ON COLORABILITY OF GRAPHS WITH FORBIDDEN MINORS ALONG PATHS AND CIRCUITS

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Abstract. Graphs distinguished by K_r-minor prohibition limited to subgraphs induced by circuits have chromatic number bounded by a function $f(r)$; precise bounds on $f(r)$ are unknown. If minor prohibition is limited to subgraphs induced by simple paths instead of circuits, then for certain forbidden configurations, we reach tight estimates.

A graph whose simple paths induce $K_{3,3}$-minor free graphs is proven to be 6-colorable; K_5 is such a graph. Consequently, a graph whose simple paths induce planar graphs is 6-colorable. We suspect the latter to be 5-colorable and we are not aware of such 5-chromatic graphs. Alternatively, (and with more accuracy) a graph whose simple paths induce $\{K_5, K_{3,3}^-\}$-minor free graphs is proven to be 4-colorable (where $K_{3,3}^-$ is the graph obtained from $K_{3,3}$ by removing a single edge); K_4 is such a graph.

Keywords: Chromatic number, bridges of circuits.

1. Introduction. Throughout, a graph is finite and simple; notation and terminology follow [2] when possible. G always denotes a graph. If H is a subgraph of G, then the subgraph of G induced by H is $G[V(H)]$.

A graph whose even circuits induce bipartite graphs is 3-colorable [5, Proposition 4]. To continue in this spirit of restricting the chromatic number of subgraphs induced by circuits and study the effect of such a restriction on the chromatic number of the host graph, we inquire as to the chromatic number of graphs whose circuits induce graphs that are K_r-minor free, for $r \in \mathbb{N}$. Indeed, K_r-minor free graphs have their chromatic number bounded by some $f(r)$ [1, 8, 9]. Lemma 2 of [3] states that: *if a graph G satisfies $\delta(G) \geq d \geq 5$, then G has a circuit C and a subgraph H satisfying $V(H) \subseteq V(C), E(H) \subseteq E(G) \setminus E(C)$, and $\delta(H) \geq 1 + d/6$.* From this lemma and the main results of [1, 8, 9] we infer that:

Theorem 1.1 *There exist constants $c_1, c_2 \in \mathbb{R}$ such that for every $r \in \mathbb{N}$:*
(a) a graph whose circuits induce graphs containing no topological K_r satisfies $\chi(G) \leq c_1 r^2$.
(b) a graph whose circuits induce K_r-minor free graphs satisfies $\chi(G) \leq c_2 r\sqrt{\log r}$.

Prior to Theorem 1.1 these asymptotic bounds on $\chi(G)$ were known [1, 8, 9] only in the case that K_r-minor prohibition extends to the entire graph. Graphs containing (topological) K_r-minors none of which is present in a subgraph induced by a circuit are easily found.

It is unknown whether the bounds in Theorem 1.1 on χ are best possible. For $r = 4$, we observe that: *a graph with minimum degree at least 3 contains a circuit with two overlapping chords* [11]. Consequently: *a graph whose circuits induce series-parallel graphs (i.e., has no K_4-minor) is 3-colorable.*

Questions of [4, 5] (mentioned here for mere inspiration) are clearly well beyond our scope and methods:
(I) *Is it true that for every $k \in \mathbb{N}$ there exists an $f(k)$ such that a graph G whose odd circuits induce k-colorable graphs has $\chi(G) \leq f(k)$?*
(II) *Is there a constant $c \in \mathbb{N}$ such that a graph G whose simple paths induce 3-colorable graphs has $\chi(G) \leq c$? Is it true that $c = 4$?*

In this note, we consider graphs characterized by forbidden minors, where the minor prohibition is restricted to the simple paths of the graph. Let $\mathcal{G}$ denote the class of graphs whose paths induce $\{K_5, K_{3,3}^-\}$-minor free graphs, and let $\mathcal{G}'$ denote the class of graphs whose paths induce $K_{3,3}$-minor free graphs. Observing that $K_4 \in \mathcal{G}$ and $K_5 \in \mathcal{G}'$ (actually, K_5-cockades of $\kappa = 2$ are also in $\mathcal{G}'$), we prove:

Theorem 1.2 *If $G \in \mathcal{G}$, then $\chi(G) \le 4$.*

Theorem 1.3 *If $G \in \mathcal{G}'$, then $\chi(G) \le 6$.*

Consequently,

Corollary 1.4 *A graph whose paths induce planar graphs is* 6*-colorable.*

Remarks:
(i) Theorem 1.2 relies on a theorem of [11].
(ii) In our proof of Theorem 1.2, the effect of the K_5-minor prohibition is negligible. The results of a forthcoming paper [6] imply that apart from K_5, the all graphs in which every path induces a $K^-_{3,3}$-minor free subgraph are 4-colorable. Let $\mathcal{G}''$ denote this class of graphs.

In [6], we describe the so called *almost series-parallel*(ASP) graphs which are the graphs containing no topological K_4 whose edges with both ends at least 3-valent induce a graph isomorphic to P_3 or P_4. It turns out that there is a class of graphs $\mathcal{C}$ such that $\mathcal{G}'' \subseteq \mathcal{C} \subset$ ASP. Through the description of the ASP graphs we are able to determine that: *apart from K_5, the members of $\mathcal{C}$ are 4-colorable.*

Sadly, the arguments of [6] are not short and employ an approach that is different and more sophisticated than that used in the proof of Theorem 1.2. The "simplicity" of the proof of Theorem 1.2, which we count as a merit, indicates that perhaps the use of a full blown structure theorem of the ASP graphs can be avoided. We do not know how to avoid it, nor do we know how to amend the arguments here so as to reach the same assertion for $\mathcal{G}''$.
(iii) Theorem 1.3 is a corollary of the Four Color Theorem and a theorem of [7, 10].
(iv) We suspect members of $\mathcal{G}'$ to be 5-colorable; or to the very least that 6-chromatic members of $\mathcal{G}'$ form a finite and probably a small subset of $\mathcal{G}'$.
(v) We are not aware of 3-connected 5-chromatic members of $\mathcal{G}'$.
(vi) We are not aware of 5-chromatic graphs of the form of Corollary 1.4.
(vii) We suspect that the replacement of "simple path" by "circuit" in Theorems 1.2 and 1.3 should not affect the bounds on the chromatic number. Our proofs do not support such a replacement.

2. Preliminaries. Let C be a circuit such that $x, y \in V(C)$ and $\emptyset \ne S \subseteq V(C) \setminus \{x, y\}$. If there is an xy-segment of C not meeting S, then it is denoted $[x, y]_S$; we omit S if it is understood from the context. The segment $[x, y] \setminus \{x, y\}$ is denoted (x, y); the segments $(x, y]$ and $[x, y)$ are defined in a similar manner whenever x and y are distinct. In addition, we define $[x, x) = (x, x] = \{x\}$.

Let G be a graph and let H be a subgraph of G. By *H-bridge* we mean either an edge $e = uv$ such that $u, v \in V(H)$ and $e \notin E(H)$, or a component C of $G - H$ together with all edges (and their ends) which have one end in H and the other in C. A bridge of the former type is called *trivial* and forms a *chord* of H. A bridge of the latter type is called *nontrivial*. If B is an H-bridge, then the vertices $V(B) \cap V(H)$[1] are called the *vertices of attachment* of B and are denoted *attach*(B). An H-bridge with $S \subseteq$ *attach*(B) is called an *(S, H)-bridge*. If $S = \{x\}$, we write *(x, H)-bridge*. We omit H if it is understood from the context.

A path is *trivial* if it consists of a single edge. A graph is *traceable* if it contains a Hamilton path.

Trivially, if B is an $(\{x, y\}, H)$-bridge, then there exists an xy-path that is internally disjoint of H. If B is nontrivial, then there exists a nontrivial such path. Further, such a nontrivial xy-path is called an *xy-ear* of H. A union of a circuit and its ear is a traceable graph. Such a union is called an *extension* of C; so that a union of C with its $\{x, y\}$-ear is a *traceable $\{x, y\}$-extension* of C. Alternatively, if C' is a traceable $\{x, y\}$-extension of C, then the *handle* of C' is the ear of C present in C'. By definition, handles are nontrivial paths.

A K_5-minor with connected subgraphs G_i, $i \in [5]$, is denoted $\{G_1, G_2, G_3, G_4, G_5\}$. A $K^-_{3,3}$-minor with connected subgraphs G_i, $i \in [6]$, is denoted $\{(G_1, G_2, G_3), (G_4, G_5, G_6)\}$ such that G_i, $i \in [3]$, form one partite set, G_i, $4 \le i \le 6$, form the other, and G_3, G_6 represent the nonadjacent vertices of $K^-_{3,3}$ on opposing parts. In any case, if $V(G_i) = \{u\}$, then instead of G_i, we write u.

[1]If B is an edge, then $V(B)$ consists of its ends.

3. Proof of 1.3. Suppose that $G \in \mathcal{G}'$ is a counterexample to Theorem 1.3 with G minimal; so that G is 7-critical. A 3-connected nonplanar graph on ≥ 6 vertices contains a circuit with 3 pairwise overlapping chords [7, 10], i.e., a subdivided $K_{3,3}$ spanned by a circuit. Thus, $\kappa(G) = 2$.

Let $\{u, v\} \subset V(G)$ be an extremal 2-disconnector of G; that is $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{u, v\}$, $V(G_i) \setminus \{u, v\} \neq \emptyset$, $i = 1, 2$, and G_2 is minimal subject to not containing 2-disconnectors of G. Clearly, $uv \notin E(G)$ and $\kappa(G_2 + uv) \geq 3$.

By the minimality of G, $\chi(G_1) \leq 6$. We may assume that $\chi(G_2) \geq 5$; for otherwise a 6-coloring of G_1 and a 4-coloring of G_2 can be combined into a 6-coloring of G. Hence, G_2 is nonplanar (by the Four Color Theorem). In addition, G_2 has order ≥ 6; for otherwise $G_2 \subseteq K_5^-$ (that is K_5 with a single edge removed) which is 4-colorable (recall that $uv \notin E(G)$).

It follows that $G_2 + uv$ contains a subdivided $K_{3,3}$, call it K, of the form described above. Let C denote the circuit contained in K. Clearly, $uv \in E(K)$ (as a chord of C or $uv \in E(C)$) otherwise $G \notin \mathcal{G}'$. Replacing uv with a uv-path internally contained in G_1 defines a traceable subgraph of G containing a subdivided $K_{3,3}$; a contradiction. ∎

4. Proof of 1.2. The following will be used to eliminate K_4-subgraphs in a minimal counterexample for Theorem 1.2.

Claim 4.1 *Let $G \in \mathcal{G}$, $\kappa(G) \geq 2$, $\delta(G) \geq 4$. If G contains a subgraph $K \cong K_4$, then $G - K$ is disconnected.*

Proof: Let $V(K) = \{y_1, y_2, y_3, y_4\}$, and assume towards contradiction that $G - K$ is connected. As $\delta(G) \geq 4$, each vertex in $V(K)$ has at least one neighbor in $V(G - K)$. Let $\mathcal{A} = \{x_1, x_2, x_3, x_4\} \subseteq V(G - K)$ such that each member in $V(K)$ has a single neighbor in $\mathcal{A}$, and let $x_i \in N_{G-K}(y_i)$. We show that we may assume that $|\mathcal{A}| = 4$. Indeed, if $|\mathcal{A}| = 1$, then G contains K_5 as a subgraph which is a contradiction. Next, if $|\mathcal{A}| = 2$, then either, without loss of generality, $x_1 = x_2$ and $x_3 = x_4$, or, without loss of generality $x_1 = x_2 = x_3$ and x_4 is a second distinct vertex. In the former case, $G[V(K) \cup \mathcal{A}]$ is isomorphic to $K_{3,3}^-$ and is traceable. In the latter case, the assumption that $G - K$ is connected implies that there exists an x_1x_4-path P in $G - K$. Let P' be a y_1y_4-path in G such that $P' = P + y_1 + y_1x_1 + y_4 + y_4x_4$, and $V(K) \cap V(P') = \{y_1, y_4\}$. There exists a path P'' such that $V(P'') = V(K) \cup V(P')$. P'' spans a K_5-minor $\{y_1, y_2, y_3, y_4, P\}$.

Finally, suppose that $|\mathcal{A}| = 3$, and assume, without loss of generality, that $x_1 = x_2$ and that x_3 and x_4 are two vertices distinct from x_1. Let P be an x_3x_4-path in $G - K$; the latter exists since $G - K$ is assumed to be connected. If $x_1 \notin V(P)$, then $G[V(P) \cup V(K) \cup \{x_1\}]$ is a traceable graph subcontractible to $K_{3,3}^-$: $\{(y_1, y_2, P), (y_3, y_4, x_1)\}$. If $x_1 \in V(P)$, then $G[V(P) \cup V(K)]$ is a traceable graph subcontractible to K_5 $\{P, y_1, y_2, y_3, y_4\}$.

We may assume that $|\mathcal{A}| = 4$. In addition, we may assume that there is no partition $(\mathcal{B}, \mathcal{C})$ of $\mathcal{A}$ into pairs such that in $G - K$ there are two vertex-disjoint $(\mathcal{B}, \mathcal{C})$-paths. Indeed, if, to the contrary and without loss of generality, P_1 is an x_1x_2-path and P_2 is an x_3x_4-path in $G - K$ such that P_1 and P_2 are vertex-disjoint, then $G[V(P_1) \cup V(P_2) \cup V(K)]$ is a traceable graph subcontractible to $K_{3,3}^-$: $\{(y_1, y_2, P_2), (y_3, y_4, P_1)\}$. Thus, there exist: a vertex $x \in V(G - K)$, an x_1x_2-path Q_1, and an x_3x_4-path Q_2, in $G - K$, such that $V(Q_1) \cap V(Q_2) = \{x\}$ (possibly $x \in \mathcal{A}$). Hence, $G[V(Q_1) \cup V(Q_2) \cup V(K)]$ is traceable and contains the K_5-minor: $\{Q_1 \cup Q_2, y_1, y_2, y_3, y_4\}$. ∎

Proof of 1.2. Suppose that $G \in \mathcal{G}$ is a counterexample to Theorem 1.2 with G minimal; so that G is 5-critical. Consequently, G is 2-connected, $\delta(G) \geq 4$, and

(1) *G has no set $S \subseteq V(G)$ such that $G - S$ is disconnected, $|S| \leq 4$, and $G[S]$ is a complete graph.*

G contains a circuit with two overlapping chords such that two end vertices of these chords are joined by an edge of the circuit [11]. Let C denote a circuit of G satisfying the following:

(2) C contains two overlapping chords xz and yw such that x, w, z, y appear in this order along C.

(3) $yz \in E(C)$.

(4) subject to (2) and (3), $|V(C)|$ is minimum.

(5) subject to (2), (3), and (4), $|[x, w]_y|$ is minimum.

Agreement. Throughout the remainder of this proof, unless otherwise stated, segments of the form $[u, v], (u, v)$ and so on always refer to segments of C where $u, v \in V(C)$. In addition, the interior of such a segment does not meet $\{x, w, z, y\}$.

At least one of the sets (w, z) or (y, x) is empty. For otherwise, $G[C]$ is subcontractible to $K^-_{3,3}$: $\{([x, w), y, (w, z)), (w, z, (y, x))\}$. Hence,

(6) *$xy \in E(C)$ or $zw \in E(C)$.*

Due to symmetry, in subsequent arguments it is assumed that

(7) $xy \in E(C)$.

A traceable $\{y, h\}$-extension of C with $h \in (w, z)$ is subcontractible to $K^-_{3,3}$: $\{(y, (w, z), [x, w)), (z, w, P - \{y, h\})\}$, where P is the handle of the extension. A symmetrical argument establishes that a traceable $\{y, h\}$-extension of C with $h \in (x, w)$ is subcontractible to $K^-_{3,3}$. Consequently,

(8) *all $\{y, h\}$-bridges of C with $h \in (x, w) \cup (w, z)$ are trivial.*

By (5) and (8),

(9) *there are no $\{y, h\}$-bridges of C with $h \in (x, w)$.*

If (w, z) is nonempty, then a traceable $\{x, y\}$-extension of C is subcontractible to $K^-_{3,3}$: $\{(x, y, (w, z)), ((x, w], z, P - \{x, y\})\}$, where P is the handle of the extension. As G is simple, (7) implies that any $\{x, y\}$-bridge of C, apart from xy, is nontrivial. Hence,

(10) *if an $\{x, y\}$-bridge of C, other than xy, exists, then $(w, z) = \emptyset$ (and thus $zw \in E(C)$).*

A symmetrical argument to the one used for (10) establishes that

(11) *if a $\{y, z\}$-bridge of C exists, then $(x, w) = \emptyset$ (and thus $xw \in E(C)$).*

By (4), C has no chord uv such that $u, v \in [x, w]$ or $u, v \in [w, z]$. The latter and since $\{xy, yz\} \subset E(C)$, it follows that

(12) *$deg_{G[C]}(w) = 3$.*

If there is a $t \in N_G(y) \cap (w, z)$, then a traceable $\{k, g\}$-extension of C with $k \in [w, t)$ and $g \in (t, z]$, is subcontractible to $K^-_{3,3}$: $\{((k, g), [x, w), P - \{k, g\}), ((g, z], [w, k), y)\}$, where P is the handle of the extension. As, by (4), all $\{u, v\}$-bridges with $u, v \in [w, z]$ are nontrivial, it follows that

(13) *if there exists a vertex $t \in N_G(y) \cap (w, z)$, then G contains no $\{k, g\}$-bridges of C with $k \in [w, t)$ and $g \in (t, z]$.*

As G is 2 connected, a bridge of C has at least two vertices of attachment in C. Let $\mathcal{B}$ denote the set of $\{\ell, r\}$-bridges of C in G such that $\ell \in [x, w)$ and $r \in (w, z]$. We propose to consider two cases. Either there exists an $\{\ell, r\}$-bridge in $\mathcal{B}$ such that $\ell \in (x, w)$ or $r \in (w, z)$, or no such bridge exists in $\mathcal{B}$.

Case (*I*). Assume that

(14) *there is no $\{\ell, r\}$-bridge in $\mathcal{B}$ such that $\ell \in (x,w)$ or $r \in (w,z)$.*

By Claim 4.1 and (1), G does not contain K_4 as a subgraph. Consequently, we suggest three cases: $(I.1)$ $(x,w) \neq \emptyset$ and $(w,z) \neq \emptyset$, $(I.2)$ $(x,w) = \emptyset$ and $(w,z) \neq \emptyset$, (I.3) $(x,w) \neq \emptyset$ and $(w,z) = \emptyset$. Since the proof of (I.2) does not rely on (5), the case (I.3) is symmetrical to case $(I.2)$.

Case (*I*.1) Assume that $(x,w) \neq \emptyset$ and $(w,z) \neq \emptyset$. If B is a nontrivial y-bridge of C, then, since $\kappa(G) \geq 2$, B has a second vertex of attachment h. By (3), (7), (8), (9), (10), and (11), $h \notin V(C) \setminus \{w\}$. Consequently, y and w are the sole vertices of attachment of B. Hence, $\{y, w\}$ is a vertex-cut; contradicting (1). It follows that G contains no nontrivial y-bridges of C. By (9) and since $\delta(G) \geq 4$,

(15) *$N_G(y) \subseteq V(C)$, and $N_G(y) \setminus \{x, w, z\}$ is nonempty and is contained in the set (w, z).*

As G is simple, any $\{x, z\}$-bridge B, apart from xz, is nontrivial. If the sole vertices of attachment of B are x and z, then $\{x, z\}$ is a vertex-cut; contradicting (1). Thus, B contains a third vertex of attachment h. By (14), $h \notin (x,w) \cup (w,z)$. By (15), $h \neq y$. Hence, $h = w$. This is a contradiction as by (13) and (15) there are no $\{w, z\}$-bridges. It follows that

(16) *the edge xz is the sole $\{x, z\}$-bridge of C.*

G admits the following description:
(i) By (3) and (7), $\{yz, xy\} \subset E(C)$. By (4), G contains no chord uv of C such that $u, v \in [x, w]$, or $u, v \in [w, z]$. By (14), G contains no edge uv such that $u \in (x, w)$ and $v \in (w, z)$. Thus, $C' = G[V(C) \setminus \{y\}]$ is an induced circuit of G.

(ii) By (15), $N_G(y) = \{d_1, d_2, \ldots, d_m\} \subseteq V(C')$, $m \geq 4$, such that $d_1 = x$, $d_2 = w$, and $d_m = z$, and $d_1, d_2, \ldots, d_m$ appear in this order along C'.

(iii) Let $[d_i, d_{i+1}]$ denote the segment of C' the interior of which meets no member of $N_G(y)$. For $1 \leq i \leq m-1$, define $\mathcal{D}_i \subseteq V(G) \setminus V(C)$ to be the set of vertices such that each belongs to a C'-bridge of G attached to $[d_i, d_{i+1}]$. Define $G_i = G[\mathcal{D}_i \cup [d_i, d_{i+1}]]$. By (13), (14), and (16), for every nontrivial uv-path P in G such that $V(C) \cap V(P) = \{u, v\}$, there exists a single $1 \leq i \leq m-1$ such that $u, v \in [d_i, d_{i+1}]$. Consequently, $\mathcal{D}_j \cap \mathcal{D}_k = \emptyset$, for every $1 \leq j < k \leq m-1$, $V(G_j) \cap V(G_k) = \{d_{j+1}\}$ if $k = j+1$, and $V(G_j) \cap V(G_k) = \emptyset$ if $k > j+1$. By (13), (14), (16), and as C' is chordless, we have that except for the edge xz that connects a vertex of G_1 and a vertex of G_{m-1}, the graph G contains no edge uv such that $u \in V(G_j) \setminus V(G_k)$ and $v \in V(G_k) \setminus V(G_j)$, for $1 \leq j < k \leq m-1$.

By the minimality of G, $\chi(G_i) \leq 4$; for $1 \leq i \leq m-1$. We propose two cases; either there exists a subgraph G_i that admits a 4-coloring μ in which $\mu(d_i) \neq \mu(d_{i+1})$, or for every subgraph G_i, each 4-coloring μ of G_i has $\mu(d_i) = \mu(d_{i+1})$. In the latter case, the minimality of G implies that there exists a 4-coloring ψ of $G - y - xz$ such that $\psi(d_i) = \psi(d_{i+1})$, for $i \in [m-1]$. Hence, in every 4-coloring of $G - y - xz$, all vertices in the set $N_G(y)$ reside in the same color class. Consequently, $\chi(G - y) \geq 5$; in contradiction to the minimality of G.

In the complementary case, that there exists a subgraph G_i that has a 4-coloring μ in which $\mu(d_i) \neq \mu(d_{i+1})$, it is easy to see that a 4-coloring of $G - y$ with $N_G(y)$ colored with at most 3 colors exists; implying that $\chi(G) \leq 4$ which is a contradiction.

Case (*I*.2) Assume now that $(x, w) = \emptyset$ and $(w, z) \neq \emptyset$. By (10), (14), simplicity of G, and since $\{xy, xw\} \subset E(C)$, it follows that $deg_{G[C]}(x) = 3$. The bound $\delta(G) \geq 4$ implies that any additional neighbor of x lies in a nontrivial x-bridge B of C. Since G is 2-connected, B has a second vertex of attachment on C, say h. As (w, z) is assumed to be nonempty, then $h \neq y$, by (10), and $h \notin (w, z)$, by (14). Consequently, $h \in \{w, z\}$. If x and h are the sole vertices of attachment of B, then the set $\{x, w\}$ or $\{x, z\}$ is a vertex-cut; contradicting (1). Hence, it follows that

(17) *a nontrivial x-bridge of C exists; and every such bridge is a $\{w, x, z\}$-bridge of C.*

If B is a nontrivial $\{y, h\}$-bridge of C, then by (10), $h \neq x$. By (8), $h \notin (x, w) \cup (w, z)$. Thus, $h \in \{w, z\}$. If y and h are the sole vertices of attachment of B, then $\{y, w\}$ or $\{y, z\}$ are vertex-cuts; contradicting (1). Hence, it follows that

(18) *a nontrivial y-bridge is a $\{w, y, z\}$-bridge of C.*

Let B be a $\{w, x, z\}$-bridge of C. If B' is a $\{w, y, z\}$-bridge of C, then by (10), there are no $\{x, y\}$-bridges, and thus $(V(B) \cap V(B')) \setminus \{w, z\} = \emptyset$. Consequently, there exist a nontrivial xw-path P_1 and a nontrivial yz-path P_2 such that $V(P_1) \cap V(P_2) = \emptyset$ and each of these paths is internally disjoint of C. Let P_3 denote the subpath of C connecting w and z on C not containing x and y. Let P denote the path $P_1 \cup P_2 \cup P_3$. $G[P]$ is subcontractible to $K^-_{3,3}$: $\{(x, [w, z), P_2 - \{y, z\}), (y, z, P_1 - \{x, w\})\}$.

By the above argument, (17), and (18), we conclude that G contains no nontrivial y-bridges; implying that $N_G(y) \subseteq V(C)$. This and the bound $\delta(G) \geq 4$ then imply that there exists a vertex $g \in N_G(y) \setminus \{w, x, z\}$. As $\{xy, yz, xw\} \subset E(C)$, it follows that $g \in (w, z)$. Thus, by (13), G contains no $\{w, z\}$-bridges of C, contradicting (17).

This completes our proof of Case (I).

Case *(II)*. Assume then that there exist $\{\ell, r\}$-bridges in $\mathcal{B}$ such that $\ell \in (x, w)$ or $r \in (w, z)$. Let $B \in \mathcal{B}$ be such a bridge. If $r \neq z$, then $\ell \in [x, w)$ and any traceable $\{\ell, r\}$-extension of C is subcontractible to $K^-_{3,3}$: $\{(w, z, P - \{\ell, r\}), ([x, w), (w, z), y)\}$, where P is the handle of the extension. It follows that

(19) *if $r \neq z$, then B is trivial.*

If $\ell \neq x$ and $r = z$, then a symmetrical argument to the one used to establish (19) implies that $z\ell \in E(G)$. However, the existence of an edge of this form contradicts (5). Hence, the case that $\ell \neq x$ and $r = z$ does not occur. Consequently, in subsequent arguments we assume that

(20) *$r \neq z$ and $r\ell \in E(G)$.*

Any traceable $\{w, h\}$-extension of C with $h \in [x, w)$ is subcontractible to $K^-_{3,3}$: $\{(y, (w, z), V(P) - \{w, h\}), ([x, w), w, z)\}$, where P is the handle of the extension. Due to (4), if G contains a $\{w, h\}$-bridge where $h \in [x, w)$, then such a bridge is nontrivial. Hence, it follows that

(21) *G contains no $\{w, h\}$-bridge, where $h \in [x, w)$.*

A symmetrical argument to the one used for (21) establishes that

(22) *if in addition to (20), $\ell \neq x$ as well, then G contains no $\{w, h\}$-bridge, where $h \in (w, z]$.*

By (20), two cases are left to be considered. Either $\ell \neq x$, or $\ell = x$. In the former case, (12), the bound $\delta(G) \geq 4$, and 2-connectivity, assert that any additional neighbor of w lies in a w-bridge of C, say B', such that a second attachment vertex h of B' satisfies $h \notin [x, w) \cup (w, z]$, by (21) and (22). By (3), (7), $\{xy, yz\} \subset E(C)$. Hence, B' has exactly two vertices of attachment which are w and y. As $yw \in E(G)$ and G is simple, B' is nontrivial implying that $\{y, w\}$ is a vertex-cut; in contradiction to (1).

We may assume $\ell = x$. By (8), if a $\{y, h\}$-bridge with $h \in (w, z)$ exists, then it is trivial. If $h \in (w, r)$, then $G[V(C)]$ is subcontractible to $K^-_{3,3}$: $\{([x, w), (w, r), (r, z]), (y, r, w)\}$. If $h \in (r, z)$, then $G[V(C)]$ is subcontractible to $K_{3,3}^-$: $\{([x,w),(r,z),[w,r)),(y,r,z)\}$. By (20), (w,z) is nonempty. Thus, by these arguments, (9), and (10) it is seen that

(23) *G contains no $\{y,h\}$-bridge, where $h \in (w,r) \cup (r,z) \cup [x,w)$.*

Let $A \in \{(w,r),(r,z)\}$. If $A \neq \emptyset$ and $yr \in E(G)$, then $G[C]$ is subcontractible to $K_{3,3}^-$: $\{([x,w),y,A),(w,[r,z),z)\}$ or the minor $\{([x,w),y,A),(z,(w,r],w)\}$. Consequently, it follows from (8) that

(24) *if a $\{y,r\}$-bridge of C exists, then $yr \in E(G)$ and $(w,r) = (r,z) = \emptyset$, so that $\{wr, rz\} \subset E(C)$.*

If $yr \in E(G)$, then a traceable $\{w,r\}$-extension of C is subcontractible to $K_{3,3}^-$: $\{([x,w),y,P-\{w,r\}),(w,r,z)\}$, where P is the handle of the extension. By (4) and as G is simple, a $\{w,r\}$-bridge of C cannot be trivial. Consequently, by (8), it follows that

(25) *if a $\{y,r\}$-bridge of C exists, then a $\{w,r\}$-bridge of C does not exist.*

As $\delta(G) \geq 4$, the set $N_G(y) \setminus \{w,x,z\}$ is nonempty. Let $u \in N_G(y) \setminus \{w,x,z\}$. Two complementary cases are considered. Either $u \in V(C)$, or $u \notin V(C)$. In the former case, it follows from (3), (7), (8), (9), and (23) that $u = r$. In addition, (24) asserts that $\{rw, rz\} \subset E(C)$. By (12) and since G is 2-connected, G contains a nontrivial $\{w,h\}$-bridge D, where $h \in V(C)$. By (21), $h \notin [x,w)$. Since $N_G(y) \cap (w,z) \neq \emptyset$, then (13) implies that $h \neq z$. By (25), $h \neq r$. It follows that D has exactly two vertices of attachments in C and these are w and y. Since $yw \in E(G)$ and G is simple, D is nontrivial. This asserts that $\{y,w\}$ is a vertex-cut; a contradiction to (1).

Consider the complementary case that $u \notin V(C)$. Let D' denote the y-bridge of C containing u. As G is 2-connected, D' has a second vertex of attachment h' in $V(C)$. By (9), (10), (23) and the above argument for the case that $u \in V(C)$, we have that $h' \in \{z,w\}$. If y and h' are the sole vertices of attachment of D', then $\{y,w\}$ or $\{y,z\}$ form vertex-cut; contradiction to (1). Thus, D' is a $\{y,w,z\}$-bridge of C. Consequently, a traceable $\{w,z\}$-extension of C exists, which is contractible to $K_{3,3}^-$: $\{(y,(w,z),P-\{w,z\}),(w,z,[x,w))\}$, a contradiction.

This completes the proof of Case (II), and of Theorem 1.2. ∎

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References.

[1] B. Bollobás and A. G. Thomason. Proof of a conjecture of Mader, Erdős, and Hajnal on topological complete subgraphs. *Euro. J. Combin.*, 19:883–887, 1998.

[2] J. A. Bondy and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[3] G. Chen, P. Erdős, and W. Staton. Proof of a conjecture of Bollobás on nested cycles. *J. Combin. Theory Ser. B*, 66(1):38–43, 1996.

[4] P. Erdős. Some recent problems and results in graph theory. *Discrete Mathematics*, 164:81–85, 1997.

[5] A. Gyárfás. Fruit salad. *Electr. J. Combin.*, 4(1), 1997.

[6] E. Horev. Almost series-parallel graphs: structure and colorability (manuscript).

[7] A. Kelmans. Graph planarity and related topics. *Contemporary Methematics*, 147:635–667, 1993.

[8] J. Komlós and E. Szemerédi. Topological cliques in graphs II. *Combin., Probab. and Comp.*, 5:79 – 90, 1996.

[9] A. V. Kostochka. The minimum Hadwiger number for graphs with given mean degree of vertices (in russian). *Metody Diskret. Analiz.*, 38:37–58, 1982.

[10] C. Thomassen. A refinement of Kuratowski's theorem. *J. Combin. Theory B.*, 47:245–253, 1984.

[11] C. Thomassen and B. Toft. Non-separating induced cycles in graphs. *J. Combin. Theory Ser. B.*, 23:199–224, 1981.