Note

Polychromatic Colorings of Rectangular Partitions

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Abstract

A *rectangular partition* is a partition of a plane rectangle into an arbitrary number of non-overlapping rectangles such that no four rectangles share a corner. In this note, it is proven that every rectangular partition admits a vertex coloring with four colors such that every rectangle, except possibly the outer rectangle, has all four colors on its boundary. This settles a conjecture of Dinitz et al. [3]. The proof is short, simple and based on 4-edge-colorability of specific class of planar graphs.

Key words: polychromatic colorings, rectangular partitions

1 Introduction

A *polychromatic k-coloring* of a plane graph G is an assignment of k colors to the vertices of G such that each face of G, except possibly the outer face, has all k colors on its boundary. More formally, a polychromatic k-coloring of a plane graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$, such that for every internal face of G there exist k vertices $\{u_1, \ldots, u_k\}$ on its boundary such that $\varphi(u_i) = i$, for $i = 1, \ldots, k$. Note that a polychromatic k-coloring allows monochromatic edges. The *polychromatic number* of a plane graph G, namely $p(G)$, is the *maximum* number k such that G admits a polychromatic k-coloring.

A general and elegant result concerning polychromatic colorings of plane graphs was recently obtained by Alon *et al.* [1]. Let g be the length of a shortest face of a plane graph G. Clearly, $p(G) \leq g$. Alon *et al.* proved that for

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any plane graph G, $p(G) \geq \lfloor (3g-5)/4 \rfloor$, and showed that this bound is sufficiently tight by presenting plane graphs G for which $p(G) \leq \lfloor (3g+1)/4 \rfloor$. In addition, they proved that for a plane graph G, determining whether $p(G) \geq 3$ is NP-hard.

Mohar and Škrekovski [11] proved that every simple plane graph admits a polychromatic 2-coloring. Their proof is short and relies on the Four-Color Theorem. Bose *et al.* [2] provided an alternative proof that does not rely on the Four-Color Theorem. Hoffmann and Kriegel [6] proved that every 2-connected bipartite plane graph can be transformed into an Eulerian triangulation by adding edges only. Since every plane Eulerian triangulation is 3-colorable in the regular sense [13], it follows that every 2-connected bipartite plane graph admits a polychromatic 3-coloring. Horev and Krakovski [9] proved that every plane graph of degree at most 3, other than K_4 and a subdivision of K_4 on five vertices, admits a polychromatic 3-coloring. Finally, Horev *et al.* [7] proved that every 2-connected cubic bipartite plane graph admits a polychromatic 4-coloring. This result is tight, since any such graph must contain a face of size four.

A *rectangular partition* is a partition of a plane rectangle into an arbitrary number of non-overlapping rectangles, such that no four rectangles meet at a common vertex (see Fig. 1 for an illustration). The *order* of a rectangular partition is the number of rectangles in the partition including the outer rectangle. One may view a rectangular partition as a plane graph whose vertices are the corners of the rectangles and edges are the line segments connecting these corners. Consequently, we refer to the corners of a rectangular partition as its vertices.

A rectangle r of a rectangular partition R may have numerous vertices of R on its boundary. However, the rectangle r is defined by a set of exactly four vertices of R, denoted $D(r)$. Two rectangles r_1 and r_2 of R are said to be *incident* in R if $D(r_1) \cap D(r_2) \neq \emptyset$. Moreover, every vertex u of R such that $u \in D(r_1) \cap D(r_2)$ is called a *common incidence vertex* of r_1 and r_2. Note that every vertex of R is a common incidence vertex between some two rectangles of R.

A stronger extension of polychromatic 4-colorings of rectangular partitions is to require a coloring that for every rectangle all four colors appear on the four vertices defining it. More formally, we define a *strong polychromatic 4-coloring* of a rectangular partition R as a vertex coloring of R with four colors such that every rectangle r of R has all four colors appearing in the vertex set $D(r)$.

Guillotine partitions (subdivisons) are a well-studied subfamily of rectangular partitions. Horev *et al.* [8] showed that every guillotine partition admits a strong polychromatic 4-coloring. Keszegh [10] extended that result for higher dimensions, showing that every n-dimensional guillotine partition, $n > 2$, admits a vertex coloring with 2^n colors, such that every n-dimensional box of the partition (except the outer one) has all 2^n colors on its corners.

Dinitz *et al.* [3] proved that every rectangular partition admits a polychromatic 3-coloring, and conjectured that every rectangular partition admits a polychromatic 4-coloring. In this note, we prove the conjecture raised by Dinitz *et al.* in [3]. Actually, we prove a stronger claim by showing that every rectangular partition admits a strong polychromatic 4-coloring. Our proof is short and simple, and is based on 4-edge-colorability of planar 4-graphs (see Definition 1). The latter result relies on the Four-Color Theorem.

2 r-graphs and Polychromatic 4-Colorings of Rectangular Partitions

For a (multi)-graph G we write $V(G)$ to denote the vertex set of G. For a vertex set $X \subseteq V(G)$ the set of edges with one endpoint in X and the other in $V(G) \backslash X$ is called an *edge-cut* of G induced by X, and is denoted $(X, V(G) \backslash X)$.

A *k-edge-coloring* of a (multi)-graph G is an assignment of k colors to the edges of G such that edges that share a common endpoint are assigned distinct colors.

Definition 1 *An r-graph is an r-regular (multi)-graph G on an even number of vertices with the property that every edge-cut which separates $V(G)$ into two sets of odd cardinality has size at least r.*

r-graphs were introduced in 1979 by Seymour [12], who conjectured that every planar 4-graph is 4-edge-colorable. This conjecture was later proved by Guenin [5], who in addition showed that the corresponding result also holds for 5-graphs.

Our proof that every rectangular partition admits a strong polychromatic 4-coloring relies on the following theorem.

Theorem 2 (Guenin) *Every planar 4-graph is 4-edge-colorable.*

Our main result is as follows.

Theorem 3 *Every rectangular partition admits a strong polychromatic 4-coloring.*

Proof. Let R be a rectangular partition. One may assume that R is of even order; otherwise, add one rectangle to R, obtaining a new rectangular partition

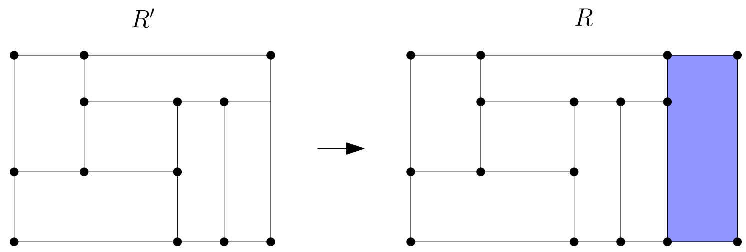

Fig. 1. Extending a rectangular partition of odd order to a rectangular partition of even order.

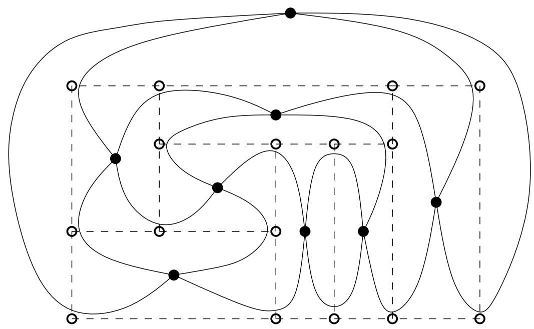

Fig. 2. Rectangular partition and its "dual" graph.

of even order. Fig. 1 illustrates this addition. Define G to be the graph obtained from R as follows. To each rectangle of R (including the outer rectangle) assign a vertex in G. For every vertex u of R add an edge between the vertices of G that correspond to the two rectangles of R for which u is a common incidence vertex. Note that G is a 4-regular planar (multi)-graph with an even number of vertices. See Fig. 2 for an illustration of the graph G.

We proceed by showing that G is a 4-graph. We will show that every edge-cut of G consists of at least four edges. For a vertex set $X \subset V(G)$, let $(X, V(G) \setminus X)$ be an edge-cut of G. Consider a maximal connected component, namely C, of $G[X]$ (the subgraph of G induced by X). Observe that in R, the component C corresponds to a union of rectangles whose boundary defines an orthogonal polygon P. Consequently, P contains at least four vertices of R on its boundary which are convex.[3] Consider an edge e in G that corresponds to a convex vertex of P. The edge e does not connect two vertices that correspond to rectangles of P, i.e., the edge e does not connect two vertices in C. Moreover, by the maximality of C, the edge e does not connect two vertices in X. Consequently, e is an edge of the edge-cut $(X, V(G) \setminus X)$. By Theorem 2, the graph G is 4-edge-colorable. Given a 4-edge-coloring of G, we color every vertex of R with the color of its corresponding edge in G. By the definition of G, the claim follows. □

[3] A vertex of an orthogonal polygon is called convex if the interior angle between its two incident edges is $\pi/2$.

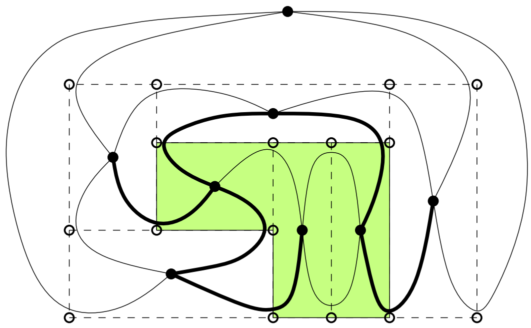

Fig. 3. Splitting a rectangular partition in to two subsets of rectangles, and the corresponding edge-cut (fat edges).

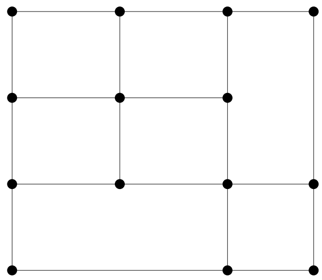

Fig. 4. A rectangular partition in which four rectangles share a common corner that does not admits a strong polychromatic 4-coloring.

Consider a rectangular partition that allows four rectangles to share a common corner. It is interesting to note that there are rectangular partitions of this type that do not admit a strong polychromatic 4-coloring. An example of such a partition is shown in Fig. 4. Note, however, that this partition admits a polychormatic 4-coloring. Hence, we asked the following question.

Is it true that every rectangular partition, of the latter form, admits a polychromatic 4*-coloring?*

A negative answer to this question was recently given by Gerbner et al. [4], who also showed that deciding whether such partition admits a polychromatic 4-coloring is NP-complete.

A natural extension to the problem considered in this note, is to consider polychromatic colorings of n-dimensional partitions. As mentioned in the introduction, it is known that any n-dimensional guillotine partition, $n \geq 2$, admits a strong polychromatic 2^n-coloring. However, the problem is still open for arbitrary n-dimensional partitions, $n > 2$. Thus, we conclude this note with the following open problem (also mentioned in [10]).

For which $n > 2$ is it true that any n-dimensional partition admits a (strong) polyhromatic 2^n-coloring?

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