concept of strong polychromatic colorings in plane graphs. Until this result, the sole two graph families that were implicitly known to admit a strong polychromatic 3-coloring were Eulerian triangulations and 2-connected bipartite graphs (by the result of [7]). These families and our result provide a solid body of evidence that strong polychromatic colorings pose an independent interest. A natural question is to ask what other non-trivial families of plane graphs admit strong polychromatic k-coloring for non-trivial values of k. Also, one may ask what are the sufficient conditions for a plane graph to a admit a strong polychromatic k-coloring for some non-trivial value of k?

Finally, it can readily be verified that our proof of Theorem 1.1 is constructive and implies a polynomial time algorithm for computing a $SP3C$ of a graph upholding the terms of Theorem 1.1.

Organization. The rest of the paper is organized as follows. In Section 2, we describe the notation and basic results on which this paper relies upon. Section 3 contains a proof for Theorem 1.1. This proof is based upon Theorem 3.1 which is proved in Section 4.

2 Preliminaries

In this paper we consider only simple graphs. Also, a *plane graph* is called a *graph*. For a graph G let $V(G)$ and $E(G)$ denote the vertex and edge set of G, respectively. For a vertex $u \in V(G)$, let $deg_G(u)$ denote the *degree* of u in G.

For a set $X \subseteq V(G)$, we write $G - X$ to denote the subgraph induced from the vertex set $V(G) \setminus X$. In case the case that $X = \{x\}$ we write $G - x$. For a set $Y \subseteq E(G)$, let $G - Y$ denote the graph induced from the edge set $E(G) \setminus Y$. In the case that $Y = \{y\}$ we write $G - y$.

Let $E \subseteq V(G) \times V(G)$ be a set of pairs of vertices where each pair is either an edge of G, or not. We write $G + E$ to denote the graph obtained by adding all the pairs of E as new edges to G. If $E = \{e = uv\}$ we write $G + e$, or $G + uv$. In addition, for a vertex $s \notin V(G)$ we write $G + s$ to denote the graph with vertex set $V(G) \cup \{s\}$ and edge set $E(G)$. The operation of *contracting* an edge e is the identification of the endpoints of e into a single vertex.

Let G be a connected graph. A vertex $u \in V(G)$ is called a *cut-vertex* of G if $G-u$ is disconnected. A pair of vertices $\{u, v\} \in V(G)$ are called a 2-*vertex-cut* if $G - \{u, v\}$ is disconnected. A graph is said to be 2-*connected* if it does not admit a cut-vertex. Analogously, an edge $e \in E(G)$ is called a *cut edge* of G if $G - e$ is disconnected. A pair of edges $\{e_1, e_2\} \in E(G)$ are called a 2-*edge-cut* if $G - \{e_1, e_2\}$ is disconnected.

For a graph G, and a pair of vertices $u, v \in V(G)$, a simple path from u to v in G is called a *uv-path*. For a simple cycle C of G we write $int(C)$ and $ext(C)$ to denote the part of the plane that lies in the *interior* and the *exterior* of C, respectively. In addition, for a simple cycle C, and two vertices $u, v \in V(C)$ such that $uv \notin E(C)$, the edge uv is called a *chord* of C.

For a face f of a graph G, let $V(f)$ and $E(f)$ denote the vertices and edges lying on the boundary of f. Two edges $e, e' \in E(f)$ are called *consecutive on* f if these edges share a common endpoint $u \in V(f)$. Two vertices $u, v \in V(f)$ are called *consecutive on* f provided that the edge $uv \in E(f)$. The boundary of f is a simple cycle C in G. The *interior* and *exterior* of the face f, denoted $int(f)$ and $ext(f)$, are defined to be $int(C)$ and $ext(C)$, respectively. For a face f of a graph G, a vertex $u \in V(f)$ is said to be *incident* to f.

For a face f of a graph G the *length* or *size* of f, denoted $|f|$, is the value $|V(f)|$. A *k-face* is a face of size k. Two faces f_1 and f_2 are said to be *adjacent* if $V(f_1) \cap V(f_2) \neq \emptyset$. In this paper we consider graphs with bounded degree. Note that for a graph G with $\Delta(G) \leq 3$, two adjacent faces f_1 and f_2 cannot have $|V(f_1) \cap V(f_2)| = 1$.

A *k-coloring* of a graph G is a function $\varphi : V(G) \to \{1, \ldots, k\}$ such that for each edge $uv \in E(G)$, $\varphi(u) \neq \varphi(v)$. Let $\chi(G)$ denote the *minimum* number k such that G admits a k-coloring. A result by Brooks [11] that relates $\Delta(G)$ with $\chi(G)$ is the following theorem.

Theorem 2.1 ***(Brooks* 1941*)***
Let G be a connected graph other than $K_{\Delta(G)+1}$ or an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Let φ be a $SkPC$ of a graph G. Let H be a proper subgraph of G. The graph H is said to *see* a set of colors $S \subseteq \{1, \ldots, k\}$ if for each color $s \in S$ there exists a vertex $u \in V(H)$ such that $\varphi(u) = s$. A face f of G is said to see a set of colors S if the simple cycle C defining the boundary of f sees S.

3 Proof of The Main Theorem

Our proof of Theorem 1.1 is done by induction. More specifically, we assume towards contradiction that Theorem 1.1 is false and consider a minimum counter example to the theorem. Theorem 3.1 lists structural characterizations of a minimum counter example to Theorem 1.1. In Section 4 we provide a proof for Theorem 3.1.

Theorem 3.1 *Let G be a counter example to Theorem 1.1 minimizing the sum $|V(G)| + |E(G)|$. Then the following properties are upheld by G.*

(a) *G is 2-connected.*

(b) *G is cubic.*

(c) *G has no 4-faces.*

(d) *Let f be a k-face of G such that $k > 4$ and k is even. Let g be a k'-face adjacent to f. Then $k' \geq 5$.*

(e) *Let f be a k-face of G such that $k > 4$ and k is even. Then f admits no chords.*

Relaying on Theorem 3.1, we proceed with our proof for Theorem 1.1.

Proof of Theorem 1.1:
Assume towards contradiction that Theorem 1.1 is false, and let G denote a counter example to Theorem 1.1 minimizing $|V(G)| + |E(G)|$. One may assume that G is not an odd cycle, for then the claim follows. Also, G is not K_4. By Theorem 2.1 $\chi(G) \leq 3$, hence it follows that G contains a face of even length; for otherwise any regular 3-coloring of G is a $SP3C$ of G in contradiction to G being a counter example.

Let f be an even face of G. By Theorem 3.1(c),(d), and (e) $|f| \geq 6$, any face that is adjacent to f is a k-face, $k \geq 5$, and f admits no chords in G. Let $E(f) = \{e_1, \ldots, e_{|f|}\}$ be the edge set of f such that the ith edge is defined by $e_i = u_i u_{i+1}$ (indices are taken modulo $|f|$). For a vertex $u_i \in V(f)$, let v_i denote its third neighboring vertex. Two complementary cases are considered. Either (A) the face f does not admit two edges that form an edge cut in G, or (B) the face f admits at least one pair of edges that forms such a cut.

Case (A). Suppose no two edges of f form an edge cut in G. As G is 2-connected and cubic, and as f admits no chords, and f does not contain a 2-edge-cut then it follows that every edge $e_i = u_i u_{i+1} \in E(G)$ lies in a distinct face f_i. Let G' be the graph obtained from G as follows. Remove the vertices $V(f)$. In the resulting graph H, the vertices v_i, $i = 1, \ldots, |f|$, exist. Add to H a set of $|f|/2$ edges $E = \{v_i v_{i+1} : i \text{ is even}\}$. Let G' be the resulting graph.

Let $\mathcal{K}$ denote the set of all the faces f_i defined above. Let $\mathcal{S} \subseteq \mathcal{K}$ denote the subset of faces f_i of G such that $f_i \in \mathcal{S}$ provided that $v_i v_{i+1} \in E$. Let $\mathcal{T} = \mathcal{K} \setminus \mathcal{S} = \{f_i : i \text{ is odd}\}$. In G', let f'_i denote the face formed from the face $f_i \in \mathcal{S}$. Since $|f_i| \geq 5$, then $|f'_i| \geq 3$. In G', let f' denote the face defined by the edge set E, and the set of boundaries of the faces in the set $\mathcal{T}$. Let $\mathcal{P}$ denote this set of vertex-disjoint boundaries, namely $\mathcal{P} = \{P_i : P_i \text{ is a boundary of } f_i \in \mathcal{T}\}$. The elements of the sets E and $\mathcal{P}$ alternate across the boundary of f'.

By the minimality of G and Theorem 1.1, the graph G' admits a $SP3C$ φ. We extend φ to a $SP3C$ of G, namely χ. Returning to G, note that φ introduces three distinct colors to every face $f_i \in \mathcal{S}$. Thus, it remains to extend φ into a proper 3-coloring so that the face f, and every face $f_i \in \mathcal{T}$ would see three distinct colors. Note that for every two distinct faces $f_i, f_j \in \mathcal{T}$ the vertices u_i, u_{i+1}, u_j and u_{j+1} are four distinct vertices.

As φ is a 3-coloring every boundary $P_i \in \mathcal{P}$ sees at least two distinct colors. Two boundaries in $\mathcal{P}$ are said to be *adjacent* on f' if in G' there exist an edge in E connecting two of their endpoints. Since under φ every boundary in $\mathcal{P}$ sees at least two colors then every two boundaries have at least one color in common. Two complementary cases are considered.
($A.I$) Either there exist two adjacent boundaries, say w.l.o.g P_1 and P_3 , both seeing a common color

c such that under φ the color c is assigned to one of the endpoints of the edge $e = v_2v_3$. W.l.o.g, assume that v_2 is the vertex assigned this common color c.
($A.II$) Or for every two adjacent boundaries the colors assigned to the endpoints of the edge in E connecting the boundaries are not seen by both boundaries.

Case (*A.I*). The vertices $\{u_1, \ldots, u_{|f|}\}$ are colored as follows, and the resulting coloring is denoted χ. Start at u_3 by defining $\chi(u_3) = \varphi(v_2)$. Proceed in a clockwise traversal on f to color the remaining vertices according to the following rule. Let u_i $i \neq 2, 3$ be the current vertex to be colored, and suppose that the vertices $u_3, \ldots, u_{i-1}$ have already been colored. The color $\chi(u_i)$ is chosen to be distinct of the colors $\{\chi(u_{i-1}), \varphi(v_i)\}$ such that the color missing from the face f_i (if i is odd) or f_{i-1} (if i is even) is assigned to u_i if this is possible. For u_2 we define its color to be distinct of $\{\chi(u_1), \varphi(v_2), \chi(u_3)\}$. Recall that $\varphi(v_2) = \chi(u_3)$.

One may verify that χ is a proper coloring of G. It remains to ensure that χ is also 3-polychromatic. It is sufficient to consider the faces $f_i \in \mathcal{T}$, and the face f. Consider a face $f_i \in \mathcal{T}$ such that $i \neq 1$. Suppose that u_i is not assigned the color missing from f_i. In that case the missing color from f_i is assigned to u_{i+1} (indices are taken modulo $|f|$). For the face f we argue as follows. When the above coloring of the vertex set $V(f)$ is done the face f sees at least two distinct colors. It remains to show that a third color can be introduced to the face f.

Suppose that after the above coloring of $V(f)$ is completed the face f is 2-colored and, w.l.o.g, sees only the colors 2 and 3 on its vertices. It is proven that the color 1 can be introduced to the face f. Since φ is a proper coloring then among the vertices $\{v_i : 1 \leq i \leq |f|\}$ there exists a vertex v_i such that $\varphi(v_i) \neq 1$. Either $f_i \in \mathcal{T}$ or $f_{i-1} \in \mathcal{T}$. Assume, w.l.o.g, that the face $f_i \in \mathcal{T}$. Since f is 2-colored then the colors $\chi(u_i), \chi(u_{i-1})$ and $\chi(u_{i+1})$ are all distinct from the color 1. Consequently, if it is not possible to change the color assigned to u_i to the color 1 then it follows that $\chi(u_i)$ is the color missing from the boundary P_i. If so, the color $\chi(u_{i+1})$ is not missing from P_i. If it is not possible to change the color of u_{i+1} to 1 then this is since $\varphi(v_{i+1}) = 1$. Consequently, since φ is a proper coloring then $\varphi(v_{i+2}) \neq 1$, and the same arguments hold for the face $f_{i+2} \in \mathcal{T}$. It follows if every vertex u_i cannot change its colors to 1 then every face $f_i \in \mathcal{T}$ obeys the following terms. (i) $\varphi(v_i) \neq 1$, and (ii) $\chi(u_i)$ is missing from P_i, and (iii) $\chi(u_{i+1})$ appears on P_i, and (iv) $\varphi(v_{i+1}) = 1$.

Recall that the face f is 2-colored with the colors 2 and 3. For a face $f_i \in \mathcal{T}$ assume, w.l.o.g, that $\chi(u_i) = 3$. By (ii) above 3 is a color missing from P_i. However, since f is 2-colored under χ it follows that $\chi(u_{i+2k}) = 3$ for $1 \leq k \leq |f|/2$ (and indices taken mosulo $|f|$). Moreover, $\chi(u_{i+2k})$ is the color missing from the boundary P_{i+2k}. Consequently, the color 3 is missing from every boundary in $\mathcal{P}$. This is a contradiction to φ being 3-polychromatic.

Case (*A.II*). The argument in this case proceeds in several stages. First, we define a coloring for the vertex set $V(f)$ that may result in edges in $E(f)$ being monochromatic, and may have the face f, and faces in $\mathcal{T}$ without three distinct colors. The second stage is to repair the resulting coloring and extend it to a $SP3C$ of G.

Let χ denote the coloring obtained as follows. Let $e = v_iv_{i+1} \in E$. Define $\chi(u_i) = \varphi(v_{i+1})$ and $\chi(u_{i+1}) = \varphi(v_i)$. This is the first stage. As mentioned above the coloring χ may be invalid. We fix χ into a valid coloring.

(*i*) Monochromatic edges in *E*(*f*). Suppose χ admits monochromatic edges on f. Let $e_i = u_iu_{i+1}$ be such an edge. By the definition of χ the edge e_i does not belong to any faces in $\mathcal{S}$. Let $f_i \in \mathcal{T}$ such that $e_i \in E(f_i)$. Note that $\varphi(v_i) = \chi(u_{i-1})$. Consequently, one may change the color of u_i to a third color not seen by its neighbors. Also, note that if in χ the face f_i does not see three distinct colors then u_i can assume the color missing from f_i. Let χ' denote the coloring resulting from applying the above correction to every monochromatic edge in a clockwise traversal along f In the coloring χ' there are no monochromatic edges in $E(f)$. Moreover, any face $f_i \in \mathcal{T}$ that in χ had u_iu_{i+1} monochromatic, in χ' f_i sees three distinct colors. This is so as under χ the colors $\chi(u_i) = \chi(u_{i+1}) = \varphi(v_{i-1}) \neq \varphi(v_i) = \chi(u_{i-1})$. Hence, under χ the face f_i sees two colors on the vertices v_i, u_i and u_{i+1}. When the color of u_i is changed it is picked such that it is distinct of the colors $\{\varphi(v_i) = \chi(u_{i-1}), \chi(u_{i+1})\}$. This change introduces a third color to f_i.

(*ii*) Introducing three distinct colors to faces in $\mathcal{T}$. Under the coloring χ' it is still possible that there are faces $f_i \in \mathcal{T}$ that see only two colors. Moreover, it is possible that several adjacent faces in $\mathcal{T}$ all see the same two colors. Consider a maximal sequence $S = \{f_i, f_{i+2}, \ldots, f_m\}$ of adjacent faces in $\mathcal{T}$ that see the same two colors, say 1 and 2. The sequence S defines a path on f, namely $Q = \langle u_i, u_{i+1}, u_{i+2}, \ldots, u_{m+1}\rangle$, such that under χ' the path Q is 2-colored using the colors 1 and 2. The set of all such maximal sequences of adjacent faces in $\mathcal{T}$ defines a division of the vertex set $V(f)$ into a set of vertex disjoint paths. This division is not necessarily a partition of $V(f)$.

Consider the path Q. This path is maximal in the sense that it does not contain vertices from faces in $\mathcal{T}$ that see three distinct colors under χ'. In addition, the color $\chi'(u_{i-1}) \neq 3$. This is so since the vertex u_{i-1} cannot assume the color 3 from the coloring argument that defined χ. In addition, the color of u_{i-1} could not have been changed in step (*i*) into 3 for even if the edge $u_{i-2}u_{i-1}$ was monochromatic under χ, it is the vertex u_{i-2} that changes its color in (*i*) and not u_{i-1}. This cannot be said regarding the vertex u_{m+2}. Consequently, consider the path Q' which is a subpath of Q defined as $Q' = \langle u_i, u_{i+1}, u_{i+2}, \ldots, u_m\rangle$ (i.e., u_{m+1} is dropped in Q'). Note that $|Q|$ is even, and thus $|Q'|$ is odd. Changing the color of every other vertex on Q' starting at u_i to the color 3 introduces three distinct colors to every face in the sequence S. Applying this argument to every such maximal sequence introduces three colors to every face in $\mathcal{T}$ that prior to that saw only two colors. In addition, no face that sees three colors under χ' have the color of its vertices changed by this argument. Define χ'' to be the resulting coloring of this correction step.

(*iii*) Three colors for f. It remains to prove that f sees three colors. By the definition of χ the face f sees at least two colors under χ. If step (*i*) is ever applied then f sees three colors. This is so since if step (*i*) is applied to a monochromatic edge $e_i = u_iu_{i+1}$ then its corresponding face in $f_i\mathcal{T}$ sees three colors due to the correction at e_i. Consequently, in f the vertices u_{i-1}, u_i, and u_{i+1} are colored in three distinct colors. Thus if step (*ii*) is never applied then the claim follows for f. If step (*ii*) is applied observe that the coloration of the vertices u_{i-1}, u_i, and u_{i+1} is left unchanged.

Consequently, one may assume that under χ there are no monochromatic edges in $E(f)$, and that f sees exactly two colors, say 1 and 2. By the definition of χ it follows that since f is 2-colored then the vertex set $\{v_i\}_{i=1}^{|f|}$ is also 2-colored. Consequently, since φ is 3-polychromatic then there is at least one boundary in $\mathcal{P}$ that sees three colors under φ. Hence, there exists at least one face $f_i \in \mathcal{T}$ that sees three colors under χ. Since f sees only the colors 1 and 2, then the face f_i contributes the colors 1 and 2 to f already in χ. If step (*ii*) is applied then the color 3 is introduced to f without changing the coloration of the face f_i. Hence, it remains to consider the case that under χ the face f has no monochromatic edges, and that every face in $\mathcal{T}$ sees all three colors on its boundary in $\mathcal{P}$, i.e., steps (*i*) and (*ii*) were never applied, and still f sees only the colors 1 and 2. As noted the boundary P_1 sees all three colors. Changing the color of the vertex u_1 to 3 completes the proof.

Case (*B*). We proceed to the case where there exist two edges of $E(f)$ that form an edge cut of G. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be such two edges. Due to Theorem 3.1(*a*) and (*b*) the edges e_1 and e_2 are not consecutive on f. Figure 1(*a*) depicts the edges e_1 and e_2, and the relative positioning of the vertices $\{u_i, v_i\}$ for $i = 1, 2$. Let P denote the v_1v_2-path along the boundary of f such that $e_1, e_2 \notin E(P)$. Similarly, let Q be the u_1u_2-path along the boundary of f such that $e_1, e_2 \notin E(Q)$. Since $|f| \geq 6$ then at least one of the paths just defined has length at least three. Assume, w.l.o.g, that $|Q| \geq 3$. The graph $G - \{e_1, e_2\}$ contains two components. Let D_1 denote the component containing the vertices u_1 and u_2, and let D_2 be the component containing the vertices v_1 and v_2. Define $G_1 = D_1 + u_1u_2$. Since f is chordless, and $|Q| \geq 3$ then G_1 is simple. In G_1, the edge u_1u_2, and the path Q define a face of size at least three. Suppose that $v_1v_2 \notin E(f)$, and define $G_2 = D_2 + v_1v_2$. Figure 1(*b*) depicts the graphs G_1 and G_2. Since f is chordless, and since that in this case $|P| \geq 3$ then the graph G_2 is simple. By the minimality of G, both of the graphs G_1 and G_2 admit $SP3C$s , namely φ_1 and φ_2, of G_1 and G_2, respectively. In addition, since $u_1u_2 \in E(D_1)$ and $v_1v_2 \in E(D_2)$ then $\varphi_1(u_1) \neq \varphi_1(u_2)$ and $\varphi_2(v_1) \neq \varphi_2(v_2)$ and thus using color renaming on φ_2 these two colorings can be combined to a $SP3C$ of G. If however, $v_1v_2 \in E(f)$ then define $G_2 = D_2$ and repeat the argument of the previous case. ▮

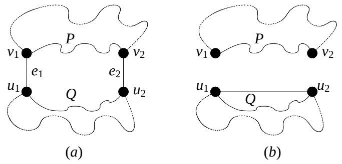

Figure 1: (a) The edges e_1 and e_2 on f. (b) Separation to the graphs G_1 and G_2.

4 Properties of a Minimum Counter Example

In this section we provide a proof for Theorem 3.1. We consider the structure of a counter example G to Theorem 1.1 minimizing the sum $|V(G)| + |E(G)|$. Also, **through this entire section we write G to denote such a counter example.** Theorem 3.1(a) can easily be verified and thus omitted. Consequently, this section is organized as follows. In Section 4.1, we consider a proof for Theorem 3.1(b). In Section 4.2, we prove Theorem 3.1(c). In Section 4.3 we prove Theorem 3.1(d), and in Section 4.4 we prove Theorem 3.1(e).

4.1 Proof of Theorem 3.1(b)

Theorem 3.1(b) *G is cubic.*

Proof: Let v be a degree two vertex in G, and let u and w be its two neighbors in G. If the vertices $\{u, v, w\}$ do not induce a 3-face in G, then contract the edge uv. By the minimality of G, the resulting graph admits a $SP3C$, which is easily extended to a $SP3C$ of G. If $\{u, v, w\}$ induce a 3-face in G then the edge uv can not be contracted (for that results in a 2-face). If $deg_G(u) = deg_G(w) = 2$ then G is a 3-cycle, and the claim follows. In addition, one may assume that $deg_G(u) = deg_G(w) = 3$, for if only one of the vertices u or w is a degree three vertex then this vertex is a cut-vertex of G; contradiction to Theorem 3.1(a).

Let z and x denote the third neighbors of u and w, respectively. The vertices z and x are distinct. If $z = x$ and of degree two the claim follows. If $z = x$ and of degree three then this forms a cut-vertex in G; a contradiction to Theorem 3.1(a). Consequently, $z \neq x$. Let $G' = G - \{u, v, w\}$. If G' consists of the single edge zx, then G is a graph on five vertices, that admits a $SP3C$. Thus one may assume that G' is of order at least three. By the minimality of G, the graph G' admits a $SP3C$, namely φ. Assume, w.l.o.g, that $\varphi(z) = 1$. We extend the coloring φ into a $SP3C$ of G as follows. (i) if $\varphi(x) \neq 3$ then color the vertices u, v, w with the colors $2, 1, 3$, respectively. (ii) if $\varphi(x) = 3$ then color the vertices u, v, w with the colors $3, 1, 2$, respectively. ▮

4.2 Proof of Theorem 3.1(c)

We proceed to proving Theorem 3.1(c). To facilitate the proof of Theorem 3.1(c) it is proven that a 3-face and a 4-face cannot be adjacent in G.

Lemma 4.1 *The graph G does not contain a 3-face and a 4-face that are adjacent.*

Proof: Assume towards contradiction that the claim is false. Let f be a 4-face with vertices u_i, $i = 1, \dots, 4$. Since by Theorem 3.1(b) G is cubic, it follows that the number of 3-faces that are adjacent to f is either one or two. We proceed with a case analysis on the number of 3-faces adjacent to f.

Case (1): Suppose f has two adjacent 3-faces g and h. One may assume that g and h are not adjacent to one another; for otherwise the vertex u_i of f that is not incident to the faces g and h, forms a cut-vertex of G, negating Theorem 3.1(a). Thus, the 3-faces g and h are incident to opposite edges u_1u_2 and u_3u_4, respectively, of the 4-face f. Let v be the third vertex of g not incident to f. Similarly, let w be the third vertex of h not incident to f. Figure 2(a) depicts the faces f and g in this case. By Theorem 3.1(***b***), the vertices v and w are both of degree three. In addition, one may assume that the edge $vw \notin E(G)$; for otherwise G is a six vertex graph that admits a $SP3C$, and the claim follows. Let v' and w' denote the neighboring vertices of the vertices v and w, respectively, that are not incident to the face f. By Theorem 3.1(a) it follows that $v' \neq w'$.

Let $G' = G - \{v, u_1, u_2, u_3, u_4, w\}$. By Theorem 3.1(***b***) the vertices v' and w' are of degree three in G. Thus, G' is of order at least three. By the minimality of G, the graph G' admits a $SP3C$, namely φ. The coloring φ is extended to a $SP3C$ of G as follows. Assume, w.l.o.g, that $\varphi(v') = 1$. (i) if $\varphi(w') = \varphi(v') = 1$ then color the vertices $\{v, u_1, u_2, u_3, u_4, w\}$ with the colors $2, 1, 3, 2, 1, 3$, respectively. (ii) Otherwise, w.l.o.g, $\varphi(w') = 2$. In this case, color the vertices $\{v, u_1, u_2, u_3, u_4, w\}$ with the colors $2, 1, 3, 3, 2, 1$, respectively.

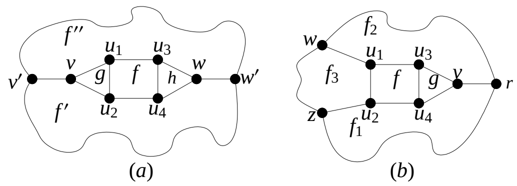

Figure 2: (a) The faces g and h are adjacent to opposite edges of the face f. (b) The faces f and g in Case (2).

Case (2): Consider the complementary case in which the face f is adjacent to exactly one 3-face g. Let u_3, u_4 and v the vertices defining the face g where $v \notin V(f)$. Also, let w and z be the vertices adjacent to the vertices u_1 and u_2 such that $w, z \notin V(f)$. By the previous case, $w \neq z$, and by Theorem 3.1(b), the vertices w, z and v are of degree three. Figure 2(b) depicts the faces f and g in this case. We use the vertex name labeling in Figure 2(b) to refer to vertices in this case. Thus, there exists a vertex r that is adjacent to the vertex v. By Theorem 3.1(a) G is 2-connected, hence, if $r = w$, but $r \neq z$, then it follows be that the edge $zr \in E(G)$, for otherwise the vertex z is a cut-vertex. In this case the claim follows as G is a graph on six vertices that admits a $SP3C$. Thus, one may assume that the vertices r, z, and w are three distinct vertices.

Let $G' = G - \{v, u_1, u_2, u_3, u_4\}$. By the minimality of G, the graph G' admits a $SP3C$, namely φ. Assume, w.l.o.g, that $\varphi(r) = 1$. The coloring φ is extended into a $SP3C$ of G as follows. (i) if $\varphi(z) = \varphi(w) = \varphi(r)$ then color the vertices $\{v, u_1, u_2, u_3, u_4\}$ with the colors $3, 2, 3, 1, 2$, respectively. (ii) if $\varphi(z) \neq \varphi(w) \neq \varphi(r)$, assume, w.l.o.g, that $\varphi(w) = 2$ and $\varphi(z) = 3$. In this case, color the vertices $\{v, u_1, u_2, u_3, u_4\}$ with the colors $2, 1, 2, 3, 1$, respectively. (iii) if $\varphi(z) = \varphi(r) \neq \varphi(w)$, assume, w.l.o.g, that $\varphi(w) = 2$, and color the vertices $\{v, u_1, u_2, u_3, u_4\}$ with the colors $3, 3, 2, 2, 1$,respectively. The case where $\varphi(w) = \varphi(r) \neq \varphi(z)$ is symmetric and thus omitted. (iv) if $\varphi(z) = \varphi(w) \neq \varphi(r)$, assume, w.l.o.g, that $\varphi(z) = \varphi(w) = 2$. Color the vertices $\{v, u_1, u_2, u_3, u_4\}$ with the colors $3, 1, 3, 2, 1$,respectively. ▮

Let $\{u, v\}$ be a 2-vertex-cut of a connected graph G such that $uv \in E(G)$, and $G - \{u, v\}$ contains exactly two components C_1 and C_2. Define $C_i' = C_i + \{u, v\} + uv$, for $i = 1, 2$. The graphs C_i' are said to be obtained from G by *separating G along the edge uv*. We proceed with proving that G does not contain a 4-face.

Theorem 3.1(c) *G does not contain 4-faces.*

Proof: Assume towards contradiction that G contains a 4-face f. Let u_i, $i = 1, \dots, 4$, denote the vertices of f, and let v_i, $i = 1, \dots, 4$ denote the neighboring vertices of the vertices of f, such that for each $i = 1, \dots, 4$, the edge $u_i v_i \in E(G)$. By Theorem 3.1(b) and Lemma 4.1, the vertices v_i exist and are all distinct. Two cases are considered. Either there exist two edges of $V(f)$ that form an edge cut, or not.

Case (1): Assume the edges $\{u_1 u_3\}$ and $\{u_2 u_4\}$ form an edge cut in G. Note that any two non-adjacent vertices of f form a separation pair of G. Consider the vertices $\{u_1, u_4\}$ as such a pair. We decompose G into the following two graphs. Consider the graph $G' = G + u_1 u_4$. In G', the face f is triangulated, containing the triangles defined by the vertex sets $\{u_1, u_2, u_4\}$, and $\{u_1, u_3, u_4\}$. Separate G' along the edge $u_1 u_4$, and let G_1 be the part of G' containing the vertex u_2, and let G_2 be the part of G' containing the vertex u_3. Also, in the graph G_2, let u_1' and u_4' denote the copies of the vertices u_1 and u_4 in this graph.

Each of the graphs G_i, $i = 1, 2$, upholds the terms of Theorem 1.1, and thus by the minimality of G, the graph G_i admits a $SP3C$ φ_i, for $i = 1, 2$. The combinations of the two colorings φ_i into a single consistent $SP3C$ of G is straightforward.

Case (2): Let f_i, $i = 1, \dots, 4$, be the four adjacent faces to f. Figure 3(a) depicts the face f and the faces f_i in this case. By Lemma 4.1, the length $|f_i| \geq 4$, for $i = 1, \dots, 4$. We proceed with a case analysis on the sizes $|f_i|$.

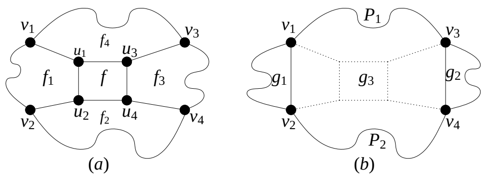

Figure 3: (a) The face f and its adjacent faces f_i in Case (2). (b) The graph G' of Case (2.a).

Case (2.a): Suppose that two opposite adjacent faces of f are of size strictly larger than four. W.l.o.g, suppose $|f_1| > 4$ and $|f_3| > 4$. Consider the graph $G' = G - V(f) + v_1 v_2 + v_3 v_4$. Figure 3($b$) depicts this construction. Let g_i, $i = 1, 2, 3$, be the faces of G' as shown in Figure 3(b). Note that G' upholds the terms of Theorem 1.1. Thus, by the minimality of G, and Theorem 1.1, the graph G' admits a $SP3C$ φ. It remains to extend the coloring φ into a $SP3C$ of G. One may observe that the same arguments used for proving Theorem 1.1, also apply here. For the sake of completeness, we provide a direct proof. Assume, w.l.o.g, that $\varphi(v_1) = 1$. We proceed with a case analysis over the colors assigned to the vertices v_2, v_3 and v_4 under φ. (i) Suppose that $\varphi(v_1) = \varphi(v_3)$. If $\varphi(v_2) = \varphi(v_4)$, assume, w.l.o.g, that $\varphi(v_2) = 2$, and color the vertices $\{u_1, u_2, u_3, u_4\}$ with colors $2, 3, 3, 1$ respectively. If $\varphi(v_2) \neq \varphi(v_4)$, assume, w.l.o.g, that $\varphi(v_2) = 2$ and $\varphi(v_4) = 3$. In this case color the vertices $\{u_1, u_2, u_3, u_4\}$ with colors $2, 3, 3, 1$. (ii) Suppsoe $\varphi(v_1) \neq \varphi(v_3)$, and assume, w.l.o.g, $\varphi(v_3) = 2$. If $\varphi(v_2) = \varphi(v_4)$, then it follows that $\varphi(v_2) = \varphi(v_4) = 3$. Color the vertices $\{u_1, u_2, u_3, u_4\}$ with colors $3, 1, 1, 3$, respectively. Next consider the case in which $\varphi(v_2) \neq \varphi(v_4)$. If $\varphi(v_2) = 2$ and $\varphi(v_4) = 1$, color the vertices $\{u_1, u_2, u_3, u_4\}$ with colors $3, 1, 2, 3$, respectively. Otherwise, assume, w.l.o.g, that $\varphi(v_2) = 3$ and $\varphi(v_4) = 1$. In this case, color the vertices $\{u_1, u_2, u_3, u_4\}$ with colors $3, 2, 1, 3$, respectively.

Case (2.b): By the previous case, one may assume that no two opposite adjacent faces of f are of size strictly larger than four. Consequently, it follows that there exist two non-opposite adjacent faces of f of size four. At least one of the faces adjacent to f is of size greater than four, for otherwise, G is a graph on eight vertices that admits a $SP3C$. Assume, w.l.o.g, that $|f_1| > 4$, $|f_3| = |f_4| = 4$. The faces f_i are presented in Figure 4(a).

Consider the graph $G' = G - \{u_1, u_2, u_3, u_4, v_3\} + v_1v_2 + v_1v_4$. The graph G' is of lesser size than G, and upholds the terms of Theorem 1.1, thus there exists a $SP3C$ φ of G'. Assume, w.l.og, that $\varphi(v_1) = 1$ and $\varphi(v_2)$=2. (i) if $\varphi(v_4) = 3$ color the vertices $\{u_1, u_2, u_3, u_4, v_3\}$ with colors $3, 1, 1, 2, 2$, respectively. (ii) if $\varphi(v_4) = 2$ color the vertices $\{u_1, u_2, u_3, u_4, v_3\}$ with colors $2, 1, 1, 3, 3$, respectively. ∎

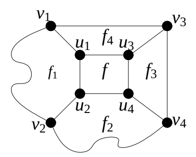

Figure 4: The face f and its adjacent faces f_i in Case (2.b).

4.3 Proof of Theorem 3.1(d)

In this section Theorem 3.1(d) is proven. By Theorem 3.1(c), the graph G contains no 4-face. Thus, in order to prove Theorem 3.1(d) it remains to eliminate adjacencies between 3-faces and even faces of length strictly larger than four. In this section, a k-face with even k is called an *even k-face*.

The following lemmas facilitate the proof of Theorem 3.1(d). Lemma 4.2 considers even k-faces, $k > 4$, with two adjacent 3-faces. We prove that these 3-faces cannot be adjacent to one another.

Lemma 4.2 *Let f be an even k-face of G, $k > 4$, having f_1 and f_2 two 3-faces adjacent to f. Then f_1 and f_2 are not adjacent to one another.*

Proof: Assume towards contradiction that the claim is false. Let $\{x, y, w\}$ and $\{y, z, w\}$ be the vertices of G defining the faces f_1 and f_2, respectively, such that the vertices $\{x, y, z\}$ are vertices of the face f. The vertices $\{x, y, z\}$ are consecutive on the boundary of f. Figure 5(a) depicts these vertices and faces. Let ℓ and r denote the vertices of f that are adjacent to the vertices x and z on the boundary of f, respectively.

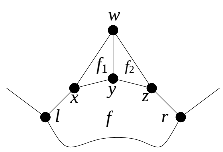

Figure 5: The faces f, f_1, and f_2.

Let G' be the graph obtained from G by identifying the vertices x, y, z, w into a single new vertex u (and removing all self-loops). By the minimality of G, the graph G' admits a $SP3C$, namely φ. Assume, w.l.o.g, that $\varphi(u) = 1$. To extend the coloring φ to G, color the vertices x, y, z, w with colors $1, 2, 1, 3$ respectively. ∎

The following lemma completes the case analysis initiated by Lemma 4.2.

Lemma 4.3 *The graph G does not admit a 3-face f defined by the vertices $\{u_1, u_2, u_3\}$, such that the following terms hold.*
(a) *The face f has three adjacent faces f_i, $i = 1, 2, 3$, with $|f_i| \geq 4$, and*
(b) *The faces f_i are pairwise adjacent. (See Figure 6(a)).*

Proof: Assume towards contradiction that the claim is false. Due to the existence of the faces f_i, there exist vertices v_i, $i = 1, 2, 3$, such that v_i is the third neighboring vertex of u_i and the vertex

$v_i \notin V(f)$. One may assume that at least one of the faces f_i is not a 4-face, for otherwise G is a graph on six vertices that admits a $SP3C$. Assume, w.l.o.g, that the face f_1 is a k-face, with $k > 4$. Also, let v_1 and v_2 lie on the boundary of f_1. See Figure 6(a) for an illustration.

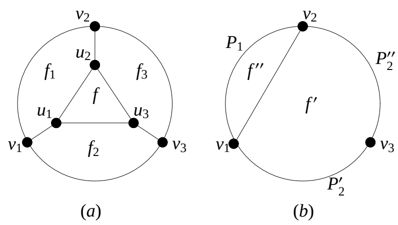

Figure 6: (a) The faces f, and f_i, $i = 1, 2, 3$ of Lemma 4.3. (b) The faces $f'$and f'' of the graph G'.

Let $G' = G - \{u_1, u_2, u_3\} + v_1v_2$. Note that the graph G' upholds the terms of Theorem 1.1, and thus, by the minimality of G, the graph G' admits a $SP3C$ φ. We extend φ to a $SP3C$ of G.

In the graph G', there exists a v_1v_2-path, namely P_1, not containing the edge v_1v_2. Also, there exists a v_1v_2-path, namely P_2, containing the vertex v_3. These two paths and the edge v_1v_2 define two faces f' and f'' in G'. These paths and faces are illustrated in Figure 6(b). Since φ is a $SP3C$ of G', the vertices of both the paths P_1 and P_2 are colored in three distinct colors. Let P_2' and P_2'' be the v_1v_3-path and v_3v_2-path, respectively, such that P_2' and P_2'' are subpaths of P_2. Assume, w.l.o.g, that $\varphi(v_1) = 1$ and $\varphi(v_2) = 2$. Returning to G, we observe that the face f_1 already sees three distict colors. If $\varphi(v_3) = 3$ color the vertices u_1, u_2, u_3 with the colors $2, 3, 1$ respectively. If $\varphi(v_3) \neq 3$, Assume, w.l.og., that $\varphi(v_3) = 1$. In this case, color the vertices u_1, u_2, u_3 with the colors $2, 1, 3$ respectively ∎

The following corollary is obtained from Lemma 4.3.

Corollary 4.4 *Let f be an even k-face of G, $k > 4$. Then no 3-face is adjacent to f*

Proof: Assume towards contradiction that there exists an even k-face f, $k > 4$, and a 3-face g such that g is adjacent to the face f. By Theorem 3.1(b), exactly two of the vertices of g lie on f, and all the vertices of g are of degree three. Let u_1 and u_2 denote these vertices. Let u_3 denote the third vertex of g not lying on f. Let v_i denote the third neighboring vertex of the vertex u_i, for $i = 1, 2, 3$. The vertices $v_1, v_2 \in V(f)$. Since $|f| > 4$, then $v_1 \neq v_2$.

Two complementary cases are considered. Either the vertex $v_3 \notin V(f)$ or $v_3 \in V(f)$. If the $v_3 \notin V(f)$ then let P_1 denote the path in G defined by the sequence of vertices $\langle v_1, u_1, u_3, v_3\rangle$, and let P_2 denote the path in G defined by the sequence of vertices $\langle v_2, u_2, u_3, v_3\rangle$. By Theorem 3.1($a$), there exist two faces, namely f_1 and f_2, such that P_i forms part of the boundary of the face f_i, for $i = 1, 2$. Note that $|f_1|, |f_2| \geq 4$, $|f| > 4$ and $|g| = 3$. Hence we get a contradiction to Lemma 4.3.

The complementary case in which $v_3 \in V(f)$, can be solved using similar arguments as in the proof of Theorem 3.1(e), and thus omitted. ∎

Finally, the combination of Theorem 3.1(c) and Corollary 4.4 yields Theorem 3.1(d).

4.4 Proof of Theorem 3.1(e)

Theorem 3.1(e). *Let f be an even k-face of G such that $k > 4$. Then f is chordless.*

Proof: Assume towards contradiction that f admits a chord $uv \in E(G)$. The vertex set $\{u, v\}$ defines a separation pair of G. Let P_{uv} be the uv-path on f that connects u and v in clockwise direction starting at u. Similarly, let P_{vu} be the vu-path along f connecting v and u in clockwise direction starting at v. Assume, w.l.o.g, that P_{vu} is embedded in the interior of the cycle $uv + P_{uv}$. By Theorem 3.1(b) $|P_{vu} - \{u, v\}| \geq 2$. Define $P' = P_{uv} \setminus u, v$. By the simplicity of G and Theorem 3.1(a) $|P'| \geq 2$. Let p

and q be the endpoints of P' such that $\{up, vq\} \subset E(G)$. The edge set $\{up, vq\}$ forms an edge cut of size two in G. Two complementary cases are considered. Either $pq \in E(G)$ or not.

In the latter case, define $G' = G - \{up, vq\}$. The graph G' contains exactly two connected components K_1 and K_2 such that $\{u, v\} \subset E(K_1)$, and $\{p, q\} \subset E(K_2)$. Define $K'_1 = K_1 + z + uz + vz$, where $z \notin E(K_1)$, and z is embedded in the exterior of X_{K_1}, and such that $P_{vu} + uz + zv$ forms a face in K'_1. In addition, define $K'_2 = K_2 + pq$, such that $pq + P'$ forms a face in K'_2. Note that both of the graphs K'_1 and K'_2 are of maximum degree at most three, and are simple. By the minimality of G the graphs K'_1 and K'_2 admit two colorings φ_1 and φ_2, respectively, such that φ_i is a $SP3C$ of K_i, $i = 1, 2$. In K'_1 the vertices $\{u, v, z\}$ form a 3-cycle. Consequently, the colors under φ_1 to $\{u, v, z\}$ are distinct. In addition, it is clear that $\varphi_2(p) \neq \varphi_2(q)$. Define φ_3 to be a coloring of K_2 obtained from φ_2 by denoting the color $\varphi_2(p)$ by the name $\varphi_1(v)$, and the color $\varphi_2(q)$ by the name $\varphi_1(z)$. The combination of φ_1 and φ_3 defines the requested coloring for G.

Consider the complementary case where $pq \in E(G)$. By Theorem 3.1(c) the edge pq cannot form a chord such that in G the chord uv is embedded in the interior of the cycle $P' + pq$. Consequently, either $pq \in E(f)$, in which case $P' = pq$, or pq forms a chord of f such that the cycle $P' + pq$ does not contain the chord uv in its interior. In the former case, applying the arguments shown above for K'_1 and K'_2 to K'_1 and K_2 hold as well. In the latter case, the path $P'' = P' - \{p, q\}$ has length at least two. In addition, the path $Q = P_{vu} - \{u, v\}$ also has length at least two. Let p' and q' be the endpoints of P'' where $\{pp', q'q\} \subset E(G)$, and let u' and v' be the endpoints of Q where $\{uu', v'v\} \subset E(G)$. Consider the graph $G' = G - \{u, v, p, q\} + \{u'p', v'q'\}$. One may verify that G' upholds the terms of our inductive claim. Consequently, G' admits a $SP3C$, namely φ. The coloring φ is extended to a $SP3C$ of G, namely φ', as follows. First, define $\varphi'(u) = \varphi(p')$ and $\varphi'(p) = \varphi(u')$. Note that $\varphi'(u) \neq \varphi'(p)$, and this accounts for two colors. It remains to argue that the third color can be assigned to either v or q. Since at most one of the vertices v' and q' is assigned the third color under φ, one can assign the third color to either v or q. The remaining vertex has to have its color distinct of three vertices two of which bare the same color, and thus the claim follows. ▮

References

[1] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann, and P. Zumstein. Polychromatic colorings of plane graphs. In 24*th ACM Symp. Computational Geometry*, 2008.

[2] P. Bose, D. Kirkpatrick, and Z. Li. Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces. *Computational Geometry: Theory and Applications*, 26(3):209–219, 2003.

[3] V. Chvátal. A combinatorial theorem in plane geometry. *J. Combin. Theory*, 18:39–41, 1975.

[4] D. Dimitrov, E. Horev, and R. Krakovski. 4-polychromatic colorings of rectangular partitions. In 24*th European Workshop Computational Geometry*, 2008.

[5] Y. Dinitz, M. J. Katz, and R. Krakovski. Guarding rectangular partitions. In 23*rd European Workshop Computational Geometry*, pages 30–33, 2007.

[6] S. Fisk. A short proof of Chvátal's watchman theorem. *J. Combin. Theory*, 24:374, 1978.

[7] F. Hoffmann and K. Kriegel. A graph-coloring result and its consequences for polygon-guarding problems. *SIAM J. Discrete Math.*, 9(2):210–224, 1996.

[8] E. Horev, M. J. Katz, and R. Krakovski. Polychormatic coloring of cubic bipartite plane graphs, manuscript, 2007.

[9] B. Mohar and R. Skrekovski. The Grötzsch theorem for the hypergraph of maximal cliques. *Electr. J. Comb.*, 6, 1999.

[10] J. Órourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1976.

[11] D. West. *Introduction to Graph Theory*. Prentice Hall, 1996.