LINEAR QUASI-RANDOMNESS OF SUBSETS OF ABELIAN GROUPS AND HYPERGRAPHS

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Abstract. We establish an equivalence between the two seemingly distant notions of quasi-randomness: small linear bias of subsets of abelian groups and uniform edge distribution for uniform hypergraphs.

For a subset $A \subset G$ of an abelian group G consider the k-uniform Cayley (sum) hypergraph $H^{(k)}(A)$. The vertex set of $H^{(k)}(A)$ is G and the edges are k-element sets $\{x_1, \dots, x_k\} \in \binom{G}{k}$ with $x_1 + \dots + x_k \in A$. For $d \in (0,1)$ we show that sets $A \subset G$ of density $d+o(1)$ have all non-trivial Fourier coefficients of order $o(|G|)$ if and only if $e(U) = d\binom{|U|}{k} + o(|G|^k)$ for all $U \subset V\big(H^{(k)}(A)\big)$.

This connects the work of Chung and Graham on quasi-random subsets of the integers and that of Conlon-Hàn-Person-Schacht on weak/linear quasi-random hypergraphs. Further, it extends the work of Chung and Graham who established the corresponding result for $k = 2$ and $G = \mathbb{Z}_n$.

1. Introduction

Quasi-random properties are deterministic properties which capture certain characteristics of random objects. The last decades have seen an extensive effort in the study of this subject which has revealed many connections between different branches of mathematics and theoretical computer science.

For graphs the systematic investigation was initiated by Thomason [29, 30] who studied deterministic graph sequences which exhibit a key property of the binomial random graph $G(n,p)$: *uniform edge distribution*. Subsequently, in a cornerstone result of the area [9], Chung, Graham and Wilson proved that many graph properties characteristic for $G(n,p)$ are *equivalent* to a qualitative version of uniform edge distribution. For example, a graph sequence has uniform edge distribution if and only if the number of labelled copies of every fixed graph is about what is expected from the binomial random graph with the same edge density. We refer to [9] for the full statement of the result and to [25] for a survey on the subject.

Linear quasi-random hypergraphs. This line of research has been extended to k-uniform hypergraphs (k-graphs for short) in various ways [3, 4, 5, 6, 7, 14, 2, 24, 17, 19, 1]. One straightforward extension of uniform edge distribution from graphs to k-graphs is the following.

$\mathrm{DISC}_d(\varepsilon)$**:** A k-graph H on n vertices satisfies $\mathrm{DISC}_d(\varepsilon)$ if

$$e(U) = d\binom{|U|}{k} \pm \varepsilon n^k \quad \text{for all subsets} \quad U \subset V(H).$$

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Rödl noted that a construction from [13] shows that this notion is not sufficient to control subhypergraph counts. In contrast and somewhat surprising, Kohayakawa, Nagle, Rödl and Schacht [22] proved that it is sufficient to control the count of linear subhypergraphs, i.e., those in which every two edges intersect in at most one vertex. Subsequently, Conlon, Hàn, Person and Schacht [10] established an extension of the theorem of Chung, Graham and Wilson [9] to k-graphs, showing that several properties are equivalent to uniform edge distribution in k-graphs. As part of their result, they construct a linear k-graph M with the following property: A sequence of k-graphs has uniform edge distribution if and only if the number of labelled copies of M is about what is expected from the random k-graph with the same density. The graph M is defined as follows. Start with M_0 which consists of the edge $(x_1, \dots, x_k)$, considered as a k-partite k-graph. For $i = 1, \dots, k$ successively, define M_i from M_{i-1} by taking two copies of M_{i-1} and identify the vertices at their i-th partition class. We state here an abridge version of the results in [10].

Theorem 1.1 (Conlon-Hàn-Person-Schacht). *Given an integer $k \geq 2$ and $d \in (0,1)$. For any sequence $(H_n)_{n\to\infty}$ of k-uniform hypergraphs with $|V(H_n)| = n$ the following properties are equivalent:*

Q_1: *for all linear k-uniform hypergraphs F with v_F vertices and e_F edges*

$$N_F\big(H_n\big) = d^{e_F} n^{v_F} + o(n^{v_F});$$

where $N_F(H_n)$ denotes the number of labelled copies of F in H_n;

Q_2: $e\big(H_n\big) \geq d\binom{n}{k} + o(n^k)$ *and* $N_M\big(H_n\big) = d^{e_M} n^{v_M} + o(n^{v_M})$.

Q_3: *for every $U \subset V\big(H_n\big)$ we have $e(U) = d\binom{|U|}{k} + o(n^k)$.* □

Due to the restriction of counting linear hypergraphs (sequences of) hypergraphs satisfying any of the properties above are called *weak/linear quasi-random (w.r.t. d)*. Later on and building on a work of Friedman and Wigderson [15], Lenz and Mubayi [26] provided an extension of the so called spectral property which characterize weak quasi-randomness in terms of hypergraph spectral gap (see Theorem 2.5 in Section 2.2).

1.1. **Quasi-random subsets of abelian groups.** Quasi-randomness has not only been studied for graphs and hypergraphs. Indeed, a very rich theory comes from the studies of subsets of abelian groups, see e.g. [27, 28, 8, 20, 21, 16, 23, 11]. This can be traced back at least to the work of Roth [27, 28] on arithmetic progressions in the integers in which he employes a notion of quasi-randomness in terms of discrete Fourier coefficients (see property R_4 of Theorem 1.2 below). In the spirit of [9] Chung and Graham showed in [8] that this property is equivalent to many others. In the following we denote the characteristic function of a subset $A \subset \mathbb{Z}_n$ by A itself, i.e., $A\colon \mathbb{Z}_n \to \{0,1\}$ is the 0-1-function which satisfies $A(x) = 1$ if and only if $x \in A$. Further, write $A + x$ for the set $\{a + x \colon a \in A\}$. The following is an abridged version of the main theorem in [8].

Theorem 1.2 (Chung-Graham). *Given $d \in (0,1)$ and an integer $k \geq 2$. Then for all sequences $(A_n)_{n\to\infty}$ of subsets $A_n \subset \mathbb{Z}_n$ the following properties are equivalent*

R_1: $|A_n| = (d + o(1))n$ *and all but $o(n)$ elements $x \in \mathbb{Z}_n$ satisfy*

$$\big|A_n \cap (A_n + x)\big| = d^2 n + o(n);$$

R_2: *for all $U \subset \mathbb{Z}_n$ all but $o(n)$ elements $x \in \mathbb{Z}_n$ satisfy*

$$\big|A_n \cap (U + x)\big| = d|U| + o(n);$$

$R_3(k)$**:** *$|A_n| = (d + o(1))n$ and all but $o(n^k)$ tuples $u_1, \dots, u_k \in \mathbb{Z}_n$ satisfy*

$$\sum_{x\in\mathbb{Z}_n}\prod_{i\in[k]} A_n(x+u_i) = d^k n + o(n);$$

R_4**:** *$|A_n| = (d + o(1))n$ and for all $j \in \mathbb{Z}_n \setminus \{0\}$ we have*

$$\Big|\sum_{x\in\mathbb{Z}_n} A_n(x)\exp\left(\frac{2\pi i}{n}jx\right)\Big| = o(n);$$

$R_5(k)$**:** *$|A_n| = (d + o(1))n$ and all but $o(n)$ elements $x \in \mathbb{Z}_n$ satisfy*

$$\sum_{u_1+\dots+u_k=x}\prod_{i\in[k]} A_n(u_i) = d^k n^{k-1} + o(n^{k-1});$$

R_6**:** *the graph sequence $\big(G(A_n)\big)_{n\to\infty}$ is quasi-random w.r.t. d where $G(A_n) = (\mathbb{Z}_n, E)$ and $E = \{xy \colon x + y \in A_n\}$.* □

As mentioned above Roth's notion of quasi-randomness is property R_4 and as it measures the correlation of A_n with the linear phase $x \mapsto \exp\left(\frac{2\pi i}{n}jx\right)$, the maximum absolute value over all $j \in \mathbb{Z}_n \setminus \{0\}$ is often referred to as the *linear bias* of A_n. Accordingly, (a sequence of) sets satisfying R_4 (and hence all properties) we shall call *linear quasi-random (w.r.t. d)*, indeed, due to a rather different reason than the one for hypergraphs.

We emphasize that the result is stated for $\mathbb{Z}_n$, and Chung and Graham asked for the analog of Theorem 1.2 in general abelian groups.

1.2. **Our results.** Our main objective is to show that the two distant seeming notions of linear quasi-randomness from above are indeed very closely related. More precisely, we establish an equivalence between small linear bias and uniform edge distribution of the corresponding Cayley hypergraphs. As such our result relates Theorem 1.1 and Theorem 1.2 and generalizes property R_6 of Theorem 1.2 to k-graphs.

Besides the generalization to k-graphs, the result is formulated in terms of discrete Fourier analysis, a straightforward analog of R_4 for arbitrary finite abelian group. This addresses the question of Chung and Graham mentioned above.

Let G be an abelian group and let $\widehat{G}$ denote the set of characters of G, i.e., the set of homomorphisms of the form $G \to \mathbb{T}$, where $\mathbb{T}$ denotes the multiplicative group of complex numbers of modulus one. The *Fourier coefficient* of a function $f \colon G \to \mathbb{C}$ at $\eta \in \widehat{G}$ is

$$\widehat{f}(\eta) = \sum_{x\in G} f(x)\overline{\eta(x)}.$$

In the case $G = \mathbb{Z}_n$ characters are functions of the form $x \mapsto \exp\left(\frac{2\pi i}{n}jx\right)$ for some fixed $j \in \mathbb{Z}_n$. Further, for any subset $A \subset G$ of a finite abelian G we have $\widehat{A}(\widehat{0}) = |A|$, where $\widehat{0}$ denotes the constant 1 character, the neutral element of $\widehat{G}$.

$\mathrm{BIAS}_d(\varepsilon)$**:** A subset of $A \subset G$ of an abelian group G satisfies $\mathrm{BIAS}_d(\varepsilon)$ if

$$|A| = (d \pm \varepsilon)|G| \quad \text{and} \quad \left|\widehat{A}(\eta)\right| \le \varepsilon|G| \quad \text{for all} \;\; \eta \in \widehat{G} \setminus \{\widehat{0}\}.$$

Property R_4 is that of BIAS_d for $G = \mathbb{Z}_n$. In accordance to $G = \mathbb{Z}_n$ we refer to this property as small linear bias and (a sequence of) subsets (of a sequence of groups) with linear bias $o(|G|)$ are called *linear quasi-random.*

Given a subset $A \subset G$ of an abelian group G. For an integer $k \geq 2$ define the *Cayley (sum) hypergraph* $H^{(k)}(A)$ as follows. The vertex set of $H^{(k)}(A)$ is G and $\{x_1, \ldots, x_k\} \in \binom{G}{k}$ forms an edge if $\sum_{i\in[k]} x_i \in A$. Our main result shows that linear quasi-random hypergraphs and linear quasi-random subsets of abelian groups are equivalent notions in the following sense.

Theorem 1.3. *Given an integer $k \geq 2$, a real $d \in (0,1)$ and a sequence of abelian groups $\big((G_n,+)\big)_{n\to\infty}$ with $|G_n| = n$. Then a sequence $(A_n)_{n\to\infty}$ of subsets $A_n \subset G_n$ of size $|A_n| = (d + o(1))n$ is linear quasi-random (w.r.t. d) if and only if the sequence of Cayley hypergraphs $\big(H^{(k)}(A_n)\big)_{n\to\infty}$ is linear quasi-random (w.r.t. d).*

In other words, for all integer $k \geq 2$ and $d, \varepsilon \in (0,1)$ there is a $\delta > 0$ and an n_0 such that the following holds for all abelian group G of order $|G| > n_0$.

- *If $A \subset G$ satisfies $\mathrm{BIAS}_d(\delta)$ then $H^{(k)}(A)$ satisfies $\mathrm{DISC}_d(\varepsilon)$ and*
- *If $H^{(k)}(A)$ satisfies $\mathrm{DISC}_d(\delta)$ then $A \subset G$ satisfies $\mathrm{BIAS}_d(\varepsilon)$.*

We shall give different proofs of the implications as they highlight different aspects of the two notions of linear quasi-randomness and their connections. Moreover, as a byproduct, the proofs allow us to deduce an extension of Theorem 1.2. First we extend the list of equivalent properties by the property $S_5(k)$ from below which generalizes $R_5(k)$. Further, we show that all equivalences continue to hold for arbitrary abelian group G once the properties R_1, R_2, R_3, R_5, R_6 are replaced by their straightforward analogs $R_1', R_2', R_3', R_5', R_6'$ (replacing $\mathbb{Z}_n$ by G of order n) and R_4 is replaced by (the asymptotic notion of) BIAS_d.

R_4'**:** $|A_n| = (d + o(1))n$ and $|\widehat{A}_n(\eta)| = o(n)$ for all $\eta \in \widehat{G}_n \setminus \{\widehat{0}\}$.

It shall be noted that except for the proofs that R_5 implies R_4 and that R_4 implies R_2 all other proofs in [8] extend to their analogs R_i' without change. We will address these two implications and the equivalence to $S_5(k)$.

Theorem 1.4. *Given $d \in (0,1)$, an integer $k \geq 2$ and a sequence of abelian groups $\big((G_n,+)\big)_{n\to\infty}$ with $|G_n| = n$. Then for all sequences $(A_n)_{n\to\infty}$ of subsets $A_n \subset G_n$ the following properties are equivalent*

R_2'**:** *for all $U \subset G_n$ all but $o(n)$ elements $x \in G_n$ satisfy*

$$\big|A_n \cap (U + x)\big| = d|U| + o(n);$$

R_4'**:** *$|A_n| = dn + o(n)$ and $|\widehat{A}_n(\eta)| = o(n)$ for all $\eta \in \widehat{G}_n \setminus \{\widehat{0}\}$.*

$R_5'(k)$**:** *$|A_n| = dn + o(n)$ and all but $o(n)$ elements $x \in G_n$ satisfy*

$$\sum_{u_1+\cdots+u_k=x} \prod_{i\in[k]} A_n(u_i) = d^k n^{k-1} + o(n^{k-1});$$

$S_5(k)$**:** *for all $U_1, \ldots, U_{k-1} \subset G$ all but $o(n)$ elements $x \in G_n$ satisfy*

$$\sum_{u_1+\cdots+u_k=x} A_n(u_k) \prod_{i\in[k-1]} U_i(u_i) = d \prod_{i\in[k-1]} |U_i| + o(n^{k-1}).$$

In particular, $R_1', \ldots R_5'$ and $S_5(k)$ are equivalent and the (extended) analog of Theorem 1.2 holds with $(\mathbb{Z}_n)_{n\to\infty}$ replaced by $(G_n)_{n\to\infty}$.

We remark that property $R_3(k)$ can be extended in a manner to that seen in extending $S_5(k)$ to $R_5(k)$. Such an extension of $R_3(k)$ property would read

$S_3(k)$**:** for all $U_1,\dots,U_{k-1}\subset G_n$ all but at most $o(n^k)$ tuples $u_1\dots u_k\in G^k$ satisfy
$$\sum_{x\in G} A_n(x+u_k)\prod_{i\in[k-1]} U_i(x+u_i)=d\Big|\bigcap_{i\in[k-1]}(U_i+x_i)\Big|+o(n).$$

It clearly implies $R_3'(2)$ and follows from R_6', for example. As this property is easily seen to be equivalent to the properties in Theorem 1.4 we do not include it to the theorem.

Organization. In the next section, Section 2, we introduce concepts and auxiliary results needed for the proofs of Theorem 1.3 and Theorem 1.4. This includes discrete Fourier analysis in Section 2.1, hypergraph eigenvalues in Section 2.2 and a combinatorial property related to uniform edge distribution in Section 2.3.

The proofs of Theorem 1.3 and Theorem 1.4 are distributed among Section 3 and Section 4. That small linear bias implies uniform edge distribution is shown in Section 3.1 using a straightforward Fourier analytic argument. Alternatively, in Section 3.2, we provide a proof relying on uniformity induction and relating linear bias with property R_2' of Theorem 1.4. We will indeed establish a more general result which will be used to prove Theorem 1.4 in Section 3.3.

For the more interesting implication, that uniform edge distribution implies small linear bias, we give a proof in Section 4.1 which uses hypergraphs eigenvalues and the work of Lenz and Mubayi [26]. In Section 4.2 we give a "more combinatorial" proof which relates the property from Section 2.3 with Fourier convolutions.

2. Notation, tools and k-graphs with loops

It will be more convenient to work with k-graphs with loops, whose edges may be multi-sets, i.e., containing vertices with multiplicity. For clarity k-graphs H_* with loops we indicate by the star subscript. Moreover, we shall work with a partite version of uniform edge distribution for k-graphs with loops.

$\mathrm{EXPAND}_d^*(\varepsilon)$**:** A k-graph H_* with loops satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$ if for all subsets $U_1,\dots,U_k\subset V(H_*)$ we have
$$e_*(U_1,\dots,U_k)=d\prod_{i\in[k]}|U_i|\pm\varepsilon n^k,$$
where $e_*(U_1,\dots,U_k)$ denote the number of tuples $(x_1,\dots,x_k)\in U_1\times\dots\times U_k$ such that $x_1\dots x_k$ forms an edge in H_*.

Fact 2.1. *For all integer $k\geq 2$, $d\in(0,1)$, $\varepsilon>0$ there is a $\delta>0$ and an n_0 such that the following holds for all $n>n_0$. Given a k-graph H_* with loops on n vertices and let H be the k-graph obtained from H_* by removing all loops.*

- *If H satisfies $\mathrm{DISC}_d(\delta)$, then H_* satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$ and*
- *if H_* satisfies $\mathrm{EXPAND}_d^*(\delta)$, then H satisfies $\mathrm{DISC}_d(\varepsilon)$.* □

This fact is easily seen by noting that the number of ordered k-tuples which may form a loop is at most $\binom{k}{2}n^{k-1}=o(n^k)$, hence negligible. Furthermore, an edge counted in $e(U)$ is counted $k!$-times in $e_*(U,\dots,U)$, implying the second property. The first property follows by noting that for all $\ell\in[k-1]$ we have
$$\begin{aligned}e_*(U_1,\dots,U_{\ell+1})&=e_*(U_1,\dots,U_{\ell-1},U_\ell\cup U_{\ell+1})\\&\quad-e_*(U_1,\dots,U_{\ell-1},U_\ell)-e_*(U_1,\dots,U_{\ell-1},U_{\ell+1}).\end{aligned}$$

We refer to [10] for more details.

Given a subset $A \subset G$ of an abelian group G. For an integer $k \geq 2$ define the *Cayley (sum) hypergraph* $H_*^{(k)}(A)$ *with loops* as follows. The vertex set of $H_*^{(k)}(A)$ is G and the multiset $x_1 \dots x_k \in G^k$ forms an edge if $\sum_{i \in [k]} x_i \in A$. For hypergraphs with loops we shall introduce the notion of spectral gap in Section 2.2 and an extended version of uniform edge distribution will be considered in Section 2.3. Prior to doing so, we collect the required fundamental notions of discrete Fourier analysis in the next section. We finish this section with the following useful fact.

Fact 2.2. *For $\beta > 0$ and $a \geq 0$ the following holds. If $\sum_{i \in [n]} a_i \geq an$ and $\sum_{i \in [n]} a_i^2 \leq a^2 n + \beta n$, then $\sum_{i \in [n]} (a_i^2 - a^2) \leq \beta n$.*

Proof. As $a \sum_{i \in [n]} (a - a_i) \leq 0$, we have $\sum_{i \in [n]} (a_i - a)^2 \leq \sum_{i \in [n]} (a_i^2 - a^2) \leq \beta n$. □

2.1. **Discrete Fourier analysis.** For a finite abelian group G let $\widehat{G}$ denote its *dual*. This consists of homomorphisms (called characters) of the form $G \to \mathbb{T}$, where $\mathbb{T}$ denotes the multiplicative group of complex numbers of modulus 1. It is known that $\widehat{G}$ with pointwise multiplication forms a group isomorphic to G.

The *Fourier coefficient* of the function $f \colon G \to \mathbb{C}$ at $\eta \in \widehat{G}$ is given by

$$\widehat{f}(\eta) = \sum_{x \in G} f(x) \overline{\eta(x)}. \tag{2.1}$$

Conversely, we have the *Fourier expansion* of f which reads

$$f(x) = \frac{1}{|G|} \sum_{\eta \in \widehat{G}} \widehat{f}(\eta) \eta(x) \quad \text{for every } x \in G. \tag{2.2}$$

Hence, $\widehat{G}$ forms a basis of the $\mathbb{C}$-vector space of complex valued functions over G. Moreover, it satisfies the following orthogonal relation. For all $\chi, \eta \in \widehat{G}$ we have

$$\sum_{x \in G} \chi(x) \overline{\eta(x)} = \begin{cases} n & \text{if } \chi = \eta \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The two sides $f \colon G \to \mathbb{C}$ and $\widehat{f} \colon \widehat{G} \to \mathbb{C}$ relate through *Parseval's identity*

$$\frac{1}{|\widehat{G}|} \sum_{\eta \in \widehat{G}} \left| \widehat{f}(\eta) \right|^2 = \sum_{x \in G} |f(x)|^2. \tag{2.4}$$

The *convolution* of $f, g \colon G \to \mathbb{C}$ is the function

$$f * g(x) = \sum_{x = y + z} f(y) \overline{g(z)} = \sum_{y} f(y) \overline{g(x - y)} \tag{2.5}$$

which satisfies

$$\widehat{f * g}(\eta) = \widehat{f}(\eta) \overline{\widehat{g}(\eta)}. \tag{2.6}$$

The following identity involving k-fold convolution is crucial for our proofs.

Lemma 2.3. *For $k \geq 2$ and any $f_1, \dots, f_k \colon G \to \mathbb{R}$ we have*

$$\frac{1}{|G|} \sum_{\eta \in \widehat{G}} \prod_{i \in [k]} |\widehat{f}_i(\eta)|^2 = \sum_{x} \Big((f_1 * \dots * f_k)(x) \Big)^2 = \sum_{x \in G} \Big(\sum_{x = x_1 + \dots + x_k} \prod_{i \in [k]} f_i(x_i) \Big)^2. \tag{2.7}$$

Proof. From Parseval (2.4) and convolution (2.6) we have

$$\sum_x \Big((f_1 * \cdots * f_k)(x)\Big)^2 = \frac{1}{|G|}\sum_{\eta\in\widehat{G}} |\widehat{f_1 * \cdots * f_k}(\eta)|^2$$

$$= \frac{1}{|G|}\sum_{\eta\in\widehat{G}} \widehat{f}_1(\eta)\cdot\ldots\cdot\widehat{f}_k(\eta)\cdot\overline{\widehat{f}_1(\eta)}\cdot\ldots\cdot\overline{\widehat{f}_k(\eta)} = \frac{1}{|G|}\sum_{\eta\in\widehat{G}}\prod_{i\in[k]}|\widehat{f}_i(\eta)|^2$$

which establishes the first equality.

For the second equality observe that for any $k \geq 2$ and any $x \in G$ we have

$$f_1 * \cdots * f_k(x) = \sum_{x=x_1+\cdots+x_k}\prod_{i\in[k]} f_i(x_i).$$

Indeed, for $k = 2$ this is the definition. Let $k \geq 3$ and let $g = (f_2 * \cdots * f_k)(x)$. Then, by induction, $f_1 * g(x)$ is equal to

$$\sum_{x=x_1+z} f_1(x_1)g(z) = \sum_{x=x_1+z} f_1(x_1)\sum_{z=x_2+\cdots+x_k}\prod_{i=2}^{k} f_i(x_i) = \sum_{x=x_1+\cdots+x_k}\prod_{i=1}^{k} f_i(x_i).$$

Hence,

$$\sum_x \Big((f_1 * \cdots * f_k)(x)\Big)^2 = \sum_{x\in G}\Big(\sum_{x=x_1+\cdots+x_k}\prod_{i\in[k]} f_i(x_i)\Big)^2,$$

as claimed. □

2.2. **Hypergraph spectral gap.** There are several notions of hypergraph eigenvalues [12, 15]. The one considered here, as well as their connections to quasi-random hypergraphs, are due to Friedman-Wigderson [15] and Lenz-Mubayi [26].

Given a k-graph H_* with loops and a vertex $v \in V(H_*) = [n] = \{1, \ldots, n\}$. Then H_* is called Δ-*coregular* if each $(k-1)$-tuple of not necessarily distinct vertices is contained in exactly Δ edges. Let $W = \mathbb{R}^n$ and let $\boldsymbol{e}_v \in W$ be the indicator vector of v. The *adjacency map of* H_*, denoted $\tau_{H_*}\colon W^k \to \mathbb{R}$, is the symmetric k-linear map determined by

$$\tau_{H_*}(\boldsymbol{e}_{v_1}, \ldots, \boldsymbol{e}_{v_k}) = \begin{cases} 1 & \text{if } \{v_1, \ldots, v_k\} \in H_*, \\ 0 & \text{otherwise.}\end{cases}$$

The map τ_{H_*} then extends to all W^k via multi-linearity, i.e., for all $\alpha \in \mathbb{R}$, $\ell \in [k]$ and all $\boldsymbol{x}_i \in \mathbb{R}^n$

$$\tau_{H_*}(\boldsymbol{x}_1, \ldots, \alpha\cdot(\boldsymbol{x}_\ell + \boldsymbol{x}'_\ell), \ldots, \boldsymbol{x}_k)$$
$$= \alpha\cdot\tau_{H_*}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell, \ldots, \boldsymbol{x}_k) + \alpha\cdot\tau_{H_*}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}'_\ell, \ldots, \boldsymbol{x}_k).$$

Let $J_*\colon W^k \to \mathbb{R}$ denote the adjacency map of the complete k-graph with loops, i.e., the k-linear map $J_*(\boldsymbol{e}_{v_1}, \ldots, \boldsymbol{e}_{v_k}) = 1$ for all $v_1, \ldots, v_k \in [n]$. The *balanced adjacency map of* H_* is defined by

$$\sigma_{H_*} = \tau_{H_*} - \frac{k!e(H_*)}{n^k}J_*.$$

For $w \in \mathbb{R}^n$ write $\|w\|$ to denote its Euclidean norm. The following definition of the second hypergraph eigenvalue is due to Friedman and Wigderson [15].

Definition 2.4. *Let H_* be a k-uniform hypergraph with loops on the vertex set $[n]$ and let σ_{H_*} denote its balanced adjacency map. The* second eigenvalue *of H_* is defined by*

$$\|\sigma_{H_*}\| = \sup_{\|\boldsymbol{x}_i\|=1} |\sigma_{H_*}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|$$

where the supremum is taken over all $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in W = \mathbb{R}^n$ of modulus 1.

Spectral gap or *eigenvalues separation* for hypergraphs then reads as follows.

$\mathrm{EIG}^*_d(\varepsilon)$**:** A k-graph H_* satisfies $\mathrm{EIG}_d(\varepsilon)$ if it is $\lfloor dn \rfloor$-coregular and

$$\|\sigma_{H_*}\| \le \varepsilon n^{k/2}.$$

One may wonder why this property is called spectral gap/eigenvalues separation. Indeed, Friedman and Wigderson also define the first eigenvalue of H_* to be $\|\tau_{H_*}\|$. They then show that a dn-coregular k-graph has first eigenvalue $\|\tau_{H*}\| = dn^{k/2}$. Thinking of ε substantially smaller than d then signifies a gap/separation between the first and the second eigenvalues, hence justifies the names given.

In the same paper Friedman and Wigderson proved an analog of the expander mixing lemma for hypergraphs. As a consequence, spectral gap implies uniform edge distribution. The converse was proven by Lenz and Mubayi [26], establishing therefore an equivalence between the two properties.

Theorem 2.5 (Lenz-Mubayi [26])**.** *For every integer $k \ge 2$, every $d, \varepsilon \in (0,1)$ there is a $\delta > 0$ such that the following holds. Every k-uniform $\lfloor dn \rfloor$-coregular hypergraph that satisfies* $\mathrm{EXPAND}^*_d(\delta)$ *also satisfies* $\mathrm{EIG}^*_d(\varepsilon)$*.* □

2.3. **An extension of uniform edge distribution.** Consider a k-graph H_* with loops on the vertex set V and let $d \in [0,1]$. Define $h_* \colon V^k \to [0,1]$, the *balanced characteristic function (w.r.t. d)*, to be

$$h_*(x_1, \dots, x_k) = \begin{cases} 1-d & \text{if } \{x_1, \dots, x_k\} \in H_* \\ -d & \text{otherwise.} \end{cases}$$

The objective of this section is to show the following which will be essential in the proof that uniform distribution implies small linear bias.

Lemma 2.6. *Given $k \ge 2$, $\varepsilon > 0$ and suppose that H_* satisfies* $\mathrm{EXPAND}^*_d(\varepsilon)$*. Then for any k-tuple $\boldsymbol{g}_k = (g_1, \dots, g_k)$ of functions $g_i \colon G \to [-1,1]$ we have*

$$\sum h_*(x_1, \dots, x_k) h_*(x_1, y_2, \dots, y_k) \prod_{i=2}^{k} g_i(x_i) g_i(y_i) \le 2^k \varepsilon n^{2k-1} \tag{2.8}$$

where the sum ranges over all $(x_1, \dots, x_k) \in V^k$ and $(y_2, \dots, y_k) \in V^{k-1}$.

Proof. The proof follows an argument by Gowers [18, Section 3]. Given H_* that satisfies $\mathrm{EXPAND}^*_d(\varepsilon)$ and suppose there exists $(g_1, \dots, g_k)$ of functions $g_i \colon G \to [-1,1]$ which certifies that (2.8) does not hold. Then, by averaging we conclude that there is a tuple $(\hat{y}_2, \dots, \hat{y}_k) \in V^{k-1}$ such that

$$\begin{aligned} 2^k \varepsilon n^k &\le \sum_{(x_1,\dots,x_k) \in V^k} h_*(x_1, \dots, x_k) h_*(x_1, \hat{y}_2, \dots, \hat{y}_k) \prod_{i=2}^{k} g_i(x_i) g_i(\hat{y}_i) \\ &= \sum_{(x_1,\dots,x_k) \in V^k} h_*(x_1, \dots, x_k) \prod_{i=1}^{k} g_i(x_i) \end{aligned} \tag{2.9}$$

where $g_1(x) = h_*(x, \hat{y}_2, \dots, \hat{y}_k) \prod_{i=2}^{k} g_i(\hat{y}_i)$ is a function of modulus at most one.

Define $V_i^+ = \{x \in V : g_i(x) > 0\}$ and $V_i^- = V \setminus V_i^+$. Further, write $g_i = g_i^+ - g_i^-$ where $g_i^+, g_i^- : V \to [0,1]$ defined via

$$g_i^+(x) = \begin{cases} g_i(x) & \text{if } x \in V_i^+ \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_i^-(x) = \begin{cases} -g_i(x) & \text{if } x \in V_i^- \\ 0 & \text{otherwise.} \end{cases}$$

For given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \{+,-\}^k$ let $|\boldsymbol{\alpha}|$ denote the number of minuses in $\boldsymbol{\alpha}$ and let $V(\boldsymbol{\alpha}) = \prod_{i \in [k]} V_i^{\alpha_i}$. Then we have

$$(-1)^{|\boldsymbol{\alpha}|} \prod_{i \in [k]} g_i^{\alpha_i}(x_i) = \begin{cases} \prod_{i \in [k]} g_i(x_i) & \text{if } (x_1, \dots, x_k) \in V(\boldsymbol{\alpha}) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\bigcup_{\boldsymbol{\alpha} \in \{+,-\}^k} V(\boldsymbol{\alpha})$ is a partition of V^k. Hence, from (2.9) and averaging over $\boldsymbol{\alpha} \in \{+,-\}^k$ we conclude that there exists an $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \{+,-\}^k$ such that

$$(-1)^{|\boldsymbol{\alpha}|} \sum_{(x_1, \dots, x_k) \in V(\boldsymbol{\alpha})} h_*(x_1, \dots, x_k) \prod_{i \in [k]} g_i^{\alpha_i}(x_i) \geq \varepsilon n^k.$$

For each $i \in [k]$ define a random subset $U_i \subset V$ by picking each $v \in V$ with probability $g_i^{\alpha_i}(v)$ independently. This is possible since $g_i^{\alpha_i} : V \to [0,1]$. Then the probability that $(x_1, \dots, x_k) \in \prod_{i \in [k]} U_i(x_i)$ is exactly $\prod_{i \in [k]} g_i^{\alpha_i}(x_i)$, thus

$$\mathbb{E}\left((-1)^{|\boldsymbol{\alpha}|} \sum_{(x_1, \dots, x_k) \in V^k} h_*(x_1, \dots, x_k) \prod_{i \in [k]} U_i(x_i) \right) \geq \varepsilon n^k.$$

Therefore, there exists a choice of $U_1, \dots, U_k$ such that

$$(-1)^{|\boldsymbol{\alpha}|} \sum_{(x_1, \dots, x_k) \in V^k} h_*(x_1, \dots, x_k) \prod_{i=1}^{k} U_i(x_i) \geq \varepsilon n^k.$$

By the definition of h_*, however, this means that $\left| e(U_1, \dots, U_k) - d \prod |U_i| \right| \geq \varepsilon n^k$ contradicting the fact that H_* satisfies $\text{EXPAND}_d^*(\varepsilon)$. □

3. Small linear bias implies uniform edge distribution and the proof of Theorem 1.4

In this section we give the proof of the first part of Theorem 1.3, that small linear bias implies uniform edge distribution, and the proof of Theorem 1.4. We give a direct Fourier analytic argument in Section 3.1 and an alternative proof in Section 3.2 which relies on the shifting property $\text{SHIFT}_{k,d}$. This property relates to uniform distribution and for $k = 1$ it is the property R_2' of Theorem 1.4. Hence, as a byproduct, we establish Theorem 1.4 in Section 3.3.

3.1. A Fourier analytic argument.

We give a direct proof that small linear bias implies uniform edge distribution. In view of Fact 2.1 it is sufficient to show the following which relates BIAS_d to EXPAND_d^* instead of DISC_d.

Lemma 3.1. *For all integer $k \geq 2$, all $d, \varepsilon \in (0,1)$ the following holds. If a subset $A \subset G$ of an abelian group G satisfies $\text{BIAS}_d(\varepsilon)$, then $H_*^{(k)}(A)$ satisfies $\text{EXPAND}_d^*(2\varepsilon)$.*

Proof. Let $A\subset G$ with density $d_A=(d\pm\varepsilon)$ be given. By definition of $H_*^{(k)}(A)$ we have

$$e_*(U_1,\dots,U_k)=\sum_{x_1,\dots,x_k\in G}A\Big(\sum_{i\in[k]}x_i\Big)\prod_{i\in[k]}U_i(x_i). \tag{3.1}$$

Using Fourier expansion (2.2) we have

$$A(x)=\frac{1}{n}\sum_{\eta_A\in\widehat{G}}\widehat{A}(\eta_A)\eta_A(x_i)\qquad\text{and}\qquad U_i(x)=\frac{1}{n}\sum_{\eta_i\in\widehat{G}}\widehat{U}_i(\eta_i)\eta_i(x).$$

Substituting the identities into (3.1), expanding and rearranging the two sums yields

$$\begin{aligned}
&e_*(U_1,\dots,U_k)=\\
&\quad=\frac{1}{n^k}\sum_{x_1,\dots,x_k\in G}\;\sum_{\eta_A,\eta_1,\dots,\eta_k\in\widehat{G}}\widehat{A}(\eta_A)\eta_A\Big(\sum_{i\in[k]}x_i\Big)\prod_{i\in[k]}\widehat{U}_i(\eta_i)\eta_i(x_i)\\
&\quad=\frac{1}{n^k}\sum_{\eta_A,\eta_1,\dots,\eta_k\in\widehat{G}}\widehat{A}(\eta_A)\prod_{i\in[k]}\widehat{U}_i(\eta_i)\sum_{x_1,\dots,x_k\in G}\eta_A\Big(\sum_{i\in[k]}x_i\Big)\prod_{i\in[k]}\eta_i(x_i).
\end{aligned}\tag{3.2}$$

As $\eta_A\colon G\to\mathbb{T}$ is a homomorphisms we have $\eta_A(\sum x_i)=\prod\eta_A(x_i)$. Together with orthogonality (2.3) this yields

$$\begin{aligned}
\sum_{x_1,\dots,x_k\in G}\eta_A\Big(\sum_{i\in[k]}x_i\Big)\prod_{i\in[k]}\eta_i(x_i)&=\sum_{x_1,\dots,x_k\in G}\prod_{i\in[k]}\eta_A(x_i)\eta_i(x_i)\\
&=\prod_{i\in[k]}\left(\sum_{x_i}\eta_A(x_i)\eta_i(x_i)\right)=\begin{cases}n^{k-1} & \text{if } \overline{\eta_A}=\eta_1=\dots=\eta_k\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Using this together with $\widehat{A}(\widehat{0})=|A|=d_An$ and $\widehat{U}_i(\widehat{0})=|U_i|$ we obtain from (3.2)

$$e_*(U_1,\dots,U_k)=\frac{1}{n}\sum_{\eta\in\widehat{G}}\widehat{A}(\eta)\prod_{i\in[k]}\widehat{U}_i(\overline{\eta})=d_A\prod_{i\in[k]}|U_i|+\frac{1}{n}\sum_{\eta\in\widehat{G}\setminus\{\widehat{0}\}}\widehat{A}(\eta)\prod_{i\in[k]}\widehat{U}_i(\overline{\eta}).$$

Hence, as $k\geq 2$ and $\big|\widehat{A}(\eta)\big|\leq\varepsilon n$ and $|\widehat{U}_i(\eta)|\leq|U_i|$ for all $\eta\in\widehat{G}\setminus\{\widehat{0}\}$ we obtain

$$\begin{aligned}
&\Big|e_*(U_1,\dots,U_k)-d_A\prod_{i\in[k]}|U_i|\Big|\\
&\qquad\leq\frac{1}{n}\max_{\eta\neq\widehat{0}}\big|\widehat{A}(\eta)\big|\prod_{i\in[k-2]}|\widehat{U}_i(\overline{\eta})|\sum_{\eta\in\widehat{G}}\big|\widehat{U}_{k-1}(\eta)\big|\big|\widehat{U}_k(\eta)\big|\\
&\qquad\leq\varepsilon\prod_{i\in[k-2]}|U_i|\left(\sum_{\eta\in\widehat{G}}\big|\widehat{U}_{k-1}(\eta)\big|^2\right)^{1/2}\left(\sum_{\eta\in\widehat{G}}\big|\widehat{U}_k(\eta)\big|^2\right)^{1/2}\\
&\qquad=\varepsilon\prod_{i\in[k-2]}|U_i|\left(\sum_{x\in G}\big|U_{k-1}(x)\big|^2\right)^{1/2}\left(\sum_{x\in G}\big|U_k(x)\big|^2\right)^{1/2}\\
&\qquad=\varepsilon\prod_{i\in[k]}|U_i|\leq\varepsilon n^k
\end{aligned}$$

where the third line follows from the Cauchy-Schwarz inequality and the fourth line from Parseval's identity (2.4). With $d_A=(d\pm\varepsilon)$ the lemma follows. □

3.2. **Uniformity induction and the $\mathrm{SHIFT}_{k,d}$ property.** In this section we introduce the following property called $\mathrm{SHIFT}_{k,d}$ and give an alternative proof that small linear bias implies uniform edge distribution. Moreover, this property relates to the properties R_2' and $S_5(k)$ and is helpful in the proof of Theorem 1.4 in Section 3.3. For a set $X \subset G$ and an element $z \in G$ let $X^z = \{x - z : x \in X\}$.

$\mathrm{SHIFT}_{k,d}(\varepsilon)$**:** A subset $A \subset G$ of an abelian group G satisfies $\mathrm{SHIFT}_{k,d}(\varepsilon)$ if for all $U_1, \dots, U_k \subset G$ all but at most $2\varepsilon n$ elements $z \in G$ satisfy

$$\sum_{x_1,\dots,x_k} A(x_1 + \dots + x_k) U_k^z(x_k) \prod_{i\in[k-1]} U_i(x_i) = d \prod_{i\in[k]} |U_i| \pm \varepsilon n^k.$$

By induction on k we relate $\mathrm{SHIFT}_{k,d}$ to EXPAND_d^*, which in view of Fact 2.1 relates to DISC_d. The base case of the induction is a relation between BIAS_d and $\mathrm{SHIFT}_{1,d}$ formulated in Corollary 3.4.

Lemma 3.2. *For all integers $k \geq 2$ and all $\varepsilon > 0$ the following holds for all abelian groups G.*

- *If $A \subset G$ satisfies $\mathrm{SHIFT}_{k-1,d}(\varepsilon)$, then $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(4\varepsilon)$.*
- *If $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$ then A satisfies $\mathrm{SHIFT}_{k,d}(\varepsilon)$.*

In particular, if $A \subset G$ satisfies $\mathrm{SHIFT}_{1,d}(\varepsilon)$, then $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(4^{k-1}\varepsilon)$.

Proof. The second property is immediate from the definitions and the fact that U^z is a subset of G is of size $|U^z| = |U|$ for all $z \in G$.

To show the first part, let $U_1, \dots, U_{k-1} \subset G$ and let $z \in G$. Then in $H_*^{(k)}(A)$ we have (writing y_{k-1} for $x_{k-1} + z$) that

$$\begin{aligned} e_*(U_1, \dots, U_{k-1}, \{z\}) &= \sum_{x_1,\dots,x_{k-1}\in G} A\Big(\sum_{i\in[k-1]} x_i + z\Big) \prod_{i\in[k-1]} U_i(x_i) \\ &= \sum_{x_1,\dots,x_{k-2},y_{k-1}\in G} A\Big(\sum_{i\in[k-2]} x_i + y_{k-1}\Big) U_{k-1}^z(y_{k-1}) \prod_{i\in[k-2]} U_i(x_i). \end{aligned} \tag{3.3}$$

We call z *good* with respect to $U_1, \dots, U_{k-1} \subset G$ if

$$e_*(U_1, \dots, U_{k-1}, \{z\}) = d \prod_{i\in[k-1]} |U_i| \pm \varepsilon n^{k-1}.$$

If A satisfies $\mathrm{SHIFT}_{k-1,d}(\varepsilon)$ then for all $U_1, \dots, U_{k-1} \subset G$ all but at most $2\varepsilon n$ elements $z \in G$ are good with respect to $U_1, \dots, U_{k-1}$. As a consequence, the sum of $e_*(U_1, \dots, U_{k-1}, \{z\})$ over all elements $z \in G$ which are not good is at most $2\varepsilon n \prod_{i\in[k-1]} |U_i| \leq 2\varepsilon n^k$. Therefore, we obtain from (3.3) that for all $U_{\ell+1} \subset G$

$$e_*(U_1, \dots, U_k) = \sum_{\text{good } z\in U_{k-1}} e_*(U_1, \dots, U_{k-1}, \{z\}) \pm 2\varepsilon n^k = d \prod_{i\in[k]} |U_i| \pm 4\varepsilon n^k.$$

This shows that $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(4\varepsilon)$. □

Remark 3.3. *It is also easy to show the converse of the first property, i.e., that if $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$ then A satisfies $\mathrm{SHIFT}_{k-1,d}(\varepsilon)$. Indeed, if $H_*^{(k)}(A)$ satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$, then for all $U_1, \dots, U_{k-1}$ all but at most $2\varepsilon n$ elements $z \in G$ are good with respect to $U_1, \dots, U_{k-1}$. By the identity (3.3) this implies that A satisfies $\mathrm{SHIFT}_{k-1,d}(\varepsilon)$.*

It is left to establish that $\mathrm{BIAS}_d(o(1))$ implies $\mathrm{SHIFT}_{1,d}(o(1))$. This is stated here in Corollary 3.4 and follows from the first part of Lemma 3.6 from below which will also be used to prove Theorem 1.4.

Corollary 3.4. *Given $d, \varepsilon \in (0,1)$ and a subset A of an abelian group G. If A satisfies $\mathrm{BIAS}_d(\varepsilon)$ then A satisfies $\mathrm{SHIFT}_{1,d}(2\varepsilon^{1/3})$.*

3.3. **Proof of Corollary 3.4 and of Theorem 1.4.** We first make a couple of observations. For a subset $U \subset G$ we have

$$\sum_{x\in G} A(x)U^z(x) = |A \cap (U - z)|. \tag{3.4}$$

Hence, $\mathrm{SHIFT}_{1,d}(o(1))$ is simply property R_2' of Theorem 1.4.

For a subset $U \subset G$ let $-U = \{-u \colon u \in U\}$. Then for any $z \in G$ we have

$$\sum_{z=x_1+x_2} A(x_1)U(x_2) = \sum_{x\in G} A(x)U(z-x) = \sum_{x\in G} A(x)(-U)^z(x).$$

Hence, $\mathrm{SHIFT}_{1,d}(o(1))$ is equivalent to the property $S_5(2)$ of Theorem 1.4, stated here for reference.

Fact 3.5. *A set $A \subset G$ satisfies $\mathrm{SHIFT}_{1,d}(\varepsilon)$ if and only if for all $U \subset G$ all but at most $2\varepsilon n$ elements $z \in G$ satisfy*

$$\sum_{z=x_1+x_2} A(x_1)U(x_2) = d|U| \pm \varepsilon n.$$

Hence, together with the first part of the following lemma applied for $k = 2$, Fact 3.5 implies Corollary 3.4.

Lemma 3.6. *For an integer $k \geq 2$ and an $\varepsilon \in (0,1)$ the following holds. Given an abelian group G of order $|G| = n$ and a subset A of G.*

- *If A satisfies $\mathrm{BIAS}_d(\varepsilon)$ then for all $U_1, \dots, U_{k-1} \subset G$ all but at most $\varepsilon^{1/3}n$ elements $z \in G$ satisfy*

$$\sum_{z=x_1+\dots+x_k} A(x_k) \prod_{i\in[k-1]} U_i(x_i) = d \prod_{i\in[k-1]} |U_i| \pm 2\varepsilon^{1/3}n^{k-1}. \tag{3.5}$$

  In particular, in view of Fact 3.5 we infer that A satisfies $\mathrm{SHIFT}_{1,d}(2\varepsilon^{1/3})$.
- *If $|A| = (d \pm \varepsilon)n$ and all but at most εn elements $z \in G$ satisfy*

$$\sum_{z=x_1+\dots+x_k} \prod_{i\in[k]} A(x_i) = d^k n^{k-1} \pm \varepsilon n^{k-1} \tag{3.6}$$

  then A satisfies $\mathrm{BIAS}_d(2\varepsilon^{1/3})$.

Proof. We apply Lemma 2.3 with $A = f_k$ and $U_i = f_i$ for $i \in [k-1]$. The lemma then states that

$$\frac{1}{n}\sum_{\eta\in\widehat{G}} |\widehat{A}(\eta)|^2 \prod_{i\in[k-1]} |\widehat{U}_i(\eta)|^2 = \sum_{z\in G}\Big(\sum_{z=x_1+\dots+x_k} A(x_k) \prod_{i\in[k-1]} U_i(x_i)\Big)^2, \tag{3.7}$$

where $|\widehat{A}(\widehat{0})|^2 \prod_{i\in[k-1]} |\widehat{U}_i(\widehat{0})|^2 = |A|^2 \prod_{i\in[k-1]} |U_i|^2$.

To show the first property of the lemma suppose that $|A|$ satisfies $\mathrm{BIAS}_d(\varepsilon)$. Then $|A| = (d \pm \varepsilon)n$. Further, Parseval's identity (2.4), and $|\widehat{U}_i(\eta)| \leq |\widehat{U}_i(\widehat{0})| =$

$|U_i| \le n$ implies

$$\begin{aligned}\sum_{z\in G}\Big(\sum_{z=x_1+\dots+x_k} A(x_k)\prod_{i\in[k-1]}U_i(x_i)\Big)^2 - \frac{1}{n}|A|^2\prod_{i\in[k-1]}|U_i|^2 \\ &\le \frac{1}{n}\sum_{\eta\in\widehat{G}\setminus\{\widehat{0}\}}|\widehat{A}(\eta)|^2\prod_{i\in[k-1]}|\widehat{U}_i(\eta)|^2\\ &\le \frac{1}{n}\max_{\eta\neq\widehat{0}}|\widehat{A}(\eta)|^2\prod_{i\in[k-2]}|\widehat{U}_i(\eta)|^2\sum_{\eta\in\widehat{G}\setminus\{\widehat{0}\}}|\widehat{U}_{k-1}(\eta)|^2\\ &\le \varepsilon^2 n^{2k-3}\cdot n\sum_{x\in G}|U_{k-1}(x)|^2 \le \varepsilon^2 n^{2k-1}.\end{aligned}$$

Hence,

$$\sum_{z\in G}\Big(\sum_{z=x_1+\dots+x_k} A(x_k)\prod_{i\in[k-1]}U_i(x_i)\Big)^2 \le n\left(\frac{|A|}{n}\prod_{i\in[k-1]}|U_i|\right)^2 + \varepsilon^2 n^{2k-1}.$$

On the other hand

$$\sum_{z\in G}\sum_{z=x_1+\dots+x_k} A(x_k)\prod_{i\in[k-1]}U_i(x_i) = |A|\prod_{i\in[k-1]}|U_i| = n\left(\frac{|A|}{n}\prod_{i\in[k-1]}|U_i|\right).$$

Fact 2.2 then implies that all but at most $\varepsilon^{1/3}n$ elements z satisfy

$$\sum_{z=x_1+\dots+x_k} A(x_k)\prod_{i\in[k-1]}U_i(x_i) = \frac{|A|}{n}\prod_{i\in[k-1]}|U_i| \pm \varepsilon^{1/3}n^{k-1}.$$

Together with $|A| = (d\pm\varepsilon)n$ this implies that (3.5) is satisfied.

To show the second property of the lemma suppose that $|A| = (d\pm\varepsilon)n$ and (3.6) holds. Then $|\widehat{A}(\widehat{0})| = (d\pm\varepsilon)n$ and together with the identity (3.7), applied for $U_i = A$ for all $i\in[k-1]$, this implies that

$$\sum_{\eta\in\widehat{G}\setminus\{\widehat{0}\}}|\widehat{A}(\eta)|^{2k} \le n\sum_{z\in G}\Big(\sum_{z=x_1+\dots+x_k} A(x_k)\prod_{i\in[k-1]}U_i(x_i)\Big)^2 - (d-\varepsilon)^{2k}n^{2k} \le 4\varepsilon n^{2k}.$$

We derive that

$$\max_{\eta\in G\setminus\{\widehat{0}\}}|\widehat{A}(\eta)| \le \Big(\sum_{\eta\in\widehat{G}\setminus\{\widehat{0}\}}|\widehat{A}(\eta)|^{2k}\Big)^{1/2k} \le 2\varepsilon^{1/2k}n.$$

Hence, A satisfies $\mathrm{BIAS}_d(2\varepsilon^{1/2k})$. □

Proof of Theorem 1.4. The first part of Lemma 3.6, applied for a general $k\ge 2$, yields that R_4' implies $S_5(k)$. It is clear that $S_5(k)$ implies $R_5'(k)$ and by the second part of Lemma 3.6 the property $R_5'(k)$ implies R_4'.

To establish the equivalence with R_2' first note that by Fact 3.5 and (3.4) the property R_2' is equivalent to $S_5(2)$. We apply Lemma 3.6 with $k=2$ which implies that if R_4' holds then so does $S_5(2)$. Clearly, $S_5(2)$ impleis $R_5'(2)$ which which implies R_4' by the second part of Lemma 3.6 This finishes the proof of the theorem. □

4. Uniform edge distribution implies small linear bias

In this section we give two proofs of the second part of Theorem 1.3, showing that uniform edge distribution implies small linear bias.

4.1. Hypergraph spectral gap implies small linear bias.

We show that spectral gap for $H_*^{(k)}(A)$ implies small linear bias for A. Due to Fact 2.1 and Theorem 2.5, this shows that uniform edge distribution implies small linear bias.

Lemma 4.1. *For all integer $k \geq 2$, all $d, \varepsilon \in (0,1)$ the following holds. If a subset $A \subset G$ of an abelian group G of order n is such that $H_*^{(k)}(A)$ satisfies $\mathrm{EIG}_d^*(\varepsilon)$, then A satisfies $\mathrm{BIAS}_d(2^k\varepsilon)$.*

Proof. Let $k \geq 2$, $d \in (0,1)$ and $\varepsilon > 0$ be given. As $H_*^{(k)}(A)$ is $\lfloor dn \rfloor$-coregular we have $|A| = \lfloor dn \rfloor = (d \pm \varepsilon)n$. We need to show that for all $\eta \in \widehat{G} \setminus \{\widehat{0}\}$

$$\left|\widehat{A}(\eta)\right| = \Big|\sum_{x \in G} A(x)\overline{\eta(x)}\Big| \leq 2^k \varepsilon n.$$

To this end first extend σ_{H_*} to a complex multilinear map $\mathbb{C}^n \to \mathbb{C}$ by defining for each $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$ and each of the k coordinates

$$\begin{aligned} \sigma_{H_*}(\dots, \boldsymbol{u} + i\boldsymbol{v}, \dots) &= \sigma_{H_*}(\dots, \boldsymbol{u}, \dots) + i \cdot \sigma_{H_*}(\dots, \boldsymbol{v}, \dots) \\ \sigma_{H_*}(\dots, (\alpha + i\beta)(\boldsymbol{u} + i\boldsymbol{v}), \dots) &= (\alpha + i\beta) \cdot \sigma_{H_*}(\dots, \boldsymbol{u} + i\boldsymbol{v}, \dots). \end{aligned} \tag{4.1}$$

Similarly we define the complex extension of τ_{H_*} and J_*.

Given $\eta \in \widehat{G} \setminus \{\widehat{0}\}$ and let $x_1, \dots, x_n$ be the elements of G. Define the unit vector

$$\boldsymbol{x}_\eta = \frac{1}{\sqrt{n}}\big(\eta(x_1), \dots, \eta(x_n)\big) = \frac{1}{\sqrt{n}}\boldsymbol{u}_\eta + \frac{i}{\sqrt{n}} \cdot \boldsymbol{v}_\eta \qquad \text{where} \qquad \boldsymbol{u}_\eta, \boldsymbol{v}_\eta \in \mathbb{R}^n.$$

Via (4.1) we have

$$\sigma_{H_*}(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta) = \sum_{j=0}^{k} \binom{k}{j} i^j \sigma_{H_*}\Big(\underbrace{\frac{\boldsymbol{u}_\eta}{\sqrt{n}}, \dots, \frac{\boldsymbol{u}_\eta}{\sqrt{n}}}_{(k-j)\text{-times}}, \frac{\boldsymbol{v}_\eta}{\sqrt{n}}, \dots, \frac{\boldsymbol{v}_\eta}{\sqrt{n}}\Big).$$

As $H_*^{(k)}(A)$ satisfies $\mathrm{EIG}_d^*(\varepsilon)$ we infer

$$\big|\sigma_{H_*}\big(\boldsymbol{u}_\eta/\sqrt{n}, \dots, \boldsymbol{u}_\eta/\sqrt{n}, \boldsymbol{v}_\eta\sqrt{n}, \dots, \boldsymbol{v}_\eta/\sqrt{n}\big)\big| \leq \|\sigma_{H_*}\| \leq \varepsilon n^{k/2}.$$

Hence,

$$|\sigma_{H_*}(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta)| \leq 2^k \|\sigma_{H_*}\| \leq 2^k \varepsilon n^{k/2}. \tag{4.2}$$

On the other hand write $\boldsymbol{x}_\eta = \frac{1}{\sqrt{n}} \sum_{i \in [n]} \eta(x_i) \boldsymbol{e}_{x_i}$ where $\boldsymbol{e}_{x_i}$ is the indicator vector of x_i. Recall that J_* is the k-linear map with $J_*(\boldsymbol{e}_{v_1}, \dots, \boldsymbol{e}_{v_k}) = 1$ for all $v_1, \dots, v_k \in G$. Multi-linearity and the fact that $\eta \colon G \to \mathbb{T}$ is a homomorphism yields

$$\begin{aligned} J_*(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta) &= \frac{1}{n^{k/2}} \sum_{v_1, \dots, v_k \in G} J_*(\boldsymbol{e}_{v_1}, \dots, \boldsymbol{e}_{v_k}) \prod_{i \in [k]} \eta(v_i) \\ &= \frac{1}{n^{k/2}} \sum_{v_1, \dots, v_k \in G} \eta\Big(\sum_{i \in [k]} v_i\Big) = \frac{n^{k-1}}{n^{k/2}} \sum_x \eta(x) = 0 \end{aligned}$$

where the last equality follows from the orthogonality relation (2.3) and $\eta \neq \widehat{0}$.

Furthermore, for indicator vectors $\boldsymbol{e}_{v_1}, \dots, \boldsymbol{e}_{v_k}$ we have $\tau_{H_*}(\boldsymbol{e}_{v_1}, \dots, \boldsymbol{e}_{v_k}) \in \{0,1\}$ and which assumes the value one if and only if $v_1 \dots v_k$ is an edge in $H_*^{(k)}(A)$, which happens if and only if $A(v_1 + \dots + v_k) = 1$. Therefore,

$$\begin{aligned}\tau_{H_*}(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta) &= \frac{1}{n^{k/2}} \sum_{v_1,\dots,v_k \in G} \tau_{H_*}(\boldsymbol{e}_{v_1}, \dots, \boldsymbol{e}_{v_k}) \prod_{i\in[k]} \eta(v_i) \\ &= \frac{1}{n^{k/2}} \sum_{v_1,\dots,v_k \in G} A\Big(\sum_{i\in[k]} v_i\Big)\eta\Big(\sum_{i\in[k]} v_i\Big) \\ &= \frac{n^{k-1}}{n^{k/2}} \sum_x A(x)\eta(x) = n^{(k-2)/2}\widehat{A}(\eta).\end{aligned}$$

In summary, $\tau_{H_*}(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta)$ is a multiple of $\widehat{A}(\eta)$, and $J_*(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta) = 0$. Together with (4.2) we obtain therefore that for all $\eta \neq \widehat{0}$

$$n^{(k-2)/2}|\widehat{A}(\eta)| = |\sigma_{H_*}(\boldsymbol{x}_\eta, \dots, \boldsymbol{x}_\eta)| \le 2^k \varepsilon n^{k/2}$$

and the lemma follows. □

4.2. **Uniform edge distribution implies small linear bias.** In this section we show that the expansion property of $H_*^{(k)}(A)$ implies small linear bias for A which, in view of Fact 2.1, shows that uniform edge distribution implies small linear bias.

Lemma 4.2. *For all integer $k \ge 2$ and all $d, \varepsilon \in (0,1)$ the following holds. If a subset $A \subset G$ of an abelian group G of order n is such that $H_*^{(k)}(A)$ satisfies* $\mathrm{EXPAND}_d^*(\varepsilon)$*, then A satisfies* $\mathrm{BIAS}_d(2\varepsilon^{1/2k})$.

Proof. For a set $A \subset G$ consider $H_* = H_*^{(k)}(A)$ with the balanced characteristic function $h_*(x_1, \dots, x_k) = \mathbf{1}_{H_*}(\{x_1, \dots, x_k\}) - d$ for all $(x_1, \dots, x_k) \in G^k$. Let $f = A - d$ and note that for every $(x_1, \dots, x_k) \in G^k$ we have

$$\mathbf{1}_{H_*}(x_1, \dots, x_k) = A\Big(\sum_{x_i \in e} x_i\Big) \quad \text{hence} \quad h_*(x_1, \dots, x_k) = f\Big(\sum_{i\in[k]} x_i\Big). \tag{4.3}$$

Further, for $\eta \in \widehat{G}$ we have

$$\widehat{f}(\eta) = \sum_{x\in G} \big(A(x) - d\big)\overline{\eta(x)} = \widehat{A}(\eta) - d\sum_{x\in G} \overline{\eta(x)} = \begin{cases} \widehat{A}(\eta) & \text{if } \eta \neq \widehat{0} \\ |A| - dn & \text{otherwise.} \end{cases}$$

To show the lemma it is therefore sufficient to show that

$$|\widehat{f}(\widehat{0})| \le \varepsilon n \quad \text{and} \quad \|\widehat{f}\|_{2k} = \Big(\sum\nolimits_{\eta\in\widehat{G}} \big|\widehat{f}(\eta)\big|^{2k}\Big)^{1/2k} \le 2\varepsilon^{1/2k} n \tag{4.4}$$

as

$$\max\Big\{|\widehat{A}(\eta)| \colon \eta \in \widehat{G} \setminus \{\widehat{0}\}\Big\} \le \max\Big\{|\widehat{f}(\eta)| \colon \eta \in \widehat{G}\Big\} = \|\widehat{f}\|_\infty \le \|\widehat{f}\|_{2k}.$$

Suppose that H_* satisfies $\mathrm{EXPAND}_d^*(\varepsilon)$. Then the first part of (4.4) follows from

$$\begin{aligned}\widehat{f}(\widehat{0}) = \sum_{x\in G} f(x) &= \frac{1}{n^{k-1}} \sum_{(x_1,\dots,x_k)\in G^k} f\Big(\sum x_i\Big) \\ &= \frac{1}{n^{k-1}} \sum_{(x_1,\dots,x_k)\in G^k} h_*(x_1, \dots, x_k) \\ &= \frac{1}{n^{k-1}}\big(e_{H_*}(G, \dots, G) - dn^k\big) = \pm\varepsilon n.\end{aligned}$$

To establish the second part of (4.4) we apply Lemma 2.3 with $f_1 = \cdots = f_k = f$ so that the left hand side of (2.7) in Lemma 2.3 is $\frac{1}{n}\|\widehat{f}\|_{2k}^{2k}$. Further, let g be the function given by $g(a) = f(-a)$ for all $a \in G$. Then the lemma yields

$$\begin{aligned}\tfrac{1}{n}\|\widehat{f}\|_{2k}^{2k} &= \sum_{x\in G}\Big(\sum_{x=x_1+\cdots+x_k}\prod_{i\in[k]} f(x_i)\Big)^2 = \sum_{x\in G}\sum_{\substack{x=\sum x_i\\ x=\sum y_i}}\prod_{i\in[k]} f(x_i)f(y_i)\\ &= \sum_{x\in G}\sum_{\substack{(x_2,\dots,x_k)\\(y_2,\dots,y_k)}} f\Big(x-\sum_{i=2}^k x_i\Big)f\Big(x-\sum_{i=2}^k y_i\Big)\prod_{i=2}^k f(x_i)f(y_i)\\ &= \sum_{\substack{(x,x_2,\dots,x_k)\in G^k\\(y_2,\dots,y_k)\in G^{k-1}}} f\Big(x+\sum_{i=2}^k x_i\Big)f\Big(x+\sum_{i=2}^k y_i\Big)\prod_{i=2}^k g(x_i)g(y_i).\end{aligned}$$

By Lemma 2.6 and (4.3), however, the last expression is at most $2^k\varepsilon n^{2k-1}$. Rearranging and taking root yields the second part of (4.4). □

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