On r-connected graphs with no semi-topological r-wheel

Elad Horev
horevel@cs.bgu.ac.il
Department of Computer Science

Michael Lomonosov*
lomonos@math.bgu.ac.il
Department of Mathematics

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Ben-Gurion University of the Negev, Beer Sheva, 84105, Israel

Abstract

A semi-topological r-wheel, denoted by S_r, is a subdivision of the r-wheel preserving the spokes; the paper describes the r-connected graphs having no S_r-subgraphs. For $r > 3$, these are shown to be only $K_{r,r}$, while the class $\mathcal{H}$ of 3-connected S_3-free graphs is unexpectedly rich.

First, every graph G in $\mathcal{H}$ has an efficiently recognizable set of "contractible edges" (sometimes empty) such that a contraction minor G/F belongs to $\mathcal{H}$ if and only if F is a part of this set. So, the subclass $\mathcal{H}^0$ of ante-contraction members of $\mathcal{H}$ plays a key role.

Second, the members of $\mathcal{H}^0$ have 3-edge cuts. The familiar cactus representation of minimum edge cuts (E. Dinits *et al.*. In: *Issledovaniya po Diskretnoy Optimizatsii* (A. A. Friedman, ed.), "Nauka", Moscow, pp. 290-306, 1976 (Russian); also A. Schrijver. Combinatorial Optimization (Polyhedra and Efficiency), *Algorithms and Combinatorics*, Vol. 24, Springer, 2003, p. 253) maps $\mathcal{H}^0$ onto the class of trees whose internal vertices have even degrees, equal to 6 for any vertex adjacent to a leaf. The description of $\mathcal{H}^0$ (quite concise as expressed in appropriate terms) refers to the explicit reconstruction of the reverse image of such a tree.

We also derive the upper bound $(2r-3)(n-r+1)$ on the number of edges in an arbitrary n-vertex S_r-free graph, $r \geq 4$, and conjecture that its maximum equals $(r-1)(n-r+1)+\lfloor\frac{r-1}{2}\rfloor$.

1 Introduction

An r-wheel with subdivided rim, denoted by S_r and called *semi-topological r-wheel*, appeared on the scene with the query about the maximal number $\text{ex}(n,r)$ of edges in an n-vertex graph having no S_r-subgraph. Development of this matter initiated with finding $\text{ex}(n,2) = \lfloor\frac{3(n-1)}{2}\rfloor$ [2, 1]; after the problem was formulated for general r and $\text{ex}(n,r)$ shown to grow linearly in n [1], two subsequent values was found: $\text{ex}(n,3) = 2n-3$ by C. Thomassen [8], together with a description of the extremal graphs, and $\text{ex}(n,4) = 3n-8$ by E. Horev [5], achieved by only $K_{3,n-3}+e$, the edge appended to the part of size 3. At this point the lack of a general view became perseptible. At least two general facts concerning the S_r-free graphs, $r \geq 3$, are however obvious. First, such graphs are interesting only when they are nonplanar. Indeed, any vertex of a planar 3-connected graph is the hub of an S_r-subgraph with $r = d(v)$, namely, the union of the face circuits incident with v (see, e. g., [3], Corollary 10.8); thus, a planar graph has an S_r-subgraph if and only if it has a vertex of degree at least r. Second, Dirac's theorem: *any r vertices of an r-connected graph lie on a circuit* (1952, see [3]), implies that a graph with no S_r-subgraph cannot be $(r+1)$-connected.

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The aim of the present work is to extend the latter fact by describing the r-connected S_r-free graphs for $r \geq 3$. Except for the case $r = 3$, the class of such graphs is trivial:

Theorem 1.1 *For $r \geq 4$, the only r-connected graph with no S_r-subgraph is $K_{r,r}$.*

Two proofs of this fact are presented in Section 4; one is quite short while the other one goes into deeper detail in order to fit countably infinite graphs as well. As an elementary consequence of this theorem, a bound $\mathrm{ex}(n, r) \leq (2r - 3)(n - r + 1)$ is derived in Subsection 4.2. We also conjecture that $\mathrm{ex}(n, r) = (r-1)(n-r+1) + \lfloor \frac{r-1}{2} \rfloor$, with the only extremal graph $K_{r-1,n-r+1}+M$ where M is a maximal matching on $(r - 1)$-part of $K_{r-1,n-r+1}$, extrapolating the results [8, 5],.

Let now $\mathcal{H}$ denote the class of 3-connected S_3-free graphs. It clearly contains $K_{3,3}$, and it is a pleasant exercise to construct other examples of cubic S_3-free graphs thereof. In general, however, nothing could be said about $\mathcal{H}$ except that all these graphs are nonplanar, with the number of edges lying within the bounds $3|V|/2$ (due to the connectivity) and Thomassen's $2|V| - 3$.

A detailed structure of $\mathcal{H}$ is presented by Theorems 1.2, 1.3 and 1.4 below; we outline now this result with minimal background, adjourning details to Sections 2 and 3. Consider $\mathcal{H}$ as consisting of a certain part $\mathcal{H}^0$ and the graphs reachable by an edge contracting process starting in $\mathcal{H}^0$:

$$(1) \qquad G_0 \text{ belongs to } \mathcal{H}^0, \quad G_i = G_{i-1}/e_i, \quad i = 1, 2, \ldots$$

where e_i is a *contractible* edge of G_{i-1}, that is, such that G_{i-1}/e_i belongs to $\mathcal{H}$ (provided G_{i-1} does so). A description of $\mathcal{H}^0$, together with a characterization of the contractible edges of a member of $\mathcal{H}$, would then serve as a dynamical description of $\mathcal{H}$. The reality has surpassed these hopes. Namely, denote by E^0 the set of edges of a given $G \in \mathcal{H}$ belonging to no nontrivial 3-disconnector (a disconnector $D \subset V \cup E$ is *trivial* if $G - D$ has an isolated vertex), and by V^0 the set of vertices incident with E^0. It turns out that contractible are exactly the edges of $G - V^0$ (Claim 3.4), and that process (1) preserves the graph $G^0 = (V^0, E^0)$ (Claim 3.5). The latter property immediately implies

Theorem 1.2 *Let $\mathcal{H}^0$ be any subclass of $\mathcal{H}$ such that every graph in $\mathcal{H} \setminus \mathcal{H}^0$ has the form G/e where G is a member of $\mathcal{H}$ and $e \in E(G)$. Then $\mathcal{H}$ consists of the graphs G/F where G ranges over $\mathcal{H}^0$, and F is an arbitrary set of edges of $G \setminus V^0$.*

It is then natural to take for $\mathcal{H}^0$ the class of *nonsplittable* members of $\mathcal{H}$, meaning that any split of a vertex into two adjacent ones (the reverse of edge contraction) expels such a graph from $\mathcal{H}$ (by producing a 2-disconnector or an S_3-subgraph). The nonsplittable members of $\mathcal{H}$ are characterized by Claims 3.6 and 3.7 implying a concise description of these graphs in terms of their 3-edge cuts. In general, the least-size cuts of any graph $G = (V, E)$ are represented by a certain cactus $\mathcal{T}$ (= graph whose blocks are circuits and copies of K_2) and a map ϕ: $V \to V(\mathcal{T})$, such that $\delta(X)$ is a least-size cut in G if and only if $\mathcal{T}$ has a cocircuit cutting $\phi(X)$ from $\phi(V \setminus X)$ (see [4] and also [7], p. 253). For graphs of odd edge connectivity, such a cactus is always a tree, and this *minimum cut tree* (MCT, for brevity) is unique, under the reasonable agreement to avoid 2-valent vertices in $\mathcal{T}$ except for those belonging to $\phi(V)$. We will see that the members of $\mathcal{H}^0$ have edge connectivity 3; for such graphs, the following direct construction of MCT is convenient.

3-CUT TREE. Given a graph $G = (V, E)$ of edge connectivity 3, consider the collection $\mathcal{Z}$ of sets $X \subset V$ with $d(X) = 3$, clearly cross-free (see, e. g., [7]), meaning that for any $X, Y \in \mathcal{Z}$ at least one of the four sets: $X \cap Y$, $X \setminus Y$, $Y \setminus X$ and $V \setminus (X \cup Y)$, is empty. Consider the subsets $\mathcal{X} \subseteq \mathcal{Z}$ satisfying $X \cap X' = \emptyset$ for any $X, X' \in \mathcal{X}$, call them *$\mathcal{Z}$-subpartitions*, and define a partial order $\mathcal{X} \preceq \mathcal{Y}$ between $\mathcal{Z}$-subpartitions meaning that for each $X \in \mathcal{X}$ there is $Y \in \mathcal{Y}$ such that $X \subseteq Y$. According to this order, the maximal $\mathcal{Z}$-subpartitions are just the pairs $\{X, V \setminus X\}$, that is, the 3-cuts.

Let now $\mathbf{C}$ denote the set of $\mathcal{Z}$-subpartitions $\mathcal{X}$ satisfying $X \cup X' \neq V$ for any $X, X' \in \mathcal{X}$ and $\preceq$-maximal subject to that. It is easy to see that $\mathcal{X} \in \mathbf{C}$ if and only if the graph $G/\mathcal{X}$ obtained by shrinking the members of $\mathcal{X}$ is essentially 4-edge connected [3]. As important examples, let us mention (i) a $\mathcal{Z}$-subpartition of the form $\{X\}$, belonging to $\mathbf{C}$ if and only if $V \setminus X$ is an inclusion-minimal member of $\mathcal{Z}$, and (ii) a $\mathcal{Z}$-subpartition $\mathcal{X}_v$, $v \in V$, $\preceq$-maximal subject to $\cup\mathcal{X} \subseteq V \setminus \{v\}$. Since $\mathcal{Z}$ is cross-free, $\mathcal{X}_v$ is unique and indeed belongs to $\mathbf{C}$. If v belongs to an inclusion-minimal $X \in \mathcal{Z}$ (in particular, if $d(v) = 3$) then $\mathcal{X}_v = \{V \setminus X\}$, otherwise $\mathcal{X}_v$ coincides with the set of all $X \in \mathcal{Z}$ *inclusion-maximal* subject to $v \notin X$.

The following property is essential.

(2) any member of $\mathcal{Z}$ belongs to a unique $\preceq$-maximal $\mathcal{Z}$-subpartition,

namely, the one of the form $\mathcal{X} = \{X\} \cup \mathcal{Y}$ where $\mathcal{Y}$ is the collection of all inclusion-maximal sets $Y \in \mathcal{Z}$ satisfying

(*) $Y \cap X = \emptyset$ and $Y \cup X \neq V$.

To see this, note first that $\{X\} \cup \mathcal{Y}$ belongs to $\mathbf{C}$. Indeed, for each $Y_1, Y_2 \in \mathcal{Y}$ one has $Y_1 \cup Y_2 \subseteq V \setminus X \neq V$ whence $Y_1 \cap Y_2 = \emptyset$, because $\mathcal{Z}$ is cross-free and the members of $\mathcal{Y}$ are maximal. The $\preceq$-maximality follows from the above requirements: the members of $\mathcal{Y}$ are inclusion-maximal subject to (*), and $\mathcal{Y}$ includes every such set from $\mathcal{Z}$.

The uniqueness of $\mathcal{X}$ follows from its maximality: if $\mathcal{X}' = \{X\} \cup \mathcal{Y}'$ satisfies * then each $Y' \in \mathcal{Y}'$ is contained by a set from $\mathcal{Z}$ inclusion-maximal subject to *, that is, by a member of $\mathcal{Y}$, whence $\mathcal{X}' \preceq \mathcal{X}$.

For a graph of edge connectivity 3, the MCT $\mathcal{T}$ is uniquely defined by the following rules. To avoid confusion, the vertices of $\mathcal{T}$ are referred to as *nodes*.

(MCT.1) The node-set of $\mathcal{T}$ is $\mathbf{C}$, and its edges are the pairs $\{X, V \setminus X\}$, $X \in \mathcal{Z}$; such an edge links the members of $\mathbf{C}$ containing X and $V \setminus X$.

(MCT.2) The map ϕ is given by $\phi(v) = \mathcal{X}_v$.

Cuboid graphs. $\mathcal{H}^0$ certainly contains the cubic members of $\mathcal{H}$, but not only: Claim 3.7 states that nonsplittable are also certain non-cubic graphs, suggesting the following

Definition A.

(A.1) An essentially 4-edge connected graph G is *cuboid* if each its vertex of degree $m > 3$ belongs to an induced $K_{2,m}$-subgraph J such that $\delta(V(J))$ is an m-matching, and $G - V(J)$ is 2-connected.

(A.2) In general, a 3-connected graph G is *cuboid* if its edge connectivity is 3 and in its MCT:

(A.2.1) each leaf is a 1-subset $\{V \setminus \{v\}\}$ of $\mathcal{Z}$ where v is a vertex of degree 3, and

(A.2.2) for each interior node $\mathcal{X}$, the essentially 4-edge connected graph $G/\mathcal{X}$ is cuboid.

Every essentially 4-edge connected cuboid graph is obtainable from a cubic one by iterating the following operation. Choose a chordless circuit C of length at least 4 spanned by the vertices of degree 3, such that $\delta(V(C))$ is a matching and $G - V(C)$ is 2-connected, add two new vertices, say u and v, and replace the edges of C with the edge-set of the complete bipartite graph on the colour classes $V(C)$ and $\{u, v\}$.

A tree T is isomorphic to MCT of a cuboid graph if and only if its interior node degrees are even. "Only if" follows from Definition A: since the vertices of degree greater than 3 in a cuboid graph appear in pairs as specified in (A.1), each interior node $\mathcal{X}$ of the MCT has $d_G(V \setminus \cup\mathcal{X})$ even, whence $|\mathcal{X}|$ is always even. Conversely, given such a tree T, the cuboid graphs with MCT isomorphic to T are as follows.

Choose an interior vertex t of T and an arbitrary essentially 4-edge connected cuboid graph H with exactly $p = d_T(t)$ vertices of degree 3. If T is a star, we are done: the MCT of H is a star isomorphic to T. Otherwise, consider the subtrees T_i of T, $i = 1, \ldots, p$, satisfying $T = \cup_{1 \le i \le p} T_i$ and $T_i \cap T_j = \{t\}$ for $i \neq j$; for each i, choose a cuboid graph H_i with the MCT isomorphic to T_i and denote by u_i its vertex represented by the leaf t (by (A.2.1)).

Choose an ordering v_i, $i = 1, \ldots, p$, of the vertices of H of degree 3, and construct G by stepwise substituting $H_i - u_i$ for v_i. The latter means removing u_i and v_i and linking their neighbourhoods by an arbitrary 3-matching.

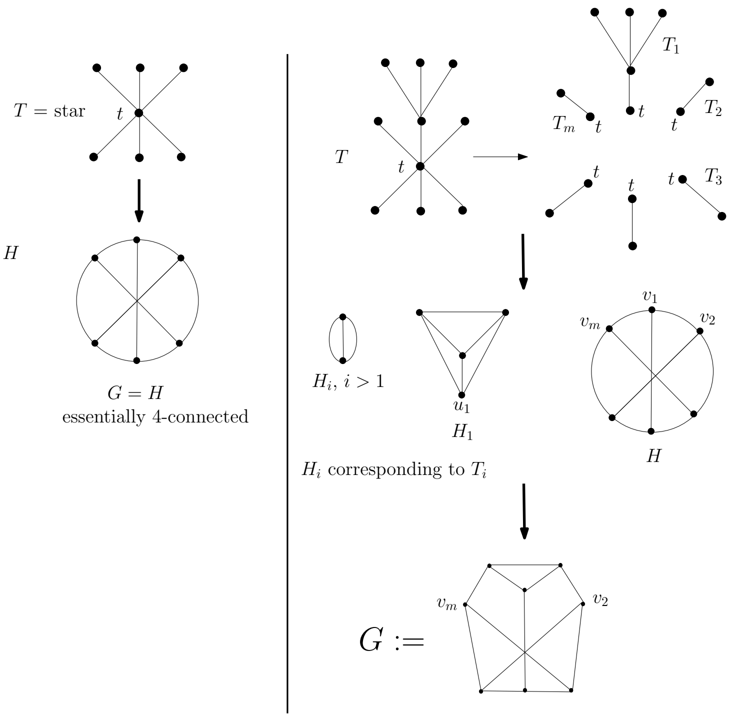


Figure 1: Cuboid graphs with a given 3-cut-tree.

Theorem 1.3 *A* 3*-connected graph is a nonsplittable member of* $\mathcal{H}$ *if and only if it is cuboid and in its MCT the nodes adjacent to leaves have degree* 6*.*

At the opposite extreme, Claim 3.4 implies that

(3) the noncontractible members of $\mathcal{H}$ are exactly those whose set $V \setminus V^0$ is stable.

Due to its role, the subgraph G^0 of a 3-connected S_3-free graph deserves more detailed clarification. Since G^0 coincides with the similar subgraph of a more transparent nonsplittable ancestor of G (see Claim 3.5), it is natural to describe G^0 in terms of $J \in \mathcal{H}^0$ such that $G = J/F$.

Theorem 1.4 *Let an* S_3*-free graph* G *of connectivity* 3 *be a contraction minor of a member* J *of* $\mathcal{H}^0$*, and consider* $\mathcal{Z}$*,* $\mathbf{C}$*, and* $\mathcal{T}$ *related to* J*. Put* $\hat{\mathbf{C}} := \{\mathcal{X} \in \mathbf{C}\colon |\mathcal{X}| = 6\}$*, and for each* $\mathcal{X} \in \hat{\mathbf{C}}$*, denote by* $S_{\mathcal{X}}$ *the subgraph of* $J/\mathcal{X}$ *induced by the short edges. Then*

(1.4.1) *for arbitrary* $\mathcal{X} \in \mathbf{C}$*,* $J/\mathcal{X} \cong K_{3,3}$ *if and only if* $\mathcal{X} \in \hat{\mathbf{C}}$*;*

(1.4.2) *a subgraph of* G *is a component of* G^0 *if and only if it coincides with some* $S_{\mathcal{X}}$*,* $\mathcal{X} \in \hat{\mathbf{C}}$*; thus, a component of* G^0 *is isomorphic to an edge-induced subgraph of* $K_{3,3}$*; in particular,* $S_{\mathcal{X}} \cong K_{3,3}$ *implies* $G = J = S_{\mathcal{X}}$*.*

Whenever possible, our terminology and notation follow [3]; otherwise, the source is [7]. A graph (not necessarily simple) is called k-connected if any two its vertices are linked by k internally disjoint paths (thus, e. g., an n-vertex graph with any pair of vertices linked by p edges is $(n+p-2)$-connected). It is convenient (and not contradictory) to regard the one-vertex graph as 2-connected (not as in [3]).

Given a connected graph $G = (V, E)$, a set $D \subset V \cup E$ is a *disconnector* if $G - D$ is disconected, and is a k-disconnector if $|D| = k$.

Given a subpartition $\mathcal{X}$ of the vertex-set of G, we denote by $G/\mathcal{X}$ the graph obtained by shrinking the members of $\mathcal{X}$ (the edges spanned by no member of $\mathcal{X}$ are preserved and may become parallel).

For a subset U of vertices, $N(U)$ and $\delta(U)$ denote, respectively, the set of vertices in $V \setminus U$ adjacent to U and the set of edges between U and $N(U)$.

For a subgraph H of G, the boundary of H, denoted by $\mathrm{bnd}_G H$ (or simply by $\mathrm{bnd} H$), is the set of vertices of H incident with $E(G) \setminus E(H)$, and $H - \mathrm{bnd}_G H$ (even if empty) is denoted by $\mathrm{int}_G H$ (or $\mathrm{int} H$). An induced subgraph with the same vertex set as H is denoted by $G(H)$.

For a set $F \subseteq E$, $V(F)$ denotes the set of vertices incident with F, and $G(F) := (V(F), F)$.

2 Separation

For a graph $G = (V, E)$, the absence of an S_r-subgraph with hub v means that $|N(v) \cap C| < r$ for any circuit C in $G - v$. When G is r-connected, this situation may always be certified by a separator, as is described in this section.

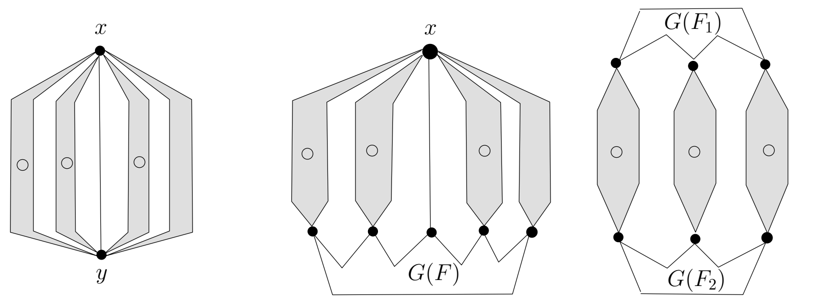
2.1 Circuit-separator alternative

A pair (X, F) with $X \subset V$ and $F \subseteq E(G - X)$ is a *separator* if the number of components of $G - X - F$ is greater than the *capacity*

(4) $$\gamma(X, F) := |X| + \sum \left\{ \lfloor \tfrac{1}{2} |\mathrm{bnd}_{G-X} Q| \rfloor \colon\ Q \text{ is a component of } G(F) \right\},$$

coinciding with the key parameter in Mader's internally disjoint paths theorem (see e. g., [7], page 1282, Corollary 73.2a). Given a set $A \subseteq V$, we refer to a circuit containing A as an *A-circuit*, and specify (X, F) as an *A-separator* if A meets more than $\gamma(X, F)$ components of $G - X - F$. Clearly, a pair G, A cannot admit both an A-circuit and an A-separator; in general, however, neither of them may exist, except for quite special cases when the "circuit-separator alternative" does hold. The simplest such case is presented by the following generalization of well-known Whitney's theorem (case $|A| = 2$) and the Dirac theorem quoted above.

Theorem 2.1 (D. M. Mesner and M. E. Watkins [9].) *The circuit-separator alternative holds for the pairs (G, A), $A \subseteq V$, satisfying $|A| \leq \kappa(G) + 1$. In such a case, together with an A-separator (X, F), the graph contains a subdivision of a complete bipartite graph with the colour classes A and Y where $X \subseteq Y \subseteq X \cup V(F)$ and $|Y| = |A| - 1$.*



(a) $F = \emptyset, X = \{x, y\}$ (b) $X, F \neq \emptyset$ (c) $X = \emptyset$,$F = F_1 \cup F_2$

Figure 2: Forms of a separator; the one presented by (b) is not an A-separator for $A = \{\circ, \circ, \circ\}$ but becomes one after extending F.

An A-separator (X, F) in an $(|A|-1)$-connected graph has $A \cap (X \cup V(F)) = \emptyset$, for otherwise a circuit containing $A' := A \setminus (X \cup V(F))$ exists by Dirac's theorem, and may always be chosen so as contain A. When $\kappa \geq 3$ and $|A| = \kappa + 1$, an A -separator is unique and has $F = \emptyset$.

When $\kappa(G) = 2$ and $|A| = 3$, an A-separator with F inclusion-minimal is unique; in this case F is not necessarily empty but the components of $G(F)$ are 2-connected. For 2-connected graphs, the assertion of Theorem 2.1 is derivable from the following useful and familiar[1] (though seemingly never published) fact:

Theorem 2.2 *A 3-connected graph has no circuit containing a given set $A \subset V \cup E$ of size 3 if and only if A consists of edges, and either is a cut or a claw, or contains two parallel edges.*

For our purpose, Theorem 2.1 should be extended so as to characterize the pairs G, U with $\kappa(G) = s$ and $U \subseteq V(G)$, $|U| > s$, satisfying

(5) $$|U \cap C| \leq s, \text{ for every circuit } C \text{ in } G.$$

With respect to this issue, the graphs of connectivity 2 appreciably differ from 3-connected ones. The case $s \geq 3$ is completely characterized by the following

Corollary 2.3 *For $s \geq 3$, (5) holds if and only if G has a U-separator X, $X \subseteq V \setminus A$, $|X| = s$. Such a separator coincides with the A-separator for an arbitrary $A \subseteq U$, $|A| = s + 1$; in particular, each component of $G - X$ contains at most one vertex from U, and G contains a subdivision of a complete bipartite graph with the colour classes U and X.*

Proof. The "if" part is obvious. To show "only if", choose $A \subseteq U$, $|A| = s+1$, and let $\mathcal{X} = (X, \emptyset)$ be an A-separator, by Theorem 2.1. Let J_i, $i = 1, 2, \dots$, be the components of $G - \mathcal{X}$ meeting U, and suppose, to the contrary, that $|U \cap J_1| > 1$. Choose $a_i \in A \cap J_i$, $1 \leq i \leq s$, and also $a_0 \in U \cap J_1 \setminus \{a_1\}$ and put $B := \{a_i\colon\ 0 \leq i \leq s\}$. By Theorem 2.1, G has a B-separator, say Y. Then the set $C := A \cap B$, of size $s \geq 3$, belongs, as a colour class, to two subgraphs homeomorphic to $K_{s,s}$, say R and S, whose opposite colour classes are X and Y respectively. Since any two members of C are disconnected in $G - X$, this happens in $S - X$ as well, so that $Y \subseteq X$. Thus $Y = X$, and the assertion follows. ∎

2.2 Case $\kappa = 2$

Though this case is also covered by Theorem 2.1, the notion of separator is now not so directly adaptable to setting (5) as we have in Corollary 2.3. Luckily, a weaker notion of U-frame is sufficient for our purpose, due to the properties presented in Theorem 2.4 and Claim 2.5 below. Certain new notions are however needed.

A 2-connected graph is a union of subgraphs to which we refer as hammocks. A *hammock* in G is a connected subgraph whose boundary consists of two vertices referred to as *ends*. An edge $e \in E$ together with the ends, and also $G - e$ are hammocks distinguished as *trivial*; by the Menger theorem, a graph has no *nontrivial* hammock if and only if it is 3-connected. Two hammocks are *internally disjoint* if their intersection consists of their common ends.

An end-to-end path in a hammock will be called a *through path*. Being not necessarily 2-connected, a hammock becomes such after an ear linking its ends is appended (because its block tree is always a path). Equivalently,

(6) any edge of a hammock lies on a through path.

Let now G be 2-connected, with a set $U \subseteq V$ satisfying (5) with $s = 2$, and let A be an arbitrary 3-subset of U. Suppose that G has no A-circuit. Instead of Theorem 2.1, this fact may be testified by applying Theorem 2.2, as follows. Refer to the hammocks of G satisfying

(7) $$|A \cap \mathrm{int} H| \leq 1$$

[1] See, e. g., [6] Chapter 6, Exercise 67. We are grateful to the referee who has informed us of this reference.

as A-*hammocks*, and let an A-hammock satisfying (7) with equality be called *heavy*. It is immediately seen that a heavy A-hammock H has no end in A, so that $|H \cap A| = 1$. For collections $\mathcal{L}$, $\mathcal{L}'$ of pairwise internally disjoint A-hammocks, let $\mathcal{L} \preceq \mathcal{L}'$ mean that each $H \in \mathcal{L}$ is a subgraph of some $H' \in \mathcal{L}'$; so, a collection $\mathcal{L}$ is $\preceq$-*maximal* if $\mathcal{L} \preceq \mathcal{L}'$ implies $\mathcal{L} = \mathcal{L}'$. Choose an arbitrary $\preceq$-maximal collection of A-hammocks and replace each its member with an edge having the same ends. Denote by $G' = (V', E')$ the resulting graph, and by A' the subset of $V' \cup E'$ composed of $A \cap V'$ and the edges representing the heavy A-hammocks. Clearly, G' is 2-connected, $|A'| = 3$, and G' has no A'-circuit, by (6). Now Theorem 2.2 can be applied, because actually G' is 3-connected. Indeed, otherwise G is either a triangle with the edge-set A' or a union of two nontrivial hammocks with the same ends. Since $|A'| = 3$, in the second case the interior of at least one of the hammocks contains at most one member of A', contradicting the definition of G'.

By Theorem 2.2, G' has no A'-circuit if and only if A' is either a cut, or a claw, or a triple of parallel edges (the latter corresponds to the third case of the theorem, because if two edges in A' are parallel then the rest of G' is an A'-hammock with the same ends). Translating this back to the initial G suggests the following notion paraphrasing the Watkins-Mesner A-separator for some 3-subset A of U. Its relation to the setting (5) is given by Theorem 2.4 below.

Definition B. Let $G = (V, E)$ be 2-connected, and U be a subset of V satisfying (5). A pair $\mathcal{X} = (X, F)$, where $X \subset V$ and $F \subset E(G - X)$, is called a U-*frame* if the subgraph $G(\mathcal{X}) := (X \cup V(F), F)$ has exactly two components, and G is the union of $G(\mathcal{X})$ and a collection $\mathcal{B}$ of at least three hammocks satisfying the following four conditions:

(B.1) a member of $\mathcal{B}$ has the ends in distinct components of $G(\mathcal{X})$;

(B.2) $U \cap \mathrm{int}H \neq \emptyset$ for at least three $H \in \mathcal{B}$;

(B.3) if $X = \emptyset$ then $|\mathcal{B}| = 3$; and

(B.4) $\mathcal{B}$ is $\preceq$-maximal subject to (B.1-3).

The members of $\mathcal{B}$ are referred to as $\mathcal{X}$-*hammocks*. Due to the maximality of $\mathcal{B}$, a frame with $F \neq \emptyset$ has the following three easily checkable properties:

(B.5) the ends of any two $\mathcal{X}$-hammocks in the same component of $G(F)$ are distinct;

(B.6) the components of $G(F)$ are 2-connected; and

(B.7) a subgraph obtained from G by removing the interiors of some $|\mathcal{B}| - 2$ $\mathcal{X}$-hammocks is 2-connected.

Remark. As long as graphs of connectivity 2 are deal with, we may assume U to be a subset of $V \cup E$, without any essential change except that if U contains an edge e then an $\mathcal{X}$-hammock containing e may be trivial.

Given an U-frame $\mathcal{X}$, denote by H_u the $\mathcal{X}$-hammock whose interior contains a member u of U. By the following theorem, H_u exists for any U-frame and $u \in U$, but $H_u = H_v$ may occur for $u \neq v$. We say that a U-frame *isolates* a vertex $u \in U$ if $U \cap \mathrm{int}H_u = \{u\}$. A U-frame isolating each member of U, and thereby being a common A-separator for any triple $A \subset U$, will be called *perfect*.

Theorem 2.4 *A subset U of vertices of a 2-connected graph G satisfies* (5) *if G has a U-frame $\mathcal{X}$ with the properties*

(2.4.1) *each member of U belongs to the interior of some $\mathcal{X}$-hammock, and*

(2.4.2) *every through path of an $\mathcal{X}$-hammock traverses at most one member of U,*

and only if U-frames exist, and every U-frame satisfies (2.4.1-2).

Proof. To show "if", let $\mathcal{X} = (X, F)$ be a U-frame satisfying (2.4.1-2). Assuming, to the contrary, that G has a circuit C with some $A = \{a, b, c\} \subseteq U \cap C$, we conclude that G has an A-separator. Indeed, the $\mathcal{X}$-hammocks H_a, H_b, H_c exist, by (2.4.1), and are all distinct, by (2.4.2). If one of X, F is empty, $\mathcal{X}$ itself is clearly an A-separator, because in such a case $\gamma(X, F) = 2$, by (B.3). If $X = \{x\}$ then we have an A-separator (X, F') where F' is the union of F and the edge-sets of $H - x$, H ranging over the $\mathcal{X}$-hammocks H with $H \cap A = \emptyset$.

"Only if". Let G and U satisfy $|U \cap C| \leq 2$ for every circuit C. Existence of a U-frame follows from Theorem 2.2, as applied to an arbitrary triple $A \subseteq U$. Let now $\mathcal{X}$ be an arbitrary U-frame.

To show (2.4.1), suppose, to the contrary, that $u \in U$ belongs to the interior of no $\mathcal{X}$-hammock, for some U-frame $\mathcal{X}$. Then u lies in some component of $G(\mathcal{X})$. Choose a vertex v in the other component, and two $\mathcal{X}$-hammocks, say H and H', whose interior meets U. By property (B.7), G has a $\{u, v\}$-circuit containing through paths of H and H'. By (6), these paths can be chosen so as to meet $U \cap \mathrm{int}H$ and $U \cap \mathrm{int}H'$, contradicting (5).

To show (2.4.2), let H be an $\mathcal{X}$-hammock with a through path P traversing two members of U, and H' be another $\mathcal{X}$-hammock whose interior meets U; by (6), H' has a through path P' meeting $U \cap \mathrm{int}H'$. Again, G has a circuit containing P and P', contradiction. ∎

Claim 2.5 *Let G and U be as in Theorem 2.4. For each $u \in U$, there exists a U-frame isolating u. In particular, U is stable.*

Proof. Given $u \in U$, let $\mathcal{X}$ be a U-frame with $H_u \cap U$ minimal; we show that $H_u \cap U = \{u\}$. If not, let v be another vertex in $H_u \cap U$. By (2.4.2), H_u has no through path traversing both u and v.

Consider the graph $J := H_u + e$ where e is a new edge linking the ends of H_u (if such an edge already exists, denote it by e and put $J := H_u$). By (6), J is 2-connected. Put $W := \{u, v, e\}$. By (2.4.2), J has no W-circuit, so it has a W-frame $\mathcal{Y} = (Y, F')$, with $Y \subset V(J)$ and $F' \subset E(J)$, with a collection $\mathcal{B}'$ of $\mathcal{Y}$-hammocks including H'_u, H'_v and H'_e, all distinct (concerning H'_e see Remark just after Definition B). Now, the subgraph $H'' := (H'_e - e) \cup (G - \mathrm{int}H_u)$ is a hammock in G, so that $\mathcal{Y}$ acts also as a U-frame in G with the collection $\mathcal{B}'' := \mathcal{B}' \setminus \{H_e\} \cup \{H''\}$ of $\mathcal{Y}$-hammocks, and $U \cap H'_u \subseteq (U \cap H) \setminus \{v\}$, contradicting the choice of $\mathcal{X}$. ∎

Remark. By definition, a U-frame has capacity 2 (see (4)), and according to (B.2), is actually a separator for some 3-subset of U. When $|U| > 3$, there may be several such subsets, but only in quite special cases a U-frame provides separation for all triples in U at once, that is, isolates each member of U. Property (2.4.2) and Theorem 2.2 seem to hint how a frame can be extended to what can be adopted for such a U-separator (as in proof of Claim 2.5), with an appropriately defined capacity. In contrast with Corollary 2.3, the later is expected to take any value in the interval $2 \leq \gamma < |U|$.

3 3-connected S_3-free graphs

From this point forth, a vertex of an r-connected graph is called *ill* if it is the hub of an S_r-subgraph, and *sound* otherwise; the graph as a whole is *sound* if each its vertex is sound.

Except for Subsection 3.1, the graph $G = (V, E)$ at hand is assumed to be sound.

The term "$N(v)$-frame in a subgraph $G - v$" is usually abbreviated to *v-frame*.

3.1 Preliminaries

This subsection presents basic tools for proving Claims 3.4 and 3.5 below; Theorem 1.2 is a direct consequence thereof.

A vertex v is sound if and only if the pair $G - v$, $N(v)$ satisfies (5). In particular, a sound vertex belongs to no triangle, by Claim 2.5.

Observation 3.1 *Let u and v be adjacent vertices of a 3-connected graph G, and H be a hammock of $G - u$ with the ends s, t and $N(u) \cap \mathrm{int}H = \{v\}$. If v is sound and adjacent to an end of H then $N(v) = \{s, t, u\}$ and H is an (s, t)-path of length 2.*

Proof. Since $d(v) \geq 3$, it suffices to show that no neighbour of v belongs to $\mathrm{int}H$. Suppose, to the contrary, that there is $x \in N(v) \cap \mathrm{int}H$. Since $G - v$ is 2-connected, it has a circuit

C traversing x and u. Since u is nonadjacent to $H - v$, C contains a through path of H, and thereby is the rim of an S_3-subgraph of G with hub v, contradiction. Thus, $N(v) = \{s, t, u\}$. Since G is 3-connected, $\text{int}H = \{v\}$. Finally, s and t are nonadjacent, for if they are linked by an edge e then a $\{u, e\}$-circuit in $G - v$ is the rim of an S_3-subgraph with hub v, contradiction. ∎

Observation 3.2 *Let v be a sound vertex of a 3-connected graph G, and $\mathcal{X} = (X, F)$ be a v-frame, with $X \neq \emptyset$. If there is an edge (x, y) with $x \in X$ and $y \in G(\mathcal{X}) - x$ then x is ill.*

Proof. Let C be a circuit in $G - x$ traversing v and y. Then C contains a path $[a, v, b]$ where $a, b \in N(v)$ and $H_a \neq H_b$, and also the ends of the $\mathcal{X}$-hammocks H_a and H_b lying in $G(\mathcal{X}) - x$. Moreover, for any $s \in H_a - x$ and $t \in H_b - x$, there exists a $\{v, y\}$-circuit C traversing both. Indeed, in the subgraph $G - x$, C has an ear P traversing s and an ear Q traversing t. Since $H_a - x$ has exactly one vertex in common with $G - x - \text{int}H_a$, we have $P \subseteq H_a - x$; similarly $Q \subseteq H_b - x$. Then $C \cup P \cup Q$ contains a circuit traversing s, t, v, and y.

Choose such s and t to be adjacent to x. Then $|N(x) \cap C| \geq 3$, so that x is ill. ∎

The following important property is a direct consequence of Claim 2.5:

(8) each edge of a sound graph belongs to a 3-disconnector.

Proof. It suffices to show that inserting an edge between nonadjacent vertices transforms a sound graph into ill. Indeed, if so then G and a subgraph $G - e$ cannot be both sound. Since G is sound, $G - e$ also has no S_3-subgraph; since $G - e$ cannot be sound, it should be non-3-connected.

So, let u, v be nonadjacent vertices of a sound graph G, and let e be a new (u, v)-edge. If $G + e$ is sound then $G + e - u = G - u$ has an $N(u) \cup \{v\}$-frame $\mathcal{X}$ isolating v, by Claim 2.5, so that the $\mathcal{X}$-hammock H_v is a nontrivial hammock in G. Then G is not 3-connected, contradiction. ∎

Let an edge be called *long* if it belongs to a nontrivial 3-disconnector, and *short* otherwise. Thus, E^0 (see Introduction) is just the set of short edges. A short edge does belong to a 3-disconnector, by (8), but only to such that isolates one of its ends (thereby having degree 3).

Lemma 3.3 *For an edge (u, v) of a sound graph, the following statements are equivalent:*

(3.3.1) *(u, v) is short;*

(3.3.2) *$d(u) = d(v) = 3$, and the hammock H_v of the u-frame has $\text{int}H_v = \{v\}$ (by symmetry, the same holds with u and v interchanged);*

(3.3.3) *there is a circuit $C \subseteq G - \{u, v\}$ and distinct vertices s, x, t, y traversed by C in this circular order, such that $s, t \in N(v)$ and $x, y \in N(u)$.*

Proof. (3.3.1) ⇒ (3.3.2). Since a short edge has an end of degree 3, let $d(u) = 3$. Put $\{x, y\} := N(u) \setminus \{v\}$, and let $\mathcal{X}$ be the u-frame, with the hammocks H_x, H_y, and H_v. Let a, b denote the ends of H_v. Since (u, v) is short, the disconnector $\{a, b, (u, v)\}$ is trivial, implying $\text{int}H_v = \{v\}$, whence $d(v) = 3$ and H_v is the path $[a, v, b]$.

(3.3.2) ⇒ (3.3.3). Let $\{x, y\} := N(u) \setminus \{v\}$ and $\{s, t\} := N(v) \setminus \{u\}$; since $G - u - v$ is 2-connected, by (B.7), it has an $\{s, t\}$-circuit, say C; then $C \cap H_x$ and $C \cap H_y$ are through paths. By (6), these paths, and thereby C, can be chosen so as to traverse x and y, as required.

(3.3.3) ⇒ (3.3.1). Let C be a circuit in $G - \{u, v\}$ as specified in (3.3.3). By (8), $G - (u, v)$ has a 2-disconnector D. Since D disconnects u and v, it either coincides with the pair of edges incident with u or v, or meets the four paths of C linking $\{s, t\}$ to $\{x, y\}$. In the latter case D coincides with one of these pairs. Thus, each 3-disconnector containing (u, v) is trivial, as required. ∎

3.2 Contracting an edge: proof of Theorem 1.2

If G is sound, contracting an edge cannot decrease $\kappa(G)$. Indeed, if an edge (u, v) is spanned by a 3-disconnector of G then there are $x, y \in N(v) \setminus \{u\}$ such that every $\{x, y\}$-circuit in $G - v$ contains u.

Claim 3.4 *An edge of G is contractible if and only if it belongs to $G - V^0$.*

Proof. "Only if" is straightforward from Lemma 3.3. Indeed,(3.3.3) implies that contracting an edge incident with V^0 creates an S_3-subgraph.

To prove "if", it suffices to show that a noncontractible edge has an end in V^0. Let e be an edge whose contraction creates an S_3-subgraph. Then G has a subgraph J isomorphic to subdivision of the 3-wheel with two spokes of length 1 and one of length 2, the latter containing e. Denote the rim of such a J by C, the hub by u, the interior vertex of the longer spoke by v, the ends of the shorter spokes in C by a and b, and of the longer one by s. It suffices to show that the edge (u, v) is short.

Consider a u-frame $\mathcal{X} = (X, F)$ isolating v. Since $a, b \notin \text{int}H_v$, we have $C \cap \text{int}H_v = \emptyset$, for otherwise $C \cap H_v$ is a through path which can be chosen so as to traverse v thus creating an S_3-subgraph. Thus, $s \in \text{bnd}H_v$; let t denote the other end of H_v. By Observation 3.1, t is adjacent to v; so it remains to choose the rim C traversing t, to show that the pairs $\{a, b\}$ and $\{s, t\}$ disconnect each other in C, and to apply Lemma 3.3. The first is possible because t belongs to $G(\mathcal{X})$ whose components are 2-connected, by (B.6).

To show the second, recall that G has no triangle, so that $t \neq a, b$. If now s and t lie in the same component of $C - \{a, b\}$ then C may be modified so as to contain a, b and v, thus creating an S_3-subgraph. ∎

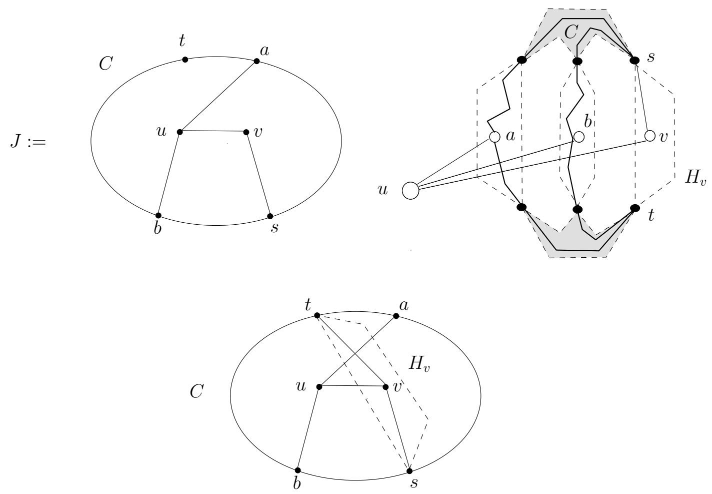


Figure 3: To proof of Claim 3.4.

Claim 3.5 *Process* (1) *preserves* G^0.

Proof. Let, to the contrary, e be a contractible edge of G whose contraction changes E^0, in any of the two ways specified below. Given an edge (u, v) with the ends of degree 3, a circuit C satisfying (3.3.3) will be called *good*.

I. $f \in E^0$ becomes long in G/e. Since a contractible edge is adjacent to no short one, the ends of f in G/e are the same as in G, say u and v. By Lemma 3.3, contracting e leaves no good circuit in G/e. This implies that each good circuit spans e.

So, in G we have $d(u) = d(v) = 3$ and a good circuit C, by Lemma 3.3, with a quadruple $Z = \{a, x, b, y\}$ of vertices traversed by C in this circular order, such that $a, b \in N(u)$ and $x, y \in N(v)$. Put $J{:}= C \cup \{u, v\} \cup \delta(u) \cup \delta(v)$. By (8) and Claim 3.4, e belongs to a nontrivial 3-disconnector, say D; we may assume that $D'{:}= D\backslash\{e\}$ is a pair of vertices, so that $G - e = G' \cup G''$ with $G' \cap G'' = G[D']$. Since e is spanned by the subgraph J, the latter is disconnected by D'. Since J is a subdivision of a 3-connected graph (namely, $K_{3,3}$), its intersection with one of G',

G'' is a path in C whose end-pair coincides with D' and the interior contains an end of e, say s, and does not meet Z. Let this path be $P = C \cap G'$. Since D is nontrivial, $\mathrm{int}(G' - s)$ in nonempty; since $G - s$ is 2-connected, there is a D'-path in $G' - s$, say Q. Then replacing P with Q yields a good circuit in G not spanning e, contradiction.

II. A long edge $f \in E(G - V^0)$ becomes short in G/e. Since in G/e the ends of f have degree 3, none of them is obtained by merging the ends of e; thus, the ends of f in G/e are the same as in G, say u and v. By Claim 3.3, contracting e creates a good circuit, so that G/e has a subgraph J as above, with $J - u - v = C$, $N_{G/e}(v) = \{s, t, u\}$ and $N_{G/e}(u) = \{a, b, v\}$, while in G one of the edges linking $\{u, v\}$ to C, say (v, s), is subdivided by a vertex w. The resulting path (v, w, s) clearly contains e. Since $d_{G/e}(v) = 3$, there should be $e = (w, s)$. Consider the v-frame in G, with the hammocks denoted by H_x, $x \in N(v) = \{u, w, t\}$. We assert that C does not meet $\mathrm{int}H_w$. Indeed, $C \cap \mathrm{int}H_w$, if nonempty, is a through path which may be chosen so as to traverse w. Since our frame isolates w (because $d(v) = 3$) such a deformation would preserve $a, b \in C$, thus creating an S_3-subgraph in G.

Since $C \cap \mathrm{int}H_w$ is empty, s is an end of H_w, so that Observation 3.1 is applicable. Thus, $d(w) = 3$ and w is the only interior vertex of H_w. By (3.3.2), the edge (v, w) is short, so that e is noncontractible, by Claim 3.4, contradiction. ∎

Theorem 1.4 follows.

3.3 Splitting a vertex

Recall that $G = (V, E)$ is sound, that is, 3-connected and S_3-free.

Splitting a vertex v is a transformation $G \mapsto \hat{G}$ by partitioning $N(v)$ into subsets N' and N'', replacing v with two adjacent vertices, v' and v'', and linking v' to each member of N' and v'' to each member of N''. The resulting graph $\hat{G}$ is 3-connected if and only if $|N'|, |N''| \geq 2$ (see [3], Theorem 9.12). In the sequel, $\hat{N}(\cdot)$ and $\hat{d}(\cdot)$ are used instead of $N_{\hat{G}}(\cdot)$ and $d_{\hat{G}}(\cdot)$. Thus, $\hat{N}(x) = N(x)$ for each $x \in V \setminus N(v) \setminus \{v\}$, $\hat{N}(v') = N' \cup \{v''\}$ and similarly for v'', and $\hat{N}(x) = N(x) \setminus \{v\} \cup \{v'\}$ for $x \in N'$ and similarly for $x \in N''$.

A vertex v of a sound graph is *splittable* if $d(v) > 3$ and v admits a split with $|N'|, |N''| \geq 2$ producing no S_3-subgraph. A sound graph with no splittable vertex is *nonsplittable*.

Claim 3.6 *A vertex v of degree greater than 3 is nonsplittable if G has a perfect v-frame (X, F) with $|X| = 1$, and only if this is the only v-frame in G.*

Proof. To show "if", consider a perfect v-frame $\mathcal{X} = (\{x\}, F)$ and choose an arbitrary partition N', N'' of $N(v)$ satisfying $|N'|, |N''| \geq 2$. An S_3-subgraph of $\hat{G}$ with hub v'' is constructed as follows. Choose $r, s \in N'$ and $t, u \in N''$, and denote by J the union of $G(F)$, the $\mathcal{X}$-hammocks H_r, H_s, and the path (r, v', s). Denote by t' and u' the ends of H_t and H_u in $G(F)$; since J is 2-connected (by remark (B.6) to Definition B), it has a (t', u')-path P traversing v'. Clearly, $x \notin P$. On the other hand, $H_t \cup H_u$ contains a (t', u')-path Q traversing t and u, by (6). Then $P \cup Q$ is the rim of an S_3-subgraph with hub v'', as required.

To prove "only if", we consider a vertex v, $d(v) > 3$, and let $\mathcal{X} = (X, F)$ be a v-frame which, to the contrary, is either imperfect (case I) or having $F = \emptyset$ (case II; by (B.3), this is the only way $|X| \neq 1$ holds), and choose a certain partition of $N(v)$ for split. The absence of an S_3-subgraph in $\hat{G}$ is established indirectly, according to the following program. Since G is sound, an S_3-subgraph J of $\hat{G}$, if exists and has rim S and hub z, certainly contains v' and v'', in a way that the edge $e = (v', v'')$ appears either as a spoke or a chord of C; in the latter case each component of $C - v' - v''$ meets $\hat{N}(z)$. The proof refutes these two possibilities, separately for cases I and II.

Note that each $\mathcal{X}$-hammock is nontrivial, by (B.3) and Observation 3.2, and therefore contains a neighbour of v in the interior.

I. Suppose that $\mathcal{X}$ is imperfect, that is, there is an $\mathcal{X}$-hammock H with $|H \cap N(v)| \geq 2$. Choose $N' = N(v) \cap H$ and $N'' = N(v) \setminus H$, and let, to the contrary, J be an S_3-subgraph of $\hat{G}$, with rim C and hub z.

(I.1) $e = (v', v'')$ is a spoke of J, so that $z \in \{v', v''\}$. Then $z = v''$, for otherwise $\hat{N}(z) \setminus \{v''\} \in \mathrm{int}H$, so that some two vertices from $N' \subset N(v)$, lie on the through path $C \cap H$ of H, contradicting (2.4.2). Further, $\mathcal{X}$ acts in $\hat{G}$ as a v''-frame, with the same hammocks H_t, $t \in N''$, as initially in G, and also $H' := \hat{G}(H \cup \{v'\})$. Thus, the circuit C lies in $\hat{G} - v''$, meets $\mathrm{int}H'$ and contains some $a, b \in N''$. Note that the number of $\mathcal{X}$-hammocks whose interior meets C is even, and its intersection with each of them can be chosen so as to traverse a neighbour of v''. Then there is exactly one hammock H_t, $t \in N''$, having $C \cap \mathrm{int}H_t \neq \emptyset$, for if there are more then contracting $e = (v', v'')$ and replacing $C \cap H'$ with a through path of H transforms J into an S_k-subgraph in G, $k \geq 3$. Thus, the through path $C \cap \mathrm{int}H_t$ traverses at least two neigbours of v (namely, the ends of spokes other than e), contradiction.

(I.2) e is a chord of C; since $\hat{G}/e = G$ is sound, each component of $C - v' - v''$ meets $N(z)$; on the other hand, it contains at most two members theoreof. We show that in G there is no place for the hub z.

(I.2.1) $z \notin G(\mathcal{X})$. Indeed, otherwise J has a spoke, say (a, z), linking the components of $G(\mathcal{X})$; refer to them as A and Z respectively. The other two spokes, say (b, z) and (c, z), are spanned by Z. Return to the initial $G = \hat{G}/e$, retain the two hammocks whose interior meets C and remove the interior of the others. Since, by (B.7), the resulting graph is 2-connected, the path $C \cap A$ may be chosen so as to traverse a; this yields an S_3-subgraph in G, contradiction.

(I.2.2) z is not an interior vertex of an $\mathcal{X}$-hammock (with respect to G). Indeed, let K be such a hammock with $z \in \mathrm{int}K$.

If $|N(v) \cap \mathrm{int}K| = 1$ then $\hat{N}(z) \setminus \{v''\} \subseteq K$. Since $C \cap K$ avoids v'' and a component of $C - v' - v''$ contains at most two members of $N(z)$, we have $\hat{d}(z) = 3$, and z is adjacent to v''. Then contracting $e = (v', v'')$ and replacing $C \cap H'$ with a trough path of H produce an S_3-subgraph in G with hub z.

If $|N(v) \cap \mathrm{int}K| > 1$ then $N(z) \setminus \{v\} \subseteq K$. Let P be a path in $G - v - \mathrm{int}K$ linking the ends of K. Then the union of P and $C \cap G(H \cup \{v\})$ is an S_3-subgraph in G, contradiction. Thus, any v-frame in G is perfect.

II. Suppose that $F = \emptyset$ and let $X = \{x, y\}$. Since X is perfect, the $\mathcal{X}$-hammocks H_r, $r \in N(v)$, are all distinct. We show that the result of every split satisfying $|N'|, |N''| \geq 2$ is sound.

Indeed, let, to the contrary, J be an S_3-subgraph of $\hat{G}$, with hub z, rim C, and spokes (z, s), (z, t), (z, u). First, $z \neq v', v''$, because if , say, $z = v''$ with $s, t \in N''$ then C contains a through path in each of three hammocks with the same ends: H_s, H_t, and also in the hammock induced by v' and H_r, $r \in N'$, impossible.

Thus, e is a chord of C. Since $J \subseteq \hat{G} - e$ and X disconnects v' and v'' in $\hat{G} - e$, we have $x, y \in J$; since J is a subdivision of a 3-connected graph (namely, $K_{3,3}$), $C - X$ has a component nonadjacent to z, say P. Suppose, without loss of generality, that s, t, x, v', y, u lie on C in this circular order. Clearly, $v' \neq x, y$ while t may coincide with x, and also u with y. Since $x, y \in C$, the hub z lies in the interior of one of the hammocks H_a, H_b where a, b are the neighbours of v' along C; indeed, otherwise replacing P with a through path of H_a yields an S_3-subgraph of $\hat{G}$ not spanning e. Suppose that $z \in H_a$. Then $z \in \mathrm{int}H_a$, so that $s, t, u \in C \cap H_a \subset P$, contradiction. ∎

Thus, related to a nonsplittable vertex v of a sound graph is, first, another vertex, say u, and, second, a set F of edges inducing a 2-connected subgraph, such that $\mathcal{X} = (\{u\}, F)$ is a perfect v-frame. Let $d(v) = m$, and denote the $\mathcal{X}$-hammocks by $H_i, \ldots, H_m$. By Claim 3.2, u and $V(F)$ are nonadjacent. By interchanging u and v, one obtains a u-frame $\mathcal{X}' = (\{v\}, F)$ with the $\mathcal{X}'$-hammocks $H'_i = G((H_i - u) \cup \{v\})$. The situation becomes quite symmetrical when u is also nonsplittable: by Claim 3.6, $\mathcal{X}'$ is perfect, so that $d(u) = m$. The picture acquires additional details when G is nonsplittable as a whole, as follows.

Claim 3.7 *Let G be nonsplittable. Then for each vertex v of degree $m > 3$ there exists a partition $\mathcal{Y}$ of V into $\{u\}$, $\{v\}$, W, and X_i, $i = 1, \ldots, m$, such that*

(3.7.1) *$d(v) = m$,*

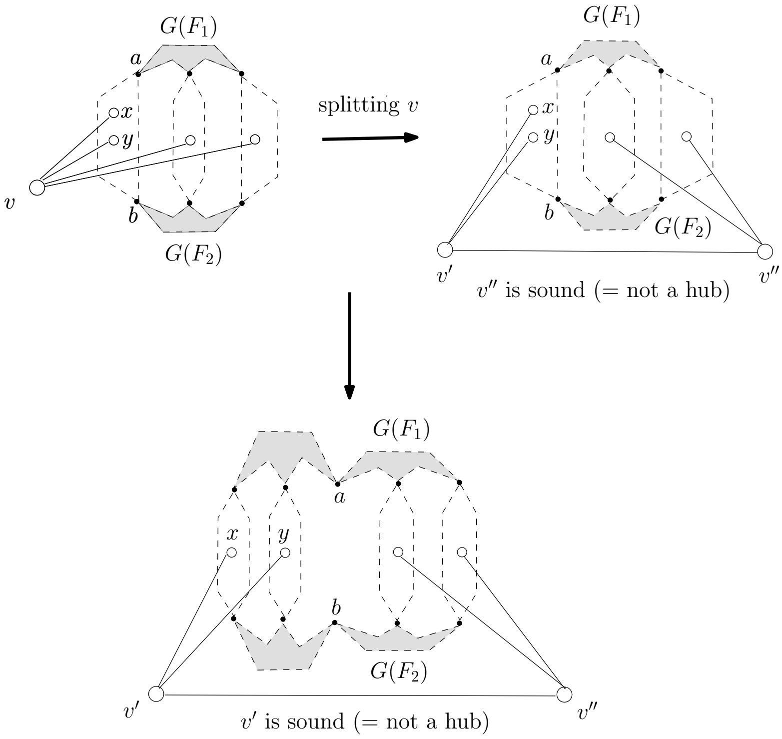


Figure 4: Splitting a vertex.

(3.7.2) $\delta(X_i)$ *is a* 3*-matching,* $i = 1, \ldots, m$,
(3.7.3) $\delta(W)$ *is an* m*-matching, and* $G(W)$ *is* 2*-connected, and*
(3.7.4) $G/\mathcal{Y} \cong K_{3,m}$.

Proof. Given a vertex v of degree $m > 3$, a partition $\mathcal{Y}$ of the required form arises from the v-frame $\mathcal{X}$ by assigning $W := V(F)$ and $X_i := \text{int}H_i$. The second assertion in (3.7.3) is just remark (B.6) to Definition B. It remains to check the rest of (3.7.2-4).

To show (3.7.2), let H be an $\mathcal{X}$-hammock, X be its interior, and s be the common vertex of H and W. Then (3.7.2) asserts that s is linked to X by exactly one edge, and X is not a singleton. If $d_H(s) > 1$ then $d(s) = n > 3$ (because $G(F)$ is 2-connected). As we have just seen, there exists a vertex t, $d(t) = n$, a set W' of vertices, and subgraphs $J_1, \ldots, J_n$ with $\text{bnd}J_i = \{r_i, a_i, t, \}$ where each a_i is linked to s and $r_1, \ldots, r_n$ are all distinct, by requirement (B.4) of Definition B. Assume that $a_1, a_2 \in X$ and $a_{n-1}, a_n \in W$. Let C be an $\{a_1, a_2\}$-circuit in $G - s$. Since each $a_i \in \text{int}_{G-s}J_i$, such a cicuit traverses t.

Suppose that $C \setminus H$ is nonempty. Then $C - X$ is a path of length at least 2. Since the subgraph $G - X$ is 2-connected, it contains an ear P of $C \setminus H$ traversing a_n. Then $C \cup P$ contains an $\{a_1, a_2, a_n\}$-curcuit lying in $G - s$, so that G has an S_3-subgraph, contradiction.

It remains to assume $C \subseteq H$; since $d_H(u) = d_H(v) = 1$, we actually have $C \subseteq X$, whence $t \in X$. Consider an $\{a_{n-1}, a_n\}$-circuit C' in $G - s$; since C' traverses t, the intesection $C' \cap H$ is a path of length at least 2. Let Q be an ear of C' in $G - s$ traversing a_1. Then $Q \subseteq X$, and $C' \cup Q$ contains an $\{a_1, a_{n-1}, a_n\}$-circuit lying in $G - s$, contradiction.

To finish with (3.7.3), $\delta(W)$ being a matching is straightforward from remark (B.5).

To show that $|X_i| > 1$, suppose, without loss of generality, that $X_4 = \{x\}$, with $N(x) = \{u, v, w\}$, $w \in W$. Remove the sets X_i with $i > 3$; since $G(W)$ is 2-connected, one easily finds a $\{u, v, w\}$-circuit in the remaining subgraph, so that G has an S_3-subgraph, contradiction. ∎

An alike property of vertices of degree 3 depicts the nodes adjacent to leaves in the mincut tree of a nonsplittable graph.

Claim 3.8 *Let G be nonsplittable, u be a vertex of G of degree 3, $\mathcal{X} = (X, F)$ be the u-frame, and H_i, $i = 1, 2, 3$, be the $\mathcal{X}$-hammocks. Then the partition of V into $\{u\}$, the components of $G(\mathcal{X})$, and* $\mathrm{int}H_i$, $i = 1, 2, 3$, *is a member of* $\mathbf{C}$ *represented in MCT by the node adjacent to the leaf* $\{V \setminus \{u\}\}$.

Proof. It suffices to show that an end of each H_i is linked to the interior by exactly one edge. The techniques of proof of Claim 3.7 works here too. Suppose, to the contrary, that an end s of an $\mathcal{X}$-hammock H has $d_H(s) > 1$. Since the components of $G(\mathcal{X})$ are 2-connected, this implies $d(s) = n \geq 4$. Since G is nonsplittable, there exists a vertex t, $d(t) = n$, and a partition of G into $\{s\}$, $\{t\}$, $Y_1, \ldots, Y_n$, and W satisfying (3.7.2-4). We may assume that Y_1, Y_2 meet $H - s$, and Y_{n-1}, Y_n meet $G - H$. Denote by a_i the vertex of Y_i adjacent to s. The subgraph $G - s$ has an $\{a_1, a_2\}$-circuit, say C, and an $\{a_{n-1}, a_n\}$-circuit C', each forcibly traversing t. Repeating the argument of proof of Claim 3.7 one concludes that $G - s$ has a circuit containing three neighbours of s. ∎

3.4 Proof of Theorem 1.3

Recall the notation: $\mathcal{Z} = \{X \subset V\colon\ d(X) = 3\}$, $\mathbf{C}$ denotes the set of $\preceq$-maximal subpartitions $\mathcal{X} \subset \mathcal{Z}$ satisfying $X \cup Y \neq V$ for each $X, Y \in \mathcal{X}$, and $\mathcal{T}$ denotes the mincut tree of G.

Note that the subgraphs $G(X)$, $X \in \mathcal{Z}$, are all 2-connected.

"Only if": *a nonsplittable sound graph* (i. e., a member of $\mathcal{H}^0$) *is cuboid, and each node of its mincut tree adjacent to a leaf has degree* 6.

Let G belong to $\mathcal{H}^0$. By Claim 3.7, G has 3-edge cuts. According to Definition A, it suffices to check that G is cuboid, and then apply Claim 3.8.

Proof of (A.2.1). We are to show that the inclusion-minimal members of $\mathcal{Z}$ are singletons. Let $Y \in \mathcal{Z}$ be inclusion-minimal. If $|Y| > 1$ then each vertex $v \in Y$ has degree greater than 3. Since v is nonsplittable, adjacent to v are pairwise disjoint sets $X_1, \ldots, X_m \in \mathcal{Z}$, $m = d(v) > 3$, having $|X_i| > 1$. By the minimality of Y, we have $X_i \setminus Y \neq \emptyset$ for each i. Thus, $Y \setminus X_i$ is nonempty (contains v) and $Y \cup X_i \neq V$ for each i. Since $\mathcal{Z}$ is cross-free, we have $Y \cap X_i = \emptyset$, so that $\delta(v) \subseteq \delta(Y)$, impossible.

Proof of (A.2.2). We are to show that for each $\preceq$-maximal subpartition $\mathcal{Y} \subset \mathcal{Z}$, the graph $G/\mathcal{Y}$ is cuboid. Since $G/\mathcal{Y}$ is essentially 4-edge connected, we only need to check (A.1). If $\cup\mathcal{Y} = V$ then $G/\mathcal{Y}$ is cubic, and (A.2.2) trivially holds. Otherwise choose $v \in V \setminus \cup\mathcal{Y}$ and put $m\colon= d(v)$. Since $\mathcal{Y}$ is $\preceq$-maximal, we have $m > 3$. By Claim 3.7, there exists a partition of V into $\{v\}$, $\{u\}$, and sets $X_1, \ldots, X_m \subset \mathcal{Z}$ and W, such that $G(W)$ is 2-connected, $\delta(W)$ is an m-matching, and each X_i is adjacent to u, v and W. We show that $X_i \in \mathcal{Y}$, $i = 1, \ldots, m$.

By (2) and the subsequent explanation, if X_1 is not a member of $\mathcal{Y}$ then it either is a proper subset of some member of $\mathcal{Y}$, say Y_1, or meets each member of $\mathcal{Y}$. In the latter case $\cup\mathcal{Y} \subseteq X_1$, because $X_1 \cup (\cup\mathcal{Y}) \subseteq G - v$, so that if some $Y \in \mathcal{Y}$ has $Y \cap X_1$ and $Y \setminus X_1$ nonempty then $X_1 \subseteq Y$, contradiction. But then $\mathcal{Y} \cup \{V \setminus X_1\}$ is a $\mathcal{Z}$-subpartition $\preceq$-majorating $\mathcal{Y}$, contradiction.

Thus, $X_i \subset Y_i$, $i = 1, \ldots, m$, where Y_i are distinct members of $\mathcal{Y}$ (because $X_i \cup X_j \subset Y \in \mathcal{Y}$ would imply that two edges from $\delta(Y)$ are incident with v, so that $\delta(Y)$ is not a matching, and therefore G is not 3-connected). This, however, leaves no place for the vertex u. Indeed, $u \notin V \setminus (\cup\mathcal{Y})$ because otherwise $|\delta(X_i) \cap \delta(Y_i)| \geq 2$ whence G is not 3-connected. On the other hand, $u \notin \cup\mathcal{Y}$, for of, say $u \in Y_1$ then $|\delta(u) \cap \delta(Y_1)| = m - 1$ whence $|\delta(Y_1)| = m > 3$, contradiction.

"If": *a cuboid graph whose MCT's nodes adjacent to leaves have degree* 6 *is sound and nonsplittable.*

Note first that an essentially 4-edge connected cuboid graph H distinct from $K_{3,3}$ has more than six vertices of degree 3. This is clearly so if H is cubic. Otherwise, H has an induced $K_{2,r}$-subgraph J, $r > 3$, such that $M\colon= \delta(V(J))$ is an r-matching linking the r-part of J to a 2-connected subgraph $H - J$. The assertion trivially holds if the outside ends of M have all degree 3. If not, let x be an end of M in $H - J$ with $d(x) = s > 3$. By (A.1), the neighbours of x have degree 3, and exacly $s - 1$ of them are outside J. Thus, H has at least $r + s - 1 > 6$ vertices of degree 6, as asserted.

Returning to the proof, let G be a cuboid graph as announced, v be a vertex of G of degree m, and $\mathcal{X}$ be the member of $\mathbf{C}$ such that v is a vertex of $G' := G/\mathcal{X}$. We are to show that v is sound and either has degree 3 or satisfies the condition of Claim 3.6.

Suppose first that $m = 3$. Then $\{v\} \in \mathcal{X}$, so that $\mathcal{X}$ is adjacent to the leaf $\{V \setminus \{v\}\}$. Then G' is an essentially 4-edge connected cuboid graph with exactly six vertices of degree 3, that is $K_{3,3}$. In the initial G, the members of $\mathcal{X}$ nonadjacent to v form a v-frame, so that v is sound.

Let now $m > 3$. By (A.2.2), $G/\mathcal{X}$ has a $K_{2,m}$-subgraph J as above, with v in its 2-part. Let u be the other such vertex of J, X_i, $i = 1, \dots, m$, be the members of $\mathcal{X}$ adjacent to u and v, and F denote the edge-set of $G - \{u, v\} - \cup_{1 \le i \le m} X_i$. Then $(\{u\}, F)$ is a perfect v-frame, so that v is just as required.

Thus, G is sound, and Claims 3.7 and 3.8 imply the “if” part. Theorem 1.3 is proved.

3.5 Proof of Theorem 1.4

By Claim 3.5, the subgraph G^0 of a member of $\mathcal{H}$ is copied from the unsplittable ancestor of G. So, assume, for simplicity, that G itself belongs to $\mathcal{H}^0$, let $e = (u, v)$ be a short edge of G, and denote by J the component of G^0 containing e. Recall that $d(u) = d(v) = 3$. Let $N(u) = \{q, r, v\}$ and $\mathcal{X}$ be the u-frame, with the hammocks H_q, H_r, H_v; the latter is a two-edge path with v between, by Claim 3.4. By Claim 3.8, $\mathrm{int} H_q$ is a 2-connected subgraph linked by a single edge to each end of H_q, and similarly for H_r. Put $Z_q := \mathrm{int} H_q$ and $Z_r := \mathrm{int} H_r$; these subgraphs are easily seen to form the v-frame, say $\mathcal{Y}$. Simmetrically, the components of $G(\mathcal{X})$ are similar 2-connected subgraphs of the $\mathcal{Y}$-hammocks. Denote these components by Z_s and Z_t where s, t are the neighbours of v other than u.

Put $\mathcal{A} := \{Z_q, Z_r, Z_s, Z_t\}$. Then $G/\mathcal{A} \cong K_{3,3}$, and the shape of J depends on which members of $\mathcal{A}$ are singletons. Namely, an edge adjacent to e, say (q, u), is short if and only if $Z_q = \{q\}$. Indeed, (q, u) belongs to a nontrivial 3-disconnector $\delta(Z_q)$ if Z_q is not a singleton, and is short otherwise by Claim 3.4.

Theorem 1.4 is proved.

4 Case $r \ge 4$: proof of Theorem 1.1

Here $G = (V, E)$ is r-connected, $r \ge 4$. As in Section 3, a vertex will be called ill if it is the hub of an S_r-subgraph, and sound otherwise. A set is considered as *ill* if it contains an ill vertex, and as *sound* otherwise.

4.1 Proof of Theorem 1.1

We use the following properties implied by Corollary 2.3. A sound vertex u belongs to a uniquely defined set $N^*(u) \subseteq V \setminus N(u)$ of size r satisfying $c(G - N^*(u)) \ge |N^*(u)|$, namely, the union of $\{u\}$ and the $N(u)$-separator in $G - u$ whose uniqueness is asserted by Corollary 2.3.

In the forthgoing Claims 4.1 and 4.2, G is an arbitrary r-connected graph, u is a sound vertex of G, and $B := N^*(u)$.

Claim 4.1 *A vertex $v \in B$ is ill if and only if $d(v) > d(u)$.*

Proof. Since each component J of $G - B$ has $N(J) = B$, we have $d(v) \ge d(u)$ for $v \in B$. So, $d(v) = d(u)$ implies that $B \setminus \{v\}$ is an $N(v)$-separator in $G - v$, and the “only if” assertion follows.

To show “if”, consider each of the two reasons for $d(v) > d(u)$, namely, $|N(v) \cap J| > 1$ for some component J of $G - B$, and $N(v) \cap B \neq \emptyset$)

Let, first, v have two (or more) neighbours in some component of $G - B$. If v is sound then there exists $B' := N^*(v)$, of size r. Choose $A \subset N(v)$, $|A| = r$, satisfying $|A \cap J| \leq 1$ for each component $G - B$. Since $B \setminus \{v\}$ is an A-separator, G contains a subdivision H of $K_{r,r}$ with the bipartition A, B, by Theorem 2.1. Since $|A \cap K| \leq 1$ also for the components K of $G - B'$, we have $B' \subset V(H)$, and each component of $H - B'$ contains at most one member of A. Then $B' = B$, contradicting Corollary 2.3.

Let, second, v be adjacent to another member of B. Form a set $A \subset N(v)$ of size $r - 1$ by choosing some $r - 1$ components of $G - B$, and one member of $N(v)$ in each. By Dirac's theorem, $G - v$ has an A-circuit, say C. Since C contains $B \setminus \{v\}$, we have $|C \cap N(v)| \geq r$, so that v is ill. ∎

Claim 4.2 *Suppose that B is sound, and let J be a component of $G - B$. Then either the (B, J)-edges form an r-matching, or J is an sigleton. In the latter case, either $G \cong K_{r,r}$ or J is ill.*

Proof. Let v be a vertex of J adjacent to more than one member of B. If $J' := J \setminus \{v\}$ is nonempty then $N(J') = (N(J) \setminus N(v)) \cup \{v\}$, because $N(N(v)) \cap J = \{v\}$ by Claim 4.1. Hence $|N(J) \setminus \{u\}| < r$, contradiction. Thus, $J' = \emptyset$ and $J = \{v\}$.

Suppose now that no circuit of $G - v$ contains $N(v) = B$, and let Z be an B-separator in $G-v$. Clearly, Z meets each component K of $G-v-B$, so that the number of these components is $|Z| = r - 1$ and each K has $|K \cap Z| = 1$. For an arbitrary K, let z be the member of $K \cap Z$; if $K \neq \{z\}$ then $K \cap N(B)$ has cardinality r, as we have just seen, and at least $r - 1$ of its members belong to distinct components of $J - z$, contradicting G being 3-connected. Thus, K is a singleton, so that $G \cong K_{r,r}$, as required. ∎

Theorem 1.1 is a consequence of a certain finiteness property of the graph. We present two proofs, exploiting finiteness of the vertex-set (the first proof) and of paths with prescribed ends (the second proof). The second proof remains valid for graphs with countably infinite vertex-set.

First proof. Suppose, to the contrary, that G is sound, and for each vertex v define

$$\alpha(v) := \min\{|J| \colon J \text{ is a component of } G - N^*(v)\}.$$

Put $\alpha := \min_{v \in V} \alpha(v)$; we have $\alpha > 1$, by Claim 4.2. Choose v with $\alpha(v) = \alpha$, let J be a component of $G - N^*(v)$ of size α, and choose $u \in J$. In what follows K stands for a component of $G - N^*(u)$.

By Claim 4.2, $|N(u) \cap J| \geq d(v) - 1 \geq r - 1$. We prove the theorem by showing that J properly contains some K, contradicting the choice of J. Indeed, otherwise each K meets $N^*(v)$. Since $c(G - N^*(u)) \geq r = |N^*(v)|$, the members of $N^*(v)$ belong to distinct K's. Since each member of $N^*(v)$ is adjacent to each component J' of $G - N^*(v)$, each J' should contain a member of $N^*(u)$. Thus, $N^*(u) \cap N^*(v) = \emptyset$ and $J \cap N^*(u) = \{u\}$. Now, there clearly exists K intersecting with J. We have then $N(K \cap J) = \{u, t\}$ where t is the member of $K \cap N^*(v)$. Since $r > 2$, this is impossible, and the theorem follows. ∎

Second proof. We actually establish a more precise fact. Sets of the form $N^*(u)$ (for some sound vertex u) will be referred to as *blocks*.

Theorem 4.3 *Let G be r-connected, $r \geq 4$. For any sound vertex v, every $N^*(v)$-path of G either contains an ill vertex, or, otherwise, meets an ill block.*

Proof. Let (u, v) be an edge of G such that v and $N^*(u)$ are sound (if no such edge exist, the assertion is trivially true). We put $B := N^*(u)$, and denote by J components of $G - N^*(v)$, and by K those of $G - B$. In particular, J_u and K_v are the components containing u and v respectively. The key observation is that

(9) J_u and K_v form a partition of V, and $E(J_u, K_v)$ is an $(N^*(u), N^*(v))$-matching

(so that B and the other K's are inside J_u, and $N^*(v)$ and the other J's are inside K_v). We show (9) in three steps, as follows.

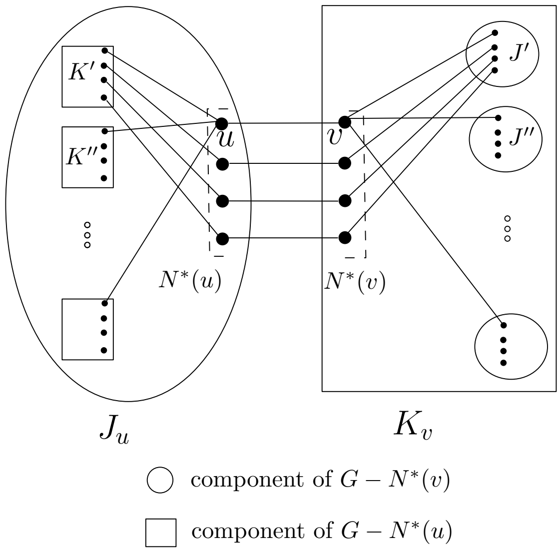


Figure 5: Assertion (9).

I. $K \neq K_v \Rightarrow K \subseteq J_u$. To show this, note first that $N(v) \setminus \{u\} \subseteq K_v$; indeed, since v is sound, we have $|N(v) \cap B| = 1$ by Claim 4.2.

Suppose to the contrary that $K \setminus J_u \neq \emptyset$ for some $K \neq K_v$. Since $N(J_u) = N^*(v)$, there exists $t \in K \cap N^*(v)$. As members of $N^*(v)$, t and v are adjacent to each J; since they lie in distinct K's, the interior of every (t, v)-path meets B; therefore each J should meet B. Moreover, since the number of J's is at least $|N^*(v)| = r$, we have $|J \cap B| = 1$ for each J; that is, the members of B belong to distinct J's. Since $K_v \subset V \setminus B$, we have $N(J \cap K_v) = \{v, s\}$ where s is the only common member of J and B, contradicting the connectivity assumption.

II. $B \cap N^*(v) = \emptyset$. Indeed, if, to the contrary, there is $t \in B \cap N^*(v)$ then t is adjacent to each J, as a member of $N^*(v)$. Since, however, $V \setminus (J_u \cup K_v) \subset B$ (by I), each $J \neq J_u$ is a subset of $K_v \cup B$, so that $|N(t) \cap (K_v \cup B)| > 1$, contradicting Claim 4.1.

III. It follows from I that $V \setminus (J_u \cup K_v) \subset B$; since $N^*(v)$ meets neither B (by II) nor J_u, we have $N^*(v) \subset K_v$. Therefore each $t \in B \setminus J_u$, if any, belongs to some $J \neq J_u$ (by II); on the other hand, t is adjacent to each $K \subset J_u$, so that J and J_u are adjacent, contradiction. Thus, $B \subset J_u$. Finally, the relation $K_v \cap J_u = \emptyset$ and the second assertion of (9) are now obvious.

Based on Claims 4.1 and 4.2 and observation (9), one may imagine a branching process starting with a sound block, say B_0, and stopping in a component J of the current $G - B$ whenever the subset $N(B) \cap J$ is ill. The idea of this proof is that every B_0-path, due to being finite, should inevitably enter a component in which the process stops.

At this point, Theorem 1.1 may already be extended to r-connected countably infinite graphs, because if such G is sound then the above process never stops, nesting within the components, so that G is a union of r disjoint trees, contradicting the connectedness assumption. We, however, continue proving Theorem 4.3.

Let B be a sound block, and P be a path in G with the ends in B. It suffices that an ill set be found by an inclusion-minimal segment of P with the ends in B; let Q be such a segment, with the ends t and u. Then $P' := Q - \{t, u\}$ lies in some component J of $G - B$. Let $t', u' \in B' := N(B) \cap J$ be the ends of P'. If B' is ill, we are done; otherwise B' is a sound block, by (9), and the assertion follows by induction in the length of P. ∎

4.2 Application to an extremal problem

Return to the extremal problem mentioned at the beginning of Introduction. The maximal number $\mathrm{ex}(n,r)$ of edges in an n-vertex simple graph with no S_r-subgraph was known to grow linearly in n [1]. A refinement of this fact follows from Theorem 1.1.

Claim 4.4 *Let $r \geq 4$ be an integer. A graph $G = (V, E)$ satisfying $|V| \geq 2r-1$ and $|E| > (2r-3)(|V|-r+1)$ has an S_r-subgraph.*

Proof, by induction in $|V|$. For $|V| = 2r-1$ we have $|E| \geq (2r-3)r+1 = \binom{2r-1}{2}$, so that G is a complete graph with more than r vertices, clearly containing S_r.

Suppose now that $|V| > 2r-1$. If G has a vertex v of degree $\leq 2r-3$ then $|E(G-v)| = |E|-d(v) > (2r-3)((|V|-1)-r+1)$, so that $G-v$, and thereby G, has an S_r-subgraph by the induction hypothesis. So, assume that the minimal vertex degree satisfies $d^{\min} \geq 2r-2$. Then $G \not\cong K_{r,r}$. So, if, to the contrary, G has no S_r-subgraph, it has a disconnector $X \subset V$ of size $\leq r-1$, by Theorem 1.1, so that $G = G_1 \cup G_2$, with $G_1 \cap G_2 = G(X)$. Let $G_i = (V_i, E_i)$. We have $|V_i| \geq d^{\min}+1 \geq 2r-1$; since neither of G_i has an S_r-subgraph, the induction hypothesis yields

$$\begin{aligned}
|E| &= |E_1| + |E_2| - |E(X)| \\
&\leq (2r-3)(|V_1| + |V_2| - 2r+2) = (2r-3)(|V| + |X| - 2r+2) \\
&\leq (2r-3)(|V|-r+1),
\end{aligned}$$

contradiction. ∎

Thus, $\mathrm{ex}(n,r) \leq (2r-3)(n-r+1)$, for $r \geq 3$ and $n \geq 2r-1$.

On the other hand, the graph $\Gamma_{n,r} := K_{r-1,n-r+1} + M$ where M is a maximal matching on the $(r-1)$-part of $K_{r-1,n-r+1}$, has no S_r-subgraph, whence

$$\mathrm{ex}(n,r) \geq (r-1)(n-r+1) + \left\lfloor \frac{r-1}{2} \right\rfloor . \tag{10}$$

Equality in (10) is known to hold for $r = 3$ (C. Thomassen [8]) and $r = 4$ (E. Horev [5]); in the latter case, the only extremal graph is $\Gamma_{n,4}$.

We conjecture that the equality in (10) holds for all $r \geq 4$ and $n \geq 2r-1$, with the only extremal graph $\Gamma_{n,r}$.

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