Extremal graphs without a semi-topological wheel

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Abstract

For $2 \leq r \in \mathbb{N}$, let S_r denote the class of graphs consisting of subdivisions of the wheel graph with r spokes in which the spoke edges are left undivided. Let $ex(n, S_r)$ denote the maximum number of edges of a graph containing no S_r-subgraph, and let $Ex(n, S_r)$ denote the set of all n-vertex graphs containing no S_r-subgraph that are of size $ex(n, S_r)$. In this paper, a conjecture is put forth stating that for $r \geq 3$ and $n \geq 2r + 1$, $ex(n, S_r) = (r - 1)n - \lceil (r - 1)(r - 3/2) \rceil$ and for $r \geq 4$, $Ex(n, S_r)$ consists of a single graph which is the graph obtained from $K_{r-1,n-r+1}$ by adding a maximum matching to the color class of cardinality $r - 1$. A previous result of C. Thomassen (A minimal condition implying a special K_4-subdivision, Archiv. Math., (25) 1974, 210 − 215) implies that this conjecture is true for $r = 3$. In this paper it is shown to hold for $r = 4$.

Keywords: Subdivisions of the wheel graph, Semi-topological minors.

1. Introduction

Throughout, a graph is always finite and simple (i.e., no loops or multiple edges). Unless otherwise stated, notation and terminology are that of [4].

Following [1], let S_r, $ex(n, S_r)$, and $Ex(n, S_r)$ be as defined in the abstract. For integers $a, b \geq 2$ let $K^+_{a,b}$ denote the graph obtained from $K_{a,b}$ by adding a maximum matching to the color class of cardinality a. In an upcoming paper [5], M. Lomonosov and the author conjecture the following.

Conjecture 1.1. *For $r \geq 3$ and $n \geq 2r+1$, $ex(n, S_r) = (r-1)n - \lceil (r-1)(r-3/2) \rceil$ and for $r \geq 4$, $Ex(n, S_r) = \{K^+_{r-1,n-r+1}\}$.*

The following is the main result of this paper.

Theorem 1.2. *For $n \geq 9$, $ex(n, S_4) = 3n - 8$ and $Ex(n, S_4) = \{K^+_{3,n-3}\}$.*

B. Bollobás and P. Erdős [3] determined $ex(n, S_2)$ and $Ex(n, S_2)$, and conjectured that $ex(n, S_3) = 2n - 3$. This was verified by C. Thomassen [9] who determined $ex(n, S_3)$ and

$Ex(n, S_3)$. The results of [3, 9] are presented in [1, pages 389 − 395]. C. Thomassen and B. Toft [10] obtained an alternative proof for determining $ex(n, S_3)$. The results [9, 10] are of special interest as S_3 is a specific subdivision of K_4. The result of C. Thomassen [9] and Theorem 1.2 show Conjecture 1.1 to be true for $r = 3, 4$, respectively.

B. Bollobás [1, pg. 389] proved that there exist functions $\alpha(r)$ and $\beta(r)$ such that $ex(n, S_r) \leq \alpha(r)n - \beta(r)$. The functions $\alpha(r)$ and $\beta(r)$ are rather crude, e.g. both are at least exponential in r; but they do provide an initial estimation of $ex(n, S_r)$. Indeed, B. Bollobás' aim was to show a linear bound on $ex(n, S_r)$. Consequently, Problem 15 page 398 of [1] prompts the reader to obtain accurate estimations of $\alpha(r)$ and $\beta(r)$. As far as the author could determine, the initial estimation of Bollobás was never improved.

The approach used in this paper (see Lemma 3.4 and Corollary 3.6) can be employed to obtain that for $r \geq 4$ and $n \geq r + 1$, $ex(n, S_r) < \binom{r}{2}n$. This is not presented in this paper since in [5], M. Lomonosov and the author have established that for $r \geq 4$ and $n \geq 2r - 1$, $ex(n, S_r) \leq 2(r - 1)n$. This upper bound and the graph $K^+_{r-1,n-r+1}$ imply that $ex(n, S_r) = \Theta(rn)$.

In [1], an interest in $ex(n, S_r)$ and $Ex(n, S_r)$ is expressed in the context of a wider concept called *semi-topological minors* (introduced in [3]) which refers to subdivisions in which certain edges are left undivided. Indeed, various forms of semi-topological K_4s have been of interest e.g. [6, 7, 10].

In what follows, the approach used in this paper to prove Theorem 1.2 is outlined. Let $\kappa(G)$ denote the vertex connectivity of G. A graph $G \in Ex(n, S_r)$ has $\kappa(G) \in \{2, \ldots, r\}$. Indeed, it is not hard to verify that graphs in $Ex(n, S_r)$ are 2-connected. In addition, a well-known theorem of G. A. Dirac [4], asserts that for $k \geq 1$, if G is k-connected and $A \subseteq V(G)$, $|A| \leq k$, then G contains a circuit passing through each element of A. By this theorem it follows that every $(r + 1)$-connected, $r \geq 1$, graph has an S_r-subgraph.

In order to prove Theorem 1.2, a slightly broader theorem is proved, that is Theorem 1.3 (see below). In this theorem, for each possible value of κ all possible graphs in $Ex(n, S_4)$ are listed. To obtain this list we show that an n-vertex graph of size specified in Theorem 1.3 without an S_4-subgraph is one of the graphs listed in Theorem 1.3(ii). Such a claim is proved by induction on n where the induction hypothesis assumes Theorem 1.3(i) and (ii). The induction step is to consider graphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$, where S is a minimum disconnector[1] of G.

This type of argument requires a few technical claims regarding 3-connected graphs with certain properties (see Section 4). These claims are needed in order to handle cases in which the order of G_i, for some $i = 1, 2$, is "small" and the inductive argument is insufficient. This is also the reason for considering graphs of order less than nine in Theorem 1.3. In addition, this approach requires a lemma to aid in counting edges in 4-connected graphs of order at least nine containing no S_4-subgraph. For $r \geq 4$, this is provided for r-connected graphs containing no S_r-subgraph in Lemma 3.4 (see Section 3). In Section 6, a proof of Theorem 1.3 is provided.

[1] See Section 2 for a definition.

The above inductive framework is that of C. Thomassen, as this is presented in [1, Theorem 3.9 pg. 393], who used it in order to determine $ex(n, S_3)$ and $Ex(n, S_3)$. The distinction between the proof of C. Thomassen and arguments made in this paper, is the treatment of graphs that are at least 3-connected. Indeed, this constitutes most of the work in this paper.

To make this distinction clear, we first overview the approach of C. Thomassen to handle 3-connected graphs having no S_3-subgraph. To that end we require the following definition. An $\{H, K\}$-cockade[2] is defined recursively as follows.
(i) H and K are $\{H, K\}$-cockades.
(ii) If G_1 and G_2 are disjoint $\{H, K\}$-cockades and $e_i = x_i y_i \in E(G_i)$, $i = 1, 2$, then the graph obtained by identifying x_1 with x_2 and y_1 with y_2 (removing multiple edges) is an $\{H, K\}$-cockade.

An equivalent definition of a $\{H, K\}$-cockade is that a graph G with $\kappa(G) = 2$ whose simplicial summands are members of $\{H, K\}$ is an $\{H, K\}$-cockade.

By the result of C. Thomassen, the extremal graphs having no S_3-subgraph are the $(K_3, K_{3,3})$-cockades, and $ex(n, S_3) = 2n - 3$. Consequently, the sole 3-connected graph in $Ex(n, S_3)$ is $K_{3,3}$. To show the latter, C. Thomassen proves, in an elegant manner, that every 3-connected graph of order at least 7 and size $2n - 3$ contains an S_3-subgraph. This argument is specifically suited for 3-connected graphs containing no S_3-subgraph. Also, the assumption that size is $2n-3$ is crucial to the argument. We were unable to generalize or extend this argument to handle extremal graphs containing no S_4-subgraph that are at least 3-connected. Consequently, we take a different approach. This is then supported by Theorem 1.3 that indicates that there is a significant difference between $Ex(n, S_3)$ and $Ex(n, S_4)$ (especially in the case that $\kappa = 3$). This also means that $Ex(n, S_3)$ is (probably) no indication as to what one might expect to see in $Ex(n, S_r)$, $r \geq 5$.

A main contribution of this paper is seen in Corollary 3.6 and its application in our proof of Theorem 1.3. Indeed, as stated above, Corollary 3.6 applies to r-connected graphs containing no S_r-subgraph with $r \geq 4$. It describes a structural property that (together with P. Turán's Theorem [1]) is rather useful for the purpose of counting edges in such graphs; this can be seen through the manner in which this corollary is applied. Indeed, one may use a simple induction and an argument almost identical to the argument that applies Corollary 3.6 here, to show that $ex(n, S_r) \leq \binom{r}{2} n$. This corollary also provides some indication that $Ex(n, S_4)$ portrays more accurately than $Ex(n, S_3)$ what one might expect to see in $Ex(n, S_r)$, $r \geq 5$.

[2]The definition of cockades used here is suited for the needs of this paper. The traditional definition is more general but not required here.

We now state Theorem 1.3. Put

$$f(n) = \begin{cases} \left\lfloor \frac{5}{2}n - 4 \right\rfloor, & 3 \le n \le 6, \\ 3n - 8, & n \ge 7 \end{cases}$$

Theorem 1.3.

(*i*) *For* $n \ge 3$, $ex(n, S_4) = f(n)$.

(*ii*) *Extremal graphs are as follows.*

- (*a*) *For* $3 \le n \le 6$, K_4*-cockades if* n *is even, and* (K_3, K_4)*-cockades with a single* K_3 *as a simplicial summand if* n *is odd.*
- (*b*) *For* $n = 7$, $K_{3,4}^+$ *and* (K_3, K_4)*-cockades with a single* K_3 *as a simplicial summand.*
- (*c*) *For* $n = 8$, $K_{4,4}$, $K_{3,5}^+$, *and* K_4*-cockades.*
- (*d*) *For* $n \ge 9$, $K_{3,n-3}^+$.

The requirement in Theorem 1.3$(ii.a - b)$ that a single simplicial summand is K_3 is a technical detail required by our proof of Claim 5.1.

2. Notation

Throughout, G is always a graph. The cardinality $|E(G)|$ is called the *size* of G and is denoted $\|G\|$. For disjoint vertex sets $X, Y \subset V(G)$, let $(X, Y) := \{xy \in E(G) : x \in X, y \in Y\}$ and let $\|X, Y\|$ denote $|(X, Y)|$. A set $X \subset V(G)$ with $\|G[X]\| = 0$ is *stable* in G.

For a pair of vertices $u, v \in V(G)$, a simple path from u to v in G is called a *uv-path*. For disjoint vertex sets $X, Y \subset V(G)$, an (X, Y)-path is a uv-path P such that $u \in X$ and $v \in Y$ and $V(P) \cap X \cup Y = \{u, v\}$. A set of k $(\{x\}, Y)$-paths with only x as a common vertex is called an $(\{x\}, Y)$*-fan of cardinality* k. We abbreviate $(\{x\}, Y)$-fan to (x, Y)-fan. A uv-path P in G is *internally-disjoint* of subgraph $H \subset G$ if $V(P) \cap V(H) = \{u, v\}$; P is called an H*-path*.

A circuit passing through each vertex of a set $A \subset V(G)$ is called an A*-circuit*.

If G is connected and $S \subset V(G)$ is a set such that $G - S$ is disconnected, then S is called a *disconnector*. If $|S| = k$, then S is called a k-disconnector.

If graph F is a subdivided graph H, then the vertices of degree at least three of F are called its *branch* vertices. For $r \ge 3$, the vertex of an S_r-subgraph corresponding to the center of the wheel graph is called the *center* of the S_r-subgraph.

Finally, a k-set of G is a vertex set of G of cardinality k.

3. r-connected graphs containing no S_r-subgraph

Throughout this section $r \geq 4$. The goal of this section is to prove Lemma 3.4 stated below and consequently obtain Corollaries 3.5 and 3.6. These corollaries are then used to prove Theorem 1.3. For that end we require some lemmas and claims that are presented next.

Let F be a subdivision of a bipartite graph L, and let U and V denote the branch vertices of F that denote the color classes of L. A *topological edge* of F is a uv-path P replacing an edge of L such that $u \in U$, $v \in V$, and $V(P) \cap (U \cup V) = \{u, v\}$. For a vertex $x \in U \cup V$, let $C_F(x)$ denote the set of all vertices that belong to a topological edge of F containing x.

Let M_r denote a bipartite graph with color classes Z and R such that $Z = \{z_1, \ldots, z_{r-1}\}$, and $R = \{u_1, \ldots, u_r\}$ where M_r has the following adjacencies. The vertex u_r is adjacent to all vertices of Z; the vertex u_{r-1} is adjacent to z_1 and z_{r-1}; and a vertex u_i, $1 \leq i \leq r-2$ is adjacent to z_i and z_{i+1}. For $0 \leq i \leq r-1$, define M_r^i to be as M_r except that vertices $\{u_1, \ldots, u_i\}$ are connected by an edge to every vertex of Z. Note that $M_r \cong M_r^0$ and that $M_r^{r-1} \cong K_{r-1,r}$. The following two claims are easy to verify.

Claim 3.1. *Let F be a topological M_r in graph G with sets of branch vertices Z and U as defined above. Let $u, v \in U$, $u \neq v$, and let $x \in C_F(u) \setminus Z$. If in G there exists a vx-path P internally-disjoint of F, then $F \cup P$ contains a U-circuit.*

In the following claim, it is noted that Claim 3.2(a) follows directly from Claim 3.1. Still, it is convenient to state it separately.

Claim 3.2. *Let F be a topological $K_{r-1,r}$ with sets of branch vertices Z and U representing the color classes of $K_{r-1,r}$ such that $|Z| = r-1$ and $|U| = r$. In addition, let $u_i \in U, i = 1, 2$, let $x_i \in C_F(u_i) \setminus Z$, $i = 1, 2$, let $\ell \in C_F(u_1), \ell \neq x$ (possibly $\ell \in Z$). If in G there exist an x_1x_2-path P and an $x_1\ell$-path Q both internally-disjoint of F, then the following holds.*

- (a) *$F \cup P$ contains a U-circuit.*
- (b) *If $t \in V(Q) \setminus V(F)$, then $F \cup Q$ contains an X-circuit, where X consists of t and some $r-1$ vertices from U.*

The following lemma is a well-known matroidal property which will be used repeatedly in Lemma 3.4.

Lemma 3.3. *[8]*
Let $x \in V(G)$, let $A \subseteq V(G) \setminus \{x\}$, $|A| \geq k \geq 1$, and let $\{a_1, \ldots, a_i\} \subseteq A$, $i \leq k$. Suppose G contains an (x, A)-fan F of cardinality k and that $Q_1, \ldots, Q_i$ are internally-disjoint xa_j-paths (not necessarily in F), $1 \leq j \leq i$. Then, G contains an (x, A)-fan of cardinality k containing paths $P_1, \ldots, P_k$ such that for $1 \leq j \leq i$, P_j is an xa_j-path (possibly $P_j \neq Q_j$).

Lemma 3.4. *Let G be a graph with $\kappa(G) \geq r$, and let $v \in V(G)$ with $|N(v)| = m \geq r$ The vertex v is not a center of an S_r-subgraph in G if and only if G contains an r-set W, $v \in W$, such that $G - W$ has m components each meeting a single member of $N(v)$.*

Proof: Indeed, if W as stated exists and $G - W$ satisfies the claim, then v is not a center in G. Assume then that v is not a center in G. In this case, we will show that $G - v$ contains a topological $K_{r-1,m}$, denoted F, such that the branch vertices corresponding to the color class of cardinality m are vertices of $N(v)$. We set W to consist of v and the branch vertices of F corresponding to the color class of $K_{r-1,m}$ of cardinality $r-1$. Then, using Claims 3.1 and 3.2 we establish that components of $G - W$ are as stated in the lemma.

Let $N(v) = \{u_0, \ldots, u_{m-1}\}$ and let $U = \{u_0, \ldots, u_{r-1}\}$. As v is not a center of an S_r-subgraph in G, it follows from Dirac's theorem (stated in the introduction) that $\kappa(G-v) = r-1$ and that $G - v$ admits a $\{u_0, \ldots, u_{r-2}\}$-circuit C. Let (u_i, u_{i+1}) denote the arc of C defined by u_i and u_{i+1} containing no vertices from $\{u_0, \ldots, u_{r-2}\}$ (indices are taken modulo $r-1$). Let $P_0, \ldots, P_{r-2}$ be a (u_{r-1}, C)-fan of cardinality $r-1$ such that $V(P_i) \cap V(C) = \{z_i\}$, for $0 \leq i \leq r-2$; let $Z = \{z_i : 0 \leq i \leq r-2\}$. Also, let (z_i, z_{i+1}) denote the arc of C defined by z_i and z_{i+1} containing no vertices from Z (indices are taken modulo $r-1$). As $G - v$ contains no U-circuit, it follows from Claims 3.1 and 3.2 that, for $0 \leq i \leq r-2$, $z_i \in (u_i, u_{i+1})$ (indices taken modulo $r-1$). The subgraph $F_0 = C \cup \bigcup_{i=0}^{r-2} V(P_i)$ is a topological $M_r \cong M_r^0$ with branch vertices Z and $\{u_0, \ldots, u_{r-1}\}$.

Suppose that for $0 \leq i < r-1$ a topological M_r^i, denoted F_i, has been defined satisfying the following assertions.

1. U and Z are the branch vertices of F_i.
2. For $0 \leq j \leq i$, each vertex $u_j \in U$ is connected to each vertex of Z in F_i by a topological edge. If $i < j \leq r-2$, then u_j is connected by a topological edge in F_i to z_{j-1} and z_j only.

We now show that G contains a topological M_r^{i+1}, denoted F_{i+1}, satisfying the above assertions. Let $Q_0, \ldots, Q_{r-2}$ be a (u_{i+1}, F_i)-fan of cardinality $r-1$ in $G - v$. Noting that F_i contains a topological M_r, Claim 3.1 and Lemma 3.3 assert that $V(Q_j) \cap V(F_i) = \{z_j\}$, for $j = 0, \ldots, r-2$. Put $F_{i+1} = F_i \cup \bigcup_{j=0}^{r-2} V(Q_j)$. Note that F_{i+1} is a topological M_r^{i+1} satisfying the above properties.

It follows that $G - v$ contains a topological $K_{r-1,r}$, denoted F', whose branch vertices are Z and U. By Claim 3.2(b) and the assumption that v is not a center of an S_r-subgraph in G, it follows that for $r+1 \leq i \leq m$, a (u_i, F)-fan in $G-v$ of cardinality $r-1$ containing paths $L_0, \ldots, L_{r-2}$, it holds that $V(L_i) \cap V(F) = \{z_i\}$, for $0 \leq i \leq r-2$. Consequently, $G - v$ contains a topological $K_{r-1,m}$ with Z and $N(v)$ as its branch vertices as claimed above. Recall that this subdivision is denoted F.

Put $W = Z \cup \{v\}$. For two vertices $u_i, u_j \in N(v)$, $i \neq j$, Claim 3.2 implies that $G - W$ contains no $(C_F(u_i) \setminus Z, C_F(u_j) \setminus Z)$-paths. In addition, for a vertex $x \in V(G-v) \setminus V(F)$, Claim 3.2 asserts that the paths of an (x, F)-fan (such a fan contains at least $r - 1 \geq 3$ paths) terminate either in Z or $C_F(u)$, where $u \in U$ for a single $u \in U$. Consequently, the claim follows. ∎

Two corollaries are derived from Lemma 3.4[3]. Indeed, this lemma implies that if G has $\kappa(G) \geq r$, and $v \in V(G)$ is not a center of an S_r-subgraph in G, then v is not contained in a triangle. Consequently, the following corollary is then obtained. We remark that Corollary 3.5 holds for $r = 3$ as well.

Corollary 3.5. *An r-connected graph containing no S_r-subgraph is triangle-free (i.e., G contains no triangles).*

Having $|W| = r = \kappa(G)$ in Lemma 3.4 implies that each component of $G - W$ meets at least one member of $N(w)$ for each $w \in W$. This in turn implies that every vertex $w \in W$ has $d(w) \geq m$, where m is as defined in Lemma 3.4. This property is utilized in the following corollary.

Corollary 3.6. *Let G be a graph with $\kappa(G) \geq r$, and let $v \in V(G)$ with $|N(v)| = \Delta(G) \geq r$. The vertex v is not a center of an S_r-subgraph in G if and only if G has a stable r-set W, $v \in W$, such that each vertex $w \in W$ has $d(w) = \Delta(G)$, and $G - W$ can be partitioned into $\Delta(G)$ components each meeting a single member of $N(w)$ for each $w \in W$.*

4. 3-connected graphs of small order and related structures

As mentioned in the introduction, our approach requires several technical claims regarding 3-connected graphs containing a 3-disconnector separating the graph into two subgraphs with at least one being of "small" order. In this section, these claims and a few others are presented.

Throughout this section, G has $\kappa(G) = 3$; the set $S = \{s_1, s_2, s_3\}$ is a 3-disconnector of G such that $G = G_1 \cup G_2$, where $G_1 \cap G_2 = G[S]$, and n_i denotes the order of G_i, $i = 1, 2$. Note that $n_1 + n_2 = n + 3$ and that $\|G\| = \|G_1\| + \|G_2\| - \|G[S]\|$.

The following claim asserts that if, in addition, G has no S_4-subgraph, then $G[S]$ has two possible configurations in G.

Claim 4.1. *If G contains no S_4-subgraph, then $\|G[S]\| \leq 1$.*

Proof: Assume towards contradiction that $\{s_1s_2, s_1s_3\} \subset E(G)$. As $|S| = \kappa(G)$, there exist vertices $x_i \in N_G(s_1) \cap (V(G_i) \setminus S)$, for $i = 1, 2$. As $\kappa(G) = 3$, there exist two vertex-disjoint $(\{x_i\}, S \setminus \{s_1\})$-paths, namely P_1^i and P_2^i, for $i = 1, 2$. These paths form a circuit in $G - s_1$ containing four neighbors of s_1, and thus defining an S_4-subgraph with s_1 at its center; a contradiction. ∎

Claims 4.2, 4.4, and 4.5, consider the configuration that S is stable in G. Each claim is suited for a different order of G.

Claim 4.2. *If $|G| = 7$, $\|G\| = f(7)$, and S is stable in G, then G has an S_4-subgraph.*

[3] Corollary 3.5 can be proved directly without Lemma 3.4.

Proof: Assume towards contradiction that G contains no S_4-subgraph. A graph with the properties of G for which $n_1 = n_2 = 5$ contains at most $f(7) + 1$ edges. Consequently, if $n_1 = n_2 = 5$, then G is one of the graphs depicted in Figure 1 each containing an S_4-subgraph.

Thus, it can be assumed that, without loss of generality, $n_1 = 4$ and $n_2 = 6$. In which case, as $\delta(G) \geq 3$, $\|G_1\| = 3$. Let $R = \{r_1, r_2, r_3\} = V(G_2) \setminus S$. As $\|S, R\| \leq 9$, the set R is not stable in G. If $\|G[R]\| = 1$, then $G_2 \cong K^+_{3,3}$, which implies that G has an S_4-subgraph with a center in R. Thus, it can be assumed that $\|G[R]\| \geq 2$. Consequently, G contains a path P such that $V(P) = R$. Let r_1 and r_2 be the endpoints of P. It can be assumed that, without loss of generality, $d_G(s_1) = 4$; for otherwise $\|G\| = \|G_1\| + \|S, R\| + \|G[R]\| \leq f(7) - 1$. Thus, s_1 is adjacent to all vertices of R. As G contains no S_4-subgraph with s_1 at its center, the graph $G'_2 = G_2 - s_1 + s_2s_3$ does not admit a circuit containing the edge s_2s_3 and R. It follows that G_2 contains a single edge from the set $\{r_1s_2, r_2s_3\}$, and a single edge from the set $\{r_1s_3, r_2s_2\}$. In addition, if $\|G[R]\| = 3$, then $N_{G_2}(s_2) = N_{G_2}(s_3)$ and $|N_{G_2}(s_2)| = 1$. In any case, $\|G_2\| \leq 9$ and thus $\|G\| \leq f(7) - 1$; a contradiction. ∎

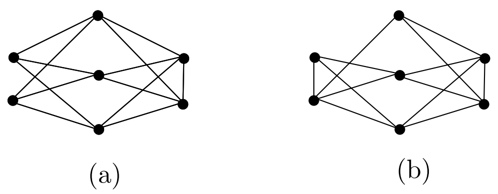

Figure 1: Graphs of order seven containing S_4.

The next lemma facilitates our proof of subsequent claims.

Lemma 4.3. *Let $x \in V(G)$ such that $d(x) \geq 4$ and let $\{y_1, y_2, y_3, y_4\} \subseteq N_G(x)$ be four distinct vertices. If $\{y_1y_2, y_3y_4\} \subset E(G)$, then G has an S_4-subgraph.*

Proof: As $\kappa(G - x) \geq 2$, $G - x$ contains a circuit containing the edges $\{y_1y_2, y_3y_4\}$. ∎

Claim 4.4. *Suppose G has order eight and satisfies the following properties.*

(a) $\|G\| = f(8)$.
(b) *S is stable*
(c) *G contains no vertex x of degree 3 where $N(x)$ is a stable disconnector in G.*

Then G has an S_4-subgraph.

Proof: Assume towards contradiction that G contains no S_4-subgraph. Let $\{x, y\} = V(G_1) \setminus S$. By (c) it can be assumed that $n_1 = 5$, $n_2 = 6$, $xy \in E(G)$, and $\|G_2 - S\| \geq 2$. As $\|G\| = \|G[\{x, y\}]\| + \sum_{s \in S} d_G(s) + \|G_2 - S\| \leq 4 + \sum_{s \in S} deg_G(s)$, it follows that S contains a vertex of degree at least four. This fact and Lemma 4.3 assert that $\|G_2 - S\| = 2$; implying that S contains a vertex of degree at least five. Hence, G has an S_4-subgraph, by Lemma 4.3; a contradiction. ∎

Claim 4.5. *Suppose G has order at least nine and satisfies the following properties.*

(*a*) $\|G\| = f(n)$.
(*b*) *S is stable.*
(*c*) $n_1 \in \{5, 6\}$.
(*d*) *G contains no vertex x of degree 3 where $N(x)$ is a stable disconnector of G.*
(*e*) *Every subgraph $H \subset G$ has $\|H\| \le f(|H|)$, and $\|G_2\| \le f(n_2) - 1$.*

Then G has an S_4-subgraph.

Proof: Assume towards contradiction that G contains no S_4-subgraph. Two cases are considered. Either $n_1 = 5$ or $n_1 = 6$. In the former case, $n_2 = n - 2 \ge 7$. As $\|G\| = \|G_1\| + \|G_2\|$, and $7 + f(n-2) - 1 = f(n)$, $n \ge 9$, it follows that $G_1 \cong K^+_{2,3}$. If S does not contain a vertex of degree at least four, then there exists a set $E \subseteq E(G)$, $|E| = 3$, such that E is an edge cut of G separating G into G_1 and a subgraph H of order $n-5$. It follows that $\|G\| = \|G_1\| + |E| + \|H\| \le 7 + 3 + f(n-5) < f(n)$, $n \ge 9$. Thus, it can be assumed that S contains a vertex of degree at least four; let s_1 be such a vertex. Let $\{x, y\} = V(G_1) \setminus S$, and let $\{\ell_1, \ell_2\} \subseteq N_{G_2}(s_1)$. By Lemma 4.3, it can be assumed that $\ell_1\ell_2 \notin E(G)$. Define $G_2' = G_2 - s_1 + s_2s_3 + \ell_1\ell_2$. The graph G_2' has $\kappa(G) \ge 2$, and thus admits a circuit C containing the edges $\{\ell_1\ell_2, s_1s_2\}$. Define $C' = C - \{\ell_1\ell_2, s_1s_2\} + s_1 + x + \{s_1\ell_1, s_1\ell_2\} + \{xs_2, xs_3\}$. As $y \notin V(C')$ and $|V(C') \cap N(y)| \ge 4$, a contradiction is obtained.

Suppose then that $n_1 = 6$ and $n_2 = n - 3 \ge 6$. It can be assumed that $\|G_1\| = f(6) - 1$. Indeed, if $n_2 \ge 7$, then the equality $f(n-3) - 1 + f(6) - 1 = f(n)$, $n \ge 9$, implies that $\|G_1\| = f(6) - 1$. If $n_2 = 6$, then since $f(9) = 19$ and $f(6) - 1 = 10$, it can be assumed, without loss of generality, that $\|G_1\| = f(6) - 1$. Let $X = \{x_1, x_2, x_3\} = V(G_1) \setminus S$. By (*d*), $\|G[X]\| \ge 2$; implying that G contains a path P such that $V(P) = X$ with endpoints x_1 and x_2. As $f(6) - 1 = \|G_1\| = \|G_1[X]\| + \|S, X\|$, and $\|G_1[X]\| \le 3$, the set S contains a vertex adjacent to all vertices in X. Let s_1 be such a vertex. As $|S| = \kappa(G)$, there exists a vertex $z \in N_{G_2}(s_1)$. In addition, since $\kappa(G) = 3$, G_2 contains a $(z, S \setminus \{s_1\})$-fan of cardinality two containing paths Q_1 and Q_2. As G does not contain an S_4-subgraph with s_1 at its center, $G_1 - s_1$ does not contain an s_2s_3-path containing X. It follows that G_1 contains at most one edge of the pair $\{x_1s_2, x_2s_3\}$, and at most one edge of the pair $\{x_1s_3, x_2s_2\}$. Thus, if $\|G_1[X]\| = 2$, then $\|G_1\| < f(6) - 1$; also, if $\|G_1[X]\| = 3$, then s_2 and s_3 may have at most one common neighbor in X and thus $\|G_1\| < f(6) - 1$; in both cases a contradiction is obtained. ∎

Claims 4.7 and 4.8 consider G in the case that $\|G[S]\| = 1$. In both these claims and Lemma 4.6 it is assumed, without loss of generality, that $s_1s_2 \in E(G)$. We require the following lemma.

Lemma 4.6. *If $d_G(s_1) \ge 4$ and there exists an edge $xy \in E(G_1)$ such that $\{x, y\} \subset N_G(s_1) \setminus \{s_2\}$, then G has an S_4-subgraph.*

Proof: As $|S| = \kappa(G)$, there exists a vertex $z \in N_{G_2}(s_1)$. The graph $G - s_1$ is 2-connected and thus admits a circuit containing xy and z. Such a circuit contains s_2 as well. Consequently, the claim follows. ∎

Claim 4.7. *Suppose G has order 7 and satisfies the following properties.*

(*a*) *G contains no S_4-subgraph.*
(*b*) $\|G\| = f(7)$.
(*c*) $\|G[S]\| = 1$.

Then $G \cong K^+_{3,4}$.

Proof: Two cases are considered. Either $n_1 = 4$ and $n_2 = 6$, or $n_1 = n_2 = 5$. In the former case, let $U = \{u_1, u_2, u_3\} = V(G_2) \setminus S$. As $\delta(G) \geq 3$, $\|G_1\| = 4$. If U is stable in G, then $G \cong K^+_{3,4}$ as required. Assume then that U is not stable in G. Consequently, Lemma 4.6 asserts that neither s_1 nor s_2 is adjacent to all of U; implying that each of these vertices has at most two neighbors in U. As $f(7) = \|G\| = \|G_1\| + \|G_2\| - 1 = 3 + \|S, U\| + \|G_2[U]\|$, and $\|G_2[U]\| \leq 3$, it follows that $\|S, U\| \geq 7$. As $d_G(s_3) \leq 4$, at least four edges in G_2 are incident with $\{s_1, s_2\}$. It follows that each of the vertices s_1 and s_2 each has exactly two neighbors in U. If $N_{G_2}(\{s_1, s_2\}) = U$, then $\|G_2[U]\| = 1$. In which case, $\|G\| < f(7)$. Assume then that, without loss of generality, $N_{G_2}(\{s_1, s_2\}) = \{u_1, u_2\}$. As $\delta(G) \geq 3$, it follows that $N_G(u_3) = \{u_1, u_2, s_3\}$. This implies that $N_{G_2}(s_3) = U$. Thus, $G - s_3$ is Hamiltonian and G has an S_4-subgraph with a center at s_3; a contradiction.

Consider the case that $n_1 = n_2 = 5$, and let $U_i = \{t^i_1, t^i_2\} = V(G_i) \setminus S$, for $i = 1, 2$. By Lemma 4.6, if $d_G(s_i) = 5$ for $i = 1$ or $i = 2$, then U_i is stable, for $i = 1, 2$. In this case, $\delta(G) \geq 3$ implies that $G \cong K^+_{3,4}$ as required. Assume then that $d_G(s_i) \leq 4$. As $f(7) = \|G\| = \|G[U_1]\| + \|G[U_2]\| + \sum_{i=1}^{3} d_G(s_i) - 1$, it follows that $d_G(s_i) = 4$, for $i = 1, 2$. This implies that $\|G[U_i]\| = 0$ for some $i = 1, 2$, and thus $\|G\| < f(7)$; a contradiction. ∎

Lemma 4.8. *Suppose G has order eight and satisfies the following properties.*

(*a*) *G contains no S_4-subgraph.*
(*b*) $\|G\| = f(8)$.
(*c*) $\|G[S]\| = 1$.
(*d*) $n_1 = 5$ *and* $n_2 = 6$.

Then $G \cong K^+_{3,5}$.

Proof: Let $U_i = V(G_i) \setminus S$, for $i = 1, 2$. Since $\delta(G) \geq 3$, if $U_1 \cup U_2$ is stable in G, then $G \cong K^+_{3,5}$ as required. Thus, it can be assumed that $U_1 \cup U_2$ is not stable. This assumption and Lemma 4.6 imply that

(1) $d_G(s_i) \leq 5$, *for* $i = 1, 2$.

In addition, it is noted that

(2) $\|G\| = \|G[U_1]\| + \|G[U_2]\| + \sum_{i=1}^{3} d_G(s_i) - 1$.

Since $\|G[U_1]\| \le 1$, $\|G[U_2]\| \le 3$, $d_G(s_3) \le 5$, and $\|G\| = f(8)$, it follows from (2) that

(3) $d_G(s_i) \ge 4$, *for some* $i = 1, 2$.

Lemma 4.6 and (3) imply that $\|G[U_1 \cup U_2]\| \le 3$. Consequently, (2) implies that $d_G(s_i) \ge 5$, for some $i = 1, 2$. Hence, (1) asserts that

(4) $d_G(s_i) = 5$, *for some* $i = 1, 2$.

Lemma 4.6 and (4) imply that $\|G[U_1 \cup U_2]\| \le 2$. In which case, (2) imply that

(5) $d_G(s_i) = 5$, *for* $i = 1$ *and* $i = 2$.

By (2), (5), and since $d_G(s_3) \le 5$, it follows that

(6) $\|G[U_1 \cup U_2]\| = 2$.

A vertex s_i, $i = 1, 2$, either has three neighbors in U_2 and one in U_1, or two neighbors in each of the sets U_i, $i = 1, 2$. Consequently, it follows from (6) that

(7) $\|G[U_1]\| = 0$ *and that* $\|G[U_2]\| = 2$.

Let $x \in U_1$. By the above properties $G - \{x, s_3\}$ is Hamiltonian and contains four neighbors of s_3. Hence, G has an S_4-subgraph; a contradiction. ∎

This section concludes with the following easy to verify claim.

Claim 4.9. *A* 4*-connected graph of order eight and size* $f(8)$ *containing no* S_4*-subgraph is isomorphic to* $K_{4,4}$.

5. Some properties of extremal graphs

We require certain properties of the graphs listed in Theorem 1.3(ii). The following three claims are easy to prove. Indeed, Claims 5.1 and 5.3 are extensions to the fact that cockades and $K_{4,4}$ are edge maximal with respect to not containing an S_4-subgraph.

Claim 5.1. *Let* G *be a* (K_3, K_4)*-cockade in which at most one of its simplicial summands is* K_3*. Let* $u, v \in V(G)$ *be two nonadjacent vertices. A graph obtained from* G *by joining* u *and* v *by a* uv*-path* P *internally-disjoint of* G *has an* S_4*-subgraph.*

Here we should explain the requirement that at most one simplicial summand is a K_3. Indeed, if there is more than one, then it is possible that the path P in Claim 5.1 connects two degree two vertices in G so that the argument to identify a center of an S_4-subgraph in $G \cup P$ becomes more difficult.

Claim 5.2. *Let $G \cong K^+_{3,n-3}$, $n-3 \geq 4$. Let $L = \{\ell_1, \ell_2, \ell_3\}$ be a stable set of G. Let H be a graph obtained from G by adding a new vertex x and adding the edge $x\ell_3$; and then joining x to $\{\ell_1, \ell_2\}$ with an $(x, \{\ell_1, \ell_2\})$-fan of cardinality two containing paths P_1 and P_2 that are internally-disjoint of G, such that $V(P_i) \cap V(G) = \ell_i$, for $i = 1, 2$. Then H has an S_4-subgraph.*

Claim 5.3. *A graph obtained from $K_{4,4}$ by joining two nonadjacent vertices of $K_{4,4}$ with a path that is internally-disjoint of the $K_{4,4}$ has an S_4-subgraph.*

6. Proof of Theorem 1.3

We note that all graphs listed in Theorem 1.3(ii) are edge-maximal with respect to not containing an S_4-subgraph. In addition, each such n-vertex graph has size $f(n)$. Consequently, $ex(n, S_4) \geq f(n)$. It remains to prove that $ex(n, S_4) \leq f(n)$ and that Theorem 1.3(ii) holds. To that end an induction on n is employed, where the induction hypothesis is that both (i) and (ii) hold for graphs of lesser order. The edge-maximality of the graphs listed in Theorem 1.3(ii) implies that it is sufficient to prove that a graph G of order n, size $f(n)$, containing no S_4-subgraph is one of the graphs listed in Theorem 1.3(ii) depending on its order. For $n = 3, 4$, this holds in a trivial manner. For $n = 5$, it is noted that K_5 with one edge removed has an S_4-subgraph. Removing two independent edges from K_5 define a graph that has an S_4-subgraph. However, removing two non-independent edges from K_5 define a graph containing no S_4-subgraph and that satisfies (i) and (ii). Thus, in subsequent arguments it is assumed that $n \geq 6$.

As a graph H with $\kappa(G) \geq 5$ has an S_4-subgraph and as a graph H of size $f(n)$ has $\kappa(H) \geq 2$, it can be assumed that $\kappa(G) \in \{2, 3, 4\}$. In what follows, three cases are considered; one for each possible value of $\kappa(G)$.

(8) *Let $S \subset V(G)$ be a disconnector of G, $|S| = \kappa(G)$, such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = G[S]$. In addition, let n_i denote the order of G_i, for $i = 1, 2$.*

Note that $\kappa(G) + 1 \leq n_i < n$ and thus the induction hypothesis applies to G_i, $i = 1, 2$. Also, note that no subgraph $G' \subset G$ has an S_4-subgraph. Thus, by the induction hypothesis $\|G'\| \leq f(|G'|)$, $|G'| \geq 3$.

Case *I*. Suppose that $\kappa(G) = 2$. It is shown that $\kappa(G) = 2$ occurs provided that $n \leq 8$. For such an order of G it is shown that S cannot be stable and then extremal graphs for such order of G are reported. Indeed, if S is stable, then since $n + 2 = n_1 + n_2$, it holds that $\|G\| = \|G_1\| + \|G_2\| \leq f(n_1) + f(n_2) < f(n)$. Thus, S is not stable implying that $\|G\| = \|G_1\| + \|G_2\| - 1$. If $n \geq 9$, then $f(n_1) + f(n_2) - 1 < f(n)$; and thus it can be assumed that $n \leq 8$. In turn, it is implied that $n_i < 7$, $i = 1, 2$.

If n is odd, the equality $n_1 + n_2 = n + 2$ implies that, without loss of generality, it can be assumed that n_1 is even and n_2 is odd. Hence, $n_1 \geq 4$ and $n_2 \geq 3$. As $f(n_1) + f(n_2) - 1 = f(n)$, induction hypothesis asserts that $G_i \in Ex(n_i, S_4)$, for $i = 1, 2$. In other words, G_1 is a

K_4-cockade and that G_2 is a (K_3, K_4)-cockade with a single simplicial summand isomorphic to K_3. Thus, G is a (K_3, K_4)-cockade with a single simplicial summand isomorphic to K_3; as required.

In the complementary case that n is even, the orders n_i, $i = 1, 2$, are of the same parity. If both orders are odd, then $f(n_1) + f(n_2) - 1 < f(n)$. Thus, it can be assumed that both orders are even. In which case, $f(n_1) + f(n_2) - 1 = f(n)$. Consequently, induction hypothesis asserts that G_i is a K_4-cockade, for $i = 1, 2$, implying that G is also of this type; as required. This concludes the argument for the case that $\kappa(G) = 2$.

Case *II*. Suppose that $\kappa(G) = 3$. Let S, G_i, and n_i, for $i = 1, 2$, be as in (8). Lemma 4.1 asserts that $\|G[S]\| \leq 1$. Consequently, two cases are considered. Either S is stable or not. The assumption of the former case is shown to lead to a contradiction; and consequently the latter case is assumed. In the latter case, extremal graphs are met.

Case *II.a*. Suppose that S is stable. The induction hypothesis, 3-connectivity of G, and Claims 5.1, 5.2, and 5.3 assert that $G_i \notin Ex(n_i, S_4)$, $i = 1, 2$, and so $\|G_i\| \leq f(n_i) - 1$. The assumption that $n \geq 6$ and Claim 4.2 assert that it can be assumed that $n \neq 7$. For remaining values of n it holds that $3 + f(n-1) - 1 < f(n)$. This implies that $n_i \geq 5$, for $i = 1, 2$, and that it can be assumed that in this case

(9) *G contains no vertex x of degree 3 where $N_G(x)$ is a stable disconnector of G.*

By (9), $n \geq 8$. Then, (9) and Claim 4.4 assert that it can be assumed that $n \geq 9$. Claim 4.5 then asserts that it can be assumed that $n_i \geq 7$. For such values of n and n_i, $i = 1, 2$, it holds that $\|G\| \leq \|G_1\| + \|G_2\| \leq f(n_1) - 1 + f(n_2) - 1 < f(n)$; a contradiction. It follows then that S cannot be stable.

Case *II.b*. Suppose that S is not stable; and thus $\|G[S]\| = 1$. We show that $n \geq 9$ can be assumed. Then graphs of such order are considered. By Claim 5.1, if $n_i \leq 6$, then $G_i \notin Ex(n_i, S_4)$ and thus

(10) *if $n_i \leq 6$, then $\|G_i\| \leq f(n_i) - 1$.*

In this case, the equality $\|G\| = \|G_1\| + \|G_2\| - 1$ and (10) imply that if $n = 6$, then, without loss of generality, $\|G_1\| = 4$ and $\|G_2\| = f(5) - 1$. Since $4 + f(5) - 1 < f(6)$, a contradiction is obtained and thus it can be assumed that $n \geq 7$. For $n = 7$, Claim 4.7 asserts that $G \cong K_{3,4}^+$ as required. Assume then that $n = 8$ and note that $4 + f(7) - 1 = f(8)$. Thus, in this case, if $n_1 = 4$ and $n_2 = n - 1 = 7$, then $G_2 \in Ex(7, S_4)$. By Claim 5.1, G_2 is not a (K_3, K_4)-cockade; implying that $G_2 \cong K_{3,4}^+$ and in turn that $G \cong K_{3,5}^+$ as required. In the complementary case, that $n_1 = 5$ and $n_2 = 6$, Claim 4.8 asserts that $G \cong K_{3,5}^+$ as required.

It follows then that it can be assumed that $n \geq 9$. Two complementary cases are considered. Either $n_i \geq 7$, for $i = 1, 2$, or, without loss of generality, $n_1 \leq 6$. In the former case, since $f(n_1) + f(n_2) - 1 = f(n)$, $n \geq 9$, induction hypothesis asserts that G_i

is extremal, for $i = 1, 2$. By Claims 5.1, and 5.3, G_i is not a (K_3, K_4)-cockade or $K_{4,4}$, for $i = 1, 2$, and thus $G_i \cong K^+_{3,n_i-3}$, $i = 1, 2$. This implies that $G \cong K^+_{3,n-3}$ as required.

Assume then the complementary case that, without loss of generality, $n_1 \leq 6$ and note that $n_1 \geq 4$. For $n_1 \in \{4, 5, 6\}$, (10) asserts that $\|G_1\| \leq f(n_1) - 1$. As for such values of n_1 the equality $f(n_1) - 1 + f(n_2) - 1 = f(n)$, $n \geq 9$, holds, it follows that

(11) $\|G_1\| = f(n_1) - 1$

Also, this equality and the induction hypothesis assert that G_2 is extremal. Hence, Claims 5.1, and 5.3 imply that

(12) $n_2 \geq 7$ *and* $G_2 \cong K^+_{3,n_2-3}$.

Let $U = V(G_1) \setminus S$. If U is stable in G, then since $\delta(G) \geq 3$, it follows that $G \cong K^+_{3,n-3}$ as required. As a result, it can be assumed that $|U| \geq 2$, and thus $n_1 \geq 5$. It is noted that

(13) $\|G_1\| = \|G[U]\| + \|S, U\| + 1$.

If $|U| = 2$ (i.e., $n_1 = 5$) and U is not stable, then Lemma 4.6 asserts that $\|S, U\| \leq 4$ and thus by (13), $\|G_1\| < f(5) - 1$; contradicting (11). Hence, it can be assumed that $n_1 = 6$. Let $U = \{u_1, u_2, u_3\}$, and let $S = \{s_1, s_2, s_3\}$ such that $s_1 s_2 \in E(G)$. If s_i for some $i = 1, 2$ is adjacent to all vertices in U, then by Lemma 4.6, U is stable and as $\delta(G) \geq 3$, it follows that $G \cong K^+_{3,n-3}$ as required. In addition, as in this case $\|G[U]\| \leq 3$, at least one of s_i, $i = 1, 2$, is adjacent to at least two vertices in U; and thus Lemma 4.6 implies that $1 \leq \|G[U]\| \leq 2$. Consequently, both s_1 and s_2 are adjacent to exactly two vertices in U. If $\|G[U]\| = 1$, then by (13), $\|G_1\| < f(6) - 1$; contradicting (11). Thus we may assume that $\|G[U]\| = 2$. This in turn implies that $N_{G_1}(s_1) = N_{G_1}(s_2) = \{u_1, u_2\}$; and thus $u_1 u_2 \notin E(G)$, by Lemma 4.6. Also follows that $N_{G_1}(u_3) = \{u_1, u_2, s_3\}$ and that $N_{G_1}(s_3) = U$. The structure of G_1 is thus determined. It is shown that in this case G has an S_4-subgraph. Let $x \in N_{G_2}(s_3)$. Define $G' = G - s_3 - u_3 + u_1 u_2$. It is noted that G' is 2-connected and thus admits a circuit C containing x and $u_1 u_2$. Define $C' = C - u_1 u_2 + u_3 + u_1 u_3 + u_2 u_3$. Note that C' is a circuit in G not containing s_3 yet contains four neighbors of s_3; implying that G has an S_4-subgraph, which is a contradiction. This concludes the argument for Case II.

Case *III*. Suppose that $\kappa(G) = 4$. Corollaries 3.5 and 3.6 are used in order to obtain a contradiction as follows. By Corollary 3.5, G is triangle-free and thus by a theorem of P. Turán [1], $f(n) = \|G\| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. This implies that it can be assumed that $n \geq 8$. If $n = 8$, then Lemma 4.9 asserts that $G \cong K_{4,4}$ and thus it can be assumed that $n \geq 9$. Let $v \in V(G)$ such that $d_G(v) = \Delta(G)$. By Corollary 3.6, G contains a vertex set W, $|W| = 4$, containing v with each vertex in W of degree $\Delta(G)$. In addition, $G - W$ can be partitioned into components $A_1, \ldots, A_{\Delta(G)}$ such that each component meets exactly one member of $N_G(w)$, for every $w \in W$. Consider A_i for some $i \in \{1, \ldots, \Delta(G)\}$. The assumptions that

$\kappa(G) = 4$ and that G is triangle-free imply that either $|A_i| = 1$ or $|A_i| > 4$. If $|A_i| > 4$, then let $X_i \subset A_i$ be the vertex set containing all vertices in A_i that are adjacent to W (note that $|X_i| = 4$) and let $Y_i \subset A_i$ be the vertex set containing all vertices in A_i that are not adjacent to W. As $\delta(G) \geq 4$ and G is triangle-free it follows that

$$\frac{(3|X_i| + 4|Y_i|)}{2} \leq \|G[A_i]\| \leq \frac{|A_i|^2}{4},$$

and thus $12 + 4(|A_i| - 4) \leq \frac{|A_i|^2}{2}$. This implies that it can be assumed that if $|A_i| > 4$, then $|A_i| > 6$. Consequently, it can be assumed that for $i \in \{1, \ldots, \Delta(G)\}$, it holds that $\|G[A_i]\| \leq \max\{0, 3|A_i| - 8\}$. Consequently, it is sufficient to assume that $|A_i| > 6$ for all $i \in \{1, \ldots, \Delta(G)\}$. Noting that $n = 4 + \sum_{i=1}^{\Delta(G)} |A_i|$ and that $\Delta(G) \geq 4$, we obtain a contradiction as follows.

$$\begin{aligned}
\|G\| &\leq |W|\Delta(G) + \sum_{i=1}^{\Delta(G)} (3|A_i| - 8) \\
&= 4\Delta(G) + 3\sum_{i=1}^{\Delta(G)} (|A_i|) - 8\Delta(G) \\
&= 3n - 12 - 4\Delta(G) \\
&< 3n - 8 \\
&= f(n);
\end{aligned}$$

This concludes the proof of Case III and of Theorem 1.3. ∎

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