FUNCTIONAL DIFFERENTIAL EQUATIONS

ISRAEL SEMINAR

EDITORS:
M.E. DRAKHLIN and E. LITSYN

VOLUME 1, 1993

THE RESEARCH INSTITUTE
THE COLLEGE OF JUDEA & SAMARIA
KEDUMIM - ARIEL, ISRAEL

This book is published with the assistance of the
Ministry of Science and the Arts of the State of Israel

Printed in Israel
ISSN 0793-1786

Production: Idit Janowsky

TABLE OF CONTENTS

FOREWORD

In December 1991 a biweekly seminar "Functional-differential equations" was started at the Research Institute of the College of Judea and Samaria, Ariel, Israel. Mathematicians working in different areas of the theory of ordinary differential equations, equations with delayed argument, integro-differential equations and on particular problems of partial differential equations participate in the seminar's sessions. Among other problems, investigation into theory of the Nemytsky superposition operator and the internal superposition operator were addressed essentially. The reason is that a lot of questions in considering different types of equations, including the aforementioned, could be reduced to some problems in Banach spaces and Frechet spaces. So, in spite of the quite wide area of the participants' interests, one could find a general basic concept and unity of the methods.

The idea of the concept is to treat the abovementioned classes of equations as particular cases of the *abstract functional-differential equation.* Therefore, every general statement about the abstract equations can be interpreted for each of the considered situations. Following N. Azbelev we will present here the notion of abstract functional-differential equation (FDE).

Let B and D be Banach spaces, such that D is isomorphic to $B \times \mathbf{R}^n$. Thus there is one-to-one correspondence between every $x \in D$ and pair $\{z, \beta\}$, $z \in B$, $\beta \in \mathbf{R}^n$. Define isomorphism $D \simeq B \times \mathbf{R}^n$ by operator $J = \{\Lambda, Y\} : B \times \mathbf{R}^n \to D$, where $\Lambda : B \to D$, $Y : \mathbf{R}^n \to D$ are linear operators, and $J\{z, \beta\} = \{\Lambda, Y\}\{z, \beta\} = \Lambda z + Y\beta$.

The operator Y will be identified with the corresponding matrix Y. The inverse operator $J^{-1} : D \to B \times \mathbf{R}^n$ possesses a representation $J^{-1}x = [\delta, r]x := \{\delta x, rx\}$ where $\delta : D \to B$, $r : D \to \mathbf{R}^n$ are linear operators. Thus for every $x \in D$ we have the following expansion $x = \Lambda\delta x + Yrx$. Applying to the last equality an operator $\mathcal{L} : D \to B$ we will get its expansion $\mathcal{L}x = Q\delta x + Arx$. The linear operator $Q = \mathcal{L}\Lambda : B \to B$ is called the *principal part* of the operator $\mathcal{L}$, the operator $A = \mathcal{L}Y : \mathbf{R}^n \to B$ is its *finite-dimensional part.*

Many basic questions in the theory of equation

$$\mathcal{L}x = f \tag{1}$$

are reducible to the corresponding problems for equation $Qz = f$ in the space B.

Let D denote the space of absolutely continuous functions $x : [a, b] \to \mathbf{R}^n$, B denote the space of summable functions $z : [a, b] \to \mathbf{R}^n$. The isomorphism $D \simeq B \times \mathbf{R}^n$ could be defined by operator $J = \{\Lambda, Y\}$, where $(\Lambda z)(t) = \int_a^t z(s)ds$, Y being the identity matrix. Then $\delta x = \dot{x}$, $rx = x(a)$, $(\mathcal{L}x)(t) = (Q\dot{x})(t) + A(t)x(a)$. Particularly for the ordinary differentiating

$$(\mathcal{L}x)(t) = \dot{x}(t) - P(t)x(t)$$

we have

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int_a^t P(t)\dot{x}(s)ds - P(t)x(a) .$$

For a wide class of equations with delayed argument the equation $Qz = f$ appears to be the classical integral equation in the space of summable functions. Some partial differential equations are convenient to be considered as equation $Mx = f$ with a linear operator $M : D \to B$ where D is isomorphic to the direct product of Banach spaces B and B_0. If operators $\Lambda : B \to D$ and $Y_0 : B_0 \to D$ define such an isomorphism $J = \{\Lambda, Y_0\}$ and $J^{-1} = [\delta, r_0]$ then the linear operator $M : D \to B$ can be expanded as $Mx = Q\delta x + A_0 r_0 x$. For investigation the equation $Mx = f$ some of the ideas used in the theory of equation (1) turn out to be helpful.

Substitution of Banach space B by Frechet space, and space $\mathbf{R}^n$ by $\mathbf{R}^\infty$ moves equation (1) to the next level of abstractness, and allows considering among equations being studied in the frames of the theory of equation (1) infinite systems of equations.

The described notion of the abstract linear FDE can be naturally generalized for the nonlinear case.

This issue consists of papers that were presented and discussed in the frames of the seminar. We are grateful to the referees who kindly agreed to review the papers or sent us comments and advice about the subjects considered in the proceedings.

Acknowledgments.

We are grateful to Professor Vitali Milman and Dr. Yakov Eshel for supporting our seminar and the idea of publishing these proceedings, and to Miriam Hercberg for preparing the articles for publication.

M. Drakhlin (The Research Institute, Ariel, Israel)
E. Litsyn (Bar-Ilan University, Israel)

GENERALIZED PROJECTION OPERATORSIN BANACH SPACES: PROPERTIES AND APPLICATIONS

Ya.I. Alber

Department of Mathematics
Technion-Israel Institute of Technology
Haifa 32000, Israel

1. Introduction

Metric projection operators in Hilbert and Banach spaces are widely used in different areas of mathematics such as functional and numerical analysis, theory of optimization and approximation and for problems of optimal control and operations research, nonlinear and stochastic programming and game theory.

Metric projection operators can be defined in a similar way in Hilbert and Banach spaces. At the same time, they differ significantly in their properties [23,27].

A metric projection operator in Hilbert space is a monotone (accretive) and nonexpansive operator. It provides an absolutely best approximation for arbitrary elements from Hilbert space by the elements of convex closed sets . This leads to a variety of applications of this operator for investigating theoretical questions in analysis and for approximation methods. (For details see [13,31,40,11,17,2,3]).

Metric projection operators in Banach space do not have the properties mentioned above and their applications are not straightforward. (See [33,9]).

On the other hand, questions of smoothness and differentiability of metric projection operators in Banach spaces were actively investigated [12,27,8,4]. New results in this field are immediately used in various applications. For example, recently established in [8,4] properties of uniform continuity of these operators were used in [8,4] to prove stability of the penalty and quasisolution methods.

Two of the most important applications of the metric projection operators in Hilbert spaces are as follows:

- solve a variational inequality by the iterative-projection method,
- find common point of convex sets by the iterative-projection method.

In Banach space these problems can not be solved in the framework of metric projection operators. Therefore, in the present paper we introduce new generalized projection operators in Banach space as a natural generalization of metric projection operators in Hilbert space. To demonstrate our approach, we apply these operators for solving two problems mentioned above in Banach space.

This research was supported in part by the Ministry of Science Grant 3481-1-91 and by the Ministry of Absorption Center for Absorption in Science.

In section 2 and section 3 we introduce notations and recall some results from the theory of variational inequalities and theory of approximation. Then in section 4 and section 5 we describe the properties of metric projection operators P_Ω in Hilbert and Banach spaces and also formulate equivalence theorems between variational inequalities and direct projection equations with these operators. In section 6 we discuss the existence of strongly unique best approximations based on Clarkson's and parallelogram inequalities.

In section 7 we introduce generalized projection operator Π_Ω which acts from Banach space B on the convex closed set Ω in the same space B. Then we state its properties and give the convergence theorem for the method of successive generalized projections used to find a common point of convex sets.

In section 8 new generalized projection operator π_Ω acting from conjugate Banach space B^* on convex closed set Ω in the space B and its properties are examined. Then a theorem of equivalence of the solutions of variational inequality and operator equation with operator π_Ω is presented. It constitutes the basis for construction of iterative-projection methods for nonlinear problems in Banach space (including smooth and nonsmooth optimization problems)

Lastly, in section 9 we establish a connection between variational inequalities and Wiener-Hopf equations in Banach spaces by means of metric and generalized projection operators.

Note that the main properties of metric and generalized projection operators in Banach spaces have been obtained by using principally new technique including Banach space geometry, parallelogram inequalities, nonstandard Lyapunov functionals and estimates of moduli of monotonicity and continuity for duality mappings.

2. Variational Inequalities. Problems of Optimization

In this section we recall some of the results from the theory of variational inequalities and formulate a problem on the equivalence between solutions of the variational inequalities and corresponding operator equations. These equations are solved by the iterative-projection methods. This yields an aproximation of solutions of the initial variational inequalities.

Let B be a real (reflexive) uniformly convex and uniformly smooth Banach space, B^* its conjugate (dual) space, $||\cdot||, ||\cdot||_{B^*}, ||\cdot||_H$ norms in the Banach spaces B, B^* and in Hilbert space H. As usually we denote $< \varphi, x >$ a dual product in B. This determines pairing between $\varphi \in B^*$ and $x \in B$ [14]. Let Ω be a nonempty convex closed set in B.

Definition 2.1. *The operator $P_\Omega : B \to \Omega \subset B$ is called metric projection operator if it yields the correspondence between an arbitrary point $x \in B$ and nearest point $\bar{x} \in \Omega$ according to minimization problem*

$$P_\Omega x = \bar{x}; \quad \bar{x} : ||x - \bar{x}|| = \inf_{\xi \in \Omega} ||x - \xi|| \,. \tag{2.1}$$

Under our conditions operator P_Ω is defined at any point $x \in B$ and it is single-valued, i.e. there exists for each $x \in B$ a unique *projection* $\bar{x}$ called best approximation [23].

Let A be an arbitrary operator acting from Hilbert space H to H , α an arbitrary fixed positive number and (φ, x) an inner product in H. Let also $f \in H$. It is well known that (see, for instance, [16])

Theorem 2.2. *The point $x \in \Omega \subset H$ is a solution of variational inequality*

$$(Ax - f, \xi - x) \geq 0, \qquad \forall \xi \in \Omega \tag{2.2}$$

if and only if x is a solution of the following operator equation in H

$$x = P_\Omega(x - \alpha(Ax - f)) \, . \tag{2.3}$$

This is an important statement, because it provides a basis for constructing approximate (iterative) methods in Hilbert spaces. The simplest method of this type can be described as follows

$$x_{n+1} = P_\Omega(x_n - \alpha_n(Ax_n - f)), \qquad n = 0, 1, 2... \tag{2.4}$$

Under suitable choice of relaxation parameters α_n , one can prove that iterative process (2.4) converges strongly to the solution of the variational inequality (2.2). It can be done for operator A which have different structures and different types of smoothness [2,3,16,11,20]. Moreover, one can establish both stability and nonasymptotic estimates of convergence rate [2,3].

We want to emphasize that the problem of solving operator equation $Ax = f$ and the problem of minimization of the functional $u(x)$ on Ω are realized as variational inequalities (2.2) for $\Omega = H$ and

$$Ax = \partial u(x), \quad x \in \Omega, \quad f = 0$$

respectively. Here $\partial u(x)$ is gradient or subgradient of the functional $u(x)$.

Now we consider more general and more complicated case of the variational inequality

$$\langle Ax - f, \xi - x \rangle \geq 0, \qquad \forall \xi \in \Omega \tag{2.5}$$

in Banach space B with operator A acting from B to B^* [9]. There is a natural problem to formulate and to prove an analogue of Theorem 2.2 in Banach space, and then to use it as a basis to construct iterative-projection methods similar to (2.4).

It is quite obvious that the Banach space analogue of the equation (2.3) has the following form

$$x = \Gamma_\Omega(Fx - \alpha(Ax - f)) \tag{2.6}$$

with operator F acting from B to B^*. The equation (2.6) is unusual because operator Γ_Ω "projects" elements from the dual space B^* on the set $\Omega \subset B$. Here one can not use metric projection operator P_Ω for this purpose because it acts from B to B. It turned out that a natural generalization of metric projection operator in Hilbert space leads to a new operator which we call generalized projection operator:

$$\pi_\Omega : B^* \to \Omega \subset B.$$

This automatically yields the following form of the equation (2.6)

$$x = \pi_\Omega(Jx - \alpha(Ax - f))$$

where $J : B \to B^*$ is a normalized duality mapping in B [14]. The operator J is one of the most significant operators in nonlinear functional analysis. In particular, it is used in the theory of optimization and in the theory of monotone and accretive operators in Banach spaces. It is determined by the expression

$$\langle Jx, x\rangle = ||Jx||_{B^*}||x|| = ||x||^2\ 1.$$

Note also that a duality mapping exists in each Banach space. In what follows we recall from [5] some of the examples of this mapping in spaces l^p, L^p , W^p_m , $\infty > p > 1$:

(i) $l^p : Jx = ||x||^{2-p}_{l^p} y \in l^q$, $x = \{x_1, x_2, ...\}$, $y = \{x_1||x_1||^{p-2}, x_2||x_2||^{p-2}, ...\}$, $p^{-1} + q^{-1} = 1$

(ii) $L^p : Jx = ||x||^{2-p}_{L^p}|x|^{p-2}x \in L^q$

(iii) $W^p_m : Jx = ||x||^{2-p}_{W^p_m} \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha(|D^\alpha x|^{p-2}D^\alpha x) \in W^q_{-m}$

Note that in Hilbert space J is an identity operator.

Now we define the iterative method similar (2.4) as follows

$$x_{n+1} = \pi_\Omega(Jx_n - \alpha_n(Ax_n - f)), \qquad n = 0, 1, 2... \tag{2.7}$$

We will give full mathematical foundation for this method including three basic aspects: convergence, stability and estimates of convergence rate, in forthcoming paper.

3. Problems of Approximation. Common Points of Convex Sets

Second important problem which is investigated in this paper using projection operators can be formulated as follows: find common point of an ordered collection of convex and closed (i.e. Chebyshev) sets $\{\Omega_1, \Omega_2, ..., \Omega_m\}$ in uniformly convex Banach space B. Here we assume that sets $\{\Omega_i, i = 1, 2, ..., m\}$ have nonempty intersection $\Omega_* = \bigcap_{i=1}^m \Omega_i$. Let us define a composition

$$P = P_1 \circ P_2 \circ \cdots \circ P_m, \qquad P_i = P_{\Omega_i} \tag{3.1}$$

and introduce method of successive projections according to a formula

$$x_{n+1} = P^{n+1}x_0, \qquad n = 0,1,2,..., \qquad x_o \in B \ . \tag{3.2}$$

Convergence of the iterative process (3.1) and (3.2) as well as of similar processes to the point $x_* \in \Omega_*$ was proved before only in Hilbert space . (See [36,13,15,17,18,22]). In the formulae (3.1) and (3.2) which describe method of successive projections one can use metric projection operators P_Ω in Banach space. However, up to this date no proof was suggested for the convergence of (3.1) and (3.2) in Banach space. The reason is that in Hilbert space H the metric projection operator satisfies the following significant inequality

$$||P_\Omega x - \xi||_H \leq ||x - \xi||_H, \qquad \forall \xi \in \Omega \tag{3.3}$$

which can be obtained from the property of nonexpansiveness of this operator in H

$$||P_\Omega x - P_\Omega y||_H \leq ||x - y||_H \ . \tag{3.4}$$

It satisfies also a much stronger property (see section 4)

$$||P_\Omega x - x||^2{}_H \leq ||x - \xi||^2{}_H - ||P_\Omega x - \xi||^2{}_H, \qquad \forall \xi \in \Omega \ . \tag{3.5}$$

But in Banach spaces these properties do not hold in general [32,33,21].

For example, in [32] it is shown that in uniformly convex Banach space with modulus of convexity $\delta(\epsilon)$ of order ϵ^q, $q \geq 2$ [19], the inequality

$$||P_\Omega x - x||^q \leq ||x - \xi||^q - \lambda||P_\Omega x - \xi||^q, \qquad \forall \xi \in \Omega \ . \tag{3.6}$$

holds. Coefficient $\lambda < 1$ in (3.6) and it depends on q. Namely, in [32] it is defined in Banach spaces of the type $L^p and W^p_m$, $1 < p < \infty$ as follows (cf. (6.6) and (6.8)):

$$1 < p \leq 2, \ \ q = 2, \ \ \lambda = (p-1)/8 \ ;$$

and

$$2 < p < \infty, \ \ q = p, \ \ \lambda = 1/p2^p \ .$$

Inequality (3.6) yields

$$||P_\Omega x - \xi||^q \leq \lambda^{-1}||x - \xi||^q - \lambda^{-1}||P_\Omega x - x||^q \ , \qquad \forall \xi \in \Omega \ . \tag{3.7}$$

This does not guarantee the nonexpansiveness of the metric projection operator in Banach space even for $\xi = y \in \Omega$ while in (3.4) $\xi = y$ is an arbitrary element of the space H. But without this property, one can not to study the method (3.1), (3.2).

Now we consider more general case. In [32] a strongly unique best approximation was defined as follows.

Definition 3.1. *$\bar{x}$ is called a strongly unique best approximation in Ω for the element $x \in B$ if there exists a constant λ and a strictly increasing function $\phi(t) : R^+ \to R^+$ such that $\phi(0) = 0$ and*

$$\phi(||P_\Omega x - x||) \le \phi(||x - \xi||) - \lambda\phi(||P_\Omega x - \xi||) , \qquad \forall \xi \in \Omega . \qquad (3.8)$$

The projection $\bar{x} = P_\Omega x$ in Hilbert space, and the projection $\bar{x}$ in Banach space under the conditions of [32], are strongly unique best approximations in Ω.

We call the projection $\bar{x}$ *absolutely best approximation* of $x \in B$ with respect to function $\phi(t)$ if $\lambda = 1$ in (3.8). In this case the inequality (3.8) can be represented in the equivalent form

$$\phi(||P_\Omega x - \xi||) \le \phi(||x - \xi||) - \phi(||P_\Omega x - x||) , \qquad \forall \xi \in \Omega .$$

It is clear from (3.5) that metric projection $\bar{x}$ in Hilbert space is absolutely best approximation with respect to function $\phi(t) = t^2$ (or with respect to functional $\phi(\xi) = ||x - \xi||^2{}_H$ with x fixed). But it is not true in Banach spaces.

Thus, metric projection operator can not be used in (3.1) and (3.2). Instead we introduce new generalized projection operator

$$\Pi_\Omega : B \to \Omega \in B$$

so that the inequalities (3.3) and (3.5) hold for some Lyapunov functional $\phi(\cdot)$. In section 7 we will provide the convergence theorem for the process (3.1) and (3.2) which now has the form

$$x_{n+1} = \Pi^{n+1} x_0, \qquad n = 0, 1, 2, ..., \qquad x_o \in B \qquad (3.9)$$

and

$$\Pi = \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_m, \qquad \Pi_i = \Pi_{\Omega_i} . \qquad (3.10)$$

4. Metric Projection Operator P_Ω in Hilbert Space

All results described in section 2 and section 3 for two basic problems were obtained only in Hilbert space. This is due to the fact that many remarkable properties of the metric projection operators can not be extended from Hilbert space to Banach space. This is why we introduce in section 7 and section 8 new generalize projection operators in Banach spaces which have all properties of metric projection operators in Hilbert space .

Before that in section 4 and section 5 we compare complete lists of the properti of the metric projection operators in Hilbert and Banach spaces.

We denote $\bar{x} = P_\Omega x$. Let $\xi \in \Omega$ and $\Omega \subset H$. The following properties are valid in Hilbert space: [11,13,16,21,23,31,5,28,39].

4.a. P_Ω is fixed at each point ξ, $P_\Omega \xi = \xi$.

4.b. P_Ω is monotone (accretive) in H, i.e. for all $x, y \in H$

$$(\bar{x} - \bar{y}, x - y) \geq 0.$$

4.c. $(x - \bar{x}, \bar{x} - \xi) \geq 0, \qquad \forall \xi \in \Omega.$
4.d. $(x - \xi, \bar{x} - \xi) \geq 0, \qquad \forall \xi \in \Omega.$
4.e. $(x - \bar{x}, x - \xi) \geq 0, \qquad \forall \xi \in \Omega.$
In fact, even stronger inequality

$$(x - \bar{x}, x - \xi) \geq ||x - \bar{x}||_H^2, \quad \forall \xi \in \Omega$$

holds (see (5.1)).

4.f. P_Ω is nonexpansive in H, i.e.

$$||\bar{x} - \bar{y}||_H \leq ||x - y||_H \ .$$

4.g. P_Ω is P-strongly monotone in H, i.e.

$$(\bar{x} - \bar{y}, x - y) \geq ||\bar{x} - \bar{y}||_H^2 \ .$$

4.h. The operator P_Ω yields an absolutely best approximation of $x \in H$ with respect to the functional $V_1(x, \xi) = ||x - \xi||_H^2$

$$||\bar{x} - \xi||_H^2 \leq ||x - \xi||_H^2 - ||x - \bar{x}||_H^2, \qquad \forall \xi \in \Omega \ .$$

4.i. Any P_Ω satisfies the inequality

$$((I - P_\Omega)x - (I - P_\Omega)y, P_\Omega x - P_\Omega y) \geq 0, \quad \forall x, y \in H \ .$$

5. Metric Projection Operator P_Ω in Banach Space

Here we show that some of the properties of the metric projection operators in Hilbert space are not satisfied in Banach space. At the same time, we describe in detail the properties of uniform continuity of the metric projection operators in Banach space.

We denote $\bar{x} = P_\Omega x$. Let $\xi \in \Omega$ and $\Omega \subset B$. The following properties hold in Banach space (the sign "—" from 5.a - 5.i denotes an absence of corresponding property):

5.a. The operator P_Ω is fixed at each point ξ, i.e. $P_\Omega \xi = \xi$.
5.b. —
5.c. $\langle J(x - \bar{x}), \bar{x} - \xi \rangle \geq 0, \qquad \forall \xi \in \Omega$ (see [27]).
5.d. —
5.e. $\langle J(x - \bar{x}), x - \xi \rangle \geq 0, \qquad \forall \xi \in \Omega$ (see [5]).
In what follows we show that a stronger statement is true [5].

Theorem 5.1. *$\bar{x} \in \Omega$ is a projection of the point $x \in B$ on Ω if and only if the inequality*

$$\langle J(x-\bar{x}), x-\xi\rangle \geq ||x-\bar{x}||^2 , \qquad \forall \xi \in \Omega \tag{5.1}$$

is satisfied.

In fact, from (5.1) it follows immediately that

$$||x-\bar{x}|| \leq ||x-\bar{x}||^{-1}\langle J(x-\bar{x}), x-\xi\rangle \leq ||x-\xi|| , \qquad \forall \xi \in \Omega$$

i.e. $\bar{x} = P_\Omega x$. Inversely, if $\bar{x} = P_\Omega x$, then by virtue of 5.c we have

$$\begin{aligned} 0 \leq \langle J(x-\bar{x}), \bar{x}-\xi\rangle &= \langle J(x-\bar{x}), \bar{x}-x\rangle + \langle J(x-\bar{x}), x-\xi\rangle = \\ &= -||x-\bar{x}||^2 + \langle J(x-\bar{x}), x-\xi\rangle \end{aligned} \tag{5.2}$$

which yields (5.1).

5.f. Now we describe the property of uniform continuity of operator $P_\Omega x$ in Banach space B. Recall that in Banach space the metric projection operator is not nonexpansive in general case. But it is uniformly continuous on each bounded set according to the following theorem.

Theorem 5.4. *Let B be an uniformly convex and uniformly smooth Banach space. If $\delta_B(\epsilon)$ is a modulus of convexity of the space B, $g_B(\epsilon) = \delta_B(\epsilon)/\epsilon$ and $g_B^{-1}(\cdot)$ is an inverse function, then*

$$||\bar{x}-\bar{y}|| \leq C g_B^{-1}(2LC^2 g_{B^*}^{-1}(2CL||x-y||)) , \tag{5.3}$$

where $1 < L < 3.18$ is Figiel's constant (see [6]) and

$$C = 2\max\{1, ||x-\bar{y}||, ||y-\bar{x}||\} .$$

Remark 5.3. If $||x-\bar{y}|| \leq R$ and $||y-\bar{x}|| \leq R$, then ($C = 2\max\{1, R\}$) is an absolute constant and (5.3) provides a quantitative description of the uniform continuity of operator P_Ω in Banach space on each bounded set.

The estimate (5.3) which was established in [8] is global in nature. Earlier, in [12] Bjernestal obtained local estimate

$$||\bar{x}-\bar{y}|| \leq 2\delta_B^{-1}(6\rho_B(2||x-y||)) , \tag{5.4}$$

where $\rho_B(\tau)$ is a modulus of smoothness of the space B [19] .

The estimate of (5.4) is better than our estimate (5.3). This is why in [4] we continued the investigation of uniform continuity of metric projection operator in Banach space. It turns out that the following global variant of (5.4) can be obtained.

Theorem 5.4. *Let B be an uniformly convex and uniformly smooth Banach space. If $\delta_B(\epsilon)$ is a modulus of convexity of the space B and $\rho_B(\tau)$ is a modulus of its smoothness, then*

$$||\bar{x}-\bar{y}|| \leq C\delta_B^{-1}(\rho_B(8CL||x-y||)) , \tag{5.5}$$

where constant L and function C are defined in Theorem 5.2.

Remark 5.5. To accuracy constants the estimates (5.3) and (5.5) give respectively

$$\|\bar{x} - \bar{y}\| \leq g_B^{-1}(g_{B^*}^{-1}(\|x - y\|)) \ ,$$
$$\|\bar{x} - \bar{y}\| \leq \delta_B^{-1}(\rho_B(\|x - y\|)) \ .$$

5.g. —
5.h. —
5.i. Any P_Ω satisfies the inequality (see [27])

$$\langle J(x - P_\Omega x) - J(y - P_\Omega y), P_\Omega x - P_\Omega y\rangle \geq 0 \ , \qquad \forall x, y \in B \ .$$

Using the properties of metric projection operator P_Ω we obtained Banach space analogue of Theorem 2.2.

Theorem 5.6. *Let A be an arbitrary operator from Banach space B to B^*, α an arbitrary fixed positive number, $f \in B^*$. Then the point $x \in \Omega \subset B$ is a solution of variational inequality*

$$\langle Ax - f, \xi - x\rangle \geq 0, \qquad \forall \xi \in \Omega$$

if and only if x is a solution of the operator equation in B

$$x = P_\Omega(x - \alpha J^*(Ax - f))$$

where $J^* : B^* \to B$ is normalized duality mapping in B^*.

Iterative process corresponding to (5.6) is the following

$$x_{n+1} = P_\Omega(x_n - \alpha_n J^*(Ax_n - f)) \ , \qquad n = 0, 1, 2... \qquad (5.7)$$

However, there are not any approaches to the investigation of (5.7).

In section 7 and section 8 we will construct the generalized projection operato Banach spaces with the additional properties 5.b, 5.d, 5.g and 5.h., and iterative methods for which one can establish convergence, stability, and nonasymptotic estimates of convergence rate.

6. Parallelogram Inequalities and Strongly Unique Best Approximations

In this section we discuss the existence of strongly unique best approximation in the spaces l^p, L^p and W^p_m where $\infty > p > 1$.

In [10] (see also [6]) we established the following upper parallelogram inequality

$$2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \leq 4\|x - y\|^2 + C_1(\|x\|, \|y\|)\rho_B(\|x - y\|),$$

$$C_1(||x||,||y||) = 2\max\{L, (||x|| + ||y||)/2\}$$

for the arbitrary points x and y from uniformly smooth Banach space B. We also obtained lower parallelogram inequality

$$2||x||^2 + 2||y||^2 - ||x+y||^2 \geq L^{-1}\delta_B(||x-y||/C_2(||x||,||y||)), \tag{6.1}$$

$$C_2(||x||,||y||) = 2\max\{1, \sqrt{(||x||^2 + ||y||^2)/2}\}$$

for the arbitrary points x and y from uniformly convex Banach space B. Analogous parallelogram inequalities for the $||x||^q$ of other orders q were obtained in [30].

If $||x|| \leq R$ and $||y|| \leq R$ then

$$C_1(||x||,||y||) = C_1 = 2\max\{L, R\}$$

and

$$C_2(||x||,||y||) = C_2 = 2\max\{1, R\}\ .$$

In this case (6.1) expresses the uniform convexity of the functional $||x||^2$ on each bounded set in B with modulus of convexity $\delta(||x-y||) = (2L)^{-1}\delta_B(||x-y||/C_2)$, and

$$||x+y||^2 \leq 2||x||^2 + 2||y||^2 - L^{-1}\delta_B(||x-y||/C_2)\ . \tag{6.2}$$

In [35] the following lemma was proved.

Lemma 6.1. *If a convex functional $\phi(x)$ defined on convex closed set Ω satisfies the inequality*

$$\phi(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y) - \kappa(||x-y||),$$

where $\kappa(t) \geq 0, \kappa(t_0) > 0$ for some $t_0 > 0$, then $\phi(x)$ is uniformly convex functional with modulus of convexity $\delta(t) = 2\kappa(t)$ and

$$\phi(y) \geq \phi(x) + \langle l(x), y - x\rangle + 2\kappa(||x-y||)\ .$$

for all $l(x) \in \partial\phi(x)$. Here $\partial\phi(x)$ is the set of all support functionals (the set of all subgradients) of $\phi(x)$ at the point $x \in \Omega$.

From this lemma and (6.2) it follows that

$$||x||^2 \leq ||y||^2 + 2\langle Jx, x-y\rangle - (2L)^{-1}\delta_B(||x-y||/C_2)\ . \tag{6.3}$$

Let $\Omega \subset B$, $\xi \in \Omega$, $\bar{x} = P_\Omega x$. We replace in (6.3) x by $(x-\bar{x})$ and y by $(x-\xi)$ and obtain

$$||x-\bar{x}||^2 \leq ||x-\xi||^2 - 2\langle J(x-\bar{x}), \bar{x}-\xi\rangle - (2L)^{-1}\delta_B(||\bar{x}-\xi||/C_2)\ .$$

The property 5.c then yields the following general formula

$$||x - \bar{x}||^2 \leq ||x - \xi||^2 - \lambda\delta_B(||\bar{x} - \xi||/C_2) , \qquad \lambda = (2L)^{-1} . \tag{6.4}$$

It is obvious that if $\delta_B(\epsilon)$ can be estimated by ϵ^2 (this occures in Hilbert spaces and in the spaces of type L^p for $1 < p \leq 2$ [7]), then projection $\bar{x}$ is a strongly unique best approximation with $\phi(t) = t^2$, at least, on each bounded set (See Def. 3.1). However, constant λ in inequality (6.4) is not exact in these cases.

In [37] it was shown that in spaces W^p_m (consequently, in L^p and l^p), $1 < p \leq 2$, the following inequality holds

$$||x + y||^2 \leq 2||x||^2 + 2||y||^2 - (p-1)||x - y||^2 . \tag{6.5}$$

Then Lemma 6.1, the property 5.c and (6.5) give the estimate

$$||x - \bar{x}||^2 \leq ||x - \xi||^2 - (p-1)||\bar{x} - \xi||^2/2 .$$

Using the inequality (2.3) from [26], we immediately obtain from Lemma 6.1 that in spaces L^p, $1 < p \leq 2$, the estimate

$$||x - \bar{x}||^2 \leq ||x - \xi||^2 - (p-1)||\bar{x} - \xi||^2 . \tag{6.6}$$

is valid. This coincides with the result of [26] (Theorem 4.1).

Furthermore, the strongly unique best approximation of the projection $\bar{x}$ in spaces B of the type l^p, L^p and W^p_m where $\infty > p > 2$, can be established from Lemma 6.1 and Clarkson's inequality

$$||x + y||^p \leq 2^{p-1}||x||^p + 2^{p-1}||y||^p - ||x - y||^p .$$

It means that functional $||x||^p$ is uniformly convex. Therefore, we can write

$$||y||^p \geq ||x||^p + p < J^\mu x, y - x > +2^{-p+1}||x - y||^p \tag{6.7}$$

where J^μ is a duality mapping with the gauge function $\mu(t) = t^{p-1}$ (see section 7). Now we substitute in (6.7) $(x - \bar{x})$ and $(x - \xi)$ for x and y, respectively. By virtue of

$$\langle J^\mu(x - \bar{x}), \bar{x} - \xi\rangle \geq 0$$

and

$$||x - \bar{x}||^p \leq ||x - \xi||^p - p\,\langle J^\mu(x - \bar{x}), \bar{x} - \xi\rangle - 2^{-p+1}||\bar{x} - \xi||^p ,$$

we have

$$||x - \bar{x}||^p \leq ||x - \xi||^p - 2^{-p+1}||\bar{x} - \xi||^p . \tag{6.8}$$

This expression improves the estimate of a strongly unique best approximation. (Compare with the corresponding inequality in [32,25]). It was obtained without any additional conditions on the modulus of convexity of the spaces and on the sets Ω. Besides, the important generalization of (6.8) is valid.

Theorem 6.2. *Let B be a space either l^p or L^p or W^p_m where $\infty > p \geq 2$. Let Ω be a closed convex set in B. Then for every point $x \in B$ there exists a unique point $\bar{x} = P_\Omega x$ such that*

$$||x - \bar{x}||^s \leq ||x - \xi||^s - 2^{-s+1}||\bar{x} - \xi||^s, \quad \forall \xi \in \Omega, \quad s \geq p \geq 2.$$

The proof follows from the inequality (see [35])

$$||x + y||^s \leq 2^{s-1}||x||^s + 2^{s-1}||y||^s - ||x - y||^s, \quad s \geq p \geq 2 .$$

7. Generalized Projection Operator Π_Ω in Banach Space

Here we introduce generalized projection operator Π_Ω and describe its properties in Banach spaces. Then we formulate theorem about convergence of the method of succesive projections given in section 3. This method yields an approximation of the common point of convex closed sets. (See second problem in section 3).

The formula (2.1) in the Definition 2.1 of the metric projection operators is equivalent to the minimization problem

$$P_\Omega x = \bar{x}; \quad \bar{x} : ||x - \bar{x}||^2 = \inf_{\xi \in \Omega} ||x - \xi||^2 . \tag{7.1}$$

Note that $V_1(x, \xi) = ||x - \xi||^2$ can be considered not only as square of distance between points x and ξ but also as Lyapunov functional with respect to ξ with fixed x. Therefore, we can rewrite (7.1) in the form

$$P_\Omega x = \bar{x}; \quad \bar{x} : V_1(x, \bar{x}) = \inf_{\xi \in \Omega} V_1(x, \xi) .$$

In Hilbert space

$$V_1(x, \xi) = ||x||^2_H - 2(x, \xi) + ||\xi||^2_H .$$

In the papers [10,7] we introduced Lyapunov functional

$$V_2(Jx, \xi) = ||Jx||^2_{B^*} - 2\langle Jx, \xi\rangle + ||\xi||^2 . \tag{7.2}$$

It is a nonstandard functional because it is defined on both the elements ξ from the primary space B and on the elements (Jx) from the dual space B^* (see also (8.1)).

Lemma 7.1. *The functional $V_2(Jx, \xi)$ has the following properties:*

1. *V_2 is continuous.*
2. *V_2 is convex with respect to $\varphi = Jx$ when ξ is fixed end with respect to ξ when x is fixed.*
3. *V_2 is differentiable with respect to φ and ξ .*

4. $grad_\varphi V_2(Jx,\xi) = 2(x-\xi)$.
5. $grad_\xi V_2(Jx,\xi) = 2(J\xi - Jx)$.
6. $V_2(Jx,\xi) > 0, \qquad \forall x,\xi \in B$.
7. $V_2(Jx,\xi) = 0$, *only if* $x = \xi$.
8. $V_2(Jx,\xi) \to \infty$, *if* $||x|| \to \infty$ *(or* $||\xi|| \to \infty$*) and vice versa.*
9. $(||x|| - ||\xi||)^2 \le V_2(Jx,\xi) \le (||x|| + ||\xi||)^2$.
10. $L^{-1}\delta_B(||x-\xi||/C) \le V_2(Jx,\xi) \le L^{-1}\rho_B(8LC||x-\xi||)$, *where the constant* L *is from Theorem 5.2 and*

$$C = 2\max\{1, \sqrt{(||x||^2 + ||\xi||^2)/2}\} .$$

11. $V_2(Jx,\xi) \to 0$ *if* $||x-\xi|| \to 0$, *and* $||x||$, $||\xi||$ *are bounded, and vice versa* .

There is a connection between the functional $V_2(Jx,\xi)$ and the Young-Fenchel transformation, because

$$||Jx||_{B^*} = \sup_{\xi \in B}\{2\langle Jx,\xi\rangle - ||\xi||^2\} .$$

Notice also that (7.2) is equivalent to

$$V_2(Jx,\xi) = ||x||^2 - 2\langle Jx,\xi\rangle + ||\xi||^2.$$

However, previous form (7.2) is necessarily used to obtain properties 4 and 10.

Definition 7.2. *Operator* $\Pi_\Omega : B \to \Omega \subset B$ *is called the generalized projection operator if it puts the arbitrary fixed point* $x \in B$ *into the correspondence with the point of minimum for the functional* $V_2(Jx,\xi)$ *according to the minimization problem*

$$\Pi_\Omega x = \hat{x}; \quad \hat{x} : V_2(Jx,\hat{x}) = \inf_{\xi \in \Omega} V_2(Jx,\xi) .$$

Remark 7.3. In Hilbert space $V_2(Jx,\xi) = V_1(x,\xi)$ and $\hat{x} = \bar{x}$.

Next we describe the properties of the operator Π_Ω:

7.a. The operator Π_Ω is fixed in each point $\xi \in \Omega$, i.e. $\Pi_\Omega \xi = \xi$.

7.b. Π_Ω is d-accretive in B , i.e. for all $x, y \in B$

$$\langle Jx - Jy, \hat{x} - \hat{y}\rangle \ge 0.$$

7.c. $\langle Jx - J\hat{x}, \hat{x} - \xi\rangle \ge 0, \qquad \forall \xi \in \Omega$.

7.d. $\langle Jx - J\xi, \hat{x} - \xi\rangle \ge 0, \qquad \forall \xi \in \Omega$.

7.e. $\langle Jx - J\hat{x}, x - \xi\rangle \ge 0, \qquad \forall \xi \in \Omega$.

7.f. $||\hat{x} - \hat{y}|| \le C g_B^{-1}(2LC^2 g_{B^*}^{-1}(2LC||x-y||))$, where constant L is from Theorem 5.2 and

$$C = 2\max\{1, ||x||, ||y||, ||\hat{x}||, ||\hat{y}||\} .$$

Remark 7.4. If $||x|| \leq R, ||\hat{x}|| \leq R, ||y|| \leq R$ and $||\hat{y}|| \leq R$, then $C = 2\max\{1, R\}$ is absolute constant and 7.f expresses the uniform continuity of operator Π_Ω in Banach space on each bounded set.

7.g. $\langle Jx - Jy, \hat{x} - \hat{y}\rangle \geq (2L)^{-1}\delta_B(||\hat{x} - \hat{y}||/C)$, where

$$C = 2\max\{1, ||\hat{x}||, ||\hat{y}||\}.$$

7.h. The operator Π_Ω gives absolutely best approximation of $x \in B$ with respect to functional $V_2(Jx, \xi)$

$$V_2(J\hat{x}, \xi) \leq V_2(Jx, \xi) - V_2(Jx, \hat{x}).$$

Consequently, Π_Ω is *conditionally nonexpansive* operator in Banach space, i.e.

$$V_2(J\hat{x}, \xi) \leq V_2(Jx, \xi)\ .$$

7.i. Any Π_Ω satisfies the inequality

$$\langle (J - J\Pi_\Omega)x - (J - J\Pi_\Omega)y, \Pi_\Omega x - \Pi_\Omega y\rangle \geq 0, \quad \forall x, y \in B.$$

Remark 7.5. If $B = H$, then the formulas 7.a - 7.e and 7.h - 7.i coincide with ones 4.a - 4.e and 4.h - 4.i, but 7.f and 7.g differ from 4.f and 4.g by only constants (on any bounded set).

Using properties of the generalized projection operator Π_Ω we obtained the theorem.

Theorem 7.6. *The following statements hold for the method of successive generalized projections (3.9) and (3.10):*

1) $V_2(Jx_{n+1}, \xi) \leq V_2(Jx_n, \xi), \qquad \forall \xi \in \Omega_* = \bigcap_{i=1}^m \Omega_i.$
2) *There exists a subsequence* $\{x_{n_k}\}$ *of the sequence* $\{x_n\}$ *such that* $x_{n_k} \to x_*$ *weakly, where* $x_* \in \Omega_*$.
3) *If* J *is sequential weakly continuous operator then* $x_n \to x_*$ *weakly.*

If $\{x_n\}$ *is an ordered sequence of the elements* x_i^j, $i = 0, 1, ..., j = m, m-1, ..., 1$, *such that* $x_i^m = \Pi_m x_{i-1}^1$; $x_i^j = \Pi_j x_i^{j+1}$, $j = m-1, m-2, ..., 2, 1$; $x_{-1}^1 = x_0$, *then, in addition to 1) - 3),*

4) $||x_n - x_{n+1}|| \to 0,$ *for* $n \to \infty$.
5) $\sum_{n=0}^\infty V_2(Jx_n, x_{n+1}) < \infty$.

In what follows we discuss statement 3 from the theorem. We recall that F is called a sequentially weakly continuous mapping if from the relation $x_n \to x$ (weakly) it follows that $Fx_n \to Fx$ (also weakly).

Theorem 7.6 is valid for dual mapping J^μ with the gauge function $\mu(t) = t^{p-1}$, defined by the relations

$$||J^\mu x||_{B^*} = ||x||^{p-1}, \qquad \langle J^\mu x, x\rangle = ||x||^p, \qquad J^\mu x = grad||x||^p/p\ .$$

(Notice that normalized dual mapping corresponds to $p=2$). We set

$$V_3(J^\mu x,\xi)=q^{-1}||J^\mu x||^q_{B^*}-\langle J^\mu x,\xi\rangle+p^{-1}||\xi||^p,\qquad p^{-1}+q^{-1}=1\ .$$

The function $q^{-1}||J^\mu x||^q_{B^*}$ is conjugate to the function $p^{-1}||x||^p$, i.e.

$$q^{-1}||J^\mu x||^q_{B^*}=\sup_{\xi\in B}\{\langle J^\mu x,\xi\rangle-p^{-1}||\xi||^p\}\ .$$

Therefore $V_3(J^\mu x,\xi)\geq 0,\quad \forall x,\xi\in B$. If we now define

$$\hat{\hat{x}}:V_3(J^\mu x,\hat{\hat{x}})=\inf_{\xi\in\Omega}V_3(J^\mu x,\xi)$$

then it can be shown that

$$\langle J^\mu x-J^\mu\hat{\hat{x}},\hat{\hat{x}}-\xi\rangle\geq 0,\qquad \forall\xi\in\Omega$$

and

$$V_3(J^\mu x,\hat{\hat{x}})\leq V_3(J^\mu x,\xi)-V_3((J^\mu\hat{\hat{x}},\xi)\ .$$

The last inequalities are used mainly in the proof of Theorem 7.6 which will be given in forthcoming paper. The property of the uniform continuity of the operator $J^\mu x$ can be obtained using the results from [30,38].

Corollary 7.7. *In Banach space $l^p, p>1$ the sequence x_n weakly converges to $x_*\in\Omega_*=\bigcap_{i=1}^m\Omega_i$.*

This holds because in $l^p,\infty>p>1$, dual mapping J^μ with the gauge function $\mu(t)=t^p$ is sequential weakly continuous [14].

Remark 7.8. $V_3(J^\mu x,\xi)$ and $V_2(Jx,\xi)$ coinside for $p=2$ (up to constant 2).

It can be shown in a way similar to the case of metric projection operator P_Ω in Banach space [8] that generalized projection operator Π_Ω is stable with respect to peturbation of the set Ω.

Let Ω_1 and Ω_2 be convex closed sets, $x\in B$ and $H(\Omega_1,\Omega_2)\leq\sigma$, where

$$H(\Omega_1,\Omega_2)=\max\{\sup_{z_1\in\Omega_1}\inf_{z_2\in\Omega_2}||z_1-z_2||,\quad \sup_{z_1\in\Omega_2}\inf_{z_2\in\Omega_1}||z_1-z_2||\}$$

is a Hausdorff distance between Ω_1 and Ω_2. Let also $\Pi_{\Omega_1}x=\hat{x}_1,\Pi_{\Omega_2}x=\hat{x}_2$.

Theorem 7.9. *If B is uniformly convex Banach space, $\delta_B(\epsilon)$ is modulus of the convexity of B and $\delta_B^{-1}(\cdot)$ is an inverse function, then*

$$||\hat{x}_1-\hat{x}_2||\leq C_1\delta_B^{-1}(4LC_2\sigma)\ ,$$

$$C_1=2\max\{1,||Jx-J\hat{x}_1||_{B^*},||Jx-J\hat{x}_2||_{B^*}\}\ ,$$
$$C_2=2\max\{||Jx-J\hat{x}_1||_{B^*},||Jx-J\hat{x}_2||_{B^*}\}\ .$$

If $||x||\leq R,||\hat{x}_1||\leq R$ and $||\hat{x}_2||\leq R$, then C_1 and C_2 are absolute constants, because operator J is bounded in any Banach space.

8. Generalized Projection Operator π_Ω in Banach Space

Here we introduce generalized projection operator π_Ω in Banach space and describe its properties. Then we use this operator to establish equivalence between solution of the variational inequality in Banach space and solution of the corresponding operator equation. In other words, we solve first problem described in section 2. Finally, we obtain a link between operators π_Ω and Π_Ω by means of the normalized duality mappings J and J^*.

We assume that φ is an arbitrary element of the space B^*.

Definition 8.1. *The generalized projection $\tilde{\varphi}$ of the element φ on the set $\Omega \subset B$ is given by means of a minimization problem*

$$\pi_\Omega \varphi = \tilde{\varphi}; \quad \tilde{\varphi} : V_4(\varphi, \tilde{\varphi}) = \inf_{\xi \in \Omega} V_4(\varphi, \xi)$$

where

$$V_4(\varphi, \xi) = ||\varphi||^2_{B^*} - 2\langle \varphi, \xi \rangle + ||\xi||^2 .$$

Remark 8.2. In Hilbert space $V_4(\varphi, \xi) = V_3((J^p \hat{x}, \xi) = V_2(Jx, \xi) = V_1(x, \xi)$ and $\tilde{\varphi} = \hat{\hat{x}} = \hat{x} = \bar{x}$.

In what follows we list properties of the generalized projection operator π_Ω in Banach Space.

8.a. The operator π_Ω is J-fixed in each point $\xi \in \Omega$, i.e. $\pi_\Omega J\xi = \xi$.

8.b. π_Ω is monotone in B^*, i.e. for all $\varphi_1, \varphi_2 \in B^*$

$$\langle \varphi_1 - \varphi_2, \tilde{\varphi}_1 - \tilde{\varphi}_2 \rangle \geq 0 ,$$

8.c. $\langle \varphi - J\tilde{\varphi}, \tilde{\varphi} - \xi \rangle \geq 0, \quad \forall \xi \in \Omega.$

8.d. $\langle \varphi - J\xi, \tilde{\varphi} - \xi \rangle \geq 0, \quad \forall \xi \in \Omega.$

8.e. $\langle Jx - J\tilde{x}, x - \xi \rangle \geq 0$, where $\tilde{x} = \pi_\Omega Jx, \quad \forall \xi \in \Omega.$

8.f. $||\tilde{\varphi}_1 - \tilde{\varphi}_2|| \leq C g_B^{-1}(2LC||\varphi_1 - \varphi_2||_{B^*})$, where the constant L is from Theorem 5.2 and

$$C = 2\max\{1, ||\tilde{\varphi}_1||, ||\tilde{\varphi}_2||\} .$$

Remark 8.3. If $||\tilde{\varphi}_1|| \leq R$, $||\tilde{\varphi}_2|| \leq R$ then $C = 2\max\{1, R\}$ is an absolute constant and 8.f expresses uniform continuity of the operator π_Ω in Banach space on each bounded set.

8.g. $\langle \varphi_1 - \varphi_2, \tilde{\varphi}_1 - \tilde{\varphi}_2 \rangle \geq (2L)^{-1}\delta_B(||\tilde{\varphi}_1 - \tilde{\varphi}_2||/C)$, where C is the constant from 8.f.

8.h. The operator π_Ω gives absolutely best approximation of $x \in B$ with respect to functional $V_4(\varphi, \xi)$

$$V_4(J\tilde{\varphi}, \xi) \leq V_4(\varphi, \xi) - V_4(\varphi, \tilde{\varphi}).$$

Consequently, π_Ω is *conditionally nonexpansive* operator in Banach space, i.e.

$$V_4(J\tilde{\varphi},\xi) \le V_4(\varphi,\xi) \ .$$

8.i. Any π_Ω satisfies the inequality

$$\langle (I_{B^*} - J\pi_\Omega)\varphi_1 - (I_{B^*} - J\pi_\Omega)\varphi_2, \pi_\Omega\varphi_1 - \pi_\Omega\varphi_2\rangle \ge 0, \ \ \forall\varphi_1,\varphi_2 \in B^*,$$

where $I_{B^*} : B^* \to B^*$ is identical operator in B^*.

Similarly to operator Π_Ω, the generalized projection operator π_Ω in Banach Space is stable with respect to perturbation of the set Ω.

Using the properties of generalized projection operator π_Ω we obtained Banach space analogue of Theorem 2.2.

Theorem 8.4. *Let A be an arbitrary operator from Banach space B to B^*, α an arbitrary fixed positive number, $f \in B^*$. Then the point $x \in \Omega \subset B$ is a solution of variational inequality*

$$\langle Ax - f, \xi - x\rangle \ge 0 \ , \qquad \forall\xi \in \Omega$$

if and only if x is a solution of the operator equation in B

$$x = \pi_\Omega\big(Jx - \alpha(Ax - f)\big) \ . \tag{8.2}$$

It is not hard to verify that

$$\Pi_\Omega = \pi_\Omega J \ , \qquad \pi_\Omega = \Pi_\Omega J^*$$

where $J : B \to B^*$ is a normalized duality mapping in B and $J^* : B^* \to B$ is normalized duality mapping in B^*. Therefore we can rewrite (8.2) in the form of

$$x = \Pi_\Omega J^*\big(Jx - \alpha(Ax - f)\big) \ .$$

Denote also that $J^* = J^{-1}$.

For the iterative process (2.7) we proved the assertions analogous to the Theorems 3 and 4 from the paper [7] (see also Remark 7 in [7]).

It is interesting to note that $\pi_\Omega = J^*$ in the case $\Omega = B$ (the problem of solving the equation $Ax = f$). Then (8.2) is rewritten as

$$x = J^*\big(Jx - \alpha(Ax - f)\big)$$

or in the form of

$$Jx = Jx - \alpha(Ax - f)$$

because $JJ^* = I$. Here I is identical operator. Iterative method for (8.3)

$$Jx_{n+1} = Jx_n - \alpha_n(Ax_n - f) \ , \quad n = 0,1,2,...,x_0 \in B, \quad x_n = J^*Jx_n$$

has been studied earlier in [7,10].

Along with (2.7) we considered the following iterative processes:

$$x_{n+1} = \pi_\Omega(Jx_n - \alpha_n(Ax_n - f)/||Ax_n - f||)\,, \qquad n = 0,1,2...$$

for the variational inequality (2.5) with nonsmooth unbounded operator A, and

$$x_{n+1} = \pi_\Omega(Jx_n - \alpha_n(\partial u(x_n)/||\partial u(x_n)||)\,, \qquad n = 0,1,2...$$

and

$$x_{n+1} = \pi_\Omega(Jx_n - \alpha_n(u(x_n) - u^*)\partial u(x_n)/||\partial u(x_n)||^2)\,, \qquad n = 0,1,2...$$

for the minimization of functional $u(x)$. Here $u^* = \min_{x\in\Omega} u(x)$.

9. Variational Inequalities and Wiener-Hopf Equations in Banach Spaces

In section 5 and section 8 we formulated new equivalence theorems between varia inequalities in Banach spaces and corresponding operator equations (5.6) and (8.2) with metric projection operator P_Ω and generalized projection operator π_Ω, respectively. It is natural to call the equations of this type *direct projection equations*. Now we establish the connection of variational inequalities with other operator equations, so called Wiener-Hopf equations.

Let P_Ω be a metric projection operator in Hilbert space H, I an identical operator, A and f operator and "right hand part" of variational inequality (2.2), $Q_\Omega = I - P_\Omega$. Then the equation $AP_\Omega z + Q_\Omega z = f$ is said to be a generalized Wiener-Hopf equations in Hilbert space.

The following theorem is valid in Banach spaces (cf. with Hilbert case in [34])

Theorem 9.1. *The variational inequality (2.5) has a (unique) solution $x \in B$ if and only if the Wiener-Hopf equation*

$$J^*(AP_\Omega z - f) + \alpha^{-1}Q_\Omega z = 0 \tag{9.1}$$

with an arbitrary fixed positive parameter α has a (unique) solution $z \in B$ for each $f \in B^*$. Moreover, $z = x - \alpha J^*(Ax - f)$ and $x = P_\Omega z$.

The simplest iterative method to approximate a solution of the equation (9.1) is the following

$$x_n = P_\Omega z_n$$

and

$$z_{n+1} = x_n - \alpha_n J^*(Ax_n - f)\,.$$

However, its convergence can not be established because of the reasons mentioned above. As before, we will obtain the convergent iterative process using generalized projection operators π_Ω. But at the beginning we need an equivalence theorem with this projection operator.

Theorem 9.2. *The variational inequality (2.5) has a (unique) solution $x \in B$ if and only if the Wiener-Hopf equation*

$$A\pi_\Omega z + \alpha^{-1} Q_\Omega z = f \;, \qquad Q_\Omega = I_{B^*} - J\pi_\Omega \;, \tag{9.2}$$

with an arbitrary fixed positive parameter α has a (unique) solution $z \in B^*$ for each $f \in B^*$. Moreover, $z = Jx - \alpha(Ax - f)$ and $x = \pi_\Omega z$. In (9.2) $I_{B^*} : B^* \to B^*$ is identical operator in B^*.

It turns out that the iterative process

$$x_n = \pi_\Omega z_n$$

and

$$z_{n+1} = Jx_n - \alpha_n(Ax_n - f)$$

converges strongly to the solution $z \in B^*$ of the Wiener-Hopf equation (9.2). On the other hand, the iterative process

$$x_{n+1} = \pi_\Omega z_{n+1}$$

and

$$z_{n+1} = Jx_n - \alpha_n(Ax_n - f)$$

for an approximation of the solution x of the variational inequality (2.5) coincides with (2.7) and converges strongly too.

Remark 9.3. It follows from Theorems 9.1 and 9.2 that

$$P_\Omega\big(x - \alpha J^*(Ax - f)\big) = \Pi_\Omega J^*\big(Jx - \alpha(Ax - f)\big)$$

where x is unique solution of the variational inequality (2.5).

Similar results can be formulated also for the quasivariational inequalities and for the complementarity problems. (See, for example, [24,29].)

In conclusion , we notice that generalized projection operators in Banach spaces constructed above and metric projection operators in Hilbert space are defined in similar way in the form of minimization problems, and these problems are of the same level of difficulty. This means that generalized projection operators in Banach spaces can be viewed as a natural generalization of the metric projection operators in Hilbert spaces .

Results of this paper were presented at the 1993 SIAM annual meeting in Philadelphia, July 12-16.

References

[1] Ya.I. Alber, The regularization method for variational inequalities with nonsmooth unbounded operator in Banach space, *Appl. Math. Letters*, 6 (1993), 63-68.

[2] Ya.I. Alber, Recurrence relations and variational inequalities, *Soviet Math. Dokl.*, 27 (1983), 511-517.

[3] Ya.I. Alber, On the solution of equations and variational inequalities with maximal monotone operators, *Soviet Math. Dokl.*, 20 (1979), 871-876.

[4] Ya.I. Alber, Global version of Bjornestal's estimate for metric projection operator in Banach space (submitted).

[5] Ya.I. Alber, On penalty method for variational inequalities with nonsmooth unbounded operator in Banach space (submitted).

[6] Ya.I. Alber and A.I. Notik, Parallelogram inequalities in Banach spaces and some properties of the duality mapping, *Ukranian Math. J.*, 40 (1988), 650-652.

[7] Ya.I. Alber and A.I. Notik, Geometric properties of Banach spaces and approximate methods for solving nonlinear operator equations, *Soviet Math. Dokl.*, 29 (1984), 611-615 .

[8] Ya.I. Alber and A.I. Notik, On some estimates for projection operator in Banach space (submitted).

[9] Ya.I. Alber and A.I. Notik, On minimization of functionals and solution of variational inequalities in Banach spaces, *Soviet Math. Dokl.*, 34 (1986), 296-300.

[10] Ya.I. Alber and A.I. Notik, Geometry of Banach spaces and approximation methods for solving nonlihear operator equations, Preprint, No. 170, Instit. Radio Phisics Researches, Gorky, 1983.

[11] A. Auslender, *Optimization. Methodes numeriques. Maitrisede Mathematiques et Applications Fundamentales* , Masson, Paris-New York-Barcelona, 1976.

[12] B.O.Bjornestal, Local Lipschitz continuity of the metric projection operator, *Approximation Theory,* Stefan Banach Internat. Math. Center Publication, Warsaw, 4 (1979), 43-53.

[13] L.M. Bregman, The method of successive projection for finding a common point of convex sets, *Soviet Math. Dokl.*, 6 (1965), 688-692.

[14] F.E. Browder, Nonlinear operators and nonlinear equations in Banach space, *Proc. Symp. Pure . Math. 18, Part II, Amer. Math. Sos.,* Providence, 1979.

[15] D. Butnariu and Y. Censor, Strong convergence of almost simultaneous block-iterative projection methods in Hilbert spaces, (to appear).

[16] J. Cea, *Optimization: Theorie et algorithmes. Methodes Mathematiques de l'Informatique, 2* , Dunod, Paris, 1971.

[17] P.L. Combettes and H.J. Trussell, The method of successive projections for finding a common point of sets in metric spaces, *J. of Optimization Theory and Applications,* 67 (1990), 487-507.

[18] F. Deutsch, Rate of convergence of method of alternating projection, *International Series of Numerical Mathematics,* 72 (1984), 96-107.

[19] J. Diestel', The Geometry of Banach Spaces, *Lecture Notes Math.,485* Springer, 1975.

[20] J.C. Dunn, Clobal and asymptotic rate estimates for a class of projected gradient processes, *SIAM J. Control and Optimization,* 19 (1981), 368-400.

[21] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry and nonexpansive mapping,* New York, N.Y., M.Dekker, 1984.

[22] L.G. Gubin, B.T.Polyak and E.V. Raik, The method of projections for finding the common point of convex sets, *USSR Computational Mathematics and Mathematical Physics,* 7 (1967), 1-24.

[23] R.B. Holmes, *A Course on Optimization and Best Approximation,* Springer-Verlag, Berlin - Heidelberg - New York, 1972.

[24] C.R. Joy and J.C. Yao, Algorithm for generealized multivalued variational inequalities in Hilbert spaces, *Computer Math. Applic.*, 25 (1993), 7-16.

[25] T.C. Lim and R. Smarzewski, On best approximation and coapproximation in L^p spaces, *Progress in Approximation Theory*, Academic Press, Inc. Harcourt Brace Jovanovich Publishers, P.Nevai and A.Pinkus, ed., 625 - 628, 1991.

[26] T.C. Lim, H.K. Xu and Z.B. Xu, An L^p inequality and its applications to fixed point theory and approximation theory, *Progress in Approximation Theory*, Academic Press, Inc. Harcourt Brace Jovanovich Publishers, P.Nevai and A.Pinkus, ed., 609-624, 1991.

[27] J.-L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod Gauthier-Villars, Paris, 1969.

[28] M.Z. Nashed, A decomposition relative to convex sets, *Proc. Am. Math. Soc.*, 19 (1968), 782-786.

[29] M.A. Noor, Quasi variational inequalities, *Appl. Math. Lett.*, 1 (1988), 367-370.

[30] A.I. Notik, Properties of a duality mapping with a scale function, *Soviet Math.*, 29 (1985), 96-98.

[31] W.V. Petryshyn, Projection methods in nonlinear numarical functional analysis, *J. Math. Mech.* , 17(1967), 353-372.

[32] B. Prus and R. Smarzewski, Strongly unique best approxination and centers in uniformly convex spaces, *J.Math. Anal. Appl.*, 121(1987), 10-21.

[33] S. Reich, Extension problems for accretive sets in Banach spaces, *J. Functional Anal.*, 26(1977), 378-395.

[34] P. Shi, Equivalence of variational inequalities with Wiener-Hopf equations, *Proc. Amer. Math. Soc.*, 111 (1991), 339-346.

[35] A.A. Vladimirov, Yu.E. Nesterov and Yu.N. Chekanov, Uniformly convex functionals, *Vestnik Moscov. Univ. Ser.15, Vychisl. Mat. Kibernet.*, 1978, no.3, 12-23.

[36] J. Von-Neumann, On rings of operators. Reduction theory, *Annals of Mathematics*, 50 (1949), 401-485.

[37] V.V. Yurgelas, Some geometric characteristics of Banach spaces and accretive operators, *Izv. Visch. Uchebn. Zaved. Math.*, 5 (1982), 63-69.

[38] Z.B. Xu and G.F. Roach, Charateristic inequalities of uniformly convex and uniformly smooth Banach Spaces, *J. Math. Anal. Appl.*, 157 (1991), 189-210.

[39] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, *Contributions to Nonlinear Functional Analysis,* Academic Press, New York - London, E.H. Zarantonello, ed., 1972.

[40] E. Zeidler, *Nonlinear Functional Analysis and its Applications 11/B: Nonlinear Monotone Operators,* Springer-Verlag, New-York Inc., 1990.

ON LINEAR NEUTRAL TYPE EQUATIONS WITH GENERALIZED RIGHT-HAND SIDE

E. Braverman

Department of Mathematics
Technion – Israel Institute of Technology
Haifa 32000, Israel

Abstract

The existence results are obtained for a functional differential equation of the type

$$B(t)\dot{x}[g(t)] + A(t)x[h(t)] + \int_a^t K(t,s)dx(s) = f(t) + \alpha_i \delta(t - t_i) \ ,$$

where differentiation and equality are understood in the distributional sense.

1. Introduction

Differential equations with the generalized right-hand side attract attention of investigators since the end of the fifties [1]. Such equations appear when describing a process that undergoes short-time perturbations that can be described by delta functions. Theory of generalized ordinary differential equations was started by Kurzweil [1] and developed later by Š. Schwabik [2], Z. Artstein [3] and many others. The complete review of the results on the subject is given in [2]. In the linear case these equations are of the type

$$\frac{dx(t)}{dt} = D[A(t)x(t) + f(t)] \ . \tag{1}$$

Here differentiation and equality are understood in the generalized sense [2], the elements of A and f are functions of bounded variation. The solution x is usually sought in the space BV of functions $x : [a,b] \to \mathbf{R}^n$ with the bounded on $[a,b]$ variation, $\| x \|_{BV} = \| x(a) \| + var_a^b x$. Here $\| \cdot \|$ is the norm of a vector

$$x = col(\xi_1, \xi_2, ..., \xi_n) : \ \|x\| = \max_{1 \le i \le n} | \xi_i |,$$

$\| \cdot \|$ will also denote the corresponding matrix norm.

Compared with the ordinary differential equation

$$\frac{dx(t)}{dt} + A(t)x(t) = f(t) \tag{2}$$

equation (1) is the more general object, but it is still "local". It means that the rate of change of the state $\frac{dx}{dt}$ depends only on the state x at the same moment t. Generally speaking it is not true. Differential equations including state $x(t)$ and the derivative $\frac{dx(s)}{dt}$ at different moments (i.e. functional differential equations) describe the process more precisely in many cases. A number of works deal with such equations; among them is the well-known monograph of J. Hail [4]. Recent results are systematically stated in [5]. In particular, in this monograph the theory of the following neutral type equation is constructed:

$$\dot{x}(t) + B(t)\dot{x}[g(t)] + A(t)x[h(t)] + \int_a^t K(t,s)dx(s) = r(t)\ , \tag{3}$$

$$t \in [a,b], x(\xi) = \varphi(\xi), \dot{x}(\xi) = \psi(\xi),\ \text{if}\ \xi \notin [a,b].$$

A neutral type equation is a differential equation in which the highest order derivative of the unknown function is included at the present state t and at one or more past or future states.

Suppose that the following assumption is valid:

$$\text{if}\ \ e \in \Sigma,\ \ m(e) = 0,\ \ \text{then}\ \ m(g^{-1}(e)) = 0 \tag{4}$$

Here m is the Lebesgue measure, $\sum$ is the $\delta-$algebra of measurable subsets of the segment $[a,b]$. By BS_g the weighted composition operator is denoted

$$(BS_g y)(t) = \begin{cases} B(t)y[g(t)], & \text{if}\ g(t) \in [a,b], \\ 0, & \text{if}\ g(t) \notin [a,b], \end{cases} \tag{5}$$

$$(Kx)(t) = \int_a^t K(t,s)x(s)ds\ . \tag{6}$$

Then equation (3) can be rewritten in the form

$$\mathcal{L}x \equiv \dot{x} + BS_g\dot{x} + AS_h x + K\dot{x} = f\ , \tag{7}$$

where functions φ, ψ are contained in the right-hand side f.

Suppose that B is a measurable matrix-valued function, and the function g has finite or countable number of monotonicity intervals

$$H_j,\ j = \overline{1,\nu},\ \nu \le \infty,\ [a,b] = \bigcup_{j=1}^{\nu} H_j, H_i \cap H_j = \emptyset \text{ for } i \ne j,$$

and such functions $u_j : [a,b] \to \mathbf{R}^1$, $j = \overline{1,\nu}$, $\nu \leq \infty$, satisfying (4) exist that $u_j(g(t)) = t$, $t \in H_j \cap g^{-1}([a,b])$. Then the function g satisfies Drakhlin $\omega(u_1, ..., u_\nu)$-condition [6] and the weighted composition operator (3) has the following representation in L [6]

$$(BS_g x)(t) = \frac{d}{dt} \int_a^b \sum_{j=1}^{\nu} \frac{d\mu_{[a,t]\cap H_j}}{dm}(s) B(u_j(s)) x(s) ds . \tag{8}$$

Here $\frac{d\mu_H}{dm}$ is the Radon derivative of the measure $\mu_H(e) = m(g^{-1}(e) \cap H)$, $e \in \Sigma$, with respect to the Lebesgue measure.

Let us introduce spaces of functions $x : [a,b] \to \mathbf{R}^n$: L is the space of summable functions: $\| x \|_L = \int_a^b \| x(s) \| ds$, $\mathcal{D}$ is the space of absolutely continuous functions, $\| x \|_{\mathcal{D}} = \| x(a) \| + \| \dot{x} \|_L$. Denote

$$\Phi(B,g)(t) = \sum_{j=1}^{\nu} \| B(u_j(t)) \| \frac{d\mu_{H_j}}{dm}(t) .$$

Theorem [7]. *Weighted composition operator BS_g continuously acts in L if and only if*

$$\lambda = \mathrm{vraisup}_{t\in[a,b]} \Phi(B,g)(t) < \infty , \tag{9}$$

with $\| BS_g \|_{L\to L} \leq \lambda$.

Here $\|T\|_{X\to Y}$ is the norm of a linear bounded operator T acting from a space X to a space Y.

In the monograph [5] the Cauchy problem and boundary value problems for the equation (7) are researched.

Theorem. *Suppose that the Cauchy problem $\mathcal{L}x = f$, $x(a) = \alpha$ has one and only one solution $x \in \mathcal{D}$ for any $f \in L$, $\alpha \in \mathbf{R}^n$. Then the solution has the following representation:*

$$x(t) = \int_a^b C(t,s) f(s) ds + X(t) \cdot \alpha ,$$

where X is the fundamental matrix of solutions of the corresponding homogeneous equation, C is the Cauchy matrix.

Theorem. *Let the inequality (9) be satisfied and let such constant $\sigma > 0$ exist that $g(t) \leq t - \sigma$, $t \in [a,b]$, $h(t) \leq t$, $t \in [a,b]$, the elements of the matrix A be summable on $[a,b]$. Suppose there exists such summable function r that $\|K(t,s)\| \leq r(t)$, $t \in [a,b]$. Then the Cauchy problem (7), $x(a) = \alpha$ has one and only one solution $x \in \mathcal{D}$ for any $f \in L$, $\alpha \in \mathbf{R}^n$.*

Since the eighties generalized functional differential systems have attracted the attention of investigators. Thus A. Anokhin [8] has shown that the linear impulse system

$$\dot{x} + BS_g\dot{x} + AS_h x + K\dot{x} = f \tag{10}$$

$$\Delta x(\tau_i) = x(\tau_i + 0) - x(\tau_i - 0) \; = \; l_i x + \beta_i, \;\; i = 1, ..., n_1,$$

τ_i are fixed points of the segment $[a, b]$, with the initial condition $x(a) = \alpha$, has one and only one solution for any $f \in L$, $\beta_i \in \mathbf{R}^n$, $\alpha_i \in \mathbf{R}^n$. Here the solution x belongs to the space of piecewise absolutely continuous functions with n_1 fixed discontinuity points, l_i are linear bounded functionals acting from this space to $\mathbf{R}^n$. Linear impulse systems of the type (10) for ordinary differential equations are investigated in many works (see [9] and bibliography). In contrast to (1) in the impulse system (10) discontinuity points are fixed and $\Delta x(\tau_i)$ are given in the explicit form, but the discontinuous character of the solution remains. Investigations of various spaces of discontinuous functions were also stimulated by the research of linear impulse systems (10) for functional differential equations. Thus the base for investigation of functional differential equations with the generalized right-hand side was prepared.

2. Statement of the Problem

Let us consider equation (7) with a generalized right-hand side F and, generally speaking, discontinuous solution z. So the derivative $\dot{z}$ has to be replaced by the generalized differentiation operator D. But the equation

$$Dz + BS_g z + AS_h z + KDz \; = \; F \tag{11}$$

is not more than a notation since the operators BS_g and K are not defined on distributions. Usually (see, for example, [2]) the right-hand side is considered $F = D\psi$, where ψ is a function of bounded variation, and the solution also belongs to space BV. It is known that space BV can be represented as the direct sum of three spaces:

$$BV = \mathcal{D} \oplus SP \oplus SG,$$

where $\mathcal{D}$ is the space of absolutely continuous functions,

$$SP = \{y \in BV \mid y = \sum_{i=1}^{\infty} \xi_i \chi_{\tau_i} \, ,$$

$$\xi_i \in \mathbf{R}^n, \tau_i \in [a, b), i \in \overline{1, \infty}, \tau_i \neq \tau_j, \text{ if } i \neq j\} \, , \tag{12}$$

$$\|y\|_{SP} = \sum_{i=1}^{\infty} \|\xi_i\| \, ,$$

χ_τ denotes the characteristic function of $(\tau, b]$, SG is the space of singular (continuous, but not absolutely continuous) functions of bounded variation. For certain systems it is natural to do without this singular component. Thus the solution remains discontinuous, but some problems are solved more easily. Define the space DR of countable sums of delta- functions:

$$DR = \{\delta = \sum_{i=1}^{\infty} \xi_i \delta_{\tau_i} \mid \sum_{i=1}^{\infty} \| \xi_i \| < \infty, \tau_i \in [a,b) \, ,$$

$$\xi_i \in \mathbf{R}^n, i = \overline{1,\infty}, \tau_i \neq \tau_j, \text{ if } i \neq j\}, \quad \|\delta\|_{DR} = \sum_{i=1}^{\infty} \|\xi_i\| \, ,$$

where $\alpha\delta_\tau(f) = f(\tau)\alpha$, f is a vector valued continuous function (row vector), $\tau \in [a,b)$, $\alpha \in \mathbf{R}^n$, $L\delta$ is the direct sum of DR and L:

$$L\delta = \{\varphi \mid (\exists f \in L, \exists \delta \in DR) : \varphi = f + \delta\}, \quad \|\varphi\|_{L\delta} = \|f\|_L + \|\delta\|_{DR} \, ,$$

DS - of spaces $\mathcal{D}$ and SP:

$$DS = \{z \in BV \mid (\exists x \in \mathcal{D}, \ \exists y \in SP) : z = x + y\}, \quad \|z\|_{DS} = \|x\|_{\mathcal{D}} + \|y\|_{SP} \, .$$

D is the generalized differentiation operator:

$$Dx = \dot{x}, \ x \in D, \ D(\alpha\chi_\tau) = \alpha\delta_\tau .$$

If in (11) $F \in L\delta$, $z \in DS$, the operator K acts on an element from DR as follows $K(\alpha\delta_\tau) = K(t,\tau)\alpha$, then in (11) only the weighted composition operator BS_g is not defined for elements $\alpha\delta_\tau \in DR$. In [10] this problem is investigated in detail. For instance, it is shown there that if we define

$$BS_g(\alpha\delta_\tau) = \sum_{j=1}^{\nu} \frac{d\mu_{H_j}}{dm}(\tau) B(u_j(\tau)) \alpha \delta_{u_j(\tau)}, \ \ \alpha \in \mathbf{R}^n \, , \tag{13}$$

then this operator has the same differential integral representation (8). The condition

$$q = \sup_{t \in [a,b)} \Phi(B,g)(t) < \infty \tag{14}$$

is necessary and sufficient for the continuous action of the operator BS_g in DR [10], $\|BS_g\|_{DR \to DR} \leq q$.

It is to be emphasized that if BS_g is not the zero operator then discontinuity points of the solution of (11) may differ from points τ_i included in the right-hand side $F = f + \sum \alpha_i \delta_{\tau_i}$, f is summable.

Example 1. Let $g(t) = t,\ t \in [0,1],\ t$ is irrational, $g(t) = t + 2,\ t \in [0,1],\ t$ is rational. Then the equation

$$Dz - S_g Dz + z(0) = \delta_{0.5}$$

has infinite set of solutions of the type $z = \delta_{0.5} + \sum \alpha_i \delta_{\tau_i}$, where τ_i are irrational.

The following example shows that the solution may be discontinuous at only one point for a right- hand side including different delta functions.

Example 2. Let $[a,b] = [0,1],\ g(t) = 2t - 1, t \in (0.5,1],\ g(t) = 4t - 1, t \in (0.25, 0.5], \ldots g(t) = 2^k t - 1, t \in (\frac{1}{2^k}, \frac{1}{2^{k-1}}], \ldots,\ g(0) = 0$.. Then the equation

$$Dz - S_g Dz + z(0) = \delta_{\frac{1}{2}} - \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{\frac{3}{2^{k+1}}}$$

has the solution $z(t) = \chi_{0.5}(t)$.

The behavior of differential equations with impulse effect may differ greatly from usual [8]. For example, the Cauchy problem $Dz = z(2)\delta_1 + f,\ z(0) = \alpha,\ [a,b] = [0,3]$ is solvable only for such f that $\int_0^2 f(t)dt = -\alpha$.

Below the conditions will be established under which the solvability of a functional differential equation implies the solvability of the same equation with a right-hand side from $L\delta$.

3. Results

I denotes the identity operator, V is the generalized integration operator

$$(Vz)(t) = \int_a^t z(t)dt, \quad z \in DS, \quad V\delta_\tau = \chi_\tau.$$

Theorem 1. *Let the bounded operator $I + BS_g$ be continuously invertible in $L\delta$, let the operator PS_h continuously act from SP to L and the operator K continuously act from DR to L. Suppose that the Cauchy problem (7), $x(a) = \alpha$ is uniquely solvable in $\mathcal{D}$ for any $f \in L,\ \alpha \in \mathbf{R}^n$. Then the Cauchy problem (11), $z(a) = \alpha$ has one and only one solution $z \in DS$ for any $F \in L\delta,\ \alpha \in \mathbf{R}^n$ that can be represented as follows*

$$z(t) = \int_a^b C(t,s)f(s)ds + \int_a^b C_1(t,s)\varphi(s)ds + X(t)\alpha\,, \tag{15}$$

where $f + \varphi = F,\ f \in L,\ \varphi \in DR,\ C, X$ are the same as in the representation of the solutions of (7), C_1 is a matrix valued function with the following properties:

1. for any $s \in [a,b]$ the columns of $C_1(\cdot, s)$ belong to DS;

2. for any $t \in [a,b]$ the elements of $C_1(t,\cdot)$ are bounded;
3. $\sup_{s\in[a,b]} \text{var}_t C_1(t,s) < \infty$.

Proof: Denote $y = V(I + BS_g)^{-1}\varphi$. Then the problem (11), $z(a) = \alpha$ is equivalent to the following problem

$$Dx + BS_gDx + AS_hx + KDx = f - PS_hy - KDy, x(a) = \alpha.$$

As the operators AS_h and K act from DR to L then $f - AS_hy - KDy \in L$. Thus this problem has one and only one solution $x \in \mathcal{D}$. By direct substitution one obtains that $z = x + y$ is the solution of the Cauchy problem (11), $z = \alpha$. Further, the solution $z = x_1 + x_2 + y$, where

$$y = V(I + BS_g)^{-1}\varphi\,, \qquad (i)$$

$x_2 \in \mathcal{D}$ is the solution of the equation

$$Dx_2 + BS_gDx_2 + AS_hx_2 + KDx_2 = -AS_hV(I+BS_g)^{-1}\varphi - K(I+BS_g)^{-1}\varphi\,, \quad (ii),$$

$x_1 \in \mathcal{D}$ is the solution of the problem

$$Dx_1 + BS_gDx_1 + AS_hx_1 + KDx_1 = f, x_1(a) = \alpha,$$

therefore

$$x_1(t) = \int_a^b C(t,s)f(s)ds + X(t)\alpha,$$

where C is the Cauchy matrix of equation (7), X is the fundamental matrix.

The hypotheses of the theorem imply that equalities (i), (ii) define a linear bounded operator T acting continuously from DR to DS:

$$x_2 + y = T\varphi.$$

In [11] it is proved that a linear bounded operator acting from DR to DS has the following representation

$$T\varphi = \int_a^b C_1(t,s)\varphi(s)ds,$$

where C_1 is a matrix-valued function satisfying conditions 1-3 of the theorem. Thus the solution z has the representation (14). The proof of the theorem is completed.

Thus it is necessary to establish invertibility conditions for $I + BS_g$ in DR. One can easily see that $q < 1$, where q is defined by (14), ensures that $(I+BS_g)^{-1}$ exists.

Theorem 2. *If there exists such $\delta > 0$ that $g(t) \leq t - \delta$ for any $t \in [a, b)$ then BS_g is a nilpotent operator in DR.*

Proof: If $g(t) \leq t - \delta$ then $BS_g(\alpha\delta_\tau)$ is the zero vector for any $\tau \in (b - \delta, b), (BS_g)^2(\alpha\delta_\tau) = 0$ for any $\tau \in (b - 2\delta, b)$, etc. Hence BS_g is a nilpotent operator.

Now the existence theorem for functional differential equations with a generalized right-hand side will be proved.

Theorem 3. *Suppose g is a piecewise monotone function, (13) is satisfied and there exist such constant $\sigma > 0$ and such summable function $r(t)$ that*

$$g(t) \leq t - \sigma, \ h(t) \leq t, \ \|K(t, s)\| \leq r(t), \ s, t \in [a, b],$$

A has summable elements. Then the Cauchy problem (11), $z(a) = \alpha$ has one and only one solution $z \in DS$ for any $F \in L\delta$, $\alpha \in \mathbf{R}^n$.

Proof: Let $y \in SP$. For each $\chi = \chi_\tau$ the measurability of $\chi[h(\cdot)]$ results from the measurability of the set $\{s \in [a, b] | \ h(s) \leq t\}$. Then $S_h y$ is measurable as the limit of measurable functions. As A has summable elements then there exists such constant $M > 0$ that

$$\|A\beta\|_L \leq M \sup_{t \in [a,b]} \|\beta(t)\|$$

for any bounded vector valued function β. Since $\|y(t)\| \leq \|y\|_{SP}$, $\|AS_h y\|_L \leq M\|y\|_{SP}$. Thus the operator AS_h continuously acts from SP to L. For any $y \in DR$

$$\|Ky\|_L \leq \|y\|_{DR} \cdot \int_a^b r(s)ds,$$

therefore the operator K also continuously acts from DR to L. By applying the theorems 1,2 and the existence theorem for the equation (7) presented in the introduction we obtain solvability of the Cauchy problem (11), $z(a) = \alpha$.. The proof of the theorem is completed.

One can easily see that there are no additional constraints in Theorem 3 compared with the corresponding existence theorem for the equation (7) with a summable right- hand side, only it is to be noted that only piecewise monotone functions g are considered.

The following example shows that functional differential equations with impulses may have solutions while the corresponding equation without impulses is unsolvable.

Example. The equation

$$Dx - x(1) + x(0) = 1 - \delta_{\frac{1}{2}}$$

has the solution

$$x(t) = t - \chi_{\frac{1}{2}}(t),$$

but the equation $\dot{x}(t) - x(1) + x(0) = 1$ has no absolutely continuous solutions.

Remarks. *1. Let the hypotheses of the theorem 3 be satisfied. Then (11) is the equation with aftereffect and [12] the solution z has the representation of the Volterra type*

$$z(t) = \int_a^t G(t,s)f(s)ds + \int_a^t C_1(t,s)\varphi(s)ds + X(t)z(a),$$

and Theorem 1 implies $G \equiv C$, where C is the Cauchy function in the representation of the solution of (7). 2. Theorem 3 can be generalized to the case of the sum of weighted composition operators, i.e. to the equation

$$Dz + \sum_{i=1}^{n_1} B_i S_{g_i} Dz + \sum_{i=1}^{n_2} A_i S_{h_i} z + KDz = F.$$

In Theorem 3 the conditions $g(t) \leq t - \delta$, $h(t) \leq t$, $\|A(t)\| \leq r(t)$ have to be replaced by $g_i(t) \leq t - \delta$, $h_i(t) \leq t$, $\|A_i(t)\| \leq r_i(t)$, where r_i are summable.

These results can be applied to the investigation of functional differential equations perturbed by discontinuous random processes.

Let $\Omega(\Lambda, \mathcal{F}, P)$ be a probability space and the equation (11) have the right-hand side $F = f + D\xi$, where f is a random process with trajectories in L, ξ is a random process with trajectories in SP,$f(\cdot,\omega) \in L$,$\xi(\cdot,\omega) \in SP$ almost surely ($a.s$). Such processes ξ include, for example, Poisson random processes. It is to be emphasized that jumps of process ξ happen at random instants, therefore it is necessary to consider ξ as an element of a certain function space with discontinuity points being not fixed. SP is exactly the space satisfying this condition.

The equation (11) perturbed by a random process of this type can be rewritten using (8) in the form

$$dz(t,\omega) + d\int_a^b \sum_{j=1}^{\nu} \frac{d\mu_{[a,t]\cap H_j}}{dm}(s)B(u_j(s))dz(s,\omega) +$$

$$+ (AS_h z)(t,\omega)dt + [\int_a^t K(t,s)dz(s,\omega)]dt = fdt + d\xi\,. \qquad (16)$$

By the solution of the equation (15) we will mean a measurable random process z with trajectories from DS satisfying the following integral relation a.s.

$$z(t,\omega) - z(a,\omega) + \int_a^b \sum_{j=1}^{\nu} \frac{d\mu_{[a,t]\cap H_j}}{dm}(s) B(u_j(s)) dz(s,\omega) +$$

$$+ \int_a^t (AS_h z)(s,\omega)ds + \int_a^t d\zeta \int_a^{\zeta} K(\zeta,s)dz(s,\omega) = \int_a^t f(s,\omega)ds + \xi(t,\omega) . \tag{17}$$

Theorem 4. *Let the hypotheses of Theorem 3 be satisfied. Then the Cauchy problem (16), $z(a) = \alpha$ has one and only one solution in the space of processes with trajectories from DS a.s. for any $\alpha \in \mathbf{R}^n$ and any processes $f(\cdot,\omega) \in L$, $\xi(\cdot,\omega) \in SP$ a.s.*

Proof: For almost all $\omega \in \Omega(\Lambda,\mathcal{F},P)$ both $f(\cdot,\omega) \in L$ and $\xi(\cdot,\omega) \in SP$. By Theorem 3 the Cauchy problem (16), $z(a) = \alpha$ has the unique solution with a trajectory from DS. Therefore $z(\cdot,\omega) \in DS$ a.s. To complete the proof of the theorem it is enough to show that the process z is measurable.

By the proof of Theorem 1 $z = x + y$, where

$$y(t,\omega) = V(I + BS_g)^{-1}D\xi = \xi(t,\omega) + \sum_{i=1}^{k}(-1)^i \int_a^t [(BS_g)^i D\xi](s,\omega)ds ,$$

since the weighted composition operator BS_g is nilpotent. Here

$$\int_a^t (BS_g D\xi)(s,\omega)ds$$

is the integral operator included in (16) (see (8)). One can easily see that y is measurable as the sum of measurable processes. Similarly by the proof of Theorem 1

$$x(t,\omega) = \int_a^t C(t,s)f(s,\omega)ds -$$

$$- \int_a^t C(t,s)[AS_h V(I + BS_g)^{-1}D\xi(\cdot,\omega) - K(I + BS_g)^{-1}D\xi(\cdot,\omega)](s)ds$$

is also measurable. Thus $z(t,\omega) = x(t,\omega) + y(t,\omega)$ is measurable, which completes the proof of the theorem.

4. Acknowledgement

The author is very grateful to Israel Ministry of Absorption for the financial support of this investigation, to Prof. M. Drakhlin and Dr. A. Ponosov for fruitful discussions and to Dr. V. Cherdak for useful remarks.

References

[1] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. Czech Math. J., 7:82 (1957), 418-449.

[2] Š. Schwabik, Generalized Differential Equations. Fundamental Results. Praha, Academia, 1985.

[3] Z. Artstein, Topological dynamics of ordinary differential equations and Kurzweil equations. J. Diff. Eq. 23 (1977), 224-243.

[4] J. Hail, Theory of Functional Differential Equations. New York. Springer-Verlag, 1977.

[5] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatulina, Introduction to the Theory of Functional Differential Equations. (in Russian) Nauka, Moscow, 1991.

[6] M.E. Drakhlin, To the theory of linear functional equations. Coll. of sci. works "Functional differential equations". (in Russian) Perm (1986), 92-96.

[7] M.E. Drakhlin, An internal composition operator in the space of summable functions. Izvestiia VUS. Matematika 5 (1986), 18-24.

[8] A.V. Anokhin , On linear impulse systems for functional differential equations. Soviet Math. Dokl. 33 (1986), 220-223.

[9] V. Lakshmikantham, D. Bainov, P. Simeonov, Theory of Impulsive Differential Equations. World Scientific, Singapore, 1989.

[10] E.Ja. Braverman, An internal composition operator in the space of discrete measures. Izvestiia VUS. Matematika 12 (1991), 80-85.

[11] A.V. Anokhin, E.Ja. Braverman, The Question of Conditions for the Continuity of a Linear Operator on a Space of Step Functions. Perm, 1985. (in Russian) (Dep. in VINITI 22.04.85, Manuscript No. 2644-85).

[12] A. Anokhin, E. Braverman, On linear impulse functional differential equations. Differential Equations 26 (1990), 1378-1385.

THE ASYMPTOTIC ATTAINABILITY AND EXTENSIONS IN THE CLASS OF FINITELY ADDITIVE MEASURES, 1

A.G. Chentsov

Institute of Mathematics of the Russian Academy of Science
Ekaterinburg, Russia

Abstract

In this paper a problem of the asymptotic attainable set's construction by the perturbation of conditions is investigated. Conditions of asymptotic non-sensitivity with respect to certain kinds of perturbations are obtained. A correct extension in a special class of finitely additive measures is suggested.

1. Introduction

The perturbation of conditions is the typical phenomenon in many practical problems. Therefore the investigation of various regularizations plays an important role in extremal problems theory (see [1, ch. III] and bibliography [1,2]). Not infrequently for regularization constructions the apparatus of extensions and relaxations is exploited [1, ch. III, IV]. In particular, this approach is applied in the present article. The immediate application of the construction considered is the problem of domain of attainability investigation in the linear control systems with integral restrictions [3]. See [4-7] in this connection.

2. General Definitions and Designations

We use quantors, propositional connectives, $\triangleq$ (the equality by definition). If X is a set, then denote by $\mathcal{P}(X)$ (by 2^X) the family of all (of all nonempty) subsets of X; $\mathrm{Fin}(X)$ stands for the family of all finite sets from 2^X. If A and B are sets, denote by B^A the set of all mappings from A to B; for $g \in B^A$ and $C \in \mathcal{P}(A)$ let $g^1(C) \triangleq \{g(x) : x \in C\}$. The expression

$$\underset{X}{\exists}\ S[X \neq \emptyset]\Big(\underset{X}{\forall}\ S[X \neq \emptyset]\Big)$$

means: exist (for any) set X, $X \neq \emptyset$. For a given set T the (DIR)$[T]$ is the set of all directions in T [8,9]. If A and B are sets $\ll \in$ (DIR)$[A]$, $f \in B^A$, then we name the triple $(A, \ll, f)$ as directedness in B; if moreover B is equipped with topology τ and $B \in B$, then expression $(A, \ll, f) \xrightarrow{\tau} B$ means the convergence of the directedness $(A, \ll, f)$ to B in (B, τ)[8, 9]. If (T, τ) is a topological space and $H \in \mathcal{P}(T)$, then we denote by $\mathrm{cl}(H, \tau)$ the closure of H in (T, τ), $\tau|_H \triangleq \{H \cap G : G \in \tau\}$. Let $\underset{T}{\forall}\ S[T \neq \emptyset]$

$$\mathcal{B}[T] \triangleq \{\mathcal{H} \in 2^{\mathcal{P}(T)} \mid \forall A \in \mathcal{H}\ \forall B \in \mathcal{H}\ \exists C \in \mathcal{H} : C \subset A \cap B\}\ .$$

If P,T are nonempty sets and τ is a topology T, then $\bigotimes^P(\tau)$ denotes the natural topology of Tichonov's product of copies of (T,τ) provided P is the set of indexes; $\bigotimes^P(\tau)$ is the topology of T^P. Let $\mathbf{R}$ be the real line, $\mathcal{N} \triangleq \{1;2;\ldots\}$;$\forall m \in \mathcal{N}$: $\overline{1,m} \triangleq \{k \in \mathcal{N} \mid k \leq m\}$. Then we define $\underset{T}{\forall}\ S[T \neq \emptyset]\ \forall m \in \mathcal{N}$:

$$T^m \triangleq T^{\overline{1,m}}$$

If $k \in \mathcal{N}$ and $(T,\tau), T \neq \emptyset$ is a topological space, then $\bigotimes^k[\tau] \triangleq \bigotimes^{\overline{1,k}}(\tau)$; $(T^k, \bigotimes^k[\tau])$ is the finite Cartesian extent of (T,τ).

3. Finitely Additive Measures

We follow the stipulations of [10,11] in connection with finitely additive measures (FAM). Fix a nonempty set E and a semialgebra [11,12] $\mathcal{L}$ of subsets E. Let $(\text{add})_+[\mathcal{L}]$ be the cone of all real-valued non-negative FAM on $\mathcal{L}$, and $\mathbb{A}(\mathcal{L})$ be the linear subspace of $\mathbf{R}^{\mathcal{L}}$ generated by $(\text{add})_+[\mathcal{L}]$. Let $\leqq$ be the pointwise order in $\mathbf{R}^{\mathcal{L}}$. If $\mu \in \mathbb{A}(\mathcal{L})$, then denote by v_μ the supremum $\{\mu; -\mu\}$ in $\mathbb{A}(\mathcal{L})$ with the order induced from $(\mathbf{R}^{\mathcal{L}}, \leq)$; $v_\mu \in (\text{add})_+[\mathcal{L}]$. Let $\mathcal{B}_*$ be the family of all sets $H \in \mathcal{P}(\mathbb{A}(\mathcal{L}))$ such that $\exists c \in [0,\infty[\forall \mu \in H : v_\mu(E) \leq c$. Fix $\eta \in (\text{add})_+[\mathcal{L}]$. Denote by $(\text{add})^+[\mathcal{L};\eta]$ the cone of all FAM $\mu \in (\text{add})_+[\mathcal{L}]$ such that $\forall L \in \mathcal{L} : (\eta(L)) = 0 \Rightarrow (\mu(L) = 0)$. Let $\mathbb{A}_\eta[\mathcal{L}]$ be the linear subspace of $\mathbb{A}(\mathcal{L})$, generated by $(\text{add})^+[\mathcal{L};\eta]$ (see [13,14]. Let $B_0(E,\mathcal{L})$ be the set of all $\mathcal{L}$-step functionals on E [10,11]; denote by $B_0^+(E,\mathcal{L})$ the positive cone of $B_0(E,\mathcal{L})$. Let $\mathbb{B}(E)$ be the space of all bounded functionals on E equipped with the sup-norm $\|\cdot\|$ [15, ch. IV]. Let $B(E,\mathcal{L})$ denote the closure of $B_0(E,\mathcal{L})$ in $(\mathbb{B}(E), \|\cdot\|)$ and $B^+(E,\mathcal{L})$ denote the positive cone of $B(E,\mathcal{L})$. Then $B(E,\mathcal{L})$ with the norm induced from $(\mathbb{B}(E), \|\cdot\|)$ is the Banach space. The topological conjugate to $B(E,\mathcal{L})$ space and the space $\mathbb{A}(\mathcal{L})$ with the norm defined as variation are isometrically isomorphic; the simplest integral [10, p. 75] (used below) defines the natural bilinear form. Let τ_* be the corresponding to duality $(B(E,\mathcal{L}), \mathbb{A}(\mathcal{L}))$ $*$-weak topology $\mathbb{A}(\mathcal{L})$. Denote by $\tau_{\mathbf{R}}$ (by τ_∂) the natural $|\cdot|$-topology (the discrete topology [8,9]) of set $\mathbf{R}$. We denote $\tau_\otimes \triangleq \bigotimes^{\mathcal{L}}(\tau_{\mathbf{R}})|_{\mathbb{A}(\mathcal{L})}$, $\tau_0 \triangleq \bigotimes^{\mathcal{L}}(\tau_\partial)|_{\mathbb{A}(\mathcal{L})}$, $\mathcal{M} \triangleq \{\tau_*; \tau_0; \tau_\otimes\}$, $\tau_*^+ \triangleq \tau_*|_{(\text{add})_+[\mathcal{L}]}$, $\tau_0^+ \triangleq \tau_0|_{(\text{add})_+[\mathcal{L}]}$, $\tau_\otimes^+ \triangleq \tau_\otimes|_{(\text{add})_+[\mathcal{L}]}$. Then $\forall\ H \in \mathcal{B}_*$: $\tau_\otimes|_H = \tau_*|_H$. Besides, $\tau_*^+ = \tau_\otimes^+ \subset \tau_0^+$.

4. The Property of Density

If $f \in B(E,\mathcal{L})$, then we denote by $f * \eta$ the indefinite η-integral of f [10, p. 76], $f * \eta \in \mathbb{A}(\mathcal{L})$. Denote by $\mathcal{J}$ the operator

$$f \mapsto f * \eta : B(E,\mathcal{L}) \to \mathbb{A}(\mathcal{L}) .$$

If $g \in \mathbf{R}^E$, then denote by $|g|$ the functional

$$x \mapsto |g(x)| : E \to [0, \infty[\; .$$

We have $(\forall f \in B_0(E,\mathcal{L}) : |f| \in B_0^+(E,\mathcal{L}))\&(\forall g \in B(E,\mathcal{L}) : |g| \in B^+(E,\mathcal{L}))$. The proof of the following statement is analogous to the arguments of [13,14]. Let $\forall\, b \in [0,\infty[$:

$$M_b^+ \triangleq \{f \in B_0^+(E,\mathcal{L}) \mid \int_E f\, d\eta \leq b\},$$

$$\mathbf{M}_b^+ \triangleq \{f \in B^+(E,\mathcal{L}) \mid \int_E f\, d\eta \leq b\},$$

$$M_+(b) \triangleq \{f \in B_0^+(E,\mathcal{L}) \mid \int_E f\, d\eta = b\},$$

$$\mathbf{M}_+[b] \triangleq \{f \in B^+(E,\mathcal{L}) \mid \int_E f\, d\eta = b\},$$

$$M_b \triangleq \{f \in B_0(E,\mathcal{L}) \mid \int_E |f| d\eta \leq b\},$$

$$\mathbf{M}_b \triangleq \{f \in B(E,\mathcal{L}) \mid \int_E |f| d\eta \leq b\},$$

$$\Xi_b^+ \triangleq \{\mu \in (\mathrm{add})^+[\mathcal{L};\eta] \mid \mu(E) \leq b\},$$

$$\Xi_+(b) \triangleq \{\mu \in (\mathrm{add})^+[\mathcal{L};\eta] \mid \mu(E) = b\},$$

$$\Xi_b \triangleq \{\mu \in \mathbb{A}_\eta[\mathcal{L}] \mid v_\mu(E) \leq b\}.$$

Then $\forall\, H \in \mathcal{P}(B(E,\mathcal{L})) : \mathcal{J}^1(H) = \{f * \eta : f \in H\}$. It can be shown that the following holds, $\forall\, b \in [0,\infty[$, $\forall\, \tau \in \mathcal{M}$:

$$\Xi_b^+ = \mathrm{cl}\,(\mathcal{J}^1(M_b^+),\tau) = \mathrm{cl}\,(\mathcal{J}^1(\mathbf{M}_b^+),\tau), \tag{4.1}$$

$$\Xi_b = \mathrm{cl}\,(\mathcal{J}^1(M_b),\tau) = \mathrm{cl}\,(\mathcal{J}^1\mathbf{M}_b),\tau)\;, \tag{4.2}$$

$$\Xi_+(b) = \mathrm{cl}\left(\mathcal{J}^1(M_+(b)),\tau\right) = \mathrm{cl}\,(\mathcal{J}^1(\mathbf{M}_+[b]),\tau)\;. \tag{4.3}$$

Relations (4.1)–(4.3) define the density property in the class of integral bounded subsets of $B(E,\mathcal{L})$. Besides, $\forall\, \tau \in \mathcal{M}$:

$$(\mathrm{add})^+[\mathcal{L};\eta] = \mathrm{cl}\left(\mathcal{J}^1(B_0^+(E,\mathcal{L})),\tau\right) = \mathrm{cl}\left(\mathcal{J}^1(B^+(E,\mathcal{L})),\tau\right), \tag{4.4}$$

$$\mathbb{A}_\eta[\mathcal{L}] = \mathrm{cl}\left(\mathcal{J}^1(B_0(E,\mathcal{L})),\tau\right) = \mathrm{cl}\left(\mathcal{J}^1(B(E,\mathcal{L})),\tau\right). \tag{4.5}$$

The relations (4.4), (4.5) have the sense of "weakened" approximate variants of Radon-Nikodym property; (4.4) and (4.5) differ from the famous Bochner statement [15, ch. IV]. The self Radon-Nikodym property is not valid in "universe" FAM in its pure form; see [16]. Let $\forall\, b \in [0,\infty[$: $U(b) \triangleq \{\mu \in \mathbb{A}(\mathcal{L}) | v_\mu(E) \le b\}$. Denote by $\tau_{\mathbb{B}}^*$ the family of all sets $G \in \mathcal{P}(\mathbb{A}(\mathcal{L}))$ such that $\forall\, c \in [0,\infty[$: $U(c) \cap G \in \tau_*|_{U(c)}$ ($\tau_{\mathbb{B}}^*$ is the bounded $*$-weak topology of $\mathbb{A}(\mathcal{L})$). Then

$$\mathbb{A}_\eta[\mathcal{L}] = \mathrm{cl}\left(\mathcal{J}^1(B_0(E,\mathcal{L})), \tau_{\mathbb{B}}^*\right) = \mathrm{cl}\left(\mathcal{J}^1(B(E,\mathcal{L})), \tau_{\mathbb{B}}^*\right).$$

Besides, $((\mathrm{add})_+[\mathcal{L}], \tau_*^+)$ is the locally compact Hausdorff topological space. Then $\forall\, H \in \mathcal{P}((\mathrm{add})_+[\mathcal{L}])$: $\mathrm{cl}(H, \tau_*) = \mathrm{cl}(H, \tau_{\mathbb{B}}^*)$. In particular,

$$\mathrm{cl}\left(\mathcal{J}^1(B_0^+(E,\mathcal{L})), \tau_{\mathbb{B}}^*\right) = \mathrm{cl}\left(\mathcal{J}^1(B^+(E,\mathcal{L})), \tau_{\mathbb{B}}^*\right) = (\mathrm{add})^+[\mathcal{L};\eta].$$

5. The Relaxations of Admissible Set and Problem of Asymptotic Attainability

Fix a nonempty set Γ, an operator

$$\gamma \mapsto S_\gamma : \Gamma \to B(E,\mathcal{L}) \tag{5.1}$$

and the closed in $(\mathbb{R}^\Gamma, \bigotimes^\Gamma(\tau_{\mathbb{R}}))$ set $Y \in \mathcal{P}(\mathbb{R}^\Gamma)$. Let us consider the condition

$$f \in B_0(E,\mathcal{L}) : \left(\int_E S_\gamma f\, d\eta\right)_{\gamma\in\Gamma} \in Y. \tag{5.2}$$

(5.1) is a very non-regular condition. Therefore it is relevant to consider the relaxations of (5.2). Let $\forall K \in \mathrm{Fin}(\Gamma) \forall \epsilon \in]0,\infty[$: $\Omega(K,\epsilon) \triangleq \{f \in B_0(E,\mathcal{L}) \mid \exists y \in Y \forall \gamma \in K : |(\int_E S_\gamma f\, d\eta) - y(\gamma)| \le \epsilon\}$. Then $\mathcal{T} \triangleq \{\Omega(K,\epsilon) : (K,\epsilon) \in \mathrm{Fin}(\Gamma) \times]0,\infty[\} \in \mathcal{B}[B_0(E,\mathcal{L})]$ (see section 2). The family $\mathcal{T}$ defines certain asymptotics of the perturbed conditions. Let $\Gamma_0 \in \mathcal{P}(\Gamma)$ be the set such that $\forall \gamma \in \Gamma_0 : S_\gamma \in B_0(E,\mathcal{L})$; then Γ_0 is the set of step-values of the operator (5.1). Denote $\forall K \in \mathrm{Fin}(\Gamma) \forall \epsilon \in]0,\infty[$:

$$\Omega_0(K,\epsilon) \triangleq \Big\{f \in B_0(E,\mathcal{L}) \mid \exists y \in Y : (\forall \gamma \in K \cap \Gamma_0 : \int_E S_\gamma f\, d\eta = y(\gamma)) \& \Big(\forall \gamma \in K \setminus \Gamma_0 : \Big|(\int_E S_\gamma f\, d\eta) - y(\gamma)\Big| \le \epsilon\Big)\Big\}.$$

Then $\mathcal{T}_0 \triangleq \{\Omega_0(K,\epsilon) : (K,\epsilon) \in \mathrm{Fin}(\Gamma)\times]0,\infty[\} \in \mathcal{B}[B_0(E,\mathcal{L})]$ is the various asymptotics of the perturbed conditions. We suppose $\forall \mathcal{H} \in \mathcal{B}[B_0(E,\mathcal{L})]\ \forall \tau \in \mathcal{M}$:

$$(\eta - \mathrm{Lim})[\mathcal{H};\tau] \triangleq \bigcap_{H\in\mathcal{H}} \mathrm{cl}\,(J^1(H),\tau). \qquad (5.3)$$

Besides, suppose that

$$\widetilde{\Omega} \triangleq \{\mu \in \mathbb{A}_\eta[\mathcal{L}] \mid (\int_E S_\gamma\, d\mu)_{\gamma\in\Gamma} \in Y\}. \qquad (5.4)$$

The connection between (5.3) and (5.4) (where $\mathcal{H} = \mathcal{T}$ and $\mathcal{H} = \mathcal{T}_0$) is set in the following statement [13,14]:

$$\widetilde{\Omega} = (\eta - \mathrm{Lim})[\mathcal{T}_0;\tau_*] = (\eta - \mathrm{Lim})[\mathcal{T};\tau_*] \subset (\eta - \mathrm{Lim})[\mathcal{T}_0;\tau_0]. \qquad (5.5)$$

The proof of (5.5) exploits the property of density from section 4 (see 4.5)). Let $\forall c \in [0,\infty[\ \forall K \in \mathrm{Fin}(\Gamma) \forall\epsilon \in]0,\infty[$:

$$\widehat{\Omega}[K;\epsilon|c] \triangleq \Omega(K,\epsilon) \cap M_{c+\epsilon}\ ,$$

$$\widehat{\Omega}_0[K;\epsilon|c] \triangleq \Omega_0(K,\epsilon) \cap M_c\ ,$$

$$\widehat{\Omega}_c(K,\epsilon) \triangleq \Omega(K,\epsilon) \cap M_c\ .$$

Then $\forall c \in [0,\infty[$:

$$\mathcal{T}_c \triangleq \{\widehat{\Omega}[K;\epsilon|c] : (K,\epsilon) \in \mathrm{Fin}(\Gamma)\times]0,\infty[\} \in \mathcal{B}[B_0(E,\mathcal{L})]\ ,$$

$$\mathcal{T}_c^0 \triangleq \{\widehat{\Omega}_0[K;\epsilon|c] : (K,\epsilon) \in \mathrm{Fin}(\Gamma)\times]0,\infty[\} \in \mathcal{B}[M_c]\ ,$$

$$\widehat{\mathcal{T}}_c \triangleq \{\widehat{\Omega}_c(K,\epsilon) : (K,\epsilon) \in \mathrm{Fin}(\Gamma)\times]0,\infty[\} \in \mathcal{B}[M_c]\ ,$$

$$\widetilde{\Omega}_*^{(c)} \triangleq \widetilde{\Omega} \cap \Xi_c = \{\mu \in \Xi_c \mid (\int_E S_\gamma\, d\mu)_{\gamma\in\Gamma} \in Y\}\ .$$

Theorem 5.1. $\forall c \in [0,\infty[\ \forall \tau \in \mathcal{M} : \widetilde{\Omega}_*^{(c)} = (\eta - \mathrm{Lim})[\mathcal{T}_c;\tau] = (\eta - \mathrm{Lim})[\mathcal{T}_c^0;\tau] = (\eta - \mathrm{Lim})[\widehat{\mathcal{T}}_c;\tau]$.

6. The Asymptotic Attainability, 1.

Fix the Hausdorff space [8,9](Θ,θ), $\Theta \neq \phi$ and the continuous (in the sense $\tau_*^{(\eta)} \triangleq \tau_* |_{\mathbb{A}_\eta[\mathcal{L}]}, \theta)$ operator $w : \mathbb{A}_\eta[\mathcal{L}] \to \Theta$. Denote by W the operator $f \mapsto w(f * \eta) : B_0(E,\mathcal{L})^{\to\Theta}$. Then $w^1(\widetilde{\Omega}_*^{(c)})$ coincide with the intersecting: 1) of all sets $\mathrm{cl}\,(W^1(H),\theta)$, $H \in \mathcal{T}_c$; 2) of all sets $\mathrm{cl}\,(W^1(H),\theta)$, $H \in \mathcal{T}_c^0$; 3) of all sets $\mathrm{cl}\,(W^1(H),\Theta)$, $H \in \widehat{\mathcal{T}}_c$. The proof exploits the construction of compactification (4.2) where $\tau = \tau_*$. Let $\underset{T}{\forall} S[T \neq \phi] : \mathbf{B}_0(T,E,\mathcal{L},\eta) \triangleq \{(g_t)_{t\in T} \in B_0(E,\mathcal{L})^T \mid \exists\, c \in [0,\infty[\forall t \in T : \int_E |g_t| d\eta \leq c\}$. We suppose $\forall \mathcal{H} \in \mathcal{B}[B_0(E,\mathcal{L})]$:

$$\begin{gathered}(\mathbf{B}W-\mathrm{as})[\mathcal{H};\theta] \triangleq \{w \in \Theta \mid \underset{T}{\exists}\, S[T \neq \phi]\exists \ll \in \\ \in (\mathrm{DIR})[T]\exists h \in \mathbf{B}_0(T,E,\mathcal{L},\eta) : (\forall H \in \mathcal{H}\, \exists\, \alpha \in T \\ \forall \beta \in T : (\alpha \ll \beta) \Rightarrow (h(\beta) \in H))\&((T,\ll,W \circ h) \xrightarrow{\theta} w)\}\ ;\end{gathered} \tag{6.1}$$

(6.1) is the corresponding $\mathcal{H}$ attractor of bounded convergence.

Theorem 6.1.

$$w^1(\widetilde{\Omega}) == (\mathbf{B}W-\mathrm{as})[\mathcal{T}_0;\theta] = (\mathbf{B}W-\mathrm{as})[\mathcal{T};\theta]\ .$$

The proof of this theorem exploits the above-mentioned property of the set $w^1(\widetilde{\Omega}_*^{(c)})$.

7. The Asymptotic Attainability, 2

Theorem 6.1 is the general statement about asymptotic insensitiveness of the attainable sets by the perturbation of part of the conditions. This statement is formulated in terms of attractors. We now consider the analogous statement in terms of neighborhoods. We denote $\forall K \in \mathrm{Fin}(\Gamma) : (\mathrm{Fin})[\Gamma \mid K] \triangleq \{\tilde{K} \in \mathrm{Fin}(\Gamma) \mid K \subset \tilde{K}\}$.

Theorem 7.1. *Let $c \in [0,\infty[$ and $H \in \mathcal{P}(\Theta)$ be the neighborhood $(in(\Theta,\theta))$ [17, ch. 1] of the set $w^1(\widetilde{\Omega}_*^{(c)})$. Then $\exists\, K_* \in \mathrm{Fin}(\Gamma)\exists\, \epsilon_* \in]0,\infty[\forall K \in (\mathrm{Fin})[\Gamma \mid K_*]\forall \epsilon \in]0,\epsilon_*] : \mathrm{cl}\,(W^1(\widehat{\Omega}[K;\epsilon|c]),\theta) \subset H$.*

The proof of Theorem 7.1 exploits the property mentioned in section 6, of the set $w^1(\widetilde{\Omega}_*^{(c)})$, the construction of compactification with concrete values of the parameter b in (4.2).

Until the end of this section let (Θ,θ) be the metrizable space. We suppose that $\rho : \Theta \times \Theta \to [0,\infty[$ is the metric Θ generating the topology θ. Let $\forall A \in \mathcal{P}(\Theta)\forall \epsilon \in]0,\infty[: U_\rho^0(A,\epsilon) \triangleq \{\omega \in \Theta \mid \exists\, a \in A : \rho(\omega,a) < \epsilon\}$. Then it is true (see Theorem 7.1):

Theorem 7.2.

$$\begin{gathered}\forall c \in [0,\infty[\forall \beta \in]0,\infty[\exists \tilde{K} \in \mathrm{Fin}(\Gamma)\exists \tilde{\epsilon} \in]0,\infty[\forall K \in (\mathrm{Fin})[\Gamma \mid \tilde{K}]\forall \epsilon \in]0,\tilde{\epsilon}] : \\ W^1(\widehat{\Omega}_0[K;\epsilon \mid c]) \subset W^1(\widehat{\Omega}[K;\epsilon \mid c]) \subset U_\rho^0(W^1(\widehat{\Omega}_0[K;\epsilon \mid c]),\beta)\ .\end{gathered}$$

References

[1] J. Warga, Optimal Control of Differential and Functional Equations. Academic Press, 1972.

[2] I. Ekeland, R. Temam, Convex analysis and variational problems. Studies in Mathematics and its Application. Amsterdam, Oxford, New York, 1987.

[3] N.N. Krasovskii, The Theory of the Control of a Motion. Nauka, Moscow, 1968.

[4] A.G. Chentsov, A.G., On the construction of solution to nonregular problems of optimal control. Problems of Control and Information Theory 20:2 (1991), 129-143.

[5] A.G. Chentsov, To the question on an extension and the result's stability in some class of extremal problems. Kibernetika 4 (1990), 122-123.

[6] A.G. Chentsov, Finitely additive measures and extensions of some extremal problems with restrictions of the asymptotic character. Kibernetika 6 (1990), 78-84.

[7] V.P. Serov, A.G. Chentsov, On some construction of an extension of the problem of control with integral restrictions. Differenzialnie Uravnenia 26:4 (1990), 607-618.

[8] J.L. Kelley, General Topology. New York, 1955.

[9] R. Engelking, General Topology. Warszawa, 1977.

[10] A.G. Chentsov, To the question of universal integrability of bounded functions. Mat. Sbornik 131:1 (1986), 73-93.

[11] A.G. Chentsov, Applications of Measure Theory to Control Problems. Sverdlovsk, 127 p.

[12] J. Neveu, Mathematical Foundations of the Calculus of Probability. (Translated from the French 1964.) Holden Day. San Francisco, California.

[13] A.G. Chentsov, The Asymptotic Optimization and Constructions of Compactifications. Sverdlovsk, 1991. 106 p. (Dep. in VINITI, 401:B91).

[14] A.G. Chentsov, The admissible sets and their relaxations. I. Kraevie zadachi. Perm (1990), 185-196.

[15] N. Dunford, J. Schwartz, Linear Operators. Part I: General Theory. New York, London, 1958.

[16] H. Mahnard, A Radon-Nikodym theorem. Pacific Journal of Mathematics. 83:2 (1970), 401-413.

[17] N. Bourbaki, Topology Generale. Paris (1961), ch. I et II.

SPECTRAL PROPERTIES OF DISJOINTNESS PRESERVING OPERATORS ON LATTICE-NORMED SPACES

Vadim B. Cherdak and **B. Cherdak**

Dept. of Math. & Comp. Sci.
University of Haifa
Haifa 31905, Israel

The Research Institute
The College of Judea and Sumaria
Ariel 44820, Israel

Abstract

The study of weighted shift operator spectral properties is the central problem in the investigation of Linear Differential Equations of Neutral Type. We present a new approach to the study of spectral properties of weighted shift operators in Banach spaces without Banach lattice structure. Our method is based on the theory of *majorized operators on lattice-normed spaces*.

The idea of majorization means that if a linear operator is majorized by a (positive) majorant, then the majorized operator should inherit "good" properties of its majorant. In this paper the hereditary spectral property for disjointness preserving majorized homomorphisms (DPH-operators) is established: the modulus of the spectrum $|\sigma(T)|$ of DPH-operator is the trace of the spectrum of its majorant on the half line:

$$|\sigma(T)| = \sigma([T]) \cap [0;\infty) .$$

1. Introduction

Many works have been dedicated in recent years to the investigation of the Linear Differential Operator of Neutral Type:

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) - \sum_{i=1}^{k} B_i(t)\dot{x}[g_i(t)] - \sum_{i=1}^{k} P_i(t)x[h_i(t)] = v(t),$$
$$t \in [a,b],$$

and

$$x(\xi) = \varphi(\xi),\ \dot{x}(\xi) = \phi(\xi)\ ,\ \text{if}\ \ \xi \notin [a,b].$$

See, for example, [2] or [7].

The main part in the investigation of such equations is the study of weighted shift operator

$$(Sy)(t) = B(t)y(g(t))\ ,$$

where $B(t)$ is a matrix-function, in different functional spaces (see, for example, [6]).

In particular, spectral properties of such operators give useful information about such properties of the operator $\mathcal{L}$ as invertibility, existence of solutions, asymptotics and others.

In the present paper we present a new approach to the study of spectral properties of weighted shift operators in functional spaces without Banach lattice structure (= spaces of Banach valued functions, spaces of tensor products, etc.). Our method is based on the theory of majorized operators on lattice-normed spaces.

The theory of majorized operators on lattice-normed spaces was founded by L. Kantorovich in the 1930s. Further development of this theory took place only in the last decade (see [10], [11], [12]). The general idea of this theory can be defined as follows: if a linear operator is majorized by a positive one, called its majorant, then the majorized operator in some way inherits "good" properties of its majorant. Within the framework of this situation the question naturally arises: which properties of the positive majorant does the majorized operator inherit? (see [12], [11]).

In this paper we discuss the hereditary spectral properties of disjointness preserving majorized homomorphisms (DPH-operator). We note that all weighted shift operators are included in this operator class.

These hereditary properties are based on the invertibility criterion for DPH-operators (section 2) and on the generalization of Arendt's theorem (1) about the spectral decomposition of disjointness preserving regular operators on Banach lattices (section 3): the DPH-operator T can be decomposed into the direct sum of its DPH parts with specific property: each DPH part is the restriction of T to the image of spectral projection canonically associated with some clopen subset of the spectrum $\sigma(T)$. We show that certain spectral projections of a DPH-operator have an ideal as image, and so one can decompose the operator while preserving its properties. As a result the next hereditary rule is established: the modulus of the spectrum $|\sigma(T)|$ of DPH-operator is the trace of the spectrum of its majorant on the half-line:

$$|\sigma(T)| = \sigma([T]) \cap [0; \infty) .$$

Some consequences of this result for compact DPH-operators are discussed in Section 4. The disjointness of eigenvectors corresponding to distinct eigenvalues is established. This fact is analogous to the classical result of the orthogonality of eigenvectors of a Hermitian operator on a Hilbert space corresponding to distinct eigenvalues. This is a generalization of the main result of Wickstead [15] for lattice homomorphisms on Banach lattices. We also obtain that the compact irreducible DPH-operator possesses a strictly positive spectral radius (i.e. it can not be a quasinilpotent operator) and the compact quasinilpotent DPH-operator T generalizes the T-invariant order ideal in the lattice-normed space (see also [4]).

Acknowledgement. The author is very grateful to Professor M. Drahlin and Professor M. Sonis for useful discussions.

2. Main Definitions and Properties

For the general theory and terminology of Banach lattices and lattice-normed spaces we refer to [14], [8] and [10]. Throughout this paper we will denote by E an order complete Banach lattice and by X a complex vector space.

An operator $p : X \to E$ is called *a vector norm on* X, if it satisfies the following conditions:

a) $p(x) \geq 0$; $p(x) = 0$ if and only if $x = 0$;

b) $p(x+y) \leq p(x) + p(y)$ $(x, y \in X)$;

c) $p(\lambda x) = |\lambda| p(x)$ $(\lambda \in C, x \in X)$;

d) for $x \in X$ and for disjoint $e_1, e_2 \in E$; from $p(x) = e_1 + e_2$ it follows that the representation $x = x_1 + x_2$ exists such that $p(x_i) = e_i$ $(i = 1, 2)$.

A vector space X endowed with the vector norm p is called *a lattice-normed space* and denoted by (X, p, E).

Example 2.1. a) If X is a Banach space, then we can consider X as a lattice-normed space $(X, \|\cdot\|, \mathcal{R})$ with vector norm $\|\cdot\|$.

b) If $X = E$ is a Banach lattice, then modulus of elements is a positive operator $|\cdot| : E \to E$ and all properties of vector norm are satisfied. Hence, we can consider $(E, |\cdot|, E)$ as a lattice-normed space with the vector norm $|\cdot|$.

c) Let Y be a Banach space, and $X = L^p(\nu, Y)$ be the space of all Banach-value functions $f : Q \to Y$, satisfying $\int_Q \|f\|^p d\nu < \infty$, where Q is a compact. It is easy to see that $L^p(\nu, Y)$ endowed with a vector norm $p : f(\cdot) \to \|f(\cdot)\|$ is the lattice-normed space, normed by order complete Banach lattice $L^p(\nu)$.

The lattice-normed space (X, p, E) is a *bo-complete* if for each sequence $(x_n) \subset X$ from $p(x_n - x_m) \xrightarrow{o} 0$ it follows that there exists such $x \in X$ that $p(x - x_n) \xrightarrow{o} 0$. The *bo*-complete lattice-normed space will be called a *Banach-Kantorovich space* (*BKS*).

Obviously, if E is a Banach lattice, then a lattice-normed space (X, p, E) endowed with the scalar norm $\|x\| = \|p(x)\|_E$ is a normed space. If (X, p, E) is a BKS then a vector space X endowed with such norm is the Banach space [11].

Definition 2.2. *Let $T \in L(X)$. The operator T is called majorized if there exists a positive operator $S \in L^r(E)$ ($L^r(E)$ is the space of all order bounded linear operators on E) such that*

$$p(Tx) \leq Sp(x)$$

for all $x \in X$.

Such positive operator S will be called a *majorant* of T. The set of all majorants of T is denoted by $maj(T) \subset L^r(E)$. This set has a lower bound in $L^r(E)$ and hence it has the infinum in the order complete Banach lattice $L^r(E)$. This infinum will be named the *exact majorant* of T and will be denoted by $[T]$. The space of all majorized operators on (X, p, E) will be denoted by $M(X)$. If (X, p, E) is a BKS, that $(M(X)\ ,\ [\cdot]\ ,\ L^r(E))$ is a BKS too, endowed with the vector norm $[\cdot]$ (see [12]).

Example 2.3. a) *On a Banach space X all linear bounded operators $T \in L(X)$ are majorized under our definition. Indeed, for bounded operator T with $\|T\| = c \geq 0$ the linear operator $Mz = cz$ for all $z \in \mathcal{R}$ is the majorant of T.*

b) On a Banach lattice E all order bounded operators $T \in L^r(E)$ are majorized by their modulus.

Two elements $x, y \in X$ are *disjoint* (denoted as xdy) if $p(x) \wedge p(y) = 0$. If A is a subset of X, then the disjoint complement of A is given by $\{A\}^d = \{y \in X : ydx \ \forall x \in A\}$. Recall that a subspace J of X is called *ideal in* X if $x \in J$, $y \in X$ and $p(y) \leq p(x)$ implies $y \in J$. A subspace B of X is called *a band* if $B^{dd} = B$. A band is always a closed ideal.

A linear operator $T \in M(X)$ is called a *disjointness preserving* operator if xdy implies $TxdTy$ and is called an *orthomorphism on* X if xdy imply $Txdy$ ($x, y \in X$). The set of all orthomorphisms on X is denoted by $Orth(X)$. We will need the following characterization of the orthomorphisms on X.

Proposition 2.4. *[5] Let $T \in M(X)$. The following are equivalent:*
a) $T \in Orth(X)$;
b) $[T] \in Orth(X)$.

3. DPH-Operators and their Invertibility

In this section we will consider a special sub-class of disjointness preserving majorized operators on a lattice-normed space, which generalizes the class of regular disjointness preserving operators on a Banach lattice.

Definition 3.1. *Let $T \in M(X)$ be a disjointness preserving operator. The operator T will be called a DPH-operator if for each $x \in X$,*

$$p(Tx) = [T]p(x).$$

DPH-operators will be the central object of our investigation in the present paper. Note that all disjointness preserving regular operators on Banach lattices are DPH-operators. Indeed, from [1], 2.4.v it follows that all order bounded disjointness preserving operators on E satisfy the following assertion: $|T|$ exists and satisfies $|Tz| = |T||z|$ for all z belonging to Banach lattice X. In particular, $|T|$ is a lattice homomorphism.

Proposition 3.2. *[5] Let $T \in M(X)$. The following assertions are equivalent:*
a) *T is a disjointness preserving operator;*
b) *$[T]$ is a lattice homomorphism.*

Example 3.3. On $L^p(\nu, Y)$ (see example 1.1.c) we consider a linear operator

$$(Af)(t) = K \circ \big(h \cdot f(\phi(t))\big) ,$$

where ϕ maps the compact Q into itself, $f \in L^p(\nu, Y), h \in L^\infty(\nu)$ and K is an isometric operator on the Banach space Y. Then A is DPH-operator. In fact,

$$p(Af) = \|K(hf(\phi))\|_Y = \|hf(\phi)\|_Y = h\|f(\phi)\|_Y = T_\phi p(f) ,$$

where $(T_\phi g)(t) = h \cdot g(\phi(t))$ is a weighted shift operator on the $L^p(\nu)$, i.e. T_ϕ is a lattice homomorphism (the sufficient conditions that operator T_ϕ acts to $L^p(\nu)$, can be found in [2]).

The dual space of a Banach space X is denoted by X'. If (X, p, E) is a lattice-normed space, then $(X', [\cdot], E')$ is also a lattice-normed space [11]. The vector norm of a linear functional x' is determined as the exact majorant of x' in E' and denoted by $[x']$. If $x \in X$ and $x' \in X'$ then by [11] 4.3,

$$\langle x, x' \rangle \le \langle p(x), [x'] \rangle \tag{1}$$

Theorem 3.4. *Let $T \in M(X)$. Then $T' \in M(X')$. Moreover, if T is a disjointness preserving operator, then $[T'] = [T]'$.*

Proof: First let us show that T' is majorized. Indeed, let $T \in M(X)$ and $S \in maj(T)$. Then for any $x' \in X'$ and $0 \le e \in E$ we have

$$< e, [T'x'] > \; = \; < e, [x' \circ T] > \; \le \; < Se, [x'] > \; = \; < e, S'[x'] > \; .$$

As e was chosen arbitrarily then $[T'x'] \le S'[x']$, i.e. $T' \in M(X')$.

Now let $[T]$ be the exact majorant of T and T be a disjointness preserving operator. Then $[T]$ is a lattice homomorphism. For each fixed $f' \in E'$

$$\sup\{\sum_i [T'x_i'] : (x_i') \subset X', \sum_i [x_i'] \le f'\} \;\le\; [T']f' \; . \tag{2}$$

On the other hand, let $e \in E_+$, $x \in X$ and $p(x) = e$. As $[T]$ is a lattice homomorphism, then

$$\begin{aligned} \Big\langle e, \sup\{\sum_i [T'x_i'] : \sum_i [x_i'] \;= f'\}\Big\rangle &\ge \langle e, \sup\{[T'x'] : [x'] = f'\}\rangle \ge \\ &\ge \sup\{|\langle x, T'x'\rangle| : [x'] = f', p(x) = e\} = \\ &= \Big\langle \sup_{p(x)\le e} p(Tx), f'\Big\rangle = \langle [T]e, f'\rangle = \langle e, [T]'f'\rangle \; . \end{aligned}$$

As e was chosen arbitrarily, we have

$$\sup\{\sum_i [T'x_i'] : (x_i') \subset X', \sum_i [x_i'] = f'\} \ge [T]'f' \tag{3}$$

It follows now from (2) and (3) that

$$[T']f' = \sup\{\sum_i [T'x_i'] : (x_i') \subset X', \sum_i [x_i'] \le f'\} .$$

Now from [12] 4.2 it can be seen that $[T'] = [T]'$.

The invertibility criterion for majorized DPH-operators can be formulated as follows (we use the scheme of proof [1] 2.7).

Theorem 3.5. *Let $T \in M(X)$ be a DPH-operator, such that T' is also DPH-operator on X', then*

a) *The operator T is invertible in $M(X)$ iff the operator $[T]$ is invertible in $L^r(E)$.*

b) *If T is invertible, then T^{-1} is a DPH-operator and $[T^{-1}] = [T]^{-1}$.*

Proof: Suppose that $[T]$ is invertible. Then for each $x \in X$

$$\|Tx\|_X = \|p(Tx)\|_E = \|[T]p(x)\|_E \ge \|[T]^{-1}\|^{-1}\|x\|_X \tag{4}$$

It follows that T is injective. Moreover, $[T]$ is a lattice isomorphism, hence $[T]'$ is a lattice isomorphism. By 3.3, $[T'] = [T]'$, hence T' is a disjointness preserving operator. It follows from invertibility of $[T']$ that T' is injective. Therefore, $(TX)^0 = \mathrm{Ker}T' = \{0\}$. This implies that $(TX)^- = X$, i.e. T has dense image in X. From (4) it follows that T is invertible. Let us show that $T^{-1} \in M(X)$. Suppose that $f \in E_+$. Then

$$\begin{aligned}[T^{-1}]f &= \sup\{[T^{-1}]g :\ 0 \le g \le f\} = \\ &= \sup\{\ p(x) :\ x \in X, p(Tx) = \ [T]p(x) \le f\} = \\ &= \ \sup\{\ p(T^{-1}x) :\ p(x) \le f\} \ = [T^{-1}]f \ .\end{aligned}$$

Hence, $[T^{-1}]$ exists, and moreover $[T^{-1}] = [T]^{-1}$.

We show now that T^{-1} is a DPH-operator. Indeed, T^{-1} is a disjointness preserving operator, as $[T^{-1}] = [T]^{-1}$ is a lattice homomorphism. To prove that T^{-1} is a DPH-operator, we show that $p(T^{-1}x) = [T^{-1}]p(x)$. In fact, for $x \in X$ we have

$$p(x) = p(TT^{-1}x) = [T]p(T^{-1}x) \le [T][T^{-1}]p(x) = [T][T]^{-1}p(x) = p(x) .$$

It follows that $p(T^{-1}x) = [T^{-1}]p(x)$ for all $x \in X$.

To conclude the proof we show that if T is invertible then $[T]$ is invertible on E. Suppose that T is invertible. Then for each $e \in p(X)(e = p(x))$ it follows from $[T]e = 0$ that $p(Tx) = 0$, that is equivalent to $x \in \mathrm{Ker}T$. Since T is invertible, $\mathrm{Ker}T = \{0\}$. It follows that $x = 0$ and $e = p(x) = 0$. Moreover, for each $f \in p(X)$ there exists $x \in X$ such that $Tx = y$ and $p(y) = f$. Hence, $f = p(Tx) = [T]p(x)$. Since $E_+ = p(X)$ (from the definition of a lattice-normed space; see, for example, [11]), $[T]$ is invertible.

4. Spectrum rule for DPH-operators

We recall some standard definitions (see, for example, [1], p. 206- 209). Let F be a Banach space and A be a bounded operator on F. We denote by $r(A)$ the spectral radius of A and by $r_m(A)$ the real number

$$r_m(A) = \inf\{|\lambda| : \lambda \in \sigma(A)\} .$$

Note that $r_m(A) = r(A^{-1})^{-1}$ if A is an invertible operator. For $\lambda \in \rho(A)$ we denote by $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent of A in point λ. If $|\lambda| > r(A)$, then $\lambda \in \rho(A)$ and $R(\lambda, A)$ is given by Neumann's series $R(\lambda, A) = \sum_{n=0}^{\infty} A^n/\lambda^{n+1}$.

A spectral subset σ_1 of $\sigma(A)$ is by definition an open and closed subset of $\sigma(A)$. To such a set the spectral projection P given by

$$P = \frac{1}{2\pi i} \int_c R(\lambda, A) d\lambda$$

is canonically associated, where c is the positively oriented boundary of a Cauchy domain having σ_1 in its interior and $\sigma_2 := \sigma(A) \setminus \sigma_1$ in its exterior. P reduces A, that is, $PA = AP$, or equivalently, PF and Ker$P$ are invariant under A. If A_1(respectively, A_2) denotes the restriction of A to PF(respectively, Ker$P$), then $\sigma(A_i) = \sigma_i$ $(i = 1, 2)$.

In general, if $T \in M(X)$, order properties of T are lost in the spectral decomposition. PX and Ker$P$ do not need to be sub lattice-normed spaces. The next theorem, however, relates Arendt's result [1] 4.1 to majorized operators on lattice-normed spaces and gives a positive result.

For $s \in [0; \infty)$, we denote $\Gamma_s = \{z \in C : |z| = s\}$, and $\Gamma = \Gamma_1$. Let $T \in L(X)$. If $\Gamma_s \cap \sigma(T) = \{\emptyset\}$ and $r_m(T) < s < r(T)$, set $\sigma_s = \{z \in \sigma(T) : |z| \leq s\}$. Thus, $\sigma_s(T)$ is a spectral subset of $\sigma(T)$.

Theorem 4.1. *Let T be a DPH-operator on X. Suppose that there exists an $s \in (r_m(T), r(T))$ such that $\Gamma_s \cap \sigma(T) = \{\emptyset\}$. Then the spectral projection belonging to $\sigma_s(T)$ has an ideal as image.*

Proof: Let

$$P = \frac{1}{2\pi i} \int_{\Gamma_s} R(\lambda, T) d\lambda , \quad X_1 = PX$$

and denote by T_1 the restriction of T to X_1. Since $r(T_1) < s$, we have for $x \in X_1$

$$R(s, T)x = R(s, T_1)x = \sum_{n=0}^{\infty} \left(\frac{T_1^n}{s^{n+1}}\right) x = \sum_{n=0}^{\infty} \left(\frac{T^n}{s^{n+1}}\right) x$$

and

$$\sum_{n=0}^{\infty} \left\| \frac{T^n}{s^{n+1}} \right\| = \sum_{n=0}^{\infty} \left\| \frac{T_1^n}{s^{n+1}} \right\| < \infty$$

Since T^n is a DPH-operator, then for all $x \in X$

$$p(T^n x) = [T^n]p(x) .$$

Hence, for $y \in X$ such that $p(y) \leq p(x)$ we have $\|T^n y\| \leq \|T^n x\|$. Indeed, $[T^n]$ is a lattice homomorphism, so

$$p(T^n y) = [T^n]p(y) \leq [T^n]p(x) = p(T^n x)$$

It follows, that $\|T^n y\| \leq \|T^n x\|$ by virtue of norm monotonity in the lattice-normed space X.

Hence, for all $\lambda \in \Gamma_s$ and all $y \in X$ such that $p(y) \leq p(x)$,

$$\sum_{n=0}^{\infty} \left\| \frac{T^n y}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{T^n x}{s^{n+1}} \right\| < \infty$$

Moreover,

$$(\lambda - T) \sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}} = \lim_{m \to \infty} (\lambda - T) \sum_{n=0}^{\infty} \frac{Ty}{\lambda^{n+1}} =$$

$$= \lim_{m \to \infty} \left(y - \frac{T^{m+1} y}{\lambda^{m+1}} \right) = y,$$

that is

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}} \qquad (\lambda \in \Gamma_s, p(y) \leq p(x)) .$$

Consequently, for $y \in X$ satisfying $p(y) \leq p(x)$,

$$Py = \frac{1}{2\pi i} \int_{\Gamma_s} \left(\sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}} \right) d\lambda = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_s} \frac{1}{\lambda^{n+1}} d\lambda \cdot T^n y = y$$

(since the series is uniformly convergent for $\lambda \in \Gamma_s$).

We have proved that $y \in X_1$ if $p(y) \leq p(x)$ for some $x \in X_1$; that is , X_1 is an ideal.

If J is a closed ideal which is invariant under a bounded operator A on X, then we denote by $A|_J$ the restriction of A to J and by A_J the operator on X/J induced by A (this is , $A_J(x + J) = Ax + J$ for all $x + J \in X/ J$). The ideal in E respective to J by vector norm p we denote by I.

Lemma 4.2. *Let T be a DPH-operator on X. Let J be a closed ideal of X. Then $TJ \subseteq J$ iff $[T]I \subseteq I$. If $TJ \subseteq J$, then $T_{|J}$ and T_J are DPH-operators. Moreover, $[T_{|J}] = [T]_{|I}, [T_J] = [T]_I$.*

Proof: The first assertion follows from the definition of a DPH-operator. In fact, suppose that $TJ \subseteq J$, we show that $[T]I \subseteq I$.

Let $x \in J(p(x) = e, e \in I)$, then $Tx \in J$, but $p(Tx) = [T]p(x) = [T]e$. Hence, as $Tx \in J$, then $[T]e \in I$. Conversely, suppose that for each $e \in I$ we have $[T]e \in I$ and let $x \in J$, then $p(x) = u \in I$ and $p(Tx) = [T]u \in I$. It follows that $Tx \in J$.

Let J be a closed ideal in X which is invariant under T. We show that $T_{|J}$ and T_J are DPH-operators. Indeed, the restriction of T on the invariant closed ideal J preserves the equality $p(Tx) = [T]p(x)$ for all $x \in J$, and $[T]$ is a lattice homomorphism on I, hence, $T_{|J}$ is a DPH-operator on J. Let $x \in J$, then

$$[T]_{|I}p(x) = p(Tx) = p(T_{|J}x) = [T_{|J}]p(x).$$

Suppose that q denotes a vector norm on X/J, then

$$\begin{aligned} q(T_J(x+J)) &= q(Tx+J) = p(Tx) + I = [T]p(x) + I \\ &= [T]_I(p(x)+I) = [T]_I(q(x+J)). \end{aligned}$$

Hence, $[T_J] = [T]_I$. Since $[T]_I$ is a lattice homomorphism, we see that T_J is a DPH-operator on X/J. The proof is complete.

To show the use of Lemma 4.2 in our context let us assume that we are in the situation of Theorem 4.1. Let $J = PX$, where P is the spectral projection associated with $\sigma_s(T)$. Then $T_{|J}$ and T_J are majorized DPH-operators and $\sigma(T_{|J}) = \sigma_s(T)$ and $\sigma(T_J) = \sigma(T) \setminus \sigma_s(T)$. In particular, $r(T_{|J}) < s < r_m(T_J)$. We have found a spectral decomposition of T into the two DPH-operators T_J and $T_{|J}$.

We will now use this property to prove a relation between $\sigma(T)$ and $\sigma([T])$. To simplify the notation we set

$$|\sigma| = \{|z| : z \in \sigma\} \subset [0, \infty)$$

if σ is a subset of C.

Theorem 4.3. *Let T and T' be DPH-operators on X and X', respectively. Then*

$$|\sigma(T)| = \sigma([T]) \cap [0, \infty)\ .$$

Proof: a) We show that $r(T) = r([T])$. First of all, for DPH-operators the operator norm in Banach algebra $L(X)$ coincides with one in $M(X)$. In fact,

$$\begin{aligned} \|T\|_{L(X)} &= \sup_{\|x\| \le 1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \le 1} \frac{\|p(Tx)\|}{\|p(x)\|} = \\ &= \sup_{\|p(x)\| \le 1} \frac{\|[T]p(x)\|}{\|p(x)\|} = \|[T]\|\ . \end{aligned}$$

On the other hand, $\|T\|_r = \|T\|_{M(X)} = \|[T]\|$ (see, for example, [3]), hence $\|T\|_{L(X)} = \|T\|_r$.

Second, for all DPH-operators $[T^n] = [T]^n$. Indeed,

$$[T]^n p(x) = [T]^{n-1} p(Tx) = \ldots = p(T^n x) = [T^n] p(x).$$

Now, from the spectral radius formula, we have

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|_{L^r(X)}} = \lim_{n\to\infty} \sqrt[n]{\|[T^n]\|_{L^r(X)}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\|[T]^n\|_{L^r(X)}} = r([T]) \,.$$

In conclusion, we have $r(T) = r([T])$.

b) We show that $r_m(T) = r_m([T])$. By 3.5, $r_m(T) = 0$ if and only if $r_m([T]) = 0$. Suppose now that $r_m(T) > 0$. Then T^{-1} is a DPH-operator and $[T^{-1}] = [T]^{-1}$. Therefore it follows from a) that $r_m(T) = r(T^{-1})^{-1} = r([T]^{-1})^{-1} = r_m([T])$.

c) We show that $\sigma([T]) \cap (r_m(T), r(T)) = |\sigma(T)| \cap (r_m(T), r(T))$. Let $s \in (r_m(T), r(T))$ and suppose that $s \notin |\sigma(T)|$. Then $\Gamma_s \cap \sigma(T) = \{\emptyset\}$. By 4.1 , there exists a closed ideal J of X which is invariant under T, such that $r(T_{|J}) < s < r_m(T_J)$. It also follows from 4.1 and b) that $[T]$ leaves invariant the respective closed ideal I of E (see 4.2) and $r([T]_{|J}) < s < r_m([T]_J)$. Since $\sigma([T]) \subseteq \sigma([T]_{|J}) \cup \sigma([T]_J)$, we conclude that $s \notin \sigma([T])$. Conversely, let $s \in (r_m(T), r(T))$ such that $s \notin \sigma([T])$. Since $[T]$ is a lattice homomorphism, $z \in \sigma([T])$ implies $|z| \in \sigma([T])$ (see, for example, [14] V.4.4). Therefore, $\Gamma_s \cap \sigma([T]) = \emptyset$. By 4.1 there exists a closed ideal I, which is invariant under $[T]$, such that $r([T]_{|I}) < s < r_m([T]_I)$. It follows from 4.3 a) and b) that $r(T_{|J}) < s < r_m(T_J)$ for respective $J \subset X$. Since $\sigma(T) \subseteq \sigma(T_{|J}) \cup \sigma(T_J)$, we can conclude that $s \notin |\sigma(T)|$.

d) We prove the equality in the theorem. Let $s \in \sigma([T]) \cap [0, \infty)$. If $s \in (r_m(T), r(T))$, then $s \in |\sigma(T)|$ by *c*). If $s = r(T)$ (respectively, $s = r_m(T)$), then $s \in |\sigma(T)|$, because $\Gamma_{r(T)} \cap \sigma(T) \neq \emptyset$ (respectively, $\Gamma_{r_m(T)} \cap \sigma(T) \neq \emptyset$). Conversely, let $s \in |\sigma(T)|$. Again, if $s \in (r_m(T), r(T))$, then $s \in \sigma([T])$ by *c*). If $s = r(T) = r_m([T])$, then $s \in \sigma([T])$ because $r([T]) \in \sigma([T])$. Finally, let $s = r_m(T)$. If $r_m(T) = 0$, then $r_m([T]) = 0$ by *b*). Therefore, $s = 0 \in \sigma([T])$. If $r_m(T) > 0$, then $r_m([T]) = r([T]^{-1})^{-1}$. Since $r([T]^{-1}) \in \sigma([T]^{-1})$, it follows that $s = r([T]^{-1})^{-1} \in \sigma([T]^{-1})^{-1} = \sigma([T])$. The proof is complete.

Recall that a bounded operator T on X is called *irreducible* if there exists no closed ideal $J \neq 0, X$, such that $TJ \subset J$.

Corollary 4.4. *Let T and T' be DPH-operators. If T is irreducible then*

$$|\sigma(T)| = [r_m(T), r(T)] \,.$$

Remark 4.5. a) The result of Theorem 4.3 extends theorem 4.4 [1] on majorized DPH-operators on the Banach-Kantorovich spaces. The proof of this one was modified for the work with majorized operators.

b) In the case of weighted shift operators on the Banach modulus over uniform algebras a similar result was announced in [9].

5. Compact DPH-operators

In this section we show some interesting consequences of Theorem 4.3.

We begin with a statement about compactness of exact majorants for compact DPH-operators. The full discussion about the existence of compact majorants for compact majorized operators can be found in [3].

Proposition 5.1. *Let T be a compact DPH-operator on the BKS X. Then $[T]$ is a compact lattice homomorphism on E.*

Proof: Let B_E be the unit ball in E and let $(e_n) \subset B_E$ be a sequence. Suppose that $(x_n) \subset X$ such that $p(x_i) = e_i (i = 1, 2, \ldots)$, then $(x_n) \subset B_X$, where B_X is a unit ball in X. Hence, since T is compact, from (x_n) we can choose a subsequence (x_{n_j}), such that (Tx_{n_j}) converges in X. We show that, for corresponding subsequence (e_{n_j}), $([T]e_{n_j})$ converges in E. In fact, for any k and m

$$\|[T]e_m - [T]e_k\|_E = \|[T]p(x_m) - [T]p(x_k)\| = \|p(Tx_m) - p(Tx_k)\| \leq$$

$$\leq \|p(Tx_m - Tx_k)\|_E = \|Tx_m - Tx_k\|_X .$$

It follows that $([T]e_{n_j})$ is Cauchy sequence in E and, by virtue of completeness of E, it converges in E.

We show that for a compact DPH-operator the eigenvectors which correspond to the different eigenvalues are disjoint.

Theorem 5.2. *Let T be a compact DPH-operator. Suppose that λ and μ ($|\lambda| \neq |\mu| \neq 0$) are eigenvalues of T and $x, y \in X$ are corresponded eigenvectors, then xdy.*

Proof: Since T is a compact DPH-operator, by virtue of 5.1, $[T]$ is a compact lattice homomorphism. It follows from 3.3, that $|\lambda|, |\mu| \in \sigma_p([T])(\sigma_p([T])$ denotes the point spectrum of the linear operator $[T]$), and $p(x), p(y)$ are the corresponding eigenvectors of $[T]$. Now from [15] we have $p(x) \wedge p(y) = 0$. It follows that xdy.

Recall that a linear bounded operator T is called *quasinilpotent* if its spectral radius $r(T) = 0$.

Proposition 5.3. *Let $T \in M(X)$. If T has quasinilpotent majorant, then T is quasinilpotent operator.*

Proof: In fact, let $S \in maj(T)$, $r(S) = 0$. For each $n \in \mathcal{N}$ we have $\|T^n\| \leq \|S^n\|$. Hence, by the spectral radius formula, $r(T) \leq r(S)$.

We notice that if T is a DPH-operator, then, by 4.3, we have $r(T) = r([T])$.

We now formulate sufficient conditions for a compact DPH-operator that guarantee $r(T) > 0$.

Theorem 5.4. *If T is a compact irreducible DPH-operator, then $r(T) > 0$.*

Proof: If $[T]$ is exact majorant of T, then $[T]$ is a compact (by 5.1), irreducible (by 4.2), positive operator and by 4.3 $r(T) = r([T])$. By virtue of de Pagter's theorem [13], $r([T]) > 0$. It follows, that $r(T) > 0$.

We conclude this section with a statement about the existence of nontrivial invariant ideals for DPH-operator (s.f. [4]).

Theorem 5.5. *Let $T \neq 0$ be a quasinilpotent compact DPH-operator on X. Then a T-invariant nontrivial ideal exists in X.*

Proof: Indeed, suppose that T is irreducible; then by 5.4, $r(T) > 0$. Hence, a T-invariant ideal exists in X.

References

[1] W. Arendt, Spectrum properties of Lamperti operators. Indiana Univ. Math. J., 32 (1983), 199-215.

[2] N.V. Azbelev, V. Maksimov, L. Rakhmatullina, Introduction to the Theory of Functional Differential Equations. (In Russian) Nauka, Moscow, 1991.

[3] V. Cherdak, On order spectrum of majorized operators on the lattice-normed spaces, Sibirski Math. J. 32 (1991), 126-132.

[4] V. Cherdak, Invariant ideals of majorized operators. (In Russian) Optimization, 47 (1990), 137-140. Institute of Mathematics, Novosibirsk.

[5] V. Cherdak, A some class of majorized operators. (In Russian) Optimization 42 (1988), 112-118. Institute of Mathematics, Novosibirsk.

[6] M.E. Drahlin, Operator of inner superposition in spaces of summable functions. Izvestija VUZov, Mathematics 5 (1986), 18-23.

[7] J.K. Hale, Theory of Functional Differential Equations. Applied Mathematical Sciences, Vol.3, Springer, 1977.

[8] L.V. Kantorovich, G.P. Akilov, Functional Analysis. Pergamon Press, 1981.

[9] A.K. Kitover, Weighted shift operators in Banach modulus over uniform algebras. Soviet Math. Dokl. 24 (1983), 214.

[10] A.G. Kusraev, Vector Duality. Nauka, Novosibirsk, 1985.

[11] A.G. Kusraev, Linear operators in the lattice-normed spaces. (In Russian) Transactions of Institute of Mathematics 9 (1987), 84-123.

[12] A.G. Kusraev, V. Strizevsky, Lattice-normed spaces and majorized operators. (In Russian) Transactions of Institute of Mathematics (Novosibirsk) 7 (1987), 132-158.

[13] B. de Pagter, Irreducible compact operators. Math. Z. 192 (1986), 149-153.

[14] H.H. Schaefer, Banach lattices and positive operators. Springer, Berlin, Heidelberg, New York, 1974.

[15] A.W. Wickstead, Spectral properties of compact lattice homomorphisms. Proc. Amer. Math. Soc. 84 (1982), 347-353.

NEW CONCEPT IN THE STUDY OF DIFFERENTIAL INEQUALITIES

Alexander Domoshnitsky

Department of Mathematics
Technion — Israel Institute of Technology
Haifa 32000, Israel

Abstract

New assertions on differential inequalities for systems of functional-differential equations are proposed. Main results are based on constructing a corresponding scalar functional-differential equation for single component of the solution vector.

Tests of positivity of elements of Green's matrix for some boundary value problems are discovered.

1. Introduction

Let us consider the following functional-differential equation (FDE)

$$(Mx)(t) \equiv \dot{x}(t) + (Tx)(t) = f(t),\ t \in [a,b]\ , \tag{1}$$

where $T : C \to L$ is a linear bounded operator, L is the space of summable functions $y : [a,b] \to R^n$, C is the space of continuous functions $x : [a,b] \to R^n$.

The main reason for consideration (1) is the use of equations with delayed argument, integro-differential equations and some of their "hybrids" in the mathematical description of real processes (see for example [9,17]). Note another use of FDEs, namely, they are an important instrument for investigations of ordinary differential systems. The principal results of this paper are connected with the idea of constructing a corresponding scalar FDE of the first order for a single component x_r of the solution vector x, and using known results for this scalar equation. In this connection the importance of the study of scalar FDE essentially increases.

Differential inequalities for FDEs were considered in the monographs of N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina [1], V. Lakshmikantham, S. Leela [13], J. Schröder [18]. Note the recent results on the scalar FDE of the first order [2,6,7,8,11,12,15]. Some analogues of theorems known for ordinary differential equations due to de la Vallee-Poussin [19], S.A. Chaplygin [4], J.E. Wilkins [21], T. Wazewsky [20] are obtained in the paper.

The research was supported in part by a grant from the Ministry of Science and Technology, Israel, and the "MA-AGARA" - special project for absorption of new immigrants, in the Department of Mathematics, Technion.

Let $l : C \to R^n$ be a linear bounded functional. If the homogeneous boundary value problem $Mx = 0$, $lx = 0$ has only trivial solution, then the boundary value problem

$$Mx = f, \quad lx = 0 \tag{2}$$

has, for each $f \in L$, a unique solution, which has the representation [1,3]

$$x(t) = \int_a^b G(t,s)f(s)ds \ , \tag{3}$$

where matrix $G(t,s)$ is called Green's matrix of problem (2).

From formula (3) it is clear that the key problem in the consideration of differential inequalities is that of the positivity of Green's matrix. If Green's matrix of boundary value problem (2) is positive, then it follows from the conditions

$$(Mx)(t) \geq (My)(t), \quad t \in [a,b], \quad lx = ly, \quad \text{that } x(t) \geq y(t), \quad t \in [a,b] \tag{4}$$

Property (4) is also known as Chaplygin's problem [4,14].

2. Main Results

Let $C(t,s), G(t,s), P(t,s)$ be Green's functions of the boundary value problems

$$(Mx)(t) = f(t), \quad t \in [a,b], \quad x(a) = 0 \ ,$$
$$(Mx)(t) = f(t), \quad t \in [a,b], \quad x(b) = 0, \tag{5}$$
$$(Mx)(t) = f(t), \quad t \in [a,b], \quad x(a) - x(b) = 0 \ . \tag{6}$$

We define an operator $K : C \to C$ by the formula $(Kx)(t) = \int_t^b (Tx)(s)ds$.

Theorem 1 determines relations between the following assertions for the scalar FDE of the first order:

1) there exists a nonnegative absolutely continuous function v such that

$$v(b) - \int_t^b (Mv)(s)ds > 0,$$

2) the spectral radius of the operator K is less than one,
3) problem (5) is uniquely solvable and its Green's function is negative for $a \leq t \leq s \leq b$ and nonpositive for $a \leq s < t \leq b$,
4) the Cauchy function $C(t,s)$ of equation (1) is positive,
5) a nontrivial solution of the homogeneous equation $Mx = 0$ has no zeros on $[a,b]$,
6) problem (6) is uniquely solvable and its Green's function $P(t,s)$ is positive for $t,s \in [a,b]$.

Theorem 2. *Let*

1) T_{ij} $(i,j=1,2)$ *be Volterra operators,*
2) T_{21} *be a negative operator,* T_{ij} *be positive operators for other indices,*
3) *a nontrivial solution of the scalar equation*

$$\dot{x}_1(t) + (T_{11}x_1)(t) = 0, t \in [a.b] , \tag{12}$$

have no zeros,

4) *there exist a function* $v = col(v_1, v_2)$ *such that* $v_2(t) > 0, (M_i v)(t) \leq 0$ *for* $t \in [a,b]$, $i = 1,2$, $v_1(a) \leq 0$.

Then we have $C_{21}(t,s) \geq 0$, $C_{22}(t,s) \leq 0$, $R_{21}(t,s) \leq 0$, $R_{22}(t,s) \geq 0$ *for* $t \in [a,b]$ *and almost all* $s \in [a,b]$.

Formulate several corollaries for system (7) of the second order ($n = 2$).

Corollary 1. *Let conditions* 1) *and* 2) *of Theorem* 2 *hold,*

$$g_{11}(t) \geq \max[g_{12}(t), g_{21}(t), g_{22}(t)]$$

and let there exist $k > 0$ *such that*

$$|p_{21}| + p_{22} \leq \frac{k}{g_{11}e^k} \leq p_{11} - p_{12}$$

Then 1) $C_{21}(t,s) \geq 0, C_{22}(t,s) \geq 0$ *for* $t \in [0,+\infty)$ *and almost all* $s \in [0,+\infty)$, 2) *for each* $b \in (0,+\infty)$ *we have* $R_{21}(t,s) \leq 0$, $R_{22}(t,s) \leq 0$ *for* $t \in [0,b]$ *and almost all* $s \in [0,b]$.

Corollary 2. *Let conditions* 2) *of Theorem* 2 *hold and*

$$|p_{21}| + p_{22} + p_{12} \leq p_{11} \ . \tag{13}$$

Then, for the system of ordinary differential equation assertions 1) *and* 2) *of Corollary* 1 *hold.*

Inequality (13) is best possible in the following sense. For arbitrary positive ε there are constant coefficients $[p_{ij}]$, $i,j = 1,2$ such that $|p_{21}| + p_{22} + p_{12} \leq p_{11} + \varepsilon$, but all roots of the characteristic equation are imaginary and so each element of the Cauchy matrix changes its sign.

For boundary value problem (5) in the case $n = 2$ we propose the following result.

Theorem 3. *Let*

1) T_{11}, T_{22} *be positive,* T_{12}, T_{21} *be negative,*
2) *Green's function* $G_1(t,s)$ *of the scalar boundary value problem, which consists of equation (12) and boundary condition* $x_1(b) = 0$, *be negative,*
3) *there exist a function* $v = col(v_1, v_2)$ *such that*

$$(M_1 v)(t) \geq 0 \ , \quad (M_2 v)(t) \leq 0, v_1(b) = 0 \ , \quad v_2 > 0 \ , \quad t \in [a,b] \ .$$

Then we have $G_{21}(t,s) \geq 0, G_{22}(t,s) \leq 0$ *for* $t \in [a,b]$ *and almost all* $s \in [a,b]$.

3. Proofs

Proof of Theorem 1. We prove the first part of Theorem 1 according to the following scheme 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1).

1) $\Rightarrow$ 2). The function v satisfies the integral equation $v = Kv + \psi$, where $\psi(t) = v(b) - \int_t^b (Mv)(s)ds, t \in [a,b]$. Since $\psi > 0$, then by virtue of [10] the spectral radius of the operator K is less than one.

2) $\Rightarrow$ 3). The equation $x = Kx + g$, where $g(t) = -\int_t^b f(s)ds$, is equivalent to problem (5), which is uniquely solvable and its solution can be represented in the form

$$x(t) = g(t) + \int_a^b (G(t,s) - G_0(t,s))f(s)ds \ ,$$

where $G_0(t,s) = -1$ for $a \le t < s \le b, G_0(t,s) = 0$ for $a \le s \le t \le b$. Here, if $f \le 0$, then $0 \le g \le x$. Consequently, $G(t,s) \le G_0(t,s)$.

3) $\Rightarrow$ 1). The function $v(t) = -\int_a^b G(t,s)ds$ satisfies condition 1.

4) $\Rightarrow$ 6). Problem (6) is uniquely solvable if and only if $x(a) \ne x(b)$ for nontrivial solutions of the equation $Mx = 0$. Since T is a nonzero operator and any nontrivial solution has no zero (let x be positive),then

$$x(b) = x(a) + \int_a^b \dot{x}(t)dt = x(a) - \int_a^b (Tx)(t)dt < x(a) \ .$$

The positivity of $P(t,s)$ follows from the representation

$$P(t,s) = C(t,s) + C(t,a)\frac{C(b,s)}{1 - C(b,a)} \ .$$

6) $\Rightarrow$ 5). Setting $t < s$ in the last formula, we obtain that the function $C(t,a)$, which is directly proportional to any solution of $Mx = 0$, cannot have zeros on $[a,b]$.

To prove the implication 5) $\Rightarrow$ 1) it is sufficient to set $v(t) = C(t,a)$. The implication 2) $\Rightarrow$ 4) has been proved in [6].

Proof of Theorem 2. Let us express x_1 in terms of x_2 from the first equation of the system (11):

$$x_1(t) = -\int_a^t C_1(t,s)(T_{12}x_2)(s)ds + \int_a^t C_1(t,s)f_1(s)ds + C_1(t,a)x_1(a) \ ,$$

where $C_1(t,s)$ is the Cauchy function of the scalar equation (12).

Substituting this expression of x_1 into (11) at $i = 2$, we obtain the first order equation of the following type:

$$\dot{x}_2(t) + (Tx_2)(t) = u(t), \ \ t \in [a,b] \tag{14}$$

where $T : C \to L$ is determined by the formula

$$(Tx)(t) = (T_{22}x)(t) - (T_{21}\int_a^t (T_{12}x)(s)ds)(t) ,$$

$$u(t) = f_2(t) - (T_{21}\int_a^t C_1(t,s)f_1(s)ds)(t) - x_1(0)(T_{21}C_1(t,0))(t) .$$

Since $v_1(0) \leq 0$, $M_i v \leq 0$ for $i = 1, 2$, the function v_2 satisfies the inequality

$$\dot{v}_2(t) + (Tv_2)(t) \leq 0, \quad t \in [a,b] .$$

Using the equivalences 1) ⇔ 5) and 1) ⇔ 3) of Theorem 1, we obtain the positivity of the Cauchy function $C_T(t,s)$ of equation (14) and the negativity of Green's function $G_T(t,s)$ of the boundary value problem

$$\dot{x}_2(t) + (Tx_2)(t) = u(t) , \quad t \in [a,b] , \quad x_2(b) = 0 .$$

Using formula (3) of the solution's representation we obtain

$$\int_a^t C_T(t,s)u(s)ds = \int_a^t C_{21}(t,s)f_1(s)ds + \int_a^t C_{22}(t,s)f_2(s)ds ,$$

$$\int_a^b G_T(t,s)u(s)ds = \int_a^b R_{21}(t,s)f_1(s)ds + \int_a^b R_{22}(t,s)f_2(s)ds .$$

If $f_1 \geq 0$, $f_2 \geq 0$, then $u \geq 0$. Since $C_T(t,s)$ is positive, then $C_{21}(t,s), C_{22}(t,s)$ are also positive. Since $G_T(t,s)$ is negative, then $R_{21}(t,s), R_{22}(t,s)$ are also negative.

To prove Corollaries 1 and 2 we substitute $v_1 = -e^{-kt}$, $v_2 = e^{-kt}$ into condition 4 of Theorem 2.

The proof of Theorem 3 is similar to the proof of Theorem 2.

References

[1] N.V. Azbelev, V.P. Maksimov, L.F. Rahmatullina, Introduction to the Theory of Functional-Differential Equations. (In Russian) Nauka, Moscow, 1991.

[2] N.V. Azbelev, A. Domoshnitsky, A question concerning linear differential inequalities, I. Differential Equation J. 27 (1991), 257-263.

[3] N.V. Azbelev, L.F. Rakhmatullina, Functional-differential equations. Differential Equations J. 14 (1978), 771-797.

[4] S.A. Chaplygin, A new method of approximate integration. (In Russian) Moscow, GosTexIzdat, 1950.

[5] A. Domoshnitsky, M.V. Sheina, Nonnegativity of Cauchy matrix and the exponential stability of systems of differential equations with time-lag. Differential Equations J. 25 (1989), 145-150.

[6] A. Domoshnitsky, S.A. Gusarenko, Asymptotic and oscillation properties of the first order linear scalar functional-differential equations, Differential Equations J. 25 (1989), 1480-1491.

[7] Y. Domshlak, Comparison theorems of Sturm type for first and second order differential equations with sign variable deviations of the argument, Ukrain. Mat. Zh. 34 (1982), 158-163.

[8] E.A. Grove, M.R.S. Kulenovic, G. Ladas, A Myshkis-type comparison result for neutral equations. Math. Nachr. 146 (1990), 195-206.

[9] V.B. Kolmanovsky, V.B. Nosov, Stability and periodic regimes of controlled systems with after-effects. (In Russian) Nauka, Moscow, 1981.

[10] M.A. Krasnoselskii et al., An approximate solution of operator equations. (in Russian) Nauka, Moscow, (1969).

[11] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation theory of differential equations with deviating arguments. Pure and Applied Mathematics, 1987.

[12] G. Ladas, Ch.G. Philos, Y.G. Sfikas, Oscillation of integro-differential equations. Differential and Integral Equations 4 (1991), 1113-1120.

[13] V. Lakshmikantham, S. Leela, Differential and integral inequalities. Academic Press, 1969.

[14] N.N. Luzin, On the method of approximate integration of S.A. Chaplygin. Uspehi Math. Nauk 6 (1951), 3-27.

[15] Man Kam Kwong, W.T. Patula, Comparison theorems for first order linear delay equations. J. of Differential Equations 70 (1987), 275-292.

[16] A.D. Myshkis, Linear differential equations with delayed argument. (in Russian) Nauka, Moscow, 1972.

[17] V. Rezvan, Absolute stability of automatic systems with time-lag. (in Russian) Nauka, Moscow, 1983.

[18] J. Schroder, Operator Inequalities. Academic Press, 1980.

[19] De La Vallee-Poussin, Sur l'equation differentielle lineaire du second ordre. J. Math. Pures et Appl., 8 (1928), 125-144.

[20] T. Wazewsky, Systems des equations et des ineag. differentialles aux deuxiemes members et leurs applications. Ann. Polon. Math., 23 (1950), 112-166.

[21] J.E. Wilkins. The conservation of the Chaplygin's theorem on differential inequalities. Proc. Amer. Math. Soc., 53 (1947).

OSCILLATION PROPERTIES OF DISCRETE DIFFERENCE INEQUALITIES AND EQUATIONS: THE NEW APPROACH

Yuri Domshlak *

Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel

Abstract

This paper presents a new approach to the investigation of oscillatory properties of discrete difference equations and inequalities.

New conditions are obtained, which guarantee that all solutions of difference equation having one or two argument delays are oscillatory (including equations with oscillating coefficients). Upper estimates are derived for sign preserving intervals of the solutions. It is shown that most of conditions and estimates cannot be improved.

0. Introduction

Recently, interest in the qualitative theory of delay difference equations (in discrete and continuous time) has increased. The research concentrates on the study of oscillation properties of these equations. In this direction the group of researchers headed by G. Ladas was very productive (see [1] for the present state of the art of their work). Research in this field was also carried out by the author and his collaborators (see [2] for their main results; detailed bibliography can be found in [1] and [2]). The present paper continues the research of oscillation properties for *discrete* difference equations started in [3]. The research method is based on elaborating discrete analogues of the classical Sturmian Comparison theorem, the Sturmian Oscillation theorem and the Sturmian Zeros-separation theorem for the second order differential equation. More exactly, the discrete analogues of the theory developed by the author in [4] for delay *differential* equations is presented here.

1. Sturmian Comparison Theorem

Consider two delay difference inequalities:

$$l[x]_i \equiv x(i+1) - a(i)x(i) + \sum_{k=1}^{N} b^{(k)}(i)x(i-k) + \sum_{l=1}^{M} c^{(l)}(i)x(i+l) \leq 0, \ i \geq i_0 \quad (1)$$

and

* Results of this paper were presented at the seminar "Functional-differential equations" (1992, Ariel, Israel). A complete version of the results will be published in two papers in the journal "Differential and Integral Equations" (Y.Domshlak, Sturmian comparison in oscillation study for discrete difference equations, Part I and Part II)

$$\widetilde{l}[y]_i \equiv y(i-1)-\widetilde{a}(i)y(i)+\sum_{k=1}^{N}\widetilde{b}^{(k)}(i+k)y(i+k)+\sum_{l=1}^{M}\widetilde{c}^{(l)}(i-l)y(i-l)\geq 0,\ i\geq i_0 \tag{2}$$

where $a(i);\widetilde{a}(i);b^k(i);\widetilde{b}^k(i)$, $k=\overline{1,N}$; $c^{(l)},\widetilde{c}^{(l)}, l=\overline{1,M}$ are defined on $i\geq i_0$.

Here we use the following notations: the segment $\langle n,m\rangle$ is defined as

$$\langle n,m\rangle \equiv \{n,n+1,\ldots,m-1,m\}\subset \mathrm{N} \text{ and } \sum_{i=p}^{p-1}k_i \stackrel{\text{def}}{=} 0. \tag{3}$$

Lemma 1. *For arbitrary sequences of numbers $\{x(i)\},\{y(i)\}$ on any segment $\langle n,m\rangle$ the following identity holds:*

$$\sum_{i=p+1}^{q} l[x]_i y(i) = \sum_{i=p+1}^{q}\widetilde{l}[y]_i x(i)+\sum_{i=p+1}^{q}[\widetilde{a}(i)-a(i)]x(i)y(i)+$$

$$+\sum_{k=1}^{N}\left\{\sum_{i=p+k+1}^{q}[b^{(k)}(i)-\widetilde{b}^{(k)}(i)]x(i-k)y(i)+\sum_{i=p+1}^{p+k}b^{(k)}(i)x(i-k)y(i)-\right.$$

$$\left.-\sum_{i=q+1}^{q+k}\widetilde{b}^{(k)}(i)x(i-k)y(i)\right\}+\sum_{l=1}^{M}\left\{\sum_{i=p+1}^{q-l}[c(i)-\widetilde{c}(i]x(i+l)y(i)+\right.$$

$$\left.+\sum_{i=q+1-l}^{q}c(i)x(i+l)y(i)-\sum_{i=p-1_l}^{p}\widetilde{c}(i)x(i+l)y(i)\right\}-$$

$$x(p+1)y(p)+x(q+1)y(q)\ . \tag{4}$$

The proof is carried out by means of the following equalities:

$$\sum_{i=p+1}^{q}x(i+1)y(i)=\sum_{i=p+1}^{q}x(i)y(i-1)-x(p+1)y(p)+x(q+1)y(q)\ ; \tag{5}$$

$$-\sum_{i=p+1}^{1}a(i)x(i)y(i)=\sum_{i=p+1}^{q}[\widetilde{a}(i)-a(i)]x(i)y(i)-\sum_{i=p+1}^{q}\widetilde{a}(i)x(i)y(i)\ ; \tag{6}$$

$$\sum_{i=p+1}^{q}b^{(k)}(i)x(i-k)y(i)=\sum_{i=p+1}^{q}\widetilde{b}^{(k)}(i+k)x(i)y(i+k)+$$

$$+\sum_{i=p+1+k}^{q}[b^{(k)}(i)-\widetilde{b}^{(k)}(i)]x(i-k)y(i)+\sum_{i=p+1}^{p+k}b^{(k)}(i)x(i-k)y(i)-$$

$$-\sum_{i=q+1}^{q+k}\widetilde{b}^{(k)}(i)x(i-k)y(i),\quad k=\overline{1,N}\ ; \tag{7}$$

$$\sum_{i=p+1}^{q}c^{(l)}(i)x(i+l)y(i)=\sum_{i=p+1}^{q}\widetilde{c}(i-l)x(i)y(i-l)+\sum_{i=p+!}^{q-l}[c^{(l)}(i)-\widetilde{c}^{(l)}(i)]\times$$

$$\times x(i+l)y(i)+\sum_{i=q+1-l}^{q}c^{(l)}(i)y(i)x(i+l)-\sum\widetilde{c}^{(l)}(i)x(i+l)y(i)\ ,\ l=\overline{1,M} \tag{8}$$

The following comparison theorem is a discrete analogue of the classical Sturmian Comparison theorem. More precisely, this is an analogue of our comparison theorem for differential inequality with mixed delay ([8], Theorem 1, or [4], Theorem 4.1)

$$x'(t)+\sum_{1}^{N}b_i(t)x[t-\tau_i(t)]+\sum_{1}^{M}c_i(t)x[t+\mu_i(t)]\leq 0\ . \tag{9}$$

Theorem 1. *Let $q-p>R\equiv\max\{N,M\}$. Suppose that the following conditions hold:*

1.
$$\widetilde{b}^{(k)}(i)\geq 0, i\in\langle q+1,q+k\rangle, k=\overline{1,N}\ ;$$
$$\widetilde{c}^{(l)}(i)\geq 0,\ i\in\langle p+1-l,p\rangle,\ l=\overline{1,M} \tag{10}$$

2. *There exists $y(i)$ satisfying (2) on the segment $\langle p,q\rangle$ such that*

$$y(i)>0,\ i\in\langle p+1,q\rangle;\ y(i)\leq 0,\ i\in\langle p-M,p\rangle\bigcup\langle q+1,q+N\rangle \tag{11}$$

3. *The following inequalities hold:*

$$a(i)\leq\widetilde{a}(i),\ i\in\langle p+1,q\rangle \tag{12}$$

$$b^{(k)}(i)\geq\begin{cases}0, & i\in\langle p+1,p+k\rangle\\ \widetilde{b}^{(k)}(i), & i\in\langle p+k+1,q\rangle\end{cases}\ ,\ k=\overline{1,N} \tag{13}$$

$$c^{(l)}(i)\geq\begin{cases}\widetilde{c}^{(l)}(i), & i\in\langle p+1,q-l\rangle\\ 0, & i\in\langle q-l+1,q\rangle\end{cases}\ ,\ l=\overline{1,M} \tag{14}$$

4. *At least one of the inequalities (12)-(14) becomes a strict inequality for some number i. Then there is no solution $x(i)$ of the inequality (1) for which $x(i)>0$ on the segment $\langle p-N,q+M\rangle$*

Proof: Suppose that there exists a solution $x(i)>0$ of (1). Let us write the identity (4) for $x(i)$ and $y(i)$, satisfying (2). In view of the conditions of Theorem 1, all terms of the right-hand side of (4) are non-negative and at least one is strictly positive. At the same time, the left-hand side of (4) vanishes.

Corollary 1.1. *If the conditions of Theorem 1 hold, then there are no solutions of the equation*

$$l[x]_i = 0, \quad i \geq i_0 \tag{15}$$

preserving the sign on the segment $\langle p-N, q+M\rangle$.

Corollary 1.2. *If there exists a sequence of segments* $\langle p_n, q_n\rangle$, $p_n \to \infty$, *satisfying the conditions of Theorem 1, then there is no positive solution of inequality (1) and non-oscillating solution of equation (15) on* $\langle i_0, \infty\rangle$.

Remark 1. We impose conditions on segments $\langle p_n - R, q_n + R\rangle$ *only*. For the segments $\langle q_n + R + 1, p_{n+1} - R - 1\rangle$ *no limitations* are imposed on the coefficients of inequality (1)!

2. Sturmian Oscillation Theorem and Sturmian Zeros-separation Theorem

Definition 1. *The segment* $\langle p, q\rangle \subset \mathrm{N}$ *is called the large half- cycle of the equation*

$$\tilde{l}[y]_i = 0, \quad i \geq i_0 \tag{16}$$

if $q - p > \max\{N, M\}$, *and if there is at least one solution* $y(i)$ *of (16) satisfying (11). The segment* $\langle p-M, q+N\rangle$ *is called extended half-cycle of (16).*

If for any p_0 there is a large half-cycle $\langle p, q\rangle$ of (16), $p_0 < p < q$, we say that equation (16) is *regularly oscillatory*.

The next statement follows from Theorem 1:

Theorem 2. *Let* $\tilde{b}(i) \geq 0$, $c^{(l)}(i) \geq 0, k = \overline{1,N}$, $l = \overline{1,M}, i = \overline{1,\infty}$. *If the equation (16) is a regularly oscillatory equation and conditions 3–4 of Theorem 1 hold, then all solutions of equation (15) are oscillatory. If there is at least one non-oscillatory solution of equation (15), then there is no regularly oscillatory solution of equation (16).*

This statement is an exact discrete analogue of the classical Sturmian Oscillation theorem for the second order differential equation, and (more exactly) is a discrete analogue of the Corollary 4.1.2 in [4] for the first order delay *differential* equation (9).

Put in (16) $\tilde{a}(i) \equiv a(i), \tilde{b}^{(k)}(i) \equiv b^{(k)}(i), \tilde{c}^{(l)}(i) \equiv c^{(l)}(i)$:

$$y(i-1) - a(i)y(i) + \sum_{k=1}^{N} b^{(k)}(i+k)y(i+k) + \sum_{i=1}^{M} c^{(l)}(i-l)x(i-l) = 0 \tag{17}$$

Theorem 3. *Let*

$$b^{(k)}(i) \geq 0, c^{(l)}(i) \geq 0, k = \overline{1,N}, l = \overline{1,M}, i = \overline{1,\infty}\,. \tag{18}$$

Then there is no extended half-cycle of equation (17) in the half- cycle of (15), and vice versa.

This statement is an exact discrete analogue of the classical Sturmian Zeros-separation theorem for the second order differential equation and of Theorem 4.5 in [4] for delay *differential* equations.

Theorem 3 makes it possible to estimate the length of the sign-preservating segments of the solutions not only from above, but *from below* as well (for *large* half-cycles). The exact formulation of the relevant statement is as follows:

Corollary 3.1. *Suppose that*

1. *(18) holds.*
2. *The segment $\langle p_1, q_1\rangle$ is a half-cycle of equation (17), and*

$$p_1 \le p - R < q + R \le q_1;$$

3. *$x(i)$ is a solution of equation (15), such that*

$$x(i) \le 0 \text{ on } \langle p-R, q\rangle \text{ and } x(i) > 0 \text{ on } \langle p+1, p+R\rangle$$

Then $x(i)$ preserves the positive sign on the segment $\langle p+R, q_1\rangle$.

Let us demonstrate this statement for a simple case, i.e. for the equation

$$x(i+1) - a(i)x(i) + b(i)x(i-1) = 0 \tag{19}$$

Corollary 3.2. *Let $b(i) \ge 0\ \forall i,\ q-p \ge 3$, and the segment $\langle p, q\rangle$ is a half-cycle for the equation*

$$y(i-1) - a(i)y(i) + b(i+1)y(i+1) = 0. \tag{19'}$$

If $p_1 \ge p+1$ and $x(i)$ is a solution of (19), for which $x(i) \le 0$ on $\langle p_1 - 1, p_1\rangle$, $x(i) > 0$ on $\langle p_1+1, p_1+2\rangle$, then $x(i)$ preserves the positive sign on the whole segment $\langle p_1+2, q\rangle$.

3. Applications to Some Special Cases

Theorems 1 and 2, as well as any comparison theorem, have relative character. The problem is that in each particular case we must formulate sufficient conditions to guarantee the fact that the inequality (2) has the solution $y(i)$, possessing the properties (11). These conditions must be formulated *in the terms of the properties of coefficients* $\widetilde{a}(i), \widetilde{b}(i), \widetilde{c}(i)$. This problem is a difficult one. A special method has been developed in [4] for the analysis of the analogical problem for delay *differential* equation. We describe below the discrete version of this method.

3.1 Inequalities and equations with one lag.

3.1.1 Consider the inequalities

$$l[x]_i \equiv x(i+1) - a(i)x(i) + b(i)x(i-k) \le 0, \quad i \ge i_0 \tag{20}$$

$$\widetilde{l}[y]_i \equiv y(i-1) - \widetilde{a}(i)y(i) + \widetilde{b}(i+k)y(i+k) \ge 0, \quad i \ge i_0 \tag{21}$$

for a given number $k \in \mathbf{N}$.

Suppose that the sequence $\{m(i)\}$, $m(i) \neq 0$, and the bounded sequence $\{\varphi(i)\}$, $i \ge i_0$ and $0 < \nu \le \nu_0$ are given.

Choose the sequences $\{\widetilde{a}(i)\}$ and $\{\widetilde{b}(i)\}$ such that the sequence

$$y(i) = m(i)\sin(\nu \sum_{n=p+1}^{i} \varphi(n)) \tag{22}$$

will be a solution of (21).

Substitute (22) into (21):

$$m(i-1)\sin(\nu \sum_{n=p+!}^{i-1} \varphi(n)) - \widetilde{a}(i)m(i)\sin(\nu \sum_{n=p+1}^{i} \varphi(n))$$

$$+\widetilde{b}(i+k)m(i+k)\sin(\nu \sum_{n=p+1}^{i+k} \varphi(n)) \ge 0$$

From this inequality one obtains

$$\left[m(i-1)\cos(\nu\varphi(i)) - \widetilde{a}(i)m(i) + \widetilde{b}(i+k)m(i+k)\cos(\nu \sum_{i+1}^{i+k} \varphi(n))\right]$$

$$\sin(\nu \sum_{p+1}^{i+k} \varphi(n)) + \left[-m(i-1)\sin(\nu\varphi(i)) + \widetilde{b}(i+k)m(i+k)\sin(\nu \sum_{i+1}^{i+k} \varphi(n))\right]$$

$$\cos(\nu \sum_{p+1}^{i+k} \varphi(n)) \ge 0 \tag{23}$$

Inequality (23) undoubtedly holds if the expressions in the square brackets are equal to zero. From these equalities we obtain:

$$\left.\begin{aligned} \widetilde{b}(i+k) &= \frac{m(i-1)}{m(i+k)} \frac{\sin \nu\varphi(i)}{\sin \nu \sum_{i+1}^{i+k} \varphi(n)}, \quad && i \ge p \\ \widetilde{a}(i) &= \frac{m(i-1)}{m(i)} \frac{\sin \nu \sum_{i+1}^{i+k} \varphi(n)}{\sin \nu \sum_{i+1}^{i+k} \varphi(n)}, \quad && i \ge p \end{aligned}\right\} \tag{24}$$

and

$$\frac{m(i-1)}{m(i+k)} = \sqcap_{l=i}^{i+k} \frac{\sin(\nu \sum_{n=l+1}^{l+k} \varphi(n))}{\sin(\nu \sum_{n=l}^{l+k} \varphi(n))} \cdot \widetilde{a}(l), i \geq p \tag{25}$$

$$\widetilde{b}(i) = \sqcap_{l=i-k}^{i} \left[\widetilde{a}(l) \frac{\sin(\nu \sum_{l+1}^{l+k} \varphi(n))}{\sin(\nu \sum_{l}^{l+k} \varphi(n))}\right] \frac{\sin(\nu\varphi(i-k))}{\sin(\nu \sum_{i+1-k}^{i} \varphi(n))}, \quad i \geq i_0 \tag{26}$$

The following statement is true:

Lemma 2. *Let $\widetilde{a}(i) \neq 0$ in (21), and the bounded sequence $\{\varphi(n)\}_1^\infty$ satisfy the following conditions:*

$$\sum_{i+1}^{i+k} \varphi(n) > 0 \text{ and } \sum_{i}^{i+k} \varphi(n) > 0 \text{ for } \forall i\ . \tag{27}$$

Then the sequence

$$y(i) = \sqcap_{l=p+1}^{i} \left(\frac{1}{\widetilde{a}(l)} \frac{\sin(\nu \sum_{l}^{l+k} \varphi(n))}{\sin(\nu \sum_{l+1}^{l+k} \varphi(n))} \right) \cdot \sin(\nu \sum_{p+1}^{i} \varphi(n)) \tag{28}$$

is the solution of (21) for sufficiently small $\nu > 0$, where $\widetilde{b}(l)$ is defined in (26).

Theorem 4. *Let $0 < \nu < \nu_0$, $q - p > k$ and the following conditions hold:*

1. The sequence $\{\varphi(n)\}_1^\infty$ is bounded and

a) $0 \leq \sum_{p+1}^{i} \varphi(n) \leq \frac{\pi}{\nu}\ ,\ i \in \langle p+1, q\rangle\ ;\ \frac{\pi}{\nu} \leq \sum_{p+1}^{i} \varphi(n) \leq \frac{2\pi}{\nu}, i \in \langle q+1, q+k\rangle$ (29)

b) $\varphi(i) \geq 0, i \in \langle q+1-k, q\rangle$ (30)

c) $\sum_{i+1}^{i+k} \varphi(n) > 0\ ,\ \sum_{i}^{i+k} \varphi(n) > 0, \forall i$ (31)

2. $$a(i) > 0\ \forall i \tag{32}$$

3. $$b(i) \geq \begin{cases} 0, & i \in \langle p+1, p+k\rangle \\ \widetilde{b}(i), & i \in \langle p+k+1, q\rangle \end{cases}, \tag{33}$$

where $\widetilde{b}(i)$ is defined in (26), $\widetilde{a}(i) \equiv a(i)$.

Then there are no solutions $x(i)$ of inequality (20) such that $x(i) > 0$ on $\langle p-k, q\rangle$.

Proof: The proof is based on Theorem 1, with $\widetilde{a}(i) \equiv a(i)$, and on Lemma 2 which guarantees the existence of a solution $y(i)$ of the inequality (2), satisfying conditions (11).

Corollary 4.1. *Let the sequence of the segments $\langle p_m, q_m\rangle$, $p_m \to \infty$ be such that all conditions of Theorem 4 hold. Then all solutions of the equation*

$$x(i+1) - a(i)x(i) + b(i)x(i-k) = 0 \tag{34}$$

are oscillatory.

Remark 2. In our opinion, it is important to mention the following three facts. First, the suggested method makes it possible *not only* to derive the conditions of oscillation of all solutions but also to estimate from above the length of the sign-preservating segments of any solution as well. Second, Theorem 4 gives the *whole family* of criteria of oscillation (for each sequence $\{\varphi(n)\}_1^\infty$ we obtain a new criterion). Third, in the theorems on oscillation of all solutions, the conditions for coefficients are required *on segments* $\langle p_m - k, q_m + k\rangle$ *only.*

Thus, Theorem 4 *solves completely* the Ladas' problem 8.6 ([1]) for equation (34): to find the conditions which imply the oscillation of all solutions of Eq. (34) with $a(i) \equiv 1$ and *oscillating* coefficients $b(i)$.

Let us rewrite condition (33) in the following form:

$$b(i) \geq C_\nu)i)\varphi(i-k) \tag{35}$$

where we designate

$$C_\nu(i) \equiv \sqcap_{l=i-k}^{i} \left[a(l) \frac{\sin(\nu \sum_{l+1}^{l+k} \varphi(n))}{\sin(\nu \sum_{l}^{l+k} \varphi(n))} \right] \frac{\nu}{\sin(\nu \sum_{i+1-k}^{i} \varphi(n))} \cdot \frac{\sin(\nu\varphi(i-k))}{\nu\varphi(i-k)} \,. \tag{36}$$

It is easy to see that $\lim_{\nu\to 0} C_\nu(i) = C_0(i)$, where

$$C_0(i) = \frac{\sqcap_{l=i+1-k}^{i} (\sum_{l+1}^{l+k} \varphi(n))}{\sqcap_{l=i-k}^{i} (\sum_{l}^{l+k} \varphi(n))} \sqcap_{i-k}^{i} a(l) \,. \tag{37}$$

Therefore, if

$$b(i) > C_0(i)\varphi(i-k) \tag{35'}$$

holds, then also (35) holds for sufficiently small $\nu > 0$.

Put

$$\varphi(i) \equiv b_0(i+k), \;\; 0 \leq b_0(i) \leq b(i) \text{ for } \forall i. \tag{38}$$

Then (35) turns into inequality

$$C_\nu(i) \equiv \sqcap_{l=i-k}^{i} \left[a(l) \frac{\sin(\nu \sum_{l+k+1}^{l+2k} b_0(n))}{\sin(\nu \sum_{l+k}^{l+2k} b_0(n))} \right] \cdot \frac{\nu}{\sin(\nu \sum_{i+1}^{i+k} b_0(n))} \cdot \frac{\sin(\nu b_0(i))}{\nu b_0(i)} \leq 1 \tag{39}$$

Let us formulate the Theorem 4 for this choice of the sequence $\{\varphi(n)\}$.

Theorem 5. *Let $q - p > k$, and the following conditions hold:*

1. $a(i) > 0$, $b(i) \geq 0$ *for* $i \in \langle p-k, q+k\rangle$
2. *There exists the sequence* $\{b_0(n)\}$, $0 \leq b_0(n) \leq b(n)$, *such that for* $0 \leq \nu \leq \nu_0$ *and* $i \in \langle p-k, q+k\rangle$, *inequality (39) holds;*
3. $\sum_{p+1+k}^{q+k} b(n) \leq \frac{\pi}{\nu}$, $\frac{\pi}{\nu} \leq \sum_{p+1+k}^{i+k} b(n) \leq \frac{2\pi}{\nu}$, $i \in \langle q+1, q+k\rangle$ (40)

Then there are no solutions $x(i)$ of inequality (20) such that $x(i) > 0$ on the segment $\langle p-k, q+k\rangle$, and any solution of the corresponding equation (34) has at least one change of sign on this segment.

Proof: It follows from (40), that (35) (and (33)) hold as long as $b(i) \geq 0$ $\forall i$.

Remark 3. In Theorem 4 we did not demand a priori that all $\varphi(i)$, $i \geq i_0$, should be positive. Condition (31) makes it possible for some of them to be negative (for $k \geq 2$). Thus in Theorem 4 we do not demand that all $b(i)$ should be positive *even in the segments* $\langle p_m - k, q_m + k\rangle$. On the segments $\langle q_m + k, p_{m+1} - k\rangle$, *no conditions are imposed at all.*

Notice also that if $\limsup_{i\to\infty} C_o(i) < 1$, then (35) holds for sufficiently small ν. Then the following statement is true:

Corollary 5.1. *Let $a(i) > 0$ $\forall i$, and there exists the sequence $b_0(i)$, $0 \leq b_0(i) \leq b(i)$ $\forall i$, such that the following conditions hold:*

1. $\sum_{i+1}^{i+k} b_0(n) > 0$ *for* $\forall i$,
2. $$\limsup_{i\to\infty} \frac{\prod_{i-k+1}^{i}\left(\sum_{l+k+1}^{l+2k} b_0(n)\right)}{\prod_{i-k}^{i}\left(\sum_{l+k}^{l+2k} b_0(n)\right)} \cdot \prod_{i-k}^{i} a(l) < 1 \qquad (41)$$
3. *The series* $\sum^{\infty} b_0(n)$ *is divergent.*

Then all the solutions of equation (34) are oscillatory.

This statement is a discrete analogue of our Corollary 4.2.4 in [4]:

Suppose that $r(t) \leq t$ is the increasing function, and there exists $b_0(t)$, $0 \leq b_0(t) \leq b(t)$ such that the following condition holds:

$$\liminf_{t\to\infty} \left\{\exp\left[\int_{r(t)}^{t} b_0(\xi)\left\{\int_{r(\xi)}^{\xi} b_0(s)ds\right\}^{-1} d\xi\right] \cdot \int_{r(t)}^{t} b_0(s)ds\right\} > 1 \qquad (42)$$

Then all solutions of the delay differential equation

$$x'(t) + b(t)x[r(t)] = 0$$

are oscillatory.

Remark 4. If, in particular, $a(i) \equiv 1$ and there exists $\lim_{i\to\infty} b_0(i) = b$, then condition (41) turns into

$$b > \frac{k^k}{(k+1)^{k+1}} \, . \tag{43}$$

It is known (see, for a example, [9]), that (43) is a sufficient and necessary condition for oscillation of all solutions of the equation (34), where $a(i) \equiv 1$ and $b(i) \equiv b$. However, making use of condition (41) we can state of oscillation of all for solutions of equation (34) in those cases when no other criteria are suitable. It applies equally to condition of Ladas-Philos-Sficas [5] (LPS-condition):

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{n+1}^{n+k} b(i) > \frac{k^k}{(k+1)^{k+1}}. \tag{44}$$

We shall discuss below in more detail the advantages and shortcomings of different conditions of oscillation of all solutions for equation (34) and for more general equations.

Example 1. Put in Theorem 4 for $k \geq 2$

$$\{\varphi(n)\}_1^\infty = \{\alpha, \beta, -\gamma, \alpha, \beta, -\gamma, \ldots\}, \ 0 < \gamma < \min\{\alpha, \beta\}. \tag{45}$$

Then $\{C_0(n)\}_1^\infty = \{p, q, r, p, q, r, \ldots\}$, where

$$p = \frac{(\beta-\gamma)(\alpha-\gamma)}{(\alpha+\beta-\gamma)^3}, q = \frac{(\alpha-\gamma)(\alpha+\beta)}{(\alpha+\beta-\gamma)^3}, r = \frac{(\alpha+\beta)(\beta-\gamma)}{(\alpha+\beta-\gamma)^3} \, .$$

Condition (35) turns into

$$b(3i-2) > p \cdot \beta, \ b(3i-1) > -q\gamma, \ b(3i) > r\alpha, \ i \geq i_0 \, . \tag{46}$$

Thus, if in equation (34) $a(i) \equiv 1$ and condition (46) holds, all solutions of (34) are oscillatory.

3.1.2 Case $k = 1$. Consider a particular case of inequality (20) with $k = 1$:

$$x(i+1) - a(i)x(i) + b(i)x(i-1) \leq 0, \ i \geq i_0 \tag{47}$$

Corollary 4.2. *Let $q - p > 1$ and $\{\varphi(n)\}_1^\infty$ be a sequence of positive numbers such that*

$$\sum_{p+1}^{q} \varphi(n) \leq \frac{\pi}{\nu}, \ \frac{\pi}{\nu} \leq \sum_{p+1}^{q+1} \varphi(n) \leq \frac{2\pi}{\nu}.$$

Then, if $b(p+1) \geq 0$ and for $i \in \langle p+2, q\rangle$

$$b(i) \geq a(i)a(i-1)\frac{\sin\nu\varphi(i-1)\cdot\sin\nu\varphi(i+1)}{\sin\nu[\varphi(i-1)+\varphi(i)]\cdot\sin\nu[\varphi(i)+\varphi(i+1)]}\,. \tag{48}$$

the inequality (47) has no solutions $x(i)$, for which $x(i) > 0$ on $\langle p-1, q+1\rangle$ and all solutions of the corresponding equation

$$x(i+1) - a(i)x(i) + b(i)x(i-1) = 0 \tag{49}$$

are oscillatory.

In particular, putting $\varphi(i) \equiv 1$ one obtains

Corollary 4.3. *Let $0 < \nu < \frac{\pi}{2}$ and the following condition hold:*

$$b(i) \geq \frac{a(i)\cdot a(i-1)}{4\cos^2\nu}\ ,\ \text{for } i \in \langle p+1, p+[\frac{\pi}{\nu}]\rangle \tag{50}$$

Then the statement of Corollary 4.2 is valid with $q = [\frac{\pi}{\nu}]$.

Corollary 4.4. *Suppose that condition (50) holds on the same sequence of segments $\langle p_n+1, p_n+[\frac{\pi}{\nu}]\rangle$, $p_n \to \infty$. Then any solution of equation (49) is oscillatory.*

It is useful to reformulate the previous statements in such a form that enables us to obtain the estimates for the lengths of the segments of which the solution preserves its sign. The first statement of the next Corollary 4.5 was obtained in [6] (Theorem 5) and in [7].

Corollary 4.5. *Suppose that the following condition holds:*

$$\liminf_{i\to\infty} \frac{b(i)}{a(i)\cdot a(i-1)} = C > \frac{1}{4}. \tag{51}$$

Then

a) *all solutions of (49) are oscillatory;*

b) *there are no solutions of (49) preserving the sign on any segment $\langle p, q\rangle$,*

$$q - p > \frac{\pi}{\nu} + 2,\quad \nu = \arcsin\sqrt{1-\frac{1}{4C}}.$$

Put in (49) $a(i) = 2$, $b(i) = 1 + p(i-1)$, and write of (49) in a canonical form

$$\Delta^2 x(i) + p(i)x(i) = 0,\quad i \geq i_0, \tag{52}$$

where $\Delta x(i) \equiv x(i+1) - x(i)$.

The following statement follows from Corollaries 4.3, 4.4 and 4.5;

Corollary 4.6. *If* $0 < \nu < \frac{\pi}{2}$ *and* $p(i) \geq \frac{1}{\cos\nu} - 1$, *then all solutions of (52) are oscillatory and there are no solutions of (52) preserving sign on any segment* $\langle p, q\rangle$, $q - p > \frac{\pi}{\nu} + 2$.

We have already mentioned that the constant $\frac{1}{4}$ in (51) cannot be improved. However, the statement of Corollaries 4.3, 4.4, 4.5 may be improved even in the case when $\lim_{n\to\infty} b(n)$ exists.

Putting in Corollary 4.2 $\varphi(n) = \frac{1}{n}$, one obtains

$$\frac{\sin\frac{\nu}{n+1}\cdot\sin\frac{\nu}{n-1}}{\sin\left(\frac{\nu}{n-1}+\frac{\nu}{n}\right)\cdot\sin\left(\frac{\nu}{n}+\frac{\nu}{n+1}\right)} = \frac{1}{4}\left(1+\frac{1+4\nu^2}{4n^2}\right)+o(n^{-2}). \tag{53}$$

Then Corollary 4.2 implies

Corollary 4.7. *If*

$$b(n) \geq \frac{1}{4}a(n)\cdot a(n-1)\left[1+\frac{d}{n^2}\right];\ d > \frac{1}{4},\ n \geq n_0\ , \tag{54}$$

then all solutions of (49) are oscillatory, and any solution has at least one change of sign on any segment $\langle p-1, [p\exp(\frac{\pi}{\nu})]+1\rangle$, *where* $\nu \equiv \sqrt{d-\frac{1}{4}}$.

For equation (52) this statement turns into an *exact discrete analogue* of the classical Kneser's Oscillation theorem for second order differential equation:

Corrolary 4.8. *Let*

$$\liminf_{n\to\infty} n^2\cdot p(n) = d > \frac{1}{4}\ . \tag{55}$$

Then Corollary 4.7 is valid for equation (52).

Remark 5. It is impossible to put $d = \frac{1}{4}$ in (54) and (55) because the sequence $x(n) = \sqrt{n}$ is a solution of (54), where $p(n) = 2\sqrt{1+\frac{1}{n}} - \sqrt{1+\frac{2}{n}} - 1$, and $\lim_{n\to\infty} n^2 p(n) = \frac{1}{4}$.

Remark 6. It is not difficult to obtain a condition which is even more accurate than (54) or (55). For example, it is sufficient to put in (54)

$$\varphi(n) = \frac{1}{n\ln n},\quad \varphi(n) = \frac{1}{n\ln n\ln n}\ \text{etc.}$$

Let us discuss now the meaning of the condition (41) for oscillation of all solutions of the equation

$$x(i+1) - x(i) + b(i)x(i-1) = 0 \tag{56}$$

and its connection with Hooker-Patula's condition (51) (see [6]):

$$\liminf_{n\to\infty} b(n) > \frac{1}{4}, \tag{57}$$

(the LPS-condition (44) with $k = 1$, is equivalent also to HP- condition (57)). Let us write condition (41) for $k = 1$:

$$\limsup d_n \equiv \limsup \frac{b_0(n+1)}{[b_0(n-1)+b_0(n)][b_0(n)+b_0(n+1)]} < 1, \; b_0(n) \leq b(n) \tag{58}$$

and consider the following

Example 2. Put in (56)

$$b(i) = \{\mu, \lambda, \mu, \lambda, \ldots\}, \quad \lambda, \mu > 0.$$

Then the HP-condition (57) becomes

$$\min\{\lambda, \mu\} > \frac{1}{4}, \tag{59}$$

and our condition (58) becomes:

$$\max\{\lambda, \mu\} < (\lambda + \mu)^2, \tag{60}$$

which is equivalent to

$$\mu > \begin{cases} \frac{1}{2} - \lambda + \sqrt{\frac{1}{4} - \lambda}, & 0 < \lambda < \frac{1}{4} \\ \sqrt{\lambda} - \lambda, & \frac{1}{4} \leq \lambda < 1 \\ 0, & \lambda \geq 1 \end{cases} \tag{61}$$

It is obvious that condition (61) is less restrictive than the HP-condition (59).

In this example it is possible to determine the *exact domain* $\{\lambda, \mu\}$ for which equation (56) is oscillatory, because it is possible to transform it into a third order difference equation with constant coefficients and to use the sufficient and necessary condition (the corresponding characteristic equation has no real positive roots):

$$\mu > \begin{cases} 1 - \frac{3\sqrt{2}}{2}\lambda^{\frac{2}{3}}, & j0 < \lambda < \frac{1}{4} \\ \frac{2\sqrt{3}}{9}(1-\lambda)^{\frac{3}{2}}, & \frac{1}{4} \leq \lambda \leq 1 \\ 0, & \lambda > 1 . \end{cases} \tag{62}$$

3.2 Inequalities and equations with two delays.

Consider the inequalities

$$l[x]_i \equiv x(i+1) - a(i)x(i) + b_1(i)x(i-k_1) + b_2(i)x(i-k_1) \le 0,\ i \in \mathrm{N} \qquad (63)$$

for $k_1 + 1 \le k_2,\ k_1, k_2 \in \mathrm{N}$, and

$$\widetilde{l}[x]_i \equiv y(i-1) - \widetilde{a}(i)y(i) + \widetilde{b}_1(i+k_1)y(i+k_1) + \widetilde{b}_2(i+k_2)y(i+k_2) \ge 0,\ i \ge 0 \quad (64)$$

Suppose that the sequence $\{m(i\}_1^\infty$, $m(i) \neq 0$, the bounded sequence $\{\varphi(i)\}_1^\infty$ and the small number $0 < \nu \le \nu_0$ are given.

Choose the sequences $\{\widetilde{a}(i)\}$ and $\{\widetilde{b}_j(i)\}$, $j = 1, 2$, such that the sequence

$$y(i) = m(i) \cdot \sin(\nu \sum_{n=p+1}^{i} \varphi(n)) \qquad (65)$$

is a solution of (64).

Substitute (65) into (64):

$$m(i-1)\sin(\nu \sum_{p+1}^{i-1} \varphi(n)) - \widetilde{a}(i)m(i)\sin(\nu \sum_{p+1}^{i} \varphi(n)) +$$

$$+\widetilde{b}_1(i+k_1)m(i+k_1)\sin(\nu \sum_{p+1}^{i+k_1} \varphi(n)) + \widetilde{b}_2(i+k_2)m(i+k_2)\cdot\sin(\nu \sum_{p+1}^{i+k_2} \varphi(n)) \ge 0 \quad (66)$$

Inequality (66) undoubtedly holds if

$$\begin{cases} \widetilde{b}_1(i+k_1) \cdot m(i+k_1) \cdot \sin(\nu \sum_{i+k_1+1}^{i+k_2} \varphi(n)) = \\ = \widetilde{a}(i)m(i)\sin(\nu \sum_{i+1}^{i+k_2} \varphi(n)) - m(i-1)sin(\nu \sum_{i}^{i+k_2} \varphi(n)) \\ \widetilde{b}_2(i+k_2) \cdot m(i+k_2) \cdot \sin(\nu \sum_{i+k_1+1}^{i+k_2} \varphi(n)) = \\ = -\widetilde{a}(i)m(i)\sin(\nu \sum_{i+1}^{i+k_1} \varphi(n)) + m(i-1)sin(\nu \sum_{i}^{i+k_1} \varphi(n)) \end{cases} \qquad (67)$$

Denote $\frac{m(i)}{m(i-1)} \equiv d(i)$ and define $r(i)$ by means of the equality

$$d(i) = \frac{1}{\widetilde{a}(i)} \cdot \frac{\sin(\frac{\nu}{2}\sum_{i}^{i+k_1} \varphi(n)) + (\frac{\nu}{2}\sum_{i}^{i+k_2} \varphi(n)) + r(i)\sin(\frac{\nu}{2}\sum_{i+k_1+1}^{i+k_2} \varphi(n))}{\sin(\frac{\nu}{2}\sum_{i+1}^{i+k_1} \varphi(n)) + (\frac{\nu}{2}\sum_{i+1}^{i+k_2} \varphi(n))}. \qquad (68)$$

Then from (68) we obtain

$$\begin{cases} \widetilde{b}_1(i+k_1) = \left(\prod_{n=i}^{i+k_1} d(n)\right)^{-1} \frac{\sin \nu\varphi(i) + r(i)\sin(\nu \sum_{i+1}^{i+k_2} \varphi(n))}{\sin(\nu \sum_{i+1}^{i+k_1} \varphi(n)) + \sin(\nu \sum_{i+1}^{i+k_2} \varphi(n))} \\ \widetilde{b}_2(i+k_2) = \left(\prod_{n=i}^{i+k_2} d(n)\right)^{-1} \frac{\sin \nu\varphi(i) - r(i)\sin(\nu \sum_{i+1}^{i+k_1} \varphi(n))}{\sin(\nu \sum_{i+1}^{i+k_1} \varphi(n)) + \sin(\nu \sum_{i+}^{i+k_2} \varphi(n))} \end{cases} \qquad (69)$$

The following statement is true:

Lemma 3. *Let in (64) $\tilde{a}(i) > 0$ and for the bounded non-negative sequence $\{\varphi(n)\}$, $n = \overline{1,\infty}$, satisfy $\sum_{i+1}^{i+k_1} \varphi(n) > 0, \quad \forall i$. Then the sequence*

$$y(i) = (\Pi_{n=p+1}^{i} d(n))^{-1} \cdot \sin(\nu \sum_{p+1}^{i} \varphi(n)) \tag{70}$$

is the solution of (64) *for sufficiently small* $\nu > 0$ (*the sequence* $d(i), \tilde{b}_1(i), \tilde{b}_2(i)$ *are defined in* (67), (68)).

Theorem 6. *Let $q - p > k_2$ and the following conditions hold:*

1. *For some sequences $\{\tilde{a}(i)\}$, $\{\varphi(i)\}$ the conditions of Lemma 3 hold;*
2. *For the sufficiently small number $\nu > 0$*

$$\sum_{p+1}^{i} \varphi(n) \leq \frac{\pi}{\nu},\ i \in \langle p+1, q\rangle;\ \frac{\pi}{\nu} \leq \sum_{p+1}^{i} \pi(n) \leq \frac{2\pi}{\nu},\ i \in \langle q+1, q+k_2\rangle \tag{71}$$

3. *The sequence $\{r(i)\}$ satisfy the condition*

$$-\frac{\sin \nu\varphi(i)}{\sin(\nu \sum_{i+1}^{i+k_2} \varphi(n))} \leq r(i) \leq \frac{\sin(\nu\varphi(i))}{\sin(\nu \sum_{i+1}^{i+k_1} \varphi(n))} \tag{72}$$

4. $$b_j(i) \geq \tilde{b}_j(i), i \in \langle p, q\rangle, \tag{73}$$
   where $\tilde{b}_j(i)$ are defined in (69), and $\tilde{a}(i) \equiv a(i)$.

Then there are no solutions $x(i)$ of (63) such that $x(i) > 0$ on the segment $\langle p - k_2, q\rangle$.

As an application one obtains the oscillatory properties of all solutions of the equation

$$x(i+1) - a(i)x(i) + b_1(i)x(i-k_1) + b_2(i)x(i-k_2) = 0, \quad i \geq i_0 \tag{74}$$

with two delays:

Corollary 6.1. *Let the sequence of the segments $\langle p_m, q_m\rangle$, $p_m \to \infty$ be such that all conditions of Theorem 6 hold. Then all solutions of equation (74) are oscillatory.*

Consider further a few particular cases.

3.2.1 Put $\varphi(n) \equiv 1$, $m(i) = m^i$ $(d(i) = m)$, $r(i) \equiv r$, $-\frac{1}{k_2} \leq r \leq \frac{1}{k_2}$. Then from (72) it follows that for sufficiently small $\nu > 0$ the condition $-\frac{\sin\nu}{\sin \nu k_2} \leq r(i) \leq \frac{\sin\nu}{\sin\nu k_1}$ holds. The equalities (68) and (69) turn into

$$\tilde{a}(i) = \frac{1}{m} \cdot \frac{\sin\frac{\nu(2+k_1+k_2)}{2} + r\sin\frac{\nu(k_2-k_1)}{2}}{\sin\frac{\nu(k_1+k_2)}{2}} \equiv A(\nu, m, r)$$

$$\widetilde{b}_1(i) = \frac{1}{m^{k_1+1}} \cdot \frac{\sin\nu + r\sin\nu k_2}{\sin\nu k_1 + \sin\nu k_2} \equiv B_1(\nu, m, r) \tag{75}$$

$$\widetilde{b}_2(i) = \frac{1}{m^{k_2+1}} \cdot \frac{\sin\nu - r\sin\nu k_1}{\sin\nu k_1 + \sin\nu k_2} \equiv B_2(\nu, m, r)$$

Theorem 7. *Let*

$$\nu > 0,\ m > 0,\ -\frac{1}{k_2} \le r \le \frac{1}{k_1},\ q - p > k_2 \tag{76}$$

and on the segment $\langle p, p+1+[\frac{\pi}{\nu}]\rangle$ *the following conditions hold:*

$$a(i) \le A(\nu, m, r),\ b_j(i) \ge B_j(\nu, m, r),\ \ j = 1, 2\,.$$

Then there are no solutions $x(i)$ *of inequality (63) such that* $x(i) > 0$ *on* $\langle p, p+[\frac{\pi}{\nu}] + 1 + k_2\rangle$, *and any solution of corresponding equation (74) has at least one change of sign on this segment.*

Corollary 7.1. *Let (76) hold, and*

$$\left.\begin{aligned} \limsup_{t\to\infty} a(i) &= a^{(0)} \le A(0, m, r) \equiv A = \frac{2 + k_1 + k_2 + r(k_2 - k_1)}{m(k_1 + k_2)} \\ \liminf b_1(i) &= b_1^{(0)} \ge B_1(0, m, r) \equiv B_1 = \frac{1 + rk_2}{m^{1+k_1}(k_1 + k_2)} \\ \liminf b_2(i) &= b_2^{(0)} \ge B_2(0, m, r) \equiv B_2 = \frac{1 + rk_1}{m^{1+k_2}(k_1 + k_2)} \end{aligned}\right\} \tag{77}$$

$$-a^{(0)} + b_1^{(0)} + b_2^{(0)} > -A + B_1 + B_2. \tag{77'}$$

Then all the solutions of the equation (74) are oscillatory.

Remark 7. When all the inequalities (77) are strict, then Corollary 7.1 turns (for $M = 2$) into Ladas-Gyori's theorem:

Theorem 8. *([3]) Consider the delay difference inequality*

$$x(i+1) - x(i) + \sum_{j=1}^{m} b_j(i)x(i - k_j) \le 0,\ i \ge i_0,\ k_j \in \mathrm{N} \tag{78}$$

and the delay difference equation with constant coefficients

$$z(i+1) - z(i) + \sum_{j=1}^{M} P_j z(i - k_j) = 0,\ \ i \ge i_0 \tag{79}$$

where $\liminf_{i\to\infty} b_j(i) \ge P_j > 0$ *for* $j = \overline{1, M}$,

Suppose that all the solutions of (79) are oscillatory. Then (78) has no eventually positive solution.

The connection between Corollary 7.1 and Theorem 8 follows from the following

Lemma 4. *The characteristic equation*

$$F(\mathrm{gl}) = \lambda - a^{(0)} + b_1^{(0)}\lambda^{-k_1} + b_2^{(0)}\lambda^{-k_2} = 0 \tag{80}$$

of the difference equation

$$x(i+1) - a^{(0)}x(i) + b_1^{(0)}x(i-k_1) + b_2^{(0)}x(i-k_2) = 0,\ i \geq 1 \tag{81}$$

has no positive roots if and only if there exist numbers $m > 0$ and r such that strict inequalities (77) hold.

Proof: Put $F_0(\lambda) = \lambda - A + B_1\lambda^{-k_1} + B_2\,\mathrm{gl}^{-k_2}$. It is easy to see that $F(\lambda) > F_0(\lambda)$ and $F_0''(\lambda) > 0$ for all $\lambda > 0$. This implies that equation $F_0'(\mathrm{gl} > 0$ has a unique (positive) root gl_0, i.e. $F_0(\lambda_0) = \min_{\lambda>0} F_0(\lambda)$. On the other hand, it is easy to see that

$$F_0'(\frac{1}{m}) = 1 - k_1B_1m^{1+k_1} - k_2B_2m^{1+k_2} = 0$$

which implies that $\lambda_0 = m^{-1}$.

One can check that $F_0(\lambda_0) = F_0(m^{-1}) = 0$. Therefore, for any $\lambda > 0$, $F(\lambda) > F_0(\lambda) \geq \min_{\lambda>0} F_0(\lambda) = F_0(\lambda_0) = 0$.

Now let us prove that the condition $F(\lambda) > 0$ for any $\lambda > 0$ is sufficient. Let $F(\lambda) > 0$ for any $\mathrm{gl} > 0$. Then $\min_{\lambda>0} F(\lambda) = F(\overline{\lambda}) > 0,\ F'(\overline{\lambda}) = 0$.

Put $\mu \equiv \overline{\lambda} - \Delta\lambda$, where $\Delta\lambda > 0$ is sufficiently small. Due to the monotonicity of $F'(\lambda)$

$$F'(\mu) = 1 - b_1^0k_1\mu^{-k_1-1} - b_2^0k_2\mu^{-k_2-1} \equiv -\varepsilon < F'(\overline{\lambda}) = 0 \tag{82}$$

Define m and r as $m = \mu^{-1}$ and $b_1^{(0)} = (\frac{1+rk_2}{k_1+k_2} + \frac{\varepsilon}{2k_1})m^{-(1+k_1)}$. Then (82) implies that

$$b_2^{(0)} = (\frac{1-rk_1}{k_1+k_2} + \frac{\varepsilon}{2k_2})m^{-(1+k_2)} \tag{83}$$

i.e.,

$$b_j^{(0)} > B_j,\ j = 1,2 \tag{84}$$

It is easy to see that $F(\mu) = F(\overline{\lambda}) + o(\Delta\lambda)$. But $F(\mu) = m^{-1} - a_0 + b_1^{(0)}m^{k_1} + b_2^{(0)}m^{k_2} = A - a_0$. Thus $a_0 < A$ as long as $F(\overline{\lambda}) > 0$.

In fact, we obtained here another proof of the Ladas-Gyori theorem.

3.2.2 Suppose (for simplicity) that $a(i) \equiv 1$ in (63) and (74) and put $\varphi(i) \equiv 1$. Then (69) turn into

$$\left.\begin{aligned} \widetilde{b}_1(i+k_1) &= (\sqcap_{l=i}^{i+k_1} d(l))^{-1}\frac{\sin\nu + r(i)\sin\nu k_2}{\sin\nu k_1 + \sin\nu k_2} \\ \widetilde{b}_2(i+k_2) &= (\sqcap_{l=i}^{i+k_2} d(l))^{-1}\frac{\sin\nu - r(i)\sin\nu k_1}{\sin\nu k_1 + \sin\nu k_2} \end{aligned}\right\} \tag{85}$$

where

$$d(i) = \frac{\sin\frac{2+k_1+k_2}{2}\nu + r(i)\sin\frac{\nu(k_2-k_1)}{2}}{\sin\frac{\nu(k_1+k_2)}{2}} \tag{86}$$

Sometimes the Corollary 7.1 (and Ladas-Gyori's theorem as well) do not provide a good result if the coefficients of (63) are *essentially nonstationary*. At the same time, a more delicate choice of the sequence $\{r(n)\}$ ($r(n) \neq const$) allows us to obtain more subtle results concerning the oscilatory properties of the solution of (63) or (74). Let us illustrate this fact:

Example 3. Consider the inequality

$$x(i+1) - x(i) + b(i)x[i-p(i)] \le 0 \tag{87}$$

and the corresponding equation

$$z(i+1) - z(i) + b(i)z[i-p(i)] = 0, \tag{88}$$

where

$$p(i) = \begin{cases} k_1, & i = 2n-1 \\ k_2, & i = 2n \end{cases} \tag{89}$$

To simplify the notation, we assume that k_1 and k_2 are odd. The inequality (87) can be written in the form (78), where $b_1(2n) = b_2(2n-1) = 0$, $b_1(2n-1) = b(2n-1)$; $b_2(2n) = b(2n)$. Since the $\liminf_{i\to\infty} b_1(i) = \liminf_{i\to\infty} b_2(i) = 0$, it is impossible to apply the Ladas-Gyori theorem and even the Corollary 7.1.

Put in (85)–(86)

$$r(i) = \begin{cases} \frac{\sin\nu}{\sin\nu k_1}, & i = 2n-1 \\ -\frac{\sin\nu}{\sin\nu k_2}, & i = 2n \end{cases} \tag{90}$$

Then $\tilde{b}_1(2n-1+k_1) = \tilde{b}_2(2n+k_2) = 0$,

$$\left.\begin{aligned} \tilde{b}_1(2n+k_1) &= \left(\frac{\sin\nu k_1 \cdot \sin\nu k_2}{\sin\nu(k_1+1)\cdot\sin\nu(k_2+1)}\right)^{\frac{k_1+1}{2}} \equiv q^{k_1+1} \\ \tilde{b}_2(2n-1+k_2) &= q^{\frac{k_2+1}{2}} \end{aligned}\right\} \tag{91}$$

and the following statements hold:

1) Suppose that for $\nu > 0$ the following condition holds:

$$b(i) \ge \begin{cases} q^{\frac{k_1+1}{2}}, & i = 2n-1 \\ q^{\frac{k_2+1}{2}}, & i = 2n\,. \end{cases} \tag{92}$$

Then there are no solutions of (87) preserving the positive sign on any interval $\langle p, -p + [\frac{\pi}{\nu}] + 1 + k_2 \rangle$;

2) Suppose that $\liminf_{n\to\infty} b(2n+1) > \left[\frac{k_1 k_2}{(k_1+1)(k_2+1)}\right]^{\frac{k_1+1}{2}}$,

$$\liminf_{n\to\infty} b(2n) > \left[\frac{k_1 k_2}{(k_1+1)(k_2+1)}\right]^{k_2+1} \tag{93}$$

Then all the solutions of the equation (88) are oscillatory.

3.3. Inequalities and equations of the mixed type with one delay and one advance.

Consider for $k, l \in \mathrm{N}$, the inequalities

$$l[x]_i \equiv x(i+1) - x(i) + b(i)x(i-k) + c(i)x(i+l) \leq 0,\ i \geq 1, \tag{94}$$

$$\tilde{l}[x]_i \equiv y(i-1) - y(i) + \tilde{b}(i+k)y(i+k) + \tilde{c}(i-l)y(i-l) \geq 0,\ i \geq 1 \tag{95}$$

and the corresponding equation

$$l[x]_i = 0. \tag{96}$$

(For simplicity, we consider only the case $a(i) \equiv \tilde{a}(i) \equiv 1$).

As in Section 3.2, one can demonstrate that the sequence

$$y(i) = \sqcap_{n=p+1}^{i} d(n) \cdot \sin(\nu \sum_{n=p+1}^{i} \varphi(n)) \tag{97}$$

is a solution of (95) if

$$\left.\begin{aligned} \tilde{b}(i+k) &= \left(\sqcap_{n=1}^{i+k} d(n)\right)^{-1} \cdot \frac{d(i)\sin(\nu\sum_{i-l+1}^{i}\varphi(n)) - \sin(\nu\sum_{i-l+1}^{i-1}\varphi(n))}{\sin(\nu\sum_{i-l+1}^{i+k}\varphi(n))} \\ \tilde{c}(i-l) &= \left(\sqcap_{n=1-l+1}^{i-1} d(n)\right) \cdot \frac{d(i)\sin(\nu\sum_{i+1}^{i+k}\varphi(n)) - \sin(\nu\sum_{i}^{i+k}\varphi(n))}{\sin(\nu\sum_{i-l+1}^{i+k}\varphi(n))} \end{aligned}\right\} \tag{98}$$

where $\{d(n)\}$ is the positive sequence. Therefore the following statement holds:

Lemma 5. *Let $\{d(n)\}$ be a positive sequence and $\{\varphi(n)\}$ is a non-negative sequence, which satisfies $\sum_{i+1}^{i+k}\varphi(n) > 0,\ \forall i \geq i_0$. Then the sequence (97) is the solution of (95) for sufficiently small $\nu > 0$ where $\tilde{b}(i)$ and $\tilde{c}(i)$ are defined in (98).*

Theorem 9. *Let $\nu > 0$ be sufficiently small, $q - p > k + l$ and the following conditions hold:*

1. *The non-negative sequence $\{\varphi(n)\}$ satisfies the conditions of Lemma 5 and*

$$\sum_{p+1}^{i}\varphi(n) \le \frac{\pi}{\nu}\ ,\ i \in \langle p+1, q\rangle\ ;\ \frac{\pi}{\nu} \le \sum_{p+1}^{i}\varphi(n) \le \frac{2\pi}{\nu}\ , i \in \langle q+1, q+k\rangle\ ;$$
$$-\frac{\pi}{\nu}\sum_{p+1}^{i}\varphi(n) \le 0\ ,\ i \in \langle p-l, p\rangle \tag{99}$$

2. *The sequence $\{d(i)\}$ satisfies the condition*

$$d(i) \ge \frac{\sin(\nu\sum_i^{i+k}\varphi(n))}{\sin(\nu\sum_{i+1}^{i+k}\varphi(n))},\quad \forall i \ge i_0 \tag{100}$$

3.
$$\begin{array}{l} b(i) \ge \widetilde{b}(i)\ \forall i \in \langle p, q\rangle \\ c(i) \ge \widetilde{c}(i) \end{array} \tag{101}$$

where $\widetilde{b}(i)$ and $\widetilde{c}(i)$ are defined in (98).

Then there are no solutions $x(t)$ of (94) such that $x(i) > 0$ on $\langle p-k, q+l\rangle$ and there is no solution of (96) preserving the sign on this interval.

The theorem on oscillation of all solutions of the equation (96) follows from Theorem 9.

Corollary 9.1. *Let the sequence of the interval $\langle p_m, q_m\rangle$, $p_m \to \infty$ be such that all conditions of Theorem 9 hold. Then all solutions of the (96) are oscillatory.*

Consider some particular cases:

3.3.1 Put $\varphi(n) \equiv 1, d(n) \equiv d \ge \frac{k+1}{k}$. Then (100) implies that for sufficiently small $\nu > 0$ $d \ge \frac{\sin\nu(k+1)}{\sin\nu k}$ holds. The equalities (98) turn into

$$\begin{cases} b(i) &= d^{-k-1}\frac{d\sin\nu l - \sin\nu(l-1)}{\sin\nu(k+l)} \equiv B(\nu, d) \\ c(i) &= d^{l-1}\frac{d\sin\nu k - \sin\nu(k+1)}{\sin\nu(k+1)} \equiv C(\nu, d) \end{cases} \tag{102}$$

Theorem 10. *Let*

$$\nu > 0, d \ge \frac{k+1}{k},\ q - p > k + l, \tag{103}$$

and on the segment $\langle p, p + \left[\frac{\pi}{\nu}\right] + 1\rangle$ the following conditions hold:

$$b(i) \ge B(\nu, d),\ c(i) \ge C(\nu, d). \tag{104}$$

Then there are no solutions $x(i)$ of inequality (94) such that $x(i) > 0$ on $\langle p - l, p + \left[\frac{\pi}{\nu}\right] + 1 + k\rangle$ and any solution of (96) has at least one change of sign on this segment.

Corollary 10.1. *Let (103) hold,*

$$\left.\begin{aligned}\liminf_{i\to\infty} b(i) = b^{(0)} \geq B(0,m) \equiv d^{-(k+1)}\frac{dl-l+1}{k+l}\\ \liminf_{i\to\infty} c(i) = c^{(0)} \geq C(O,m) \equiv d^{l-1}\frac{dk-k-1}{k+l},\end{aligned}\right\} \tag{105}$$

and $b^{(0)} + c^{(0)} > B(0,m) + C(0,m)$. Then all the solutions of equation (96) are oscillatory.

Lemma 6. *The characteristic equation*

$$F(\lambda) \equiv \lambda - 1 + b^{(0)}\lambda^{-k} + c^{(0)}\lambda^{l} = 0 \tag{106}$$

of the difference equation

$$z(i+1) - z(i) + b^{(0)}z(i-k) + c^{(0)}z(i+l) = 0\ ,\ i \geq 0$$

has no positive roots if and only if there exists $d > 0$ such that the strict inequalities (105) hold.

The proof is similar to the proof of Lemma 2.

3.2.2 Put in (98), $\varphi(n) \equiv 1$. Then (98) turn into

$$\left.\begin{aligned}\tilde{b}(i+k) = \left(\sqcap_{n=1}^{i+k} d(n)\right)^{-1}\frac{d(i)\sin\nu l - \sin\nu(l-1)}{\sin\nu(k+l)}\\ \tilde{c}(i-l) = \left(\sqcap_{n=1-l+1}^{i-1} d(n)\right)\frac{d(i)\sin\nu k - \sin\nu(k+1)}{\sin\nu(k+l)}\end{aligned}\right\} \tag{107}$$

The choice $d(n) = const$ (as in 3.3.1) is often too crude and does not lead to the desired results in those cases when the sequence $\{b(n)\}$ and $\{c(n)\}$ are essentially non-stationary. For these cases, the choice of $d(n)$ must be more subtle.

Example 4. Consider the inequality

$$x(i+1) - x(i) + b(i)x(i-k) + c(i)x(i+l) \leq 0 \tag{108}$$

and the corresponding equation

$$z(i+1) - z(i) + b(i)z(i-k) + c(i)z(i+l) = 0 \tag{109}$$

where $c(i) = \begin{cases} C\,, & i = 2n-1 \\ 0\,, & i = 2n \end{cases}, C > 0.$

Since $\liminf_{i\to\infty} c(i) = 0$ the application of Corollary 10.1 will demand necessarily that $d = \frac{k+1}{k}$, and condition (105) becomes

$$\liminf_{i\to\infty} b(i) > \frac{k^k}{(k+1)^{k+1}}. \tag{110}$$

In other words, nothing new has been obtained comparing to the case $c(i) \equiv 0$. The same can be said about applicability of the Theorem 10.

On the other hand, let l be even and k odd. Then putting in (107)

$$d(i) = \begin{cases} d\,, & i = 2n-1 \\ \frac{\sin\nu(k+1)}{\sin\nu k}\,, & i = 2n \end{cases}, \tag{111}$$

where $d > 0$ is the positive root of the equation

$$d^{\frac{l}{2}} - \frac{\sin\nu(k+1)}{\sin\nu k}\cdot d^{\frac{l}{2}-1} - C\cdot\frac{\sin\nu(k+l)}{\sin\nu k}\cdot\left(\frac{\sin\nu k}{\sin\nu(k+1)}\right)^{\frac{l}{2}} = 0 \tag{112}$$

One obtains $\widetilde{c}(i) \equiv c(i)$, and

$$\widetilde{b}(i) = \begin{cases} \left(\frac{\sin\nu k}{d\sin\nu(k+1)}\right)^{\frac{k+1}{2}}\cdot\frac{d\sin\nu l - \sin\nu(l-1)}{\sin\nu(k+l)}\,, & i = 2n-1 \\ \left(\frac{\sin\nu k}{d\sin\nu(k+1)}\right)^{\frac{k+1}{2}}\cdot\frac{\sin\nu}{\sin\nu k}\,, & i = 2n\,. \end{cases} \tag{113}$$

Therefore, the following statement holds: Suppose that k and l are odd and even numbers, respectively, and $b(i) \geq \widetilde{b}(i)$, $\forall i$, where $\widetilde{b}(i)$ is defined in (113). Then for (108) and (109) the statement of Theorem 10 holds.

Example 5. Put in (109) $l = 2$, and suppose that

$$\liminf_{n\to\infty} b(2n-1) > \frac{k^k}{(k+1)^{k+1}}\cdot\frac{k+1}{k+2}\left[1 + C\cdot\frac{k(k+2)}{k+1)^2}\right]^{-\frac{k-1}{2}} \tag{114}$$

$$\liminf_{n\to\infty} b(2n) > \frac{k^k}{(k+1)^{k+1}}\cdot\frac{k+1}{+2}\left[1 + C\cdot\frac{k(k+2)}{(k+1)^2}\right]^{-\frac{k+1}{2}}$$

Then all the solutions of (109) are oscillatory.

References

1. G. Ladas, Recent developments in the oscillation of delay difference equation., International Conference on Differential Equations. Colorado Springs, USA, 1989, 12 pp.
2. A. Aksijev, Y. Domshlak and A. Matakajev, Oscillatory properties of the difference equations. (In Russian) Ilim, USSR, Frunze, 1990, 96 p.
3. Y. Domshlak, Discrete analogue of Sturmian comparison theorem for non-symmetric difference equations. Doklady Acad. Sci. Azerb., 11 (1981), 11-15.
4. Y. Domshlak, Sturmian comparison method in investigation of behavior of solutions for differential-operator equations. (In Russian) Elm, Baku, USSR, 1986, 154 p.
5. G. Ladas, G. Philos and Y.G. Sficas, Sharp conditions for the oscillation of delay difference equation. Journal of Applied Mathematics and Simulations 2:2 (1989), 101-111.
6. I.W. Hooker and W.J. Patula, Riccati type tranformations for second-order linear difference equations. J. Math. Analysis and Appl. 82 (1981), 451-462.
7. I.W. Hooker, M. Kwong, W.T. Patula, Oscillatory second order linear difference equations and Riccati equations. SIAM J. Math. Analysis 18 (1987), 54-63.
8. Y. Domshlak, Sturmian comparison theorems for first and second order differential equations with mixed delay of argument. Ukraina Math. Journ. 2 (1982).
9. L.M. Erbe and B.G. Zhang, Oscillation of discrete analogues of delay equations. Proceedings of the International Conference on Theory and Applications of Differential Equations. March 1988, Ohio University, USA.
10. M.A. Skalkina, On oscillation of solutions of difference equations. Izvestia VUS'ov, seria "Mathematica" 6 (1959), 138-144.

ON CONVERGENCE OF SEQUENCES OF INTERNAL SUPERPOSITION OPERATORS

Michael E. Drakhlin

The Research Institute
The College of Judea and Samaria
Ariel 44820, Israel

Conditions for convergence of sequences of internal superposition operators in some spaces of measurable functions are considered.

1. Introduction

The theory of functional-differential equations is based on studies of the properties of an internal superposition operator in different function spaces [1]. In particular, the question of correct solvability of boundary value problems for functional-differential equations requires the establishment of conditions for convergence of sequences of such operators.

For instance, investigation of dependence of absolutely continuous solution $x : [0,\infty) \to \mathbf{R}^n$ of the Cauchy problem for a functional-differential equation of neutral type

$$\dot{x}(t) + B(t)\dot{x}\big(g(t)\big) + A(t)x\big(h(t)\big) = f(t)\ , \quad t \in [0,\infty)\ ,$$

$$\dot{x}(\xi) = x(\xi) = 0\ , \quad \text{if} \quad \xi < 0$$

on function $g : [0,\infty) \to \mathbf{R}$, leads to the necessity of determinating conditions of convergence of sequences of internal superposition operators in spaces of summable functions. Analogous problems arise in the study of integro-differential equations in these spaces. For instance, one could consider the following equation

$$y(t) + B(t)y\big(g(t)\big) + \int_0^t \mathbf{R}(t,s)y(s)ds = f(t)\ , \qquad t \in [0,\infty)\ ,$$

$$y(\xi) = 0\ , \quad \text{if} \quad \xi < 0\ .$$

In this paper we generalize and extend the results of our research [2,3,4] on the convergence of sequences of internal superposition operators in some spaces of measurable functions.

For the formulation of the central statement we need the following notation and definitions.

Denote by $\mathbf{R}^n$ the space of n-dimensional real vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ with the norm $\|\alpha\| = \max_{1\le i\le n} |\alpha_i|$. The same symbol $\|\cdot\|$ will be used for the norm of $n \times n$-matrix coordinated with the norm in $\mathbf{R}^n$. The triple $(\mathbf{E}, \Sigma, m)$, consisting of a set $\mathbf{E} \subset \mathbf{R}^n$, some σ-algebra Σ of subsets of $\mathbf{E}$ and a measure m defined on Σ, will be called the space with measure. Assume measure m to be complete positive σ-finite and non-atomic. Let $L_p(\mathbf{E}, \mathbf{R}^n, \Sigma, m)$ (or briefly L_p^n), $1 \le p < \infty$, be the space of functions $x : E \to \mathbf{R}^n$ with components summable on $\mathbf{E}$ with degree p,

$$\|x\|_{L_p} = \left(\int_E \|x(t)\|^p dm(t) \right)^{1/p} .$$

Define an internal superposition operator $S : L_p \to L_r$ by the equality

$$(Sx)(t) = B(t)(S_g x)(t) , \qquad t \in E , \tag{1}$$

where

$$(S_g x)(t) = \begin{cases} x(g(t)) , & g(t) \in E , \\ 0 , & g(t) \notin E . \end{cases} \tag{2}$$

Here B is an $n \times n$-matrix of measurable functions $b_{ij} : E \to \mathbf{R}$, $i, j = 1, 2, \ldots, n$, and function $g : \mathbf{E} \to \mathbf{R}^n$ satisfies the condition

$$(\forall e \in \Sigma) \quad m(e) = 0 \Rightarrow m(g^{-1}(e)) = 0 . \tag{3}$$

Let a measurable function $z : \mathbf{E} \to [0, \infty[$ be defined in space $(\mathbf{E}, \Sigma, m)$, and $H \in \Sigma$. Let us define on Σ a function $\mu_H(z, g, m)$ by the equality

$$\mu_H(z, g, m)(e) = \int_{\{t \in H : g(t) \in e\}} z(s)\, dm(s) , \qquad e \in \Sigma .$$

Below (see Lemma 1) it will be shown that there exists a measurable function $\psi_H(z, g, m) : \mathbf{E} \to [0, \infty]$ connecting the measures $\mu_H(z, g, m)$ and m by the equality

$$\mu_H(z, g, m)(e) = \int_e \psi_H(z, g, m)(s)\, dm(s) . \tag{4}$$

Consider a sequence of operators

$$\{S_k : L_p^n \to L_r^n\}_{k=1}^{\infty} , \qquad 1 \le r \le p < \infty ,$$

$$(S_k x)(t) \stackrel{\text{def}}{=} B_k(t)(S_{g_k} x)(t) , \qquad t \in E .$$

Here B_k, $k = 1, 2, \ldots$, are $n \times n$-matrices of measurable scalar functions, g_k, $k = 1, 2, \ldots$, satisfying condition (3). Denote by $\|B_k(t)\| \stackrel{\text{def}}{=} b_k(t)$, $\|B(t)\| \stackrel{\text{def}}{=} b(t)$ for $t \in E$. Let $\triangle$ be the symbol of symmetric difference.

The central statement of the paper is established by the following

Theorem 1. *Let* ($\forall H \in \Sigma$, $m(H) < \infty$) *the following conditions be valid:*

1) $\lim_{k\to\infty} m(\{t\in H : b(t) \neq 0, g_k(t)\in E\} \triangle \{t\in H : b(t) \neq 0, g(t)\in E\}) = 0;$
2) *for any* $\sigma > 0$

$$\lim_{k\to\infty} m\left(\{t \in H \cap g^{-1}(E) : b(t) \neq 0\ ,\ \|g_k(t) - g(t)\| \geq \sigma\}\right) = 0\ ;$$

3) *the sequence* $\{\psi_H(1, g_k, m)\}_{k=1}^{\infty}$ *has equipotentially absolutely continuous integrals;*
4) *for any* $\sigma > 0$

$$\lim_{k\to\infty} m\left(\{t \in H \cap g^{-1}(E) : \|B_k(t) - B(t)\| \geq \sigma\}\right) = 0\ ;$$

5)
$$\sup_{k\geq 1} \left\{\left\|\psi_E(b^r, g_k, m)\right\|_{L^1_{p/(p-r)}}\right\} < \infty\ .$$

Then the sequence of operators

$$\{S_k : L_p^n \to L_r^n\}_{k=1}^{\infty}\ , \qquad 1 \leq r \leq p < \infty\ ,$$

strongly converges to the operator $S : L_p^n \to L_r^n$.

The proof of the theorem consists of several steps, each of them will be formulated below as a separate statement.

Let us point out that criteria of continuity of the operator $S : L_p^n \to L_r^n$ were derived in our paper [5].

2. Convergence of the Sequence in Measure

We begin with the proof of aforementioned

Lemma 1. *Let functions* $z : \mathbf{E} \to [0, \infty]$, $g : \mathbf{E} \to \mathbf{R}^n$ *be measurable, where* g *satisfies condition* (3), *then* ($\forall H \in \Sigma$):

1) *there exists a measurable function* $\psi_H(z, g, m) : \mathbf{E} \to [0, \infty]$, *connecting measures* $\mu_H(z, g, m)$ *and* m *by equality* (4);
2) *for any measurable function* $f : \mathbf{E} \to \mathbf{R}^n$ *the following equality holds:*

$$\int_{\{t\in H:g(t)\in e\}} z(t)\|f(g(t))\|dm(t) = \int_e \|f(s)\|\psi_H(z, g, m)(s)dm(s)\ . \tag{5}$$

Proof: If $m(H) < \infty$, then the statements of the lemma are a consequence of the Radon-Nikodym theorem and the rule for change of variables. If the measure H is σ-finite, then H may be represented as

$$H = \bigcup_{i=1}^{\infty} H_i\ , \quad H_i \in \Sigma\ , \quad H_1 \subset H_2 \subset \ldots\ , \quad m(H_i) < \infty\ , \quad i = 1, 2, \ldots$$

Let us construct on Σ a sequence of functions $\{\mu_{H_i}(z,g,m)\}_{i=1}^{\infty}$. The measures $\mu_{H_i}(z,g,m)$ are σ-finite, since z is an m-almost everywhere finite function. By virtue of the Radon-Nikodym theorem, there exists a sequence $\{\psi_{H_i}(z,g,m)\}_{i=1}^{\infty}$ such that for every $i = 1, 2, ...$

$$\mu_{H_i}(z,g,m)(e) = \int_e \psi_{H_i}(z,g,m)(s)dm(s) .$$

It is worth pointing out that we do not claim here the integrability of $\psi_{H_i}(z,g,m)$, $i = 1, 2, ...$ Since $\mu_{H_i}(z,g,m) \leq \mu_{H_j}(z,g,m)$ if $i \leq j$, then m-almost everywhere

$$\psi_{H_i}(z,g,m)(s) \leq \psi_{H_j}(z,g,m)(s) , \quad s \in \mathbf{E} .$$

This implies that the function

$$\psi_H(z,g,m)(s) = \lim_{i\to\infty} \psi_{H_i}(z,g,m)(s) , \quad s \in \mathbf{E} ,$$

is measurable on $\mathbf{E}$, and for every $e \in \Sigma$ equality (4) holds.

For the proof of the second statement of the lemma it is enough to use the B. Levi theorem, by virtue of which the passage to the limit while $n \to \infty$ in the equality

$$\int_{\{t\in H_i : g(t)\in e\}} z(t)\|f(g(t))\|dm(t) = \int_e \|f(s)\|\psi_{H_i}(z,g,m)(s)dm(s)$$

is possible and leads to the equality (5). Q.E.D.

Denote by $\mathbf{M}(\mathbf{E}, \mathbf{R}^n, \Sigma, m)$ (or $\mathbf{M}^n$ in short) the space of functions $x : \mathbf{E} \to \mathbf{R}^n$ with the topology of convergence in measure m on every set $H \in \Sigma$, $m(H) < \infty$.

Lemma 2. *For convergence* $(\forall\, x \in \mathbf{M}^n)$ *of a sequence of functions* $\{S_{g_k}x\}_{k=1}^{\infty}$ *to the function* $S_g x$ *in the space* $\mathbf{M}^n$, *it is necessary and sufficient, that on every set* $H \in \Sigma$, $m(H) < \infty$, *the following conditions hold:*

1) $\lim_{k\to\infty} m\big(\{t \in H : g_k(t) \in \mathbf{E}\} \,\triangle\, \{t \in H : g(t) \in \mathbf{E}\}\big) = 0$;
2) *for any* $\sigma > 0$

$$\lim_{k\to\infty} m\big(\{t \in H \cap g^{-1}(\mathbf{E}) : \|g_k(t) - g(t)\| \geq \sigma\}\big) = 0 ;$$

3) *the sequence* $\{\psi_H(1, g_k, m)\}_{k=1}^{\infty}$ *has equipotentially absolutely continuous integrals.*

Proof: **Necessity.** Let $x_1(t) = (1, 1, ..., 1), t \in \mathbf{E}$. Then $\big(\forall\, \sigma > 0, \sigma < 1, \forall\, H \in \Sigma, m(H) < \infty, \forall\, k \geq 1\big)$

$$\{t \in H : \|(S_{g_k}x_1)(t) - (S_g x_1)(t)\| \geq \sigma\} = \{t \in H : g_k(t) \in \mathbf{E}\} \triangle \{t \in H : g(t) \in \mathbf{E}\} .$$

Consequently, from the convergence of a sequence $\{S_{g_k}x_1\}_{k=1}^{\infty}$ to the function $S_g x_1$ in measure m on every set $H \in \Sigma$, $m(H) < \infty$ we get that $(\forall\, \sigma > 0, \forall\, H \in \Sigma, m(H) < \infty)$

$$\lim_{k\to\infty} m\big(\{t \in H : g_k(t) \in \mathbf{E}\} \,\triangle\, \{t \in H : g(t) \in \mathbf{E}\}\big) = 0 ,$$

i.e. the first condition is satisfied.

Let $x_2(t) = (\|t\|, ..., \|t\|)$, $t \in \mathbf{E}$. Then

$$\begin{aligned}
&\{t \in H : \|(S_{g_k}x_2)(t) - (S_g x_2)(t)\| \geq \sigma\} \subset \\
&\subset \{t \in H \cap g^{-1}(\mathbf{E}) \cap g_k^{-1}(\mathbf{E}) : \|g_k(t) - g(t)\| \geq \sigma\} \cup \\
&\cup \{t \in H : g_k(t) \notin \mathbf{E}, g(t) \in \mathbf{E}, \|g(t)\| \geq \sigma\} \cup \\
&\cup \{t \in H : g_k(t) \in \mathbf{E}, g(t) \notin \mathbf{E}, \|g_k(t)\| \geq \sigma\} .
\end{aligned}$$

Since,

$$\begin{aligned}
&\{t \in H : g_k(t) \notin \mathbf{E}, g(t) \in \mathbf{E}, \|g(t)\| \geq \sigma\} \cup \\
&\cup \{t \in H : g_k(t) \in \mathbf{E}, g(t) \notin \mathbf{E}, \|g_k(t)\| \geq \sigma\} \subset \\
&\subset \big(\{t \in H : g_k(t) \in \mathbf{E}\} \,\triangle\, \{t \in H : g(t) \in \mathbf{E}\}\big) ,
\end{aligned}$$

and the measure of the last set tends to zero, then

$$\lim_{k\to\infty} m\big(\{t \in H \cap g^{-1}(\mathbf{E}) : \|g_k(t) - g(t)\| \geq \sigma\}\big) = 0 ,$$

i.e. the second condition is satisfied.

Consider the function $x_3(t) = \chi_e(t)$, $t \in \mathbf{E}$, $e \in \Sigma$. The sequence $\{S_{g_k}x_3\}_{k=1}^{\infty}$ converges in measure to $S_g x_3$ on every set $H \in \Sigma$, $m(H) < \infty$, moreover, $\|S_{g_k}x_3\| \leq 1$, $t \in \mathbf{E}$. Then from the Lebesque theorem we get

$$\begin{aligned}
\lim_{k\to\infty} \int_e \psi_H(1, g_k, m)(s)dm(s) &= \lim_{k\to\infty} \int_H \|(S_{g_k}x_3)(t)\|dm(t) = \\
&= \int_H \|(S_g x_3)(t)\|dm(t) = \int_e \psi_H(1, g, m)(t)dm(t) .
\end{aligned}$$

Since $e \subset \mathbf{E}$ is arbitrary, the sequence $\{\psi_H(1, g_k, m)\}_{k=1}^{\infty}$ has equipotentially absolutely continuous integrals.

Sufficiency. Let $H \in \Sigma$, $m(H) < \infty$. We will show that providing the conditions of the lemma hold, the sequence $\{S_{g_k}x\}_{k=1}^{\infty}$ for arbitrary function $x \in \mathbf{M}^n$ converges on H in measure to function $S_g x$. Evidently, w.l.o.g. the set H is bounded. Given $\varepsilon > 0$, choose a bounded set $Q \in \Sigma$, $m(Q) < \infty$ in such a way, that

$$m(\{t \in H : g(t) \in \mathbf{E}\backslash Q\}) < \varepsilon/4 .$$

For an arbitrary function $x \in \mathbf{M}^n$ there exists the function $y \in \mathbf{M}^n$ with uniformly continuous on Q components, which differs from x only on a set $e_0 \subset Q$ of arbitrary small measure.

The following inequality holds:

$$\|(S_{g_k}x)(t) - (S_g x)(t)\| \le \|(S_{g_k}x)(t) - (S_{g_k}y)(t)\| + \\ + \|(S_{g_k}y)(t) - (S_g y)(t)\| + \|(S_g y)(t) - (S_g x)(t)\| .$$

As a consequence of the equality

$$\mu_H(1, g_k, m)(e) = \int_e \psi_H(1, g_k, m)(s) dm(s) ,$$

we get that the equipotential absolute continuity of integrals of the sequence $\{\psi_H(1, g_k, m)\}_{k=1}^{\infty}$ is equivalent to condition: $\mu_H(1, g_k, m)(e) \to 0$ uniformly in k when $m(e) \to 0$, $e \in \Sigma$. Hence, starting from some K_1,

$$m\big(\{t \in H : \|(S_{g_k}x)(t) - (S_{g_k}y)(t)\| \ge \sigma/3\}\big) < \varepsilon/4$$

for any given σ, $\varepsilon > 0$. Since $\mu_H(1, g, m)(e) = 0$ for $m(e) = 0$, $e \in \Sigma$, then for small enough measure of the set e_0

$$m\big(\{t \in H : \|(S_g x)(t) - (S_g y)(t)\| \ge \sigma/3\}\big) < \varepsilon/4 .$$

The uniform continuity of function y on Q, and conditions 1) and 2) imply an existence of such K_2, that for $k > K_2$

$$m\big(\{t \in H : \|(S_{g_k}y)(t) - (S_g y)(t)\| \ge \sigma/3\}\big) < \varepsilon/4 .$$

Therefore, for arbitrary $x \in \mathbf{M}^n$, $H \in \Sigma$, $m(H) < \infty$, and arbitrary given $\sigma, \varepsilon > 0$ there exists $K = \max\{K_1, K_2\}$ such that for $k > K$

$$m\big(\{t \in H : \|(S_{g_k}x)(t) - (S_g x)(t)\| \ge \sigma\}\big) < \varepsilon .$$

The lemma has been proven.

3. Conditions of Strong Convergence of the Sequence

A criteria of the pointwise converge of operators is determined in the following

Theorem 2. *A sequence of operators $\{S_{g_k} : L_p^n \to L_r^n\}_{k=1}^{\infty}$, $1 \le r \le p < \infty$, strongly converges to the operator $S_g : L_p^n \to L_r^n$ if and only if the conditions 1) and 2) of Lemma 2 are valid, and*

$$\sup_{k \ge 1}\{\|\psi_{\mathbf{E}}(1, g_k, m)\|_{L^1_{p/(p-r)}}\} = c < \infty . \tag{6}$$

Proof: **Necessity**. Since the sequence of operators $\{S_{g_k}\}_{k=1}^{\infty}$ converges to the operator S_g in every point $x \in L_p^n$, then the sequence of functions $\{S_{g_k}x\}_{k=1}^{\infty}$ converges to the function $S_g x$ in measure on every set $H \in \Sigma$, $m(H) < \infty$. Therefore the conditions 1) and 2) of Lemma 2 hold. By virtue of the equality (see [5])

$$\|S_{g_k}\|_{L_p^n \to L_r^n} = \|\psi_{\mathbf{E}}(1, g_k, m)\|_{L^1_{p/(p-r)}}^{1/r}$$

we get from the Banach-Steinhaus theorem the boundedness of the sequence of norms

$$\{\|\psi_{\mathbf{E}}(1, g_k, m)\|_{L^1_{p/(p-r)}}\}_{k=1}^{\infty} .$$

Sufficiency. The boundedness of the sequence of norms $\{\|\psi_{\mathbf{E}}(1, g_k, m)\|_{L^1_{p/(p-r)}}\}_{k=1}^{\infty}$ implies, by virtue of the Valle-Poussin theorem, that the sequence of functions $\{\psi_{\mathbf{E}}(1, g_k, m)\}_{k=1}^{\infty}$ has equipotentially absolutely continuous integrals. Since

$$(\forall\, H \in \Sigma)\psi_H(1, g_k, m)(s) \le \psi_{\mathbf{E}}(1, g_k, m)(s) , \qquad s \in \mathbf{E} , \tag{7}$$

the conditions of Lemma 2 hold, hence the sequence $(\forall\, x \in L_p^n)$ of functions $\{S_{g_k}x\}_{k=1}^{\infty}$ converges in measure to the function $S_g x$ on every set $H \in \Sigma$, $m(H) < \infty$. Besides, the sequence of functions $\{\|(S_{g_k}x(\cdot)\|^r\}_{k=1}^{\infty}$ has equipotentially absolutely continuous integrals. It is justified using (6) and (7) by the following arguments. Let $e \in \Sigma$ then

$$\int_e \|(S_{g_k}x)(t)\|^r dm(t) = \int_{\{t \in e : g_k(t) \in \mathbf{E}\}} \|x(g_k(t))\|^r dm(t) =$$

$$= \int_e \|x(s)\|^r \psi_e(1, g_k, m)(s) dm(s) \le$$

$$\le \int_e \|x(s)\|^r \psi_{\mathbf{E}}(1, g_k, m)(s) dm(s) \le$$

$$\le \left(\int_e \|x(s)\|^p dm(s)\right)^{r/p} \left(\int_e [\psi_{\mathbf{E}}(1, g_k, m)(s)]^{p/(p-r)} dm(s)\right)^{(p-r)/p} \le$$

$$\le c\left(\int_e \|x(s)\|^p dm(s)\right)^{r/p} . \tag{8}$$

Thus, while $m(e)$ tends to zero, the left hand side of (8) tends to zero uniformly with respect to k. Let us choose a set $e = \mathbf{E} \backslash H$ ($H \in \Sigma$, $m(H) < \infty$) from the condition

$$\int_e \|x(s)\|^p dm(s) < \left(\frac{\varepsilon}{c}\right)^{p/r} .$$

Then, (8) implies $(\forall \varepsilon > 0)$ an existence of set $H \in \Sigma$, $m(H) < \infty$ such that simultaneously for all $k \in \{1, 2, ...\}$ the following inequality holds

$$\int_{\mathbf{E} \backslash H} \|(S_{g_k}x)(t)\|^r dm(t) < \varepsilon .$$

Hence, the sequence of functions $\{S_{g_k}x\}_{k=1}^{\infty}$ satisfies all the conditions of [6, Theorem 6, chapter 3-3], and therefore converges in the norm of the space L_r^n to the limit function $S_g x$. That completes the proof of the theorem.

Lemma 3. *Let* $(\forall H \in \Sigma,\ m(H) < \infty)$ *the following conditions hold:*

1) $\lim\limits_{k\to\infty} m(\{t \in H : b(t) \neq 0\ ,\ g_k(t) \in \mathbf{E}\} \triangle \{t \in H : b(t) \neq 0\ ,\ g(t) \in \mathbf{E}\}) = 0;$
2) *for any* $\sigma > 0$

$$\lim_{k\to\infty} m(\{t \in H \cap g^{-1}(\mathbf{E}) : b(t) \neq 0\ ,\ \|g_k(t) - g(t)\| \geq \sigma\}) = 0\ ;$$

3) *the sequence* $\{\psi_H(1, g_k, m)\}_{k=1}^{\infty}$ *has equipotentially absolutely continuous integrals;*
4) *for any* $\sigma > 0$

$$\lim_{k\to\infty} m(\{t \in H \cap g^{-1}(E) : \|B_k(t) - B(t)\| \geq \sigma\}) = 0\ .$$

Then, $(\forall x \in \mathbf{M}^n)$ *the sequence of functions* $\{(S_k x)\}_{k=1}^{\infty}$ *converges in the space* $\mathbf{M}^n$ *to the function* Sx.

Proof: Use imbedding:

$$\{t \in H : \|(S_k x)(t) - (Sx)(t)\| \geq \sigma\} \subset \{t \in H : (S_k x)(t) - B(t)(S_{g_k}x)(t)\| \geq \sigma/2\} \cup$$

$$\cup \{t \in H : \|B(t)(S_{g_k}x)(t) - (Sx)(t)\| \geq \sigma/2\} \subset$$

$$\subset \{t \in H \cap g_{-1}(\mathbf{E}) : \|B_k(t) - B(t)\| \|x(g_k(t))\| \geq \sigma/2\} \cup$$

$$\cup \{t \in H : b(t)\|(S_{g_k}x)(t) - (S_g x)(t)\| \geq \sigma/2\}\ .$$

Conditions 3) and 4) imply

$$\lim_{k\to\infty} m(\{t \in H \cap g^{-1}(\mathbf{E}) : \|B_k(t) - B(t)\| \|x(g_k(t))\| \geq \sigma/2\}) = 0\ .$$

Conditions 1), 2), 3) by virtue of Lemma 2 imply

$$\lim_{k\to\infty} m(\{t \in H : b(t)\|(S_{g_k}x)(t) - (S_g x)(t)\| \geq \sigma/2\}) = 0\ .$$

Thus, $(\forall \sigma > 0,\ \forall H \in \Sigma,\ m(H) < \infty)$

$$\lim_{k\to\infty} m(\{t \in H : \|(S_k x)(t) - (Sx)(t)\| \geq \sigma\}) = 0\ .$$

Q.E.D.

4. Proof of the Main Result (Theorem 1)

A consequence of Lemma 3 is that $(\forall x \in L_p^n)$ a sequence of functions $\{S_k x\}_{k=1}^{\infty}$ converges in measure to the function Sx on every set $H \in \Sigma$, $m(H) < \infty$. Similarly to the proof of Theorem 2 one could justify that $(\forall e \in \Sigma)$

$$\int_e \|(S_k x)(t)\|^r dm(t) \le d\Big(\int_e \|x(s)\|^p dm(s)\Big)^{r/p}$$

analogous to inequality (8). This implies that for any $\varepsilon > 0$ there exists a set $H \in \Sigma$, $m(H) < \infty$, such that simultaneously for all $k \in \{1, 2, ...\}$

$$\int_{\mathbf{E}\setminus H} \|(S_k x)(t)\|^r dm(t) < \varepsilon \ .$$

Reference to [6, Theorem 6, chapter 3-3] completes the proof of Theorem 1.

5. Conditions for Uniform Convergence

Theorems 1 and 2 establish conditions of the strong (pointwise) convergence of sequences of operators $\{S_k : L_p^n \to L_r^n\}_{k=1}^{\infty}$, $\{S_{g_k} : L_p^n \to L_r^n\}_{k=1}^{\infty}$, $1 \le r \le p < \infty$. Let us consider now conditions for uniform convergence of these sequences.

Let $\mathfrak{J}$ be a pre-compact family of functions in space L_p^n. Then, under the conditions of Theorem 2 the sequence of operators $\{S_{g_k} : L_p^n \to L_r^n\}_{k=1}^{\infty}$, $1 \le r \le p < \infty$, converges uniformly on the set $\mathfrak{J}$ to the operator S_g. The assumption about the pre-compactness of the set $\mathfrak{J}$ is essential for the uniform convergence of the sequence $\{S_{g_k}\}_{k=1}^{\infty}$ to S_g.

Example. Let $\mathbf{E} = [0,1]$, $g_k(t) = t - 1/k$, $t \in [0,1]$, $k = 1, 2, ...$ Then, since the family of functions is compact in space $L_p^1([0,1])$, $1 \le p < \infty$, if the sequence of operators $\{S_{g_k}\}_{k=1}^{\infty}$ uniformly converges on the set $\mathfrak{J} \subset L_p^1([0,1])$ to the identity operator, then the set $\mathfrak{J}$ is pre-compact.

In [2] it is shown that the assumption about compactness of the family of functions $\mathfrak{J}$ in measure is essential for uniform convergence $S_{g_k} \to S_g$ on every element $x \in \mathfrak{J}$ in measure.

Definition. Let us say that a measurable function $g : \mathbf{E} \to \mathbf{R}^n$ possesses property N_0 if there exists a set $\overline{G} \subset g^{-1}(\mathbf{E})$ such that $m(\overline{G}) = m(g^{-1}(\mathbf{E}))$ and every measurable set $e \subset \overline{G}$, $m(e) = 0$ then $m(g(e)) = 0$.

Note that a function g possesses property N_0 if Luzin's N-condition holds. Particularly, it holds for absolutely continuous functions.

For a measurable function $g : \mathbf{E} \to \mathbf{R}^n$ possessing the property N_0 the set $g(e \cap g^{-1}(\mathbf{E}))$ is measurable for any measurable $e \subset \mathbf{E}$. According to the theorem of Luzin $(\forall H \in \Sigma\ ,\ m(H) < \infty)$ there exists a sequence of functions

$g_k : \mathbf{E} \to \mathbf{R}^n$,continuous on H, such that $m\{t \in H : g(t) \neq g_k(t)\} < m(H)/2^k$, and by virtue of continuity of the measure m there exist sets H_k such that

$$H_k \subset \left\{ t \in H \Big\backslash \bigcup_{i=1}^{k-1} H_i : g(t) = g_k(t) \right\} \quad \text{and} \quad m(H_k) = \frac{m(H)}{2^k} .$$

Note that $H_i \cap H_j = \emptyset$, $i \neq j$, and $m\left(\bigcup_{k=1}^{\infty} H_k\right) = m(H)$. Denote by $H_0 \stackrel{\text{def}}{=} H \backslash \bigcup_{k=1}^{\infty} H_k$. Then $m(H_0) = 0$. Let $e \in \Sigma$, $e \subset H$. Then

$$g(e) = \bigcup_{k=0}^{\infty} g(e \cap H_k) = \left[\bigcup_{k=1}^{\infty} g_k(e \cap H_k) \right] \cup g(e \cap H_0) .$$

Since $m(e \cap H_0) = 0$ we conclude using [6, ch. 3] that every set $g_k(e \cap H_k)$ is measurable. Then $g(e)$ is measurable.

6. A Sequence of Operators in L_∞^n

In Theorems 1 and 2 we exploited the restriction $p < \infty$. In what follows we will show that the question about convergence in L_∞^n has a negative answer.

Consider a sequence of operators S_{g_k}, $k = 1, 2, ...$, acting in space L_∞^1, where $\mathbf{E} \subset \mathbf{R}^1$, $g_k : \mathbf{E} \to \mathbf{R}^1$, $k = 1, 2, ...,$. The convergence in space L_∞^1 is the uniform convergence m-almost everywhere on $\mathbf{E}$. Assume that for functions $g_k, g : \mathbf{E} \to \mathbf{R}^1$, $k = 1, 2, ...$, condition N_0 holds. Choose such $e \in \Sigma$ that $e \subset g(G)$, $m(e) > 0$ and $m(g(G) \backslash e) > 0$. Consider the function x:

$$x(t) = \begin{cases} \alpha , & t \in e , \\ \beta , & t \notin e , \end{cases} \qquad \alpha \neq \beta .$$

For the uniform convergence of the sequence $\{S_{g_k} x\}_{k=1}^{\infty}$ almost everywhere on $\mathbf{E}$ to $S_g x$ it is necessary that, starting from some number, the following equality holds

$$m(g_k^{-1}(e) \bigtriangleup g^{-1}(e)) = 0 , \tag{9}$$

otherwise $\lim\limits_{k \to \infty} \|S_{g_k} x - S_g x\|_{L_\infty^1} = |\alpha - \beta| > 0$.

Since e is an arbitrary set from $g(G)$, then for convergence (pointwise and uniform) of the sequence $\{S_{g_k} x\}_{k=1}^{\infty}$ to S_g on some "good" set (for example, the sphere, a densely closed set in L_∞^1, etc.) it is necessary for the number K to exist, such that for $k > K$ condition (9) holds for every measurable set $e \subset g(G)$.

Let us show that condition (9) implies that beginning from some number K for every $H \subset G$

$$m(g_k(H) \bigtriangleup g(H)) = 0 \tag{10}$$

Assume that (10) does not hold for some measurable set $H \subset G$, $m(H) > 0$. Let $g(H) = U$, $g_k(H) = U_k$, $m(U \triangle U_k) > 0$. Let $m(U') > 0$, where $U' = U \backslash U_k$. By virtue of (9) $m(g_k^{-1}(U') \triangle g^{-1}(U)) = 0$, and thus the sets $g_k^{-1}(U') \cap H$, and $g^{-1}(U') \cap H$ coincide to within a set of zero measure. Moreover, $m(g^{-1}(U') \cap H) > 0$, since the function g possesses property N_0. Then $m(g_k^{-1}(U') \cap H) > 0$ and $g_k(g_k^{-1}(U') \cap H) = U' \subset U_k$, this contradicts $g_k(H) = U_k$. Therefore, $m(U') = 0$. Consequently, for every measurable set $H \subset G$ for $k > K$ equality (10) holds.

Now we will show that providing condition (9) holds, for $k > K$, $g^k(t) = g(t)$ m-almost everywhere on G. Assume the contrary, i.e. that for every number $K_1 > K$ there exists $k > K_1$ such that $m(\{t \in G : g_k(t) \neq g(t)\}) \geq \varepsilon > 0$. Denote $H' \stackrel{\text{def}}{=} \{t \in G : g^k(t) > g(t)\}$, $H'' = \{t \in G : g^k(t) < g(t)\}$. Then either $m(H') \geq \varepsilon/2$ or $m(H'') \geq \varepsilon/2$. Assume the first, i.e. $m(H') \geq \varepsilon/2$. Using definition (4), consider functions of the set $\mu_{H'}(1, g_k, m)$ and $\mu_{H'}(1, g, m)$. From (10) we have that $g(H')$ coincides with $g_k(H')$ to within a set of zero measure. Condition (9) implies

$$\int_e \psi_{H'}(1, g_k, m)(s)dm(s) = \int_e \psi_{H'}(1, g, m)(s)dm(s)\,, \quad e \subset g(H')\,.$$

Hence $\psi_{H'}(1, g_k, m)(s) = \psi_{H'}(1, g, m)(s)$ almost everywhere on $g(H')$. Since $g_k(t) > g(t)$, $t \in H'$, the following inequality holds

$$\int_{H'} g_k(t)dm(t) > \int_{H'} g(t)dm(t)$$

or

$$\int_{g_k(H')} s\psi_{H'}(1, g_k, m)(s)dm(s) > \int_{g(H')} s\psi_{H'}(1, g, m)(s)dm(s)\,.$$

On the other hand,

$$\int_{g_k(H')} s\psi_{H'}(1, g_k, m)(s)dm(s) = \int_{g(H')} s\psi_{H'}(1, g, m)(s)dm(s)\,.$$

The contradiction shows that for $k > K$, $g_k(t) = g(t)$ almost everywhere on G, and consequently almost everywhere on $g^{-1}(\mathbf{E})$, since $m(G) = m(g^{-1}(\mathbf{E}))$.

Thus, the sequence of operators $\{S_{g_k}\}_{k=1}^{\infty}$ does not converge in the space $L_\infty(\mathbf{E}, \mathbf{R}^1, \Sigma, m)$ excluding the trivial case when functions g_k beginning from some number K equals g. The same holds (using the same arguments) for the sequence of operators $\{S_{g_k}\}_{k=1}^{\infty}$ in the space $L_\infty(\mathbf{E}, \mathbf{R}^n, \Sigma, m)$ of vector-functions $x : \mathbf{E} \to \mathbf{R}^n$, where $\mathbf{E} \subset \mathbf{R}^n$, $g_k, g : \mathbf{E} \to \mathbf{R}^n$, $k = 1, 2, \ldots$.

Acknowledgement

This research was supported by the Ministry of Science and Technology, State of Israel.

References

1. N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional-Differential Equations, Moscow, Nauka, 1991 (in Russian).
2. M. Drakhlin, T. Plyshevskaya, On the theory of functional-differntial equations, Differential Equations 14:8 (1978), 1347-1361.
3. M. Drakhlin, L. Kultysheva, Non-linear equations with internal superposition operator, Differential Equations 16:12 (1980), 2219-2229.
4. M. Drakhlin, Properties of internal superposition operators sequences, in "Boundary Value Problems", Perm, 1989, 179-184 (in Russian).
5. M. Drakhlin, Internal superposition operator in the spaces of summable functions, Izv. VUZov - Matematika 5 (1986), 18-24.
6. N. Dunford, J.T. Schwartz, Linear Operators. Part 1: General Theory, Interscience Publishers, 1958.

POPOV-TYPE STABILITY CRITERION FOR THE FUNCTIONAL-DIFFERENTIAL EQUATIONS DESCRIBING PULSE MODULATED CONTROL SYSTEMS

A.Kh. Gelig
Dept. Mathematics & Mechanics
St. Petersburg State University
Bibliotechnaya Pl. 2, Peterhof
St. Petersburg 190008, Russia

A.N. Churilov
Dept. Shipbuilding & Ocean Technology
Naval Technical University
Lotsmanskaya St., 3
St. Petersburg, 198904, Russia

Abstract

We study the stability of equilibrium states of nonlinear functional-differential equations describing pulse modulated control systems. The frequency-domain criteria of the Popov type are obtained.

1. Introduction

In the past three decades, a lot of effort has been devoted to the study of sample-data systems such that their mathematical description can be reduced to discrete-time models. However, numerous pulse-modulated systems of theoretical and practical interest do not admit a discrete-time reduction. The systems of this kind involve a modulated parameter (e.g. frequency, width, amplitude, phase) which depends nonlinearly on a modulated input function. So their study leads to functional-differential or functional-integral equations.

The present communication is concerned with the stability of the equilibrium states of nonlinear closed-loop pulse-modulated systems. It follows in the footsteps of papers [1]–[3]. The approach based on averaging of a modulator output function and on the Yakubovich-Kalman lemma is proposed for solving the problem. The frequency conditions obtained generalize the classical Popov stability criterion. The method proposed here resembles the equivalent areas in principle but avoids the mathematical non-rigorism of the latter. It is applicable to all kinds of modulation where an upper bound of a sampling period is known. The advantage of the present approach is that the form of the impulse may not be known exactly, so the criteria provided guarantee good robustness to variations of the modulator's parameters.

2. Problem Formulation

The main element of a pulse-modulated system is the modulator. It is described with a nonlinear operator M which transforms a continuous output function $\sigma(t)$ into a piecewise continuous output function $f(t)$:

$$f = M\sigma \qquad (1)$$

The operator M satisfies the causality condition: the value of f at a point t depends on values of σ at preceding points τ, $\tau \leq t$, only. The description of $f(t)$ involves an increasing sequence of sampling moments $t_0, t_1, t_2 \ldots (t_n \to +\infty$ as $n \to +\infty)$. When $t_n < t < t_{n+1}$ the function $f(t)$ presents the form on the n-th impulse. Suppose that $f(t)$ does not change its sign on a sampling interval (t_n, t_{n+1}), and a sampling period $T_n = t_{n+1} - t_n$ can be estimated

$$\kappa_0 T \leq T_n \leq T \,, \tag{2}$$

where κ_0, T are positive constants; i.e. upper and lower bounds of T_n exist.

Let vector $x(t)$ of dimension m describe the system's state at a moment t. Consider a functional equation (1) together with a linear differential equation with constant coefficients written in vector form

$$\frac{dx}{dt} = Ax + bf \,, \sigma = c^* x \,. \tag{3}$$

Here A is a real constant $m \times m$ matrix, b, c are real constant m-dimensional vectors, and $*$ denotes vector transpose. In technical applications equation (3) is commonly described by the rational complex-valued function $W(p)$ of a complex variable p: $W(p) = c^*(A - pI_m)^{-1} b$, with I_m being the identity matrix of order m. It is called the transfer function from $-f$ to σ. Suppose that polynomials $\det(pI_m - A)$ and $W(p)\det(pI_m - A)$ have no common roots (such transfer functions are said to be nonsingular). Denote

$$\rho = \lim_{p\to 0} pW(p)\,, \qquad \kappa = \lim_{p\to\infty} pW(p)\,, \qquad \chi(p) = pW(p) - \kappa \,.$$

Let us consider the sequence $\{v_n\}$ of mean values of $f(t)$ at sampling intervals

$$v_n = \frac{1}{T_n} \int_{t_n}^{t_{n+1}} f(t)\, dt \,.$$

Require v_n to be bounded for $n \geq 0$. For many kinds of modulation a nonlinear function $\phi(\sigma)$ can be constructed so that for any sufficiently large n, $n \geq 0$, a moment $\tilde{t}_n$ satisfying $t_n \leq \tilde{t}_n \leq t_{n+1}$ and

$$v_n = \phi\big(\sigma(\tilde{t}_n)\big) \tag{4}$$

exists. This function $\phi(\sigma)$ will be called an equivalent nonlinearity. Let us show that an equivalent nonlinearity exists for commonly used kinds of modulation.

1. Pulse-amplitude modulation (PAM). In this case $t_n = nT$,

$$f(t) = \begin{cases} a\big(\sigma(nT)\big)/\tau \,, & nT < t \leq nT + \tau \,, \\ 0 \,, & nT + \tau < t \leq (n+1)T \,, \end{cases}$$

with τ, T positive constant, $0 < \tau < T$. Here $a(\sigma)$ is a continuous bounded function, $a(0) = 0$. Obviously $v_n = a(\sigma(nT))/T$, and (4) holds for $\tilde{t}_n = nT$ and $\phi(\sigma) = a(\sigma)/T$.

2. Pulse-frequency modulation of the first kind (PFM-1). Here $t_{n+1} = t_n + \Phi(|\sigma(t_n)|)$,

$$f(t) = \begin{cases} \lambda(\sigma(t_n))/\tau\,, & t_n < t \le t_n + \tau\,, \\ 0\,, & t_n + \tau < t \le t_{n+1}\,, \end{cases}$$

$\lambda(\sigma) = 0$ for $|\sigma| \le \Delta$ and $\lambda(\sigma) = sign(\sigma)$ for $|\sigma| > \Delta$, τ, Δ are positive constants. The function $\Phi(\mu)$ is continuous, nonincreasing for $\mu \ge 0$ and $\Phi(\mu) \to \Phi_\infty = \text{const} > 0$ as $\mu \to +\infty$, $0 < \tau < \Phi_\infty$. Apparently $v_n = \lambda(\sigma(t_n))/\Phi(|\sigma(t_n)|)$ and (4) holds for $\tilde{t}_n = t_n$ and $\phi(\sigma) = \lambda(\sigma)/\Phi(|\sigma|)$.

3. Pulse-frequency modulation of the second kind (PFM-2). In this case t_{n+1} is the minimal root of the equation $t_{n+1} = t_n + \Phi(|\sigma(t_{n+1})|)$, the functions f, λ and Φ are the same as those for PFM-1. Evidently $v_n = \lambda(\sigma(t_n))/\Phi(|\sigma(t_{n+1})|)$. Since $\lambda(\sigma(t_n)) \ne \lambda(\sigma(t_{n+1}))$, relation (4) does not hold directly. To solve this problem let us consider the sequence $\hat{t}_n = t_n + \tau$ instead of t_n. Then $\hat{t}_{n+1} - \hat{t}_n = T_n = \Phi(|\sigma(t_{n+1})|)$ and $\phi(\sigma) = \lambda(\sigma)/\Phi(|\sigma|)$ turns out to be an equivalent nonlinearity with $\tilde{t}_n = t_{n+1}$.

4. Pulse-width modulation of the first kind (PWM-1). Here $t_n = nT$ ($T = \text{const} > 0$),

$$f(t) = \begin{cases} sign\,(\sigma(nT))\,, & nT < t \le nT + \tau_n\,, \\ 0\,, & nT + \tau_n < t \le (n+1)T\,, \end{cases}$$

$\tau_n = F(|\sigma(nT)|)$. The function $F(\mu)$ is continuous and nondecreasing when $\mu \ge 0$, $F(0) = 0$, $F(\mu) \le T$ for all $\mu > 0$. Evidently $v_n = F(|\sigma(nT)|)\, sign\,(\sigma(nT))/T$ and (4) holds for $\phi(\sigma) = F(|\sigma|)\, sign(\sigma)/T$ and $\tilde{t}_n = nT$.

5. Pulse-width modulation of the second kind (PWM-2). In this case $t_n = nT$, τ_n is the minimal nonnegative root, satisfying $\tau_n \le T$, of the equation $\tau_n = F(|\sigma(nT + \tau_n)|)$, if it exists; otherwise, $\tau_n = T$. The functions $f(t)$ and $F(\mu)$ are the same as those for PWM-1. It is obvious that $v_n = F(|\sigma(nT + \tau_n)|)\, sign\,(\sigma(nT))/T$. The modulation law implies that $\tau < F(|\sigma(nT + \tau)|)$ for $\tau \in [0, \tau_n)$. Hence $\sigma(nT + \tau) \ne 0$ when $\tau \in [0, \tau_n]$ and therefore $sign\,(\sigma(nT)) = sign\,\sigma(nT + \tau_n))$. So (4) holds when $\phi(\sigma) = F(|\sigma|)\, sign(\sigma)/T$ and $\tilde{t}_n = nT + \tau_n$.

6. Integral pulse-width modulation (IPWM). In this case $t_n = nT$,

$$f(t) = \begin{cases} 0\,, & nT < t \le nT + \tau_n\,, \\ sign\,(\mu_n(\tau_n))\,, & nT + \tau_n < t \le (n+1)T\,, \end{cases} \qquad \mu_n(\tau) = \int_0^\tau \sigma(nT+s)\, ds\,.$$

Here τ_n is the minimal positive root, belonging to the interval $(0, T]$, of the equation $|\mu_n(\tau_n)| = \Delta$ (with Δ a positive constant). If such a root does not exist, then $f(t) \equiv 0$ for $nT < t \leq (n+1)T$. It is evident that $v_n = (T - \tau_n)\, sign\,(\mu_n(\tau_n))/T$, if the root τ_n exists, and $v_n = 0$ otherwise. Apparently $\mu_n(\tau_n) = \sigma(nT + \zeta_n)\tau_n$ for some middle point ζ_n, $0 \leq \zeta_n \leq \tau_n$. Hence, if the root τ_n exists, then $\tau_n = \Delta/|\sigma(nT + \zeta_n)|$ and

$$v_n = \left[1 - \frac{\Delta}{T|\sigma(nT + \zeta_n)|}\right] sign\,(\sigma(nT + \zeta_n)) \ .$$

If the root τ_n does not exist, then a number η_n such that $nT < \eta_n < (n+1)T$ and $|\sigma(\eta_n)| < \Delta/T$ can be found. Therefore (4) follows by setting

$$\phi(\sigma) = \begin{cases} (1 - \Delta/(T|\sigma|))\, sign(\sigma)\,, & |\sigma| \geq \Delta/T\,, \\ 0\,, & |\sigma| < \Delta/T\,, \end{cases}$$

with either $\widetilde{t}_n = nT + \zeta_n$ or $\widetilde{t}_n = \eta_n$.

The equivalent nonlinearity can also be constructed for some more complicated types of modulation (combined, linear integral pulse-width, phase) [4].

3. The Main Results

Suppose that for any sufficiently large n, $n \geq 0$, a number $\widetilde{t}_n$ exists such that $t_n \leq \widetilde{t}_n \leq t_{n+1}$ and

$$\left(\sigma(\widetilde{t}_n) - \sigma_* v_n\right) v_n \geq 0\,, \tag{5}$$

where σ_* is a nonnegative constant. If an equivalent nonlinearity $\phi(\sigma)$ exists, then (5) is evidently guaranteed by the conditions: $\phi(0) = 0$ and

$$0 \leq \frac{\phi(\sigma)}{\sigma} \leq \frac{1}{\sigma_*} \tag{6}$$

for all $\sigma \neq 0$ (if $\sigma_* = 0$ then $1/\sigma_* = +\infty$). The restriction (6) means that the graphic of the function $y = \phi(\sigma)$ lies in the plane sector bounded by the straight lines $y = 0$ and $y = \sigma/\sigma_*$. The Popov stability criterion is known for systems with nonlinearities satisfying (6) [5]. We will formulate the stability criterion for pulse-modulated systems which is converted to that of Popov as T tends to zero. The first theorem is concerned with systems having stable linear part.

Theorem 1. *Let matrix A be stable, i.e. all its eigenvalues lie in the open left half-plane, and let numbers τ, ε, θ, (τ, ε positive) exist such that the following conditions hold.*

1. *Provided $\theta \neq 0$, an equivalent nonlinearity $\phi(\sigma)$ exists which satisfies (4), with $\widetilde{t}_n$ being the same as in (5), and the Lipschitz condition*

$$|\phi(\sigma_1) - \phi(\sigma_2)| \leq L|\sigma_1 - \sigma_2| \tag{7}$$

for all σ_1, σ_2. (Here L is a positive constant.)

2. *The algebraic inequality*

$$\sigma_* + \theta\kappa > \tau + \varepsilon_2 + \varepsilon_1\kappa^2 \tag{8}$$

takes place.

3. *For all real numbers ω, $0 \le \omega \le +\infty$, the frequency-domain inequality*

$$\sigma_* + ReW(i\omega) + \theta Re\big(i\omega W(i\omega)\big) - \tau - \varepsilon_2 - \varepsilon_1\omega^2|W(i\omega)|^2 -$$

$$-(T^2/(12\tau))|\chi(i\omega)|^2[4\varepsilon_1(\sigma_* - \tau - \varepsilon_2)\omega^2 + \theta^2\omega^2 + 1] > 0 \tag{9}$$

holds. (For $\omega = +\infty$ this inequality is considered as a limit relation.) Here the numbers ε_1, ε_2 are defined by formulas

$$\varepsilon_1 = \frac{T^2}{\pi^2\varepsilon}\big[1 + \frac{\pi}{2}|\theta|L|\kappa|\big]^2 + \frac{2|\theta|LT}{\pi}, \quad \varepsilon_2 = \varepsilon + T|\kappa|. \tag{10}$$

Then for all the solutions of (1) and (3) the limit relations $v_n \to 0$ as $n \to \infty$, $\sigma(t) \to 0$ as $t \to +\infty$ take place.

It is of interest to observe that if $T \to 0$, $\tau \to 0$, $\varepsilon \to 0$ so that $T^2/\varepsilon \to 0$, $T^2/\tau \to 0$, then (8) and (9) are reduced to the Popov frequency-domain condition: $\sigma_* + ReW(i\omega) + \theta Re\big(i\omega W(i\omega)\big) > 0$ as $0 \le \omega \le +\infty$.

We proceed by treating the case of a system with a neutral linear part.

Theorem 2. *Assume that matrix A has one zero eigenvalue while all then other eigenvalues have negative real parts. Let $\rho > 0$ and numbers $\tau, \varepsilon, \theta$ (τ, ε positive) exist such that conditions 1–3 of Theorem 1 hold. (For $\omega = 0$ inequality (9) is considered as a limit relation.) Then for all the solutions of (1) and (3), the limit relations $v_n \to 0$ as $n \to 0$, $\sigma(t) \to \sigma_\infty$ as $t \to +\infty$ take place. Here σ_∞ is a finite number depending on initial conditions, if an equivalent nonlinearity exists, then $\phi(\sigma_\infty) = 0$.*

The above formulated theorems can be strengthened for some important special cases.

Theorem 3. *If for all sufficiently large n, $n \ge 0$, either $\tilde{t}_n = t_n$ or $\tilde{t}_n = t_{n+1}$ holds, then Theorems 1 and 2 remain valid if we use the formulas*

$$\varepsilon_1 = \frac{T^2}{\pi^2\varepsilon} + \frac{2|\theta|LT}{\pi}, \quad \varepsilon_2 = \varepsilon$$

instead of formulas (10).

As shown previously, the condition $\tilde{t}_n = t_n$ takes place for PAM, PFM-1 and PWM-1.

We shall define for positive numbers n, such that $t_n < \widetilde{t}_n < t_{n+1}$, two values

$$M_n = \frac{1}{\widetilde{t}_n - t_n} \int_{t_n}^{\widetilde{t}_n} |f(t)|\, dt\,, \qquad N_n = \frac{1}{t_{n+1} - \widetilde{t}_n} \int_{\widetilde{t}_n}^{t_{n+1}} |f(t)|\, dt\,.$$

Theorem 4. *Let for all sufficiently large n, $n \geq 0$, satisfying $t_n < \widetilde{t}_n < t_{n+1}$, one of the following conditions holds:*

1. $\kappa > 0$ and $M_n \geq N_n$;

2. $\kappa < 0$ and $M_n \leq N_n$.

Then Theorems 1 and 2 remain true for $\varepsilon_2 = \varepsilon$.

Evidently M_n, N_n are mean values of a function $|f(t)|$ on intervals $(t_n, \widetilde{t}_n)$ and $(\widetilde{t}_n, t_{n+1})$, respectively. The inequality $M_n \geq N_n$ takes place when $|f(t)|$ is nonincreasing on (t_n, t_{n+1}) or $f(t) = 0$ on $(\widetilde{t}_n, t_{n+1})$. The inequality $M_n \leq N_n$ holds when $|f(t)|$ is nondecreasing on (t_n, t_{n+1}) or $f(t) = 0$ on $(t_n, \widetilde{t}_n)$.

Example. Consider the equation of the first order $dx/dt = -\alpha x - kf$, $\sigma = x$, where k, α are positive parameters. In this case $W(p) = k/(p + \alpha)$. We shall apply Theorem 1 with $\theta = 0$. Frequency inequality (9) can be reduced to two algebraic inequalities:

$$\tau(\sigma_* - \tau - \varepsilon_2 - \varepsilon_1 k^2) - \varepsilon T^2 k^2 \alpha^2 (\sigma_* - \tau - \varepsilon_2)/3 > 0\,,$$
$$\tau(\sigma_* - \tau - \varepsilon_2)\alpha^2 + \tau k \alpha - T^2 k^2 \alpha^2/12 > 0\,.$$

Selecting $\tau = \alpha k T^2/(\pi\sqrt{3})$, $\varepsilon = (\sigma_* - Tk)/2$, it is straightforward to check that the conditions of Theorem 1 hold if

$$\sigma_* > (1 + 2/\pi)kT + 2\alpha k T^2/(\pi\sqrt{3})\,.$$

If either Theorem 3 or Theorem 4 is applicable, then by choosing $\varepsilon = \sigma_*/2$, the inequality

$$\sigma_* > 2kT/\pi + 2\alpha k T^2/(\pi\sqrt{3})$$

is obtained.

4. Proofs of the Theorems

Let us consider a piecewise constant function $v(t)$: $v(t) = v_n$ as $t_n < t \leq t_{n+1}$. We shall change the variables in equation (3) with the help of Liénard-type transformation:

$$u = \int_{t_0}^{t} \big(f(s) - v(s)\big)\, ds\,, \qquad y = x - bu\,, \qquad \widetilde{\sigma} = \sigma + \kappa u\,.$$

Taking into account that $\kappa = -c^* b$, it leads to the equations

$$\frac{dy}{dt} = Ay + Abu + bv\,, \qquad \widetilde{\sigma} = c^* y \tag{11}$$

for $t \geq t_0$. Let us define piecewise continuous functions $\xi(t) = \widetilde{\sigma}(t) - \widetilde{\sigma}(\widetilde{t}_n)$, $\eta(t) = \widetilde{\sigma}(t) - \sigma(\widetilde{t}_n)$ when $t_n < t \leq t_{n+1}$. Evidently $\eta(t) = \xi(t) + \kappa u(\widetilde{t}_n)$ for $t_n < t \leq t_{n+1}$. First we prove three lemmas.

Lemma 1. *The inequality*

$$|u(t)| \le T_n |v_n| \tag{12}$$

holds for any t, $t_n \le t \le t_{n+1}$, and the following relation holds

$$\int_{t_n}^{t_{n+1}} u(t)^2\,dt \le \frac{1}{3} v_n^2 T_n^3 . \tag{13}$$

Proof of Lemma 1. Under the assumptions above the function $f(t)$ does not change its sign on (t_n, t_{n+1}). We shall consider the case when $f(t) \ge 0$ only; otherwise the proof is analogous. Let an interval (α, β), $t_n \le \alpha < \beta \le t_{n+1}$ exist such that $u(\alpha) = u(\beta) = 0$ and $u(t)$ does not change its sign on (α, β). Provided $f(t) \ge 0$, the following inequalities hold when $\alpha \le t \le \beta$:

$$u(t) = \int_\alpha^t \left(f(s) - v_n\right) ds \ge -v_n(t - \alpha) , \tag{14}$$

$$u(t) = -\int_t^\beta \left(f(s) - v_n\right) ds \le v_n(\beta - t) . \tag{15}$$

If $u(t) \ge 0$ on (α, β), then (15) implies $0 \le u(t) \le v_n(\beta - t)$, so

$$\int_\alpha^\beta u(t)^2\,dt \le v_n^2 \int_\alpha^\beta (\beta - t)^2\,dt = \frac{1}{3} v_n^2 (\beta - \alpha)^3 .$$

If $u(t) \le 0$ on (α, β), then we obtain from (14)

$$\int_\alpha^\beta u(t)^2\,dt \le v_n^2 \int_\alpha^\beta (t - \alpha)^2\,dt = \frac{1}{3} v_n^2 (\beta - \alpha)^3 .$$

Consider the set of points t belonging to (t_n, t_{n+1}) for which $u(t) \ne 0$. Since $u(t)$ is a continuous function, this set is open and can be partitioned in not more than countable number of intervals (α_k, β_k), $k = 1, 2, \ldots$, which do not intersect each other. Then

$$\int_{t_n}^{t_{n+1}} u(t)^2\,dt = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} u(t)^2\,dt .$$

Obviously $u(\alpha_k) = u(\beta_k) = 0$ and $u(t)$ does not change its sign on (α_k, β_k). Therefore, as shown above,

$$\int_{\alpha_k}^{\beta_k} u(t)^2 \le \frac{1}{3} v_n^2 (\beta_k - \alpha_k)^3 .$$

For an arbitrary integer N, $N \geq 1$, the estimate

$$\sum_{k=1}^{N}(\beta_k - \alpha_k)^3 \leq \left[\sum_{k=1}^{N}(\beta_k - \alpha_k)\right]^3 \leq T_n^3$$

holds, whence

$$\sum_{k=1}^{\infty}(\beta_k - \alpha_k)^3 \leq T_n^3 \ .$$

Thus, inequality (13) is proved. Inequality (12) follows immediately from the estimate $-v_n(t - t_n) \leq u(t) \leq v_n(t_{n+1} - t)$ as $t_n \leq t \leq t_{n+1}$.

Lemma 2. *The following inequality holds*

$$\int_{t_n}^{t_{n+1}} \xi(t)^2 \, dt \leq \frac{4T_n^2}{\pi^2} \int_{t_n}^{t_{n+1}} \left[\frac{d\widetilde{\sigma}(t)}{dt}\right]^2 dt \ .$$

This inequality represents a version of the known Wirtinger inequality. It may be proved by slightly modified arguments given in [6].

Consider two real quadratic forms

$$F_1(u, v) = \tau(v^2 - 3u^2/T^2) \ ,$$

$$F_2(y, u, v) = (c^* y - \sigma_* v)v + \theta v c^*(Ay + Abu + bv) +$$
$$+\varepsilon_1 \left(c^*(Ay + Abu + bv)\right)^2 + \varepsilon_2 v^2 \ .$$

Lemma 3. *Let n be a sufficiently large positive integer, and conditions of any theorem of those formulated above hold. Then for functions $y(t)$, $u(t)$, $v(t)$ satisfying (11) the following integral-quadratic relations are valid:*

$$\int_{t_n}^{t_{n+1}} F_1\left(u(t), v(t)\right) dt \geq 0 \ , \tag{16}$$

$$\int_{t_n}^{t_{n+1}} F_2\left(y(t), u(t), v(t)\right) dt \geq \theta \int_{\widetilde{\sigma}(t_n)}^{\widetilde{\sigma}(t_{n+1})} \phi(\sigma) \, d\sigma \ . \tag{17}$$

Proof of Lemma 3. Inequality (16) evidently follows from Lemma 1. Denote $\Delta = 2T/\pi$. In order to prove (17), it suffices to show that there exists a nonnegative number θ_0 such that

$$F_2(y, u, v) \geq \theta\phi(\widetilde{\sigma})\frac{d\widetilde{\sigma}}{dt} + \theta_0 \left[\Delta^2 \left[\frac{d\widetilde{\sigma}}{dt}\right]^2 - \xi^2\right] \tag{18}$$

for sufficiently large t. Then Lemma 2 can be applied.

Rewrite (18) in the form of

$$(\tilde{\sigma} - \sigma_* v)v + \theta\big(v - \phi(\tilde{\sigma})\big)\frac{d\tilde{\sigma}}{dt} + (\varepsilon_1 - \theta_0\Delta^2)\left[\frac{d\tilde{\sigma}}{dt}\right]^2 + \varepsilon_2 v^2 + \theta_0\xi^2 \geq 0 \,. \qquad (19)$$

Taking (5) and (7) into account, one sees that

$$(\tilde{\sigma} - \sigma_* v)v \geq \eta v \,, \qquad |v - \phi(\tilde{\sigma})| \leq L|\eta| \,.$$

So (19) is ensured by the inequality

$$\eta v - |\theta| L |\eta| \left|\frac{d\tilde{\sigma}}{dt}\right| + (\varepsilon_1 - \theta_0\Delta^2)\left[\frac{d\tilde{\sigma}}{dt}\right]^2 + \varepsilon_2 v^2 + \theta_0\xi^2 \geq 0 \,. \qquad (20)$$

Let the conditions of Theorem 1 be fulfilled. In view of (12), the estimate

$$|\eta| \leq |\xi| + T|\kappa||v| \qquad (21)$$

can be obtained. Hence for inequality (20) to hold, it suffices to ensure the inequality

$$(\varepsilon_1 - \theta_0\Delta^2)\left[\frac{d\tilde{\sigma}}{dt}\right]^2 + (\varepsilon_2 - T|\kappa|)v^2 + \theta_0\xi^2 - |\xi||v| -$$

$$-|\theta| L |\xi| \left|\frac{d\tilde{\sigma}}{dt}\right| - |\kappa||\theta| LT|v| \left|\frac{d\tilde{\sigma}}{dt}\right| \geq 0 \,.$$

Using $\varepsilon = \varepsilon_2 - T|\kappa|$ we derive

$$\varepsilon v^2 - \left[|\xi| + |\kappa||\theta| LT \left|\frac{d\tilde{\sigma}}{dt}\right|\right] |v| \geq -\frac{1}{4\varepsilon}\left[|\kappa||\theta| LT \left|\frac{d\tilde{\sigma}}{dt}\right| + |\xi|\right]^2 \,.$$

So (20) holds if the condition

$$\left[\varepsilon_1 - \frac{1}{4\varepsilon}\theta^2 L^2 T^2 \kappa^2 - \theta_0\Delta^2\right]\left[\frac{d\tilde{\sigma}}{dt}\right]^2 + \left[\theta_0 - \frac{1}{4\varepsilon}\right]\xi^2 -$$

$$-\left[|\theta| L + \frac{1}{2\varepsilon}|\theta| LT|\kappa|\right]\left|\frac{d\tilde{\sigma}}{dt}\right| |\xi| \geq 0$$

is fulfilled. Ensure the quadratic form, with arguments $|\xi|$, $|d\tilde{\sigma}/dt|$, written in the left-hand side of this inequality to be positive semidefinite. Then θ_0 should be selected so that

$$\theta_0 \geq 1/(4\varepsilon) \,, \qquad \varepsilon_1 - \big(1/(4\varepsilon)\big)\theta^2 L^2 T^2 \kappa^2 \geq \theta_0\Delta^2 \,, \qquad (22)$$

$$4\big(\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/(4\varepsilon) - \theta_0 \Delta^2\big)\big(\theta_0 - 1/(4\varepsilon)\big) \geq \theta^2 L^2 \big(1 + T|\kappa|/(2\varepsilon)\big)^2 . \quad (23)$$

Define θ_0 by the formula $\theta_0 = (4\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/\varepsilon + \Delta^2/\varepsilon)/(8\Delta^2)$. Substituting the right-hand side of this equality for θ_0 in (23) provides

$$(4\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/\varepsilon - \Delta^2/\varepsilon)^2 \geq 4\Delta^2 \theta^2 L^2 (2 + T|\kappa|/\varepsilon)^2 . \quad (24)$$

Inequalities (22) and (24) are satisfied if

$$4\varepsilon_1 \geq \theta^2 L^2 T^2 \kappa^2/\varepsilon + \Delta^2/\varepsilon + 2\Delta|\theta| L (2 + T|\kappa|/\varepsilon),$$

which is equivalent to $\varepsilon_1 \geq \Delta|\theta|L + (\Delta + |\theta|LT|\kappa|)^2/(4\varepsilon)$. This inequality is obviously valid when ε_1 is defined by (10). So relation (17) holds.

Under the conditions of Theorem 3 we obtain $u(\widetilde{t}_n) = 0$ and $\xi(t) = \eta(t)$. So we can use the relation $|\xi| = |\eta|$ instead of (21) and the preceding arguments can be repeated as if $\kappa = 0$.

Let the assumptions of Theorem take place. It is easily seen that when $t_n \leq t \leq t_{n+1}$ the relation

$$T_n u(t) = (t_{n+1} - t) \int_{t_n}^{t} f(s)\,ds - (t - t_n) \int_{t}^{t_{n+1}} f(s)\,ds \quad (25)$$

holds. Under the assumptions imposed on $f(t)$ we have $v_n f(t) \geq 0$ when $t_n < t < t_{n+1}$. Put $t = \widetilde{t}_n$. Relation (25) and the conditions of Theorem 4 yield $\kappa u(\widetilde{t}_n) v_n \geq 0$ when n is sufficiently large. Then $v\eta \geq v\xi$ and the above proof is valid for $\varepsilon_2 = \varepsilon$. The proof of Lemma 3 is completed.

Proof of Theorem 1. Consider the real quadratic form $F(y,u,v) = F_1(u,v) + F_2(y,u,v)$, where F_1, F_2 are introduced earlier. Denote by $\widetilde{F}(\widetilde{y},\widetilde{u},\widetilde{v})$ the hermitian form (defined for complex $\widetilde{y},\widetilde{u},\widetilde{v}$) obtained from $F(y,u,v)$ by extending it to a hermitian form. For any real ω we define the 2×2 matrix function $\Pi(i\omega)$ by the formula

$$\widetilde{F}\big(-(A - i\omega I_m)^{-1}(b\widetilde{v} + Ab\widetilde{u}), \widetilde{u}, \widetilde{v}\big) = -\begin{bmatrix}\widetilde{u}\\ \widetilde{v}\end{bmatrix}^* \Pi(i\omega) \begin{bmatrix}\widetilde{u}\\ \widetilde{v}\end{bmatrix} .$$

(Here * denotes hermitian conjugation.) By direct computations we obtain that

$$\Pi(i\omega) = \begin{bmatrix}\Pi_{11}(i\omega) & \Pi_{12}(i\omega)\\ \Pi_{12}(i\omega) & \Pi_{22}(i\omega)\end{bmatrix},$$

where

$$\Pi_{11}(i\omega) = 3\tau/T^2 - \varepsilon_1\omega^2|\chi(i\omega)|^2 ,$$
$$\Pi_{22}(i\omega) = \sigma_* - \tau - \varepsilon_2 + Re W(i\omega) + \theta Re\big(i\omega W(i\omega)\big) - \varepsilon_1\omega^2|W(i\omega)|^2 ,$$
$$\Pi_{12}(i\omega) = \chi(-i\omega)\big(1/2 - \theta i\omega/2 - \varepsilon_1\omega^2 W(i\omega)\big) .$$

Condition 2 of Theorem 1 means that $\Pi_{22}(\infty) > 0$. It is straightforward to show that condition 3 ensures the inequality $\det \Pi(i\omega) > 0, \quad 0 \le \omega \le +\infty$. By the Sylvester criterion and the continuity of $\Pi(i\omega)$, the hermitian matrix $\Pi(i\omega)$ is found to be positive definite for $0 \le \omega \le +\infty$. Then, by the Yakubovich-Kalman lemma in the nonsingular case (see Theorem 1.2.7 [7]), there exist a real symmetric matrix H and a positive number δ_0 such that

$$2y^*H(Ay + Abu + bv) + F(y, u, v) \le -\delta_0 v^2 \tag{26}$$

for all real m-dimensional vectors y and real numbers u, v. Define a Lyapunov function

$$V(y) = y^*Hy + \theta \int_0^{c^*y} \phi(\sigma)\, d\sigma \,.$$

It follows from (26) and Lemma 3 that for any solution of (11) and for sufficiently large n, $n \ge 0$, the inequality

$$V\big(y(t_{n+1})\big) - V\big(y(t_n)\big) \le -\delta_0 T_n v_n^2 \tag{27}$$

holds. Under the above assumptions a sequence v_n is bounded and, consequently, so are functions $v(t)$, $u(t)$ for $t \ge t_0$. Therefore the stability of the matrix A ensured that all the solutions of (11) are bounded for $t \ge t_0$. Hence the function $V\big(y(t)\big)$ is bounded as well, and (27) implies that

$$\sum_{n=1}^{\infty} T_n v_n^2 < +\infty \,.$$

Since, from (2), $T_n \ge \kappa_0 T > 0$, we conclude that $v_n \to 0$ as $n \to \infty$. Then inequality (12) yields $u(t) \to 0$ as $t \to +\infty$. As the matrix A is stable, we obtain $y(t) \to 0$, $\sigma(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem 1.

The next lemma seems to be generally known. It will be useful in proving Theorem 2.

Lemma 4. *Let $\nu(t)$ be an absolute continuous function, $g(t)$ be a piecewise continuous one, and*

$$\big(\nu(t) - g(t)\big)\frac{d\nu}{dt} \le 0 \tag{28}$$

for all $t \ge \tau_0$. (At discontinuity points, (28) holds for both one-side limits.) Then the following statements are valid:

1. *If $g(t)$ is bounded for $t \ge \tau_0$, then so is $\nu(t)$.*
2. *If $g(t) \to 0$ as $t \to +\infty$, then $\nu(t)$ has a finite limit as $t \to +\infty$.*

Proof of Lemma 4. Let B be a number such that $g(t) \leq B$ for $t \geq \tau_0$ and $\nu(\tau_0) \leq B$. Show that $\nu(t) \leq B$ for $t \geq \tau_0$. Suppose not; then there exist numbers τ_1, τ_2 for which $\nu(\tau_1) = B$, $\nu(t) > B$ as $\tau_1 < t < \tau_2$. Then $\nu(t) - g(t) > 0$ for $\tau_1 < t < \tau_2$ and, consequently, (28) implies $d\nu/dt \leq 0$. So $\nu(t) \leq \nu(\tau_1) = B$ as $\tau_1 \leq t \leq \tau_2$. The contradiction proves B to be an upper bound of $\nu(t)$. The lower bound of $\nu(t)$ may be obtained similarly.

Further, let $g(t) \to 0$ as $t \to +\infty$. If $d\nu/dt$ does not change its sign for sufficiently large t then $\nu(t)$ becomes monotone, and hence it has a finite limit as $t \to +\infty$. Let $d\nu/dt$ take both positive and negative values when t is large. Then, in view of (28), the function $\nu(t) - g(t)$ cannot preserve its sign as t increases. Let for some positive numbers ε and τ_1, $\tau_1 > \tau_0$, we have $g(t) < \varepsilon$ as $t \geq \tau_1$. Then for some sufficiently large τ_2, $\tau_2 \geq \tau_1$, the inequality $\nu(\tau_2) - g(\tau_2) \leq 0$ holds. Hence $\nu(\tau_2) < \varepsilon$ and, as shown in the first part of the proof, $\nu(t) \leq \varepsilon$ for $t \geq \tau_2$. Similarly it can be shown that for all sufficiently large t the inequality $\nu(t) \geq -\varepsilon$ holds. Since ε may be chosen arbitrarily small, we obtain $\nu(t) \to 0$ as $t \to +\infty$.

Proof of Theorem 2. The transfer function $W(p)$ admits a representation

$$W(p) = W_1(p) + \rho/p\,, \tag{29}$$

with the function $W_1(p)$ having all the poles in the open left half- plane. Then $ReW(i\omega) = ReW_1(i\omega)$ and it becomes obvious that the matrix function $\Pi(i\omega)$ defined in the proof of Theorem 1 depends continuously on ω when $0 \leq \omega \leq +\infty$. So, as shown in the proof of Theorem 1, $\Pi(i\omega)$ is positive definite for $0 \leq \omega \leq +\infty$. However, the Yakubovich-Kalman lemma in the nonsingular case is not applicable, since $W(p)$ has a zero pole. We use the singular case of this lemma (see Theorem 1.2.6 [7]). Let us consider the 2×2 matrix $\widehat{\Pi}(i\omega)$ for which $\widehat{\Pi}_{11} = \Pi_{11}$, $\widehat{\Pi}_{12} = \Pi_{12}$, $\widehat{\Pi}_{22} = \Pi_{22} - \delta_0$ with δ_0 being a positive number. It is easily seen that $\widehat{\Pi}(i\omega)$ is positive definite for $0 \leq \omega \leq +\infty$ when δ_0 is sufficiently small. So there exist a matrix H satisfying (26), and therefore (27) takes place. Further, by changing variables, the system (11) can be reduced to the system

$$\frac{dy_1}{dt} = A_1 y_1 + A_1 b_1 u + b_1 v\,, \qquad \frac{d\nu}{dt} = v\,, \qquad \sigma = c_1^* y_1 - \rho\nu - \kappa u\,. \tag{30}$$

Here y_1 is a vector function of dimension $m-1$, A is a stable $m \times m$ matrix, b_1, c_1 are m-dimensional vectors, $\kappa = \rho - c_1^* b_1$, $c_1^*(A_1 - pI_{m-1})^{-1} b_1 = W_1(p)$ where $W_1(p)$ satisfies (29). Show that all the solutions of (30) are bounded when $t \geq t_0$. Since a sequence v_n is bounded, we obtain that so are functions $u(t)$, $v(t)$ for $t \geq t_0$. The matrix A_1 being stable, we conclude that $y_1(t)$ is also bounded. Under assumption (5), we have $\sigma(\tilde{t}_n)v_n \geq 0$ for sufficiently large n. Since $\nu(\tilde{t}_n) = \nu(t) + (\tilde{t}_n - t)v_n$ for $t_n \leq t \leq t_{n+1}$ and $\rho > 0$, inequality (28) holds with

$$g(t) = \big(c_1^* y_1(\tilde{t}_n) - \kappa u(\tilde{t}_n)\big)/\rho + v_n(\tilde{t}_n - t)$$

as $t_n < t < t_{n+1}$. By the first statement of Lemma 4, the function $\nu(t)$ is bounded for $t \geq t_0$. Thus, the solutions of (30) are bounded, and so are the solutions of (11). Similarly, as in the proof of Theorem 1, it can be shown that $v(t) \to 0$, $u(t) \to 0$ as $t \to +\infty$. Then $y_1(t) \to 0$ as $t \to +\infty$. Applying the statement of Lemma 4, we conclude that $\nu(t)$ has a finite limit as $t \to +\infty$ and, consequently, so does $\sigma(t)$.

The assertions of Theorem 3 and Theorem 4 follow immediately from Lemma 3 and the previous proofs.

References

1. A.Kh. Gelig, The averaging method in the stability theory of sampled-data systems. (In Russian) Autom. Telemekh 5 (1883), 55–64. (English translation in Autom. Remote Control (1983)).
2. A.Kh. Gelig, Frequency criterion for nonlinear pulse systems stability. Systems Control Lett.1, 6 (1982), 409–412.
3. A.N. Churilov, Frequency-domain criterion for stability of nonlinear sampled-data systems with even pulsation law. (In Russian) Autom. Telemekh. 4 (1992) 93–101.
4. A.Kh. Gelig, A.N. Churilov, Oscillations and Stability of Nonlinear Sampled-Data Systems. (In Russian) To appear.
5. V.M. Popov, A problem in the theory of absolute stability of control systems. (In Russian) Autom. Telemekh. 25, 9 (1964), 1257- 1262. (English translation in Autom. Remote Control 25, 9 (1964) 1129- 1134).
6. G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities. Cambridge University Press, London, 1951.
7. A.Kh. Gelig, G.A. Leonov, V.A. Yakubovich, Stability of Nonlinear Systems with a Nonunique Equilibrium State. (In Russian) Nauka, Moscow, 1978.

THE PROCESS OF NORMALIZATION AND SOLUTION OF BIFURCATION PROBLEMS OF THE OSCILLATION AND STABILITY THEORY: A SYNOPSIS

Yakov M. Goltser

The Research Institute
The College of Judea and Samaria
Ariel 44824, Israel

Abstract

A synopsis is presented describing a new method of normalization for some classes of ordinary differential equations including autonomous and non autonomous systems continuously depending on a vector parameter. The article especially concentrates on the application of a new normalization algorithm to the study of the birth of steady resonance modes, of stability changes during systems passing through the resonance, as well as the resonance danger.

1. Introduction

Consider a system of ordinary differential equations in $\mathbf{R}^n$

$$\frac{dx}{dt} = A(\mu)x + F(t,\mu,x), \qquad F(t,\mu,0) = \frac{\partial F}{dx}(t,\mu,0) = 0 \tag{1.1}$$

where the matrix $A(\mu)$ and the vector-function F are continuously dependent on a vector parameter $\mu \in \mathbf{D} \subset \mathbf{R}$, where $\mathbf{D}$ is a closed domain in $\mathbf{R}$ containing zero.

Let $\mathbf{C}$ be a set of functions from t, μ, continuous and bounded in $\mathbf{R} \times \mathbf{D}$. Assume that the nonlinearity F could be represented by a Taylor series with respect to x and with $\mathbf{C}$-coefficients.

Let $\Lambda(\mu) = \big((\lambda_1(\mu), \dots \lambda_n(\mu)\big)$ be a spectrum of the matrix $A(\mu)$. For each fixed value of $\mu \in \mathbf{D}$ this spectrum could be divided into two parts: a critical part $\{\lambda_s(\mu) | Re\lambda_s(\mu) = 0\}$, and a non-critical one. In this paper we concentrate on the study of systems (1.1) with a non-empty critical part. Different stability cases could be realized in the system (1.1) depending on the amount of spectrum points from the critical part of spectrum, i.e., situated on the imaginary complex plane axis, and on the structure of the elementary divisors of the matrix A. Investigations of the different aspects of the stability theory for critical cases could be found, for instance, in [1-7]. Recently special attention was paid to the study of stability of solutions for systems with internal resonance [3, 8-11].

Our paper deals with a study of bifurcation phenomena when the parameter μ is moving within the domain $\mathbf{D}$. Let us assume that for $\mu = 0$ the critical part of the spectrum for the system (1.1) is non-empty. When the parameter μ is moving from 0 into the domain $\mathbf{D}^* = \mathbf{D} \backslash \{0\}$, two main situations are possible:

a) the amount of the eigenvalues from the critical part of the spectrum and the structure of the elementary divisors of $A(\mu)$ are the same;

b) the change is taking place in the critical part of spectrum and in the structure of the elementary divisors.

In both situations different bifurcation phenomena in the neighborhood of the stationary point are possible. In situation a) the Hopf bifurcation was studied more thoroughly for autonomous and periodical systems (see [12], [13]).

The main results of our study are connected with the more general bifurcation phenomena:

1) the birth of steady modes;
2) the stability changes on passing through the inner resonance.

To solve such problems the system (1.1) can be presented in a quasi-linear form

$$\frac{dx}{dt} = A(\mu)x + F_0(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j F^{(j+1)}(t,\mu,x) \tag{1.2}$$

where ϵ is a small parameter, $F^{(j+1)}$ is a vector- form in x of the degree j with coefficients from $\mathbf{C}$.

System (1.2) could be immersed into a family of quasi-linear parametrically perturbed systems

$$\frac{dx}{dt} = A(\mu)x + F_0(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j F_j(t,\mu) \tag{1.3}$$

where $F_0(t,\mu) \in \mathbf{C}$, $F_j(t,\mu,x)$ is a vector-polynomial in x of the degree n_j with coefficients from $\mathbf{C}$. (Under the vector-polynomial degree we mean a maximal degree of its components).

System (1.3) emerges in different problems of the oscillation theory [4, 14, 15, 16] where autonomous and periodical systems were most deeply studied. Even for F_j with $n_j = 1$, a parametric resonance problem is an interesting one [17].

One of the most effective methods of studying bifurcation phenomena proves to be that of normal forms which were introduced by A. Poincaré (see [3], [18]). This method was developed in detail for the autonomous and periodical systems (1.1), (1.2) independent of the parameter μ. However, the notion of resonance terms used in the Poincaré process of normalization is not applicable for the non-autonomous systems with coefficients from $\mathbf{C}$. For example, the resonance normalization cannot be applied when the spectrum consists only of zero eigenvalues. Let us underline further that the resonance normalization of the autonomous and periodical systems (1.1), (1.2) do not in general possess the continuity in μ.

In Section 2 a new normalization process suitable for obtaining the normal form of the system (1.3) continuous in $\mathbf{D}$ is described. It is possible to obtain such a normal form of the resonance type continuous in μ for the autonomous and almost periodical systems with zero spectrum (see Section 3).

In Sections 4 and 5 we produce some results of solving problems 1 and 2 in which a developed normalization method was used. Systems of the type (1.3) will further be referred to as the **C**-systems and the **P**-system in all cases when the coefficients of (1.3) are almost periodical in t functions uniformly in μ.

2. The Normal Form of C-Systems

In this Section a complete description of a new normalization process is given, based on the generalization of the Poincaré homological equations.

2.1 General scheme of normalization. Consider a **C**-system (1.3) and the following **C**-transformation:

$$x = y + \Phi_0(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j \Phi_j(t,\mu,y) \tag{2.1}$$

where F_j and Φ_j are the vector-polynomials of n_j and m_j degree $(m \geq n_j)$. The **C**-transformation (2.1) transforms the system (1.3) into the system

$$\frac{dy}{dt} = A(\mu)y + G_0(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j G_j(t,\mu,y) \tag{2.2}$$

where F_j are vector-polynomials of m_j degree.

Let us introduce operators $\mathcal{L}_A$, $\mathcal{L}_{AG}$ such that:

$$\mathcal{L}_A\Phi = A(\mu)\Phi - \frac{\partial\Phi}{dy}A(\mu)y - \frac{\partial\Phi}{\partial t}$$

$$\mathcal{L}_{AG}\Phi = \mathcal{L}_A\Phi - \frac{\partial\Phi}{\partial y}G_0(t,\mu)$$

where $\frac{\partial\Phi}{\partial y}$ is the Jacoby matrix. Both operators convert each vector-polynomial into a vector-polynomial of the same degree.

Let us assume that (2.1) transforms (1.3) into (2.2). This means that for every $\mu \in \mathbf{D}$ the following equations should be fulfilled:

$$\mathcal{L}_A\Phi_0 = -F_0 + G_0\ , \quad \mathcal{L}_{AG}\Phi_j = \Psi_j + G_j\ , \quad j = 1,2,\ldots \tag{2.3}$$

where $\Psi_1 = -F_1(t,\mu,y+\Phi_0)$,

$$\Psi_j = -\hat{F}_j(t,\mu,y) + \sum_{s=1}^{j-1} \frac{\partial\Phi_s}{\partial y} G_{j-s}(t,\mu,y)$$

In the formation of $\hat{F}_j$, only the functions F_ℓ and Φ_q for $\ell \leq j$, $q \leq j-1$ participate.

Definition 2.1. The **C**-system (1.3) is formally equivalent in **D** to **C**-system (2.2), if there exists a **C**-transformation (2.1), such as that for each j the vector-polynomials Φ_j satisfy the identities (2.3). □

Note. In the construction of a normalization process all series are considered formal.

Definition 2.2. Equations

$$\mathcal{L}_A \Phi = H(t, \mu, y)\,, \quad \mathcal{L}_{AG} \Phi = H(t, \mu, y) \tag{2.4}$$

with a given vector-polynomial $H \in \mathbf{C}$ are called $\mathbf{C}$-solvable if they allow the existence of the solution in a form of vector-polynomial form with coefficients from $\mathbf{C}$. □

Equations (2.4) are analogous to the homological equations that appear in the Poincaré resonance normalization (see [3]). We will call them the generalized homological system of equations. The proposed normalization procedure is based on the consequent use of these equations, starting from a given concrete system (1.3), determining Φ_0 and G_0 for $j = 0$, and calculating consequently the vector-polynomials Φ_j and G_j for $j \geq 1$. On each step the $\mathbf{C}$-solvability on $\mathbf{D}$ is ensured. Because of the non-uniqueness of solutions for the generalized homological system (2.3), for the determination of the normal form of the system (1.3) one should choose among all formally equivalent systems the system with a minimal number of non-zero terms in each polynomial G_j.

Thus, the construction of a normal form continuous in $\mathbf{D}$ is reduced to obtaining $\mathbf{C}$-solvability conditions of the generalized homological equations. They are dependent essentially on the spectrum properties of the $A(\mu)$ matrix and should meet the demand of choice of the transformation class with the above-mentioned minimality conditions.

The first step of the normalization process consists of reducing $A(\mu)$ to some normal form. For a fixed μ it is usually the Jordan normal form. For a parameter-dependent matrix, the Jordan form does not always have the continuity property in μ [3]. Thus the normalization process is essentially dependent on the possible structure of the matrix $A(\mu)$ normal form continuous in $\mathbf{D}$.

2.2 A case of the Jordan form continuous in μ. Let us assume that matrix $A(\mu)$ in $\mathbf{D}$ has a continuous Jordan form

$$A(\mu) = \left\| \begin{matrix} \lambda_1(\mu) & c_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2(\mu) & c_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_n(\mu) \end{matrix} \right\|, \qquad c_j = 0 \text{ or } 1\,.$$

To obtain C-solvability conditions for generalized homological equations (2.4), let us consider the equations for the vector-polynomial Φ coefficients. A set of coefficients could be ordered in such a way that the corresponding equation system will have the following form:

$$\frac{\partial \widehat{\Phi}}{dt} = \widehat{A}(\mu)\widehat{\Phi} + \widehat{H}(t, \mu) \tag{2.5}$$

where $\widehat{\Phi}$ is a vector of vector-polynomial Φ coefficients, $\widehat{A}(\mu)$ is a triangular matrix and $\widehat{H}$ is a known vector. The triangularity of the matrix $\widehat{A}(\mu)$ allows reducing the problem to obtaining the C-solvability conditions for a scalar equation:

$$\frac{d\phi}{dt} = \alpha(\mu)\phi + h(t,\mu) , \qquad h(t,\mu) \in \mathbf{C} . \tag{2.6}$$

Definition 2.3. Equation (2.6) is called a regular equation in $\mathbf{D}$ if it has one and only one solution in $\mathbf{C}$ for any heterogeneity $h(t,\mu) \in \mathbf{C}$. □

It is quite obvious that the regularity criterion in $\mathbf{D}$ is that for each $\mu \in \mathbf{D}$

$$Re\, a(\mu) \neq 0 . \tag{2.7}$$

Let us describe some sufficient conditions of the $\mathbf{C}$-solvability for non-regular equations. Assuming that

$$a(\mu) = \eta(\mu) + ik(\mu) , \qquad \mathbf{D} = \mathbf{D}_0 \cup \mathbf{D}_+ \cup \mathbf{D}_-$$

$$\mathbf{D}_0 = \{\mu \in \mathbf{D} \mid \eta(\mu) = 0\} , \qquad \mathbf{D}_{+(-)} = \{\mu \in \mathbf{D} \mid \eta(\mu) > 0 \; (< 0)\}$$

let us introduce the following functions

$$f_\pm(t,\mu) = \int_0^t \exp\big(\pm a(\mu)\tau\big)h(\tau,\mu)d\tau , \qquad \mu \in \mathbf{D}_{+(-)}$$

$$f(t,\mu) = \int_0^t \exp\big(-ik(\mu)\tau\big)h(\tau,\mu)d\tau , \qquad \mu \in \mathbf{D}$$

and a limit

$$M_\pm = \lim_{T\to\infty} \frac{1}{2T} \int_0^T f(t,\mu)\, dt .$$

Lemma 2.1. *Let the function $h(t,\mu) \in \mathbf{C}$ in (2.6) be such that*

i) *$f(t,\mu) \in \mathbf{C}$,*

ii) *there is an average uniformly in μ and ℓ*

$$M(\mu) = \lim_{T\to\infty} \frac{1}{2T} \int_{\ell-T}^{\ell+T} f(t,\mu)\, dt .$$

Then (2.6) has the only one solution in $\mathbf{C}$-class and this solution is defined by the equality

$$\phi(t,\mu) = \exp\big(a(\mu)t\big)\Big(E(\mu) + \int_0^t \exp\big(-a(\mu)\tau\big)h(\mu,\tau)d\tau\Big) \tag{2.8}$$

where

$$E(\mu) = \mp M_{\pm}(\mu)\,, \quad \text{for} \quad \mu \in \mathbf{D}_{\pm}\,, \qquad E(\mu) = M(\mu)\,, \quad \text{for} \quad \mu \in \mathbf{D}_0\,. \ \square$$

Consider a vector-polynomial F_j, where $f_{ps}^{(j)}(t,\mu)x^p$ is some term of its s-th component; $g_{ps}^{(j)}(t,\mu)y^p$, $\phi_{ps}^{(j)}(t,\mu)y^p$ have the analogous meaning for the vector-polynomials G_j, Φ_j. Here x^p, y^p are of a multidegree, $p = (p_1, \ldots, p_n)$. In the process of normalization for each term of the equation (1.3) there is a corresponding equation (2.6):

$$\frac{d\phi_{ps}^{(j)}}{dt} + \langle p - \delta_s, \Lambda(\mu)\rangle\, \phi_{ps}^{(j)} = \widehat{h}_{ps}^{(j)}(t,\mu) + f_{ps}^{(j)}(t,\mu) - g_{ps}^{(j)}(t,\mu)\,, \tag{2.9}$$

where $\widehat{h}_{ps}^{(j)}(t,\mu) \in \mathbf{C}$ is a known function.

Definition 2.4. The vector $p \in \mathbf{Z}_n^+$, the correspondent to it terms of the s-th equation systems (1.3), (2.2), as well as the transformations (2.1) are called regular (non-regular) in $\mathbf{D}$ if the corresponding equation (2.9) is regular (non-regular) in $\mathbf{D}$. □

To normalize in $\mathbf{D}$ we should find for each j, p, s a consequent solution (2.9) in $\mathbf{C}$-class according to the following alternative:

1. If p is a regular vector, then we assume in (2.9) $g_{ps}^{(j)}(t,\mu) = 0$ and find $\phi_{ps}^{(j)}(t,\mu)$ as its only $\mathbf{C}$-solution,
2. If p is a non-regular vector, then we choose $g_{ps}^{(j)}(t,\mu) \in \mathbf{C}$ to satisfy the conditions of Lemma 2.1 then determine according to (2.8). (In general,the choice of $g_{ps}^{(j)}(t,\mu)$ is not unique).

Let us formulate a normal form theorem, defining the set of p non-regular vectors of j-th order in ϵ by $N_{js}(0 \le |p| \le m_j)$ from the s-th equations.

Theorem 2.1. *Consider the equation (1.3) with the $A(\mu)$ matrix which Jordan form is continuously dependent on $\mu \in \mathbf{D}$. Then there exists a formal $\mathbf{C}$-transformation (2.1) which transforms (1.3) into its normal form*

$$\frac{dy_s}{dt} = \lambda_s(\mu)y_s + c_s y_{s+1} + g_0^{(s)}(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j \sum_{p\in N_{sj}} g_{ps}^{(j)}(t,\mu)y^p \tag{2.10}$$

containing only the non-regular in D terms. □

The structure of the normal form is closely dependent on the domain $\mathbf{D}$. Let us consider two domains $\mathbf{D}$ and $\mathbf{D}^*$ such that $\mathbf{D} \cap \mathbf{D}^* \neq 0$ and to which correspond different non-regular vector-sets. Then the system (1.3) will have different normal forms in $\mathbf{D}$ and $\mathbf{D}^*$ and in the domain $\mathbf{D} \cap \mathbf{D}^*$. This appears to be very important for the study of bifurcation problems (see Section 5).

3. The Normalization of Almost Periodical and Autonomous Systems

3.1 P-systems with a Jordan matrix. Consider the systems (1.3) assuming that all its coefficients belong to the class $\mathbf{P}$-almost periodic functions t, uniformly in $\mu \in \mathbf{D}$. It is known that the spectrum of such functions does not depend on μ. The normalization problem will then be considered in the class of $\mathbf{P}$-transformations (2.1). In this case in the equations (2.6) $h(t,\mu) \in \mathbf{P}$ and Lemma 2.1 provide conditions for the existence of $\mathbf{P}$-solutions.

In the procedure of finding the normal form coefficients from (2.9) the alternatives for their solutions change. This is connected with the appearance of new possibilities of choosing $g_{ps}^{(j)}(t,\mu)$ in the non-regular equations (2.9).

All non-regular equations (2.9) in an almost periodical case could be divided into resonance and non-resonance ones. Here are the necessary definitions.

Let us consider the spectrum of the heterogeneous and non-linear part of the system (1.3) in the j-th approximation in ϵ. It is formed as a union of spectrum S_j of corresponding coefficients. Let $\overline{S}_j$ be their closures and $M(\overline{S}_j)$ be a minimum module whose elements γ could be represented as follows: $\gamma = \sum_r k_r \nu_r, \quad \nu_r \in \overline{S}_j$, k_j are integers. Let, further, $M_q(\overline{S}_j)$ be such γ-sets that $\sum_r |k_r| \leq q$.

Definition 3.1. The vector p and the corresponding terms of the s- th equations of systems (1.3), (2.2) as well as a transformation 2.1 are called the resonance ones in $\mathbf{D}$ in the j-th approximation in ϵ, only if for some $\mu_0 \in \mathbf{D}$ the following condition holds:

$$< p - \delta_s \, , \Lambda(\mu_0) > \in M_{m_j}(\overline{S}_j) \tag{3.1}$$

where δ_s is a s-th unit vector in $\mathbf{R}^n, <,>$ is a scalar product and m_j is a degree of the vector-polynomial Φ_j . □

Denote by P_{sj} the set of resonance vectors of an s-th equation in the j-th approximation in ϵ. The alternative for the equation solution is as follows:

1) If p is a non-resonance vector, then assume that $g_{ps}^{(j)}(t,\mu) = 0$ in (2.9) and determine $\phi_{ps}^{(j)}(t,\mu)$ from the formulas (2.8).
2) If p is a resonance vector then let us choose $g_{ps}^{(j)}(t,\mu) \in \mathbf{P}$ in (2.9) so that the conditions of Lemma 2.1 are met. Then $\phi_{ps}^{(j)}(t,\mu)$ are determined in accordance with (2.8).

The above-mentioned choice is performed according to the following scheme: Let $\overline{S}_{ps}^{(j)}$ be the closure of the spectrum of the function $\hat{h}_{ps}^{(j)}(t,\mu) + f_{ps}^{(j)}(t,\mu)$ in (2.9). Divide it into resonance $R_{ps}^{(j)}$ and non-resonant parts. The choice should be made in such a way that the resonance part of spectrum function $\hat{h}_{ps}^{(j)}(t,\mu) + f_{ps}^{(j)}(t,\mu) - g_{ps}^{(j)}(t,\mu)$ will remain empty.

Theorem 3.1. *If the matrix $A(\mu)$ in the $\mathbf{P}$-system (1.3) has a continuous Jordan form in $\mathbf{D}$, then there is such formal $\mathbf{P}$-transformation which transforms it into* *a normal form continuous in* **D**:

$$\frac{dy_s}{dt} = \lambda_s(\mu)y_s + c_s y_{s+1} + g_0^{(s)}(t,\mu) + \sum_{j=1}^{\infty} \epsilon^j \sum_{p\in p_{sj}} g_{ps}^{(j)}(t,\mu)y^p \qquad (3.3)$$

which contains the resonance members only. □

Consider a situation when the **P**-system (1.3) in **D** has only one internal resonance connecting with purely imaginary eigenvalues of matrix $A(\mu)$. Let us also assume that for all p, j, s the resonance part of the heterogeneity spectrum has no limit points in (2.9). In such a situation the choice can be accomplished in such a way that makes it possible to reduce the normal form (3.3) to an autonomous form. For systems (1.1) not dependent on μ the similar reduction is established for periodic systems and for the systems with coefficients that could be represented as finite trigonometric sums [19].

3.2 Autonomous systems. The normal form systems with a nilpotent matrix. Let system (1.3) be an autonomous one. Then **C** is the set of functions continuous in **D**. The problem of normal form construction is reduced to study of **C**-solvability in **D** of the generalized homological equations

$$\mathcal{L}_{\mathbf{A}}\Phi = H(t,\mu), \quad \text{where} \quad \mathcal{L}_{\mathbf{A}}\Phi \equiv A(\mu)\Phi - \frac{\partial\Phi}{\partial y}A(\mu) \qquad (3.4)$$

If the matrix $A(\mu)$ is a non-singular one in **D** and has a Jordan form continuous in μ in **D**, it is possible to construct a resonance type normal form continuous in **D**, similarly in the case of **P**-systems.

Consider the case when $A(0)$ is a singular matrix and has two zero eigenvalues with a non-simple elementary divisor. Therefore, without the loss of generality, the situation can be considered only for the two-dimensional system.

Consider the following normal form of the $A(\mu)$ matrix:

$$A(\mu) = \begin{Vmatrix} \lambda_1(\mu) & 1 \\ \lambda_2(\mu) & 0 \end{Vmatrix}, \qquad \lambda_1(0) = \lambda_2(0) = 0\,, \quad \lambda_1^2(\mu) + \lambda_2^2(\mu) \neq 0\,, \quad \mu \neq 0 \qquad (3.5)$$

The specifics of this case lie in the fact that the resonance normalization does not simplify this system within the **C**-transformation class. The study of **C**-solvability of equation (3.4) is complicated since the system of equations for the determination of the normal form coefficients is an algebraic system of linear heterogeneous equations with a singular matrix. For the effective normal form construction we need an elaboration of a suitable algorithm for the solution of these systems. In the case under investigation such an algorithm has been obtained. Here is the structure of the corresponding algebraic systems and the theorem of the normal form.

Consider the equation

$$\mathcal{L}_{\mathbf{A}}\Phi^{(j)} = H^{(j)}(\mu, y) \tag{3.6}$$

where $H^{(j)}$ is a vector-form of the j-th order with the coefficients from $\mathbf{C}$. To determine a form $\Phi_s^{(j)}$, $s = 1, 2$, from (3.6) one obtains the following algebraic system:

$$A_{2j+2}(\mu)\Phi_{2j+2} = \Psi_{2j+2}(\mu) + G_{2j+2}(\mu) \tag{3.7}$$

where $A_{2j+2}(\mu)$ is a $(2j+2)$-matrix, Φ_{2j+1}, G_{2j+2} are the $(2j+2)$-vectors depending on the coefficients of forms $\Phi_s^{(j)}$, $G_s^{(j)}$, $s = 1, 2$, Ψ_{2j+2} is a known $\mathbf{C}$-vector. Matrix $A_{2j+2}(\mu)$, corresponding to $A(\mu)$ has the following block structure:

$$A_{2j+2}(\mu) = \begin{Vmatrix} L_{j+1} & E_{j+1} \\ K_{j+1} & P_{j+1} \end{Vmatrix}$$

where $K_{j+1} = \lambda_2(\mu)E_{j+1}$, $P_{j+1} = L_{j+1} - \lambda_2(\mu)E_{j+1}$ and all the blocks are square $(j+1)$-matrices, E_{j+1} is an identity matrix, and L_{j+1} is a three-diagonal matrix

$$L_{j+1} = \begin{Vmatrix} -(j-1)\lambda_1 & -\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ -j & -(j-1)\lambda_1 & -2\lambda_2 & \cdots & 0 & 0 & 0 \\ 0 & -(j-1) & -(j-3)\lambda_1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -2 & 0 & -j\lambda_2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda_1 \end{Vmatrix}$$

Note that the matrix $A_{2j+2}(0)$ is a singular one for all j.

The structure of the matrix $A_{2j+2}(\mu)$ made it possible to obtain an algorithm for a construction of the $\mathbf{C}$-solution for system (3.7) and prove the following:

Theorem 3.2. *There exists a formal* $\mathbf{C}$- *transformation*

$$x = y + \sum_{j=2}^{\infty} \Phi^{(j)}(\mu, y)$$

which reduced the two-dimensional autonomous system (1.1) continuously in $\mathbf{D}$ *with a matrix (3.5) to the one form the normal form types:*

i) $\dfrac{dy_1}{dt} = \lambda_1(\mu)y_1 + y_2\,,\qquad \dfrac{dy_2}{dt} = \lambda_2(\mu)y_1 + \tau(\mu, y_1) + y_2\alpha(\mu, y_1)$

ii) $\dfrac{dy_1}{dt} = \lambda_1(\mu)y_1 + y_2 + \gamma(\mu, y_1)\,,\qquad \dfrac{dy_2}{dt} = \lambda_2(\mu)y_1 + \tau(\mu, y_1) + y_2\gamma(\mu, y_1)$

iii) $\dfrac{dy_1}{dt} = \lambda_1(\mu)y_1 + y_2 + \beta(\mu, y_1)\,,\qquad \dfrac{dy_2}{dt} = \lambda_2(\mu)y_1 + \tau(\mu, y_1)$

where $y_2\alpha$, $y_2\gamma$, β, τ are formal series with coefficients from **C** *which do not contain any free or linear terms.* □

Let us note that for $\mu = 0$ the existence of normal form types i) and ii) is found in [20, 21] and of the type iii) in [22]. The normal forms obtained in **D**, together with those obtained in $\mathbf{D}\backslash\mathbf{U}_\epsilon(0)$ where $\mathbf{U}_\epsilon(0)$ is an open neighborhood of zero , could be used for studying bifurcation phenomena, in particular, the cases of the birth of limit cycles.

4. Birth of Steady Resonance Modes

Consider a **P**-system (1.3) such as that $F_j(t,\mu) = F_j(t,\mu,0) = 0$, and suppose that the normal form can be reduced to an autonomous form. Let us further suppose that:

i) A matrix $A(\mu)$ in **D** has l pairs of different eigenvalues

$$\sigma_s(\mu) \pm i\rho_s(\mu)\,, \qquad s = 1,2,\ldots,l$$

where $\sigma_s(0) = 0$, $\rho_s(\mu) \neq 0$ for each $\mu \in \mathbf{D}$ and the other eigenvalues have negative real parts for each $\mu \in \mathbf{D}$.

ii) For $\mu = 0$ the critical eigenvalues are connected by a single internal resonance $\langle k, \rho(0)\rangle \in M_{m_1}(S_1)$ of the order $Q = |k| \leq m_1 + 1$.

Under these assumptions the "polar coordinates" normal form can be presented as

$$\frac{dr_s}{dt} = \sigma_s(\mu)r_s + \sum_{j=1}^{\infty} \epsilon^j R_{sj}(\mu, r, \psi) \tag{4.1}$$

$$\frac{d\psi}{dt} = \delta(\mu) + \sum_{j=1}^{\infty} \epsilon^j \Psi_j(\mu, r, \psi)$$

$$\frac{d\phi_s}{dt} = \Delta_s(\mu) + \sum_{j=1}^{\infty} \epsilon^j \Phi_{sj}(\mu, r, \psi)\,, \qquad s = 1,2,\ldots,l$$

where $\psi =< k, \Delta(\mu) >$ is a resonance phase, r_s are the radial variables, R_{sj}, Ψ_j, Φ_{sj} are polynomials in $r = (r_1, \ldots, r_\ell)$, whose coefficients are trigonometrical functions ψ, $\delta(\mu) =< k, \Delta(\mu) >$ is a detuning of the resonance, $\Delta_s(\mu) = \rho_s(\mu) - \rho_s(0)$, $\Delta(\mu) = \big(\Delta_1(\mu), \ldots, \Delta_\ell(\mu)\big)$. The equations for non-critical variables are not given.

The following statement gives us the conditions of birth in (1.3) of a steady mode in a situation when the main role belongs to a vector-polynomial $F_1(t,\mu,x)$. A steady mode could be presented as

$$r_s(\mu,\beta,\epsilon) = r_s^*(\mu,\beta) + \sum_{j=1}^{\infty} \epsilon^j r_s^{(j)}(\mu,\beta) \tag{4.2}$$

$$\psi(\mu,\beta,\epsilon) = \psi^*(\mu,\beta) + \sum_{j=1}^{\infty} \epsilon^j \psi^{(j)}(\mu,\beta)$$

where $\beta = [\|\sigma(\mu)\|\epsilon^{-1}]^{\frac{1}{Q-2}}$

Functions $r_s^* > 0$, ψ^* satisfy the equations

$$\frac{\sigma_s(\mu)}{\epsilon} r_s + R_{s1}(\mu,r,\psi) = 0\,, \qquad \frac{\delta(\mu)}{\epsilon} + \Psi_1(\mu,r,\psi) = 0 \qquad (4.3)$$

and r_s^* have the forms: $r_s^* = d_s(\mu)\beta + o(\beta)$. Functions $r_s(\mu,\beta)$, $\psi^{(j)}(\mu,\beta)$ are consecutively defined in the following form

$$r_s^{(j)} = \beta^{q_j} d_s^{(j)}(\mu) + o(\beta^{q_j})$$

$$\psi^{(j)} = \beta^{q_j} \psi_*^{(j)}(\mu) + o(\beta^{q_j})\,, \qquad q_j\text{-natural numbers}\,,$$

from the equations

$$\frac{\sigma_s(\mu)}{\epsilon} r_s^{(j)} + \frac{\partial R_{s1}^*}{\partial r} r^{(j)} + \frac{\partial R_{s1}^*}{\partial \psi} \psi^{(j)} = F_s^{(j)}$$

$$\frac{\partial \Psi_1^*}{\partial r} r^{(j)} + \frac{\partial \Psi_1^*}{\partial \psi} \psi^{(j)} = F_{l+1}^{(j)}$$

where $F_s^{(j)}$ $(s = 1,2,\ldots,l+1)$ are the functions of the same structure as R_{sj}, Ψ_j which depend on $r_s^{(i)}$, $\psi^{(i)}$ for $i < j$; $*$ - means that all derivatives are calculated for $r = r^*$, $\psi = \psi^*$. Denote by Q_j the minimum function $F_s^{(j)}$ degree in β.

Theorem 4.2. *Consider a system (1.3) satisfying all the above requirements, and suppose that its normal form (4.1) is such that*

a) *The following limits exist*

$$\lim_{\mu\to 0} \frac{\sigma_s(\mu)}{\|\sigma(\mu)\|} = \rho_s \neq 0\,, \qquad s = 1,\ldots,l$$

$$\lim_{\mu\to\infty} \frac{\delta(\mu)}{\|\sigma(\mu)\|} = \rho_{l+1}\,;$$

b) *The lower polynomials R_{s1}, Ψ_l terms have the order of $Q-1$, $Q-2$ correspondingly; denote them $R_{s1}^{(Q-1)}$, $\Psi_1^{(Q-2)}$;*

c) *The system of equations*

$$\rho_s d_s + R_{s1}^{(Q-1)}(0,d,\psi) = 0\,, \quad s = 1,\ldots,l$$
$$\rho_{l+1} + \Psi_1^{(Q-2)}(0,d,\psi) = 0$$

has a solution $d_s^0 > 0$, $\psi = \psi^0$;

d) *The Jacobi matrix of the system of functions $R_{s1}^{(Q-1)}$, $\Psi_1^{(Q-2)}$ is invertible in zero:*

$$\det J(0, d^0, \psi^0) \neq 0;$$

e) *$Q_j > Q - 2$ for each j;*

f) *The eigenvalues of the matrix $J(0^0, d, \psi)$ are different and have negative real parts.*

Then for a sufficiently small μ and all $\epsilon < \epsilon_0 = 0(\|\sigma\|^{1/2})$ the system (1.3) has a stable steady resonance mode. This mode has a formal representation (4.2) in which $d_s(0) = d_s^0$, $\psi^*(0) = \psi^0$. □

5. Stability Change During Passing Through the Internal Resonance

5.1 Two types of normal forms. Consider the **P**-system (1.1) such that for $\mu = 0$ the spectrum of the matrix $A(\mu)$ is purely imaginary and the system has a single internal resonance. It is also assumed that the normal form can be reduced to an autonomous form. Consider the situation when during the change of parameter μ in **D**, the critical part of the spectrum remains on the imaginary axis (situation b) in Section 1). Let $\pm i\rho_s(\mu)$ be critical eigenvalues connected with the following resonance

$$\nu =< k, \rho(0) >\in M(S) \ .$$

Consider the set of the parameter resonance values $\mathbf{\Gamma} : \langle k, \rho(\mu)\rangle = \nu$ and define $\mathbf{D}^* = \mathbf{D} \backslash \mathbf{\Gamma}$.

Our aim is to describe the stability property behaviour caused by the change of parameter μ in **D**. We will describe this behavior with the help of an example of the four-dimensional system (1.1) ($n = 4$) with the resonance of the third order

$$\nu = \rho_1(0) + 2\rho_2(0) \tag{5.1}$$

Let us assume that the original system is normalized on the sets **D** and $\mathbf{D}^*$ up to the third order terms inclusive. After the transfer to the autonomous system, one receives the following normal form in **D** in complex variables:

$$\overline{z}_s \frac{dz_s}{dt} = i\Delta_s(\mu)\omega_s + \alpha_s(\mu)\overline{z}_1\overline{z}_2 + \omega_s(\alpha_{s1}\omega_1 + \alpha_{s2}\omega_2) + O(\|z\|^5) \tag{5.2}$$

and the following normal form in $\mathbf{D}^*$:

$$\overline{z}_s^* \frac{dz_s}{dt} = i\Delta_s(\mu)\omega_s^* + \omega_s^*(\alpha_{s1}^*\omega_1^* + \alpha_{s2}^*\omega_2) + O(\|z^*\ \|) \tag{5.3}$$

where $\omega_s = z_s\overline{z}_s$, $\omega_s = z_s^*\overline{z}_s^*$, $s = 1, 2$.

Introduce the notations: $\delta(\mu) = \Delta_1(\mu) + 2\Delta_2(\mu)$ is a detuning of the resonance, $\delta(\mu) \to 0$ for $\mu \to \Gamma$, $A(\mu) = Im(\alpha_1(\mu)\overline{\alpha}_2(\mu))$, $B(\mu) = Re\alpha_1(\mu)Re\alpha_2(\mu)$ (or$B(\mu) = Im\alpha_1(\mu)Im\alpha_2(\mu)$ if $Re\alpha_1(\mu)Re\alpha_2(\mu) = 0$).

There is the following connection between the coefficients of both normal forms:

$$\alpha_{11}^*(\mu) = \alpha_{11}(\mu), \qquad \alpha_{12}^*(\mu) = \alpha_{12}(\mu) + 2i\alpha_1(\mu)\overline{\alpha}_2(\mu)\delta^{-1} \tag{5.4}$$
$$\alpha_{21}^*(\mu) = \alpha_{21}(\mu) + i\alpha_2(\mu)\overline{\alpha}_2(\mu)\delta^{-1}, \qquad \alpha_{22}^*(\mu) = \alpha_{22} + i\overline{\alpha}_1(\mu)\alpha_2(\mu)\delta^{-1}$$

One sees from (5.4) that the coefficients of the normal form (5.3) are unbounded if $\mu \to \Gamma$ and (5.3) could not be used on Γ. At the same time, both normal forms there exist in $\mathbf{D}^*$.

5.2 Strict resonance stability ($\mu \in \Gamma$). To study stability on the resonance set Γ the normal form (5.2) can be used. We begin with the study of a model system obtained from (5.2) by retaining its first nonlinear terms. In this study, use is made of the resonance real integrals of the linear approximation

$$V = C\exp(-i\delta t)z_1 z_2^2 + \overline{C}\exp(i\delta t)\overline{z}_1\overline{z}_2^2, \qquad C = \text{const}\,,$$

which are dependent on time on Γ and are the particular solutions of the "instable ray" type. Very important here is the proof of the "roughness" of the unstable model system.

Theorem 5.1. *If in (5.2):* **a)** $A(\mu) \neq 0$ *or* **b)** $A(\mu) = 0, \quad B(\mu) > 0$ *then the zero solution of the system is unstable.*

If **c)** $A(\mu) = 0, \quad B(\mu) < 0$ *then the zero solution is stable in the second approximation, the system (5.2) permits the positive definite integral*

$$V = |\alpha_2(\mu)|\,\omega_1 + |\alpha_1(\mu)|\,\omega_2\,.$$

If the domain **D** *is sufficiently small then conditions* **a)** *or* **b)** *are fulfilled for* $\mu = 0$, *and the zero solution of system (5.2) is unstable for all* $\mu \in \Gamma$. □

5.3 Near-resonance stability. In the following analysis, use is made of the normal form (5.3) and the equalities (5.4). Define

$$a_{sj}(\mu) = Re\alpha_{sj}(\mu), \qquad h(\mu) = a_{11}(\mu) + 2a_{21}(\mu)$$

We restrict ourselves to the description of the results in the following basic case:

$$A(0)h(0)a_{11}(0) \neq 0 \tag{5.5}$$

Theorem 5.2. *If the system (5.3) satisfies the conditions (5.5) then there exists a neighbourhood $U(0)$ of the point $\mu = 0$ such that the existence of the asymptotic stability of zero solution for $\mu \in \mathbf{D}^* \cap U(0)$ is equivalent to the conditions (necessary and sufficient conditions):*

a) $a_{11}(0) < 0$,   *b)* $A(0)\delta^{-1}(\mu) > 0$,   *c)* $h(0) < 0$. □

For the analysis of the system (5.3) with a fixed $\mu \in \mathbf{D}^*$ the Molchanov theorem [7] could be used. However, in the case of $n = 2$ the following theorem is correct:

Theorem 5.3. *a) The necessary and sufficient conditions for the asymptotic stability of a zero solution of the system (5.3) with a fixed $\mu \in \mathbf{D}^*$ independent of terms of order ≥ 4 are: $a^*_{11}(\mu) < 0, \quad a^*_{22}(\mu) < 0$ and, moreover, one of the following conditions is fulfilled:*

1) $K_1K_2 < 0$;

2) $K_1K_2 \geq 0$ *and if* $K_1 < 0, \quad K_2 < 0$ *that* $K_1K_2 < 1$.

(Here $K_1 = a^*_{21}a^{*-1}_{11}(\mu), \quad K_2 = a^*_{12}(\mu)a^{*-1}_{22}(\mu)$*).*

b) In the above conditions there is a Lyapunov function

$$V = -\gamma_1(\mu)\omega_1^* - \gamma_2(\mu)\omega_2^*$$

with the positive definite derivative. □

5.4 Stability change. In the basic case (5.5) the above formulated theorems allow the description of the stability property behaviour in a sufficiently small neighbourhood of point $\mu = 0$. This description is presented in the following Table 1.

Table 1. Stability changes for the basic case (5.5)

sgn $a_{11}(0)$	sgn $h(0)$	$\mathbf{D}^*_-$	Γ	$\mathbf{D}^*_+$
$-$	$-$	I	I	AS
$-$	$+$	I	I	I
$+$	$\pm$	I	I	I

Here $\mathbf{D}^*_{+(-)} = \{\mu | \mu \in \mathbf{D}^* \cap U(0), \quad A(0)\delta(\mu) > 0 (< 0)\}, I$ - denotes the instability, AS denotes the asymptotical stability.

As can be seen from the two last lines of Table 1, there are no instability changes here. However, the analysis shows that the instability type might change. The third line of Table 1 singles out the case when full instability in $\mathbf{D}^*$ changes for the instability in a zero neighbourhood zone.

5.5 Resonance stabilization. The basic case (5.5) is characterized by the fact that on the resonance set the system is instable. However, there might be cases when for $\delta(\mu) \to 0$ a system, instable near the resonance, becomes stable (asymptotically) with a strict resonance. This phenomena appears under certain limitations of the normal form coefficients. Let us give one of the possible conditions for the appearance of such resonance stabilization.

Theorem 5.4. *Let the conditions*

a) $A(0) = 0, \quad B(0) < 0, \quad a_{11}(0)h(0) \neq 0$;

b) $\lim_{\mu \to 0} A(\mu)\delta^{-1}(\mu) = \infty$ *be fulfilled for the system (5.2), and*

c) *Quadratic form*

$$|\alpha_2|a_{11}\omega_1^2 + (|\alpha_2|a_{12} + |\alpha_1|a_{12})\omega_1\omega_2 + |\alpha_1|a_{22}\omega_2^2$$

with $\mu = 0$ negatively definite in the cone $\{\omega_1 \geq 0, \quad \omega_2 \geq 0\}$.

Then, if $\mathbf{D}$ is a domain small enough, the instability in $\mu \in \mathbf{D}^*$ changes to the asymptotic stability on the resonance set Γ. □

5.6 The resonance danger. The results described above are correct only under the conditions where domain $\mathbf{D}$ is sufficiently small (or, more precisely, when the detuning of the resonance $\delta(\mu)$ is sufficiently small). Let us assume that the system (5.2) is strongly unstable in $\mathbf{D}$, i.e. the zero solution of the system is unstable for each $\mu \in \mathbf{D}$.

If the system be asymptotically stable "far from the resonance", then the problem arises to determine the critical value δ^* of the resonance detuning at which the change in stability (as the system approaches the resonance) is taking place.

For the derivation of the conditions of asymptotic stability "far from the resonance" one may act in the following way. Let us exclude the internal resonance terms from the normal form (5.2) and demand that the remaining system be stable asymptotically (the same as presuming that in (5.3) and (5.4) $\delta(\mu) = \infty$). The necessary and sufficient conditions for the asymptotic stability of the system are either

α : 1) $a_{11} < 0$, 2) $a_{22} < 0$, 3) $a_{12} < 0$ or $a_{21} < 0$;

or

β : 1) $a_{11} < 0$, 2) $a_{22} < 0$, 3) $a_{12} > 0$, $a_{21} > 0$, $\Delta < 0$;

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$.

The fulfillment of one of these conditions (α or β) implies the asymptotic stability for the sufficiently large values $\delta(\mu)$. The decrease of $\delta(\mu)$ implies a change of asymptotical stability for instability at some time. The values of δ^* corresponding to this time are presented in Table 2.

Table 2. The conditions of transfer from asymptotical stability to instability

<table>
<tr><td rowspan="2">α</td><td>$A\delta < 0$</td><td>$|\delta^*| = |Aa_{22}^{-1}|$</td></tr>
<tr><td>$A\delta > 0$, $a_{21} > 0$, $h > 0$</td><td>$|\delta^*| = \min\{2|A^{-1}|a_{21}, hA\Delta^{-1}\}$</td></tr>
<tr><td rowspan="2">β</td><td>a) $A\delta < 0$, $h \geq 0$
b) $A\delta < 0$, $h < 0$, $2|a_{22}| \geq a_{21}$</td><td>$|\delta^*| = |Aa_{21}^{-1}|$</td></tr>
<tr><td>c) $A\delta < 0$, $h < 0$, $2|a_{22}| < a_{21}$
d) $A\delta > 0$, $h > 0$</td><td>$|\delta^*| = |hA\Delta^{-1}|$</td></tr>
</table>

Table 2 does not include the cases when $\delta^* = 0$. The system works as follows: the asymptotical stability for $A\delta > 0$ transfers to the instability for $\delta = 0$; this instability is preserved for $A\delta < 0$ until the condition $|\delta| < |\delta^*|$ holds, where δ^* (for $A\delta < 0$)) is determined according to the corresponding line in Table 2.

The value of δ^* could be considered as a quantitative characteristic of the resonance "danger". Moreover, in the system (5.2) there is an additional constructive parameter whose variations greatly influence the value of δ^*.

To reveal this parameter, let us write down the coefficients $\alpha_1(\mu)$, $\alpha_2(\mu)$ with the internal resonance terms in the system (5.2) in the following way:

$$\alpha_s(\mu) = i\,|\alpha_s(\mu)| \exp i\theta_s(\mu), \qquad s = 1, 2$$

Then, in all cases we will have $\delta^*(\mu) = C(\mu)\sin(\theta_1 - \theta_2)$, where $C(\mu)$ is a known function.

A shift $\Delta\theta = \theta_1 - \theta_2$ in the resonance phases regulates the changes in the value of $\delta^*(\mu)$. The most "dangerous" resonance corresponds to the resonance phase shift $\Delta\theta = \frac{\pi}{2}$. For $\Delta\theta = 0$ the stability (instability) of the system near the resonance does not depend on the presence of the resonance in the system. In this case the resonance is the weakest and its existence is guaranteed on the resonance set $\mathbf{\Gamma}$ only.

Conclusion

The main subject of the paper is the non-autonomous (in particular almost periodic) system of ordinary differential equations, dependent on two types of parameters - the vector parameter $\mu \in \mathbf{D}$, describing the equation family continuous in $\mathbf{D}$, and a small parameter ϵ, defining the quasi-linear structure of the system. With the fixed degrees of ϵ the non-linearity is represented as polynomial of phase variables with the coefficients continuous in t and μ.

For the indicated class of equations the normalization procedure is developed, generalizing the Poincaré resonance normalization process and allowing the normal forms continuous in $\mathbf{D}$ to be obtained. For the class of non-autonomous system these normal forms contain the non-regular members only. The proposed procedure allows normal forms of the resonance type for the autonomous and almost periodical systems to be built, as well as normalizing the systems with nilpotent matrices of linear approximation that could not be simplified with the help of the Poincaré resonance normalization.

The normalization procedure is illustrated by two main examples: 1) The almost periodical systems, when a linear approximation matrix has a non-singular Jordan form continuous in $\mathbf{D}$, and 2) The autonomous systems with nilpotent (for $\mu = 0$) matrix of the 2nd order.

The normal forms method has been further applied to the study of the following bifurcation problems. Consider the case when the spectrum of the linear approximation matrix $A(\mu)$ for $\mu = 0$ contains n pairs of purely imaginary eigenvalues. Then the following situations arise:

a) under the change of μ the spectrum leaves the imaginary axis, acquiring small real parts;

b) with the changing of μ the spectrum stays on the imaginary axis.

In both situations, it is supposed that the A(0) matrix spectrum resonates with the non-linearity spectrum and that the normal form could be reduced to an autonomous type.

In situation a) the conditions for the appearance of the stable stationary mode and its asymptotic representation under resonance of the arbitrary Q order have been obtained. It is only natural that for the resonance to be substantial,the conditions of the degeneration (identical in μ) of the normal form coefficients up to the $Q-2nd$ order hold. It could not be excluded that the replacement of the identical degeneration condition by the degeneration with $\mu = 0$ only will give the conditions for the appearance of several established modes.

In situation b) a four-dimensional system was taken as an example to describe the results of stability and changes instability investigation during systems passing through a 3rd order resonance. In this investigation substantial use was made of two normal forms: one of them is continuous in $\mathbf{D}$, while the other in $\mathbf{D}\backslash\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a set of parameters' resonance values.

An important point of this investigation is the establishing of interconnections between these two normal forms. This allows us to find the stability conditions near the resonance ($\mu \in \mathbf{D}\backslash\mathbf{\Gamma}$) under the strict resonance ($\mu \in \mathbf{\Gamma}$) and, as a result, to describe the behavior of the stability properties at the set $\mathbf{D}$ according to the normal form properties for $\mu = 0$.

In the process of investigation, we reveal the resonance stabilization effect: a system which is unstable near the resonance becomes asymptotically stable under a strict resonance. This is an unexpected result since, as a rule, the system under the strict resonance is non-stable.

It is also important to mention an attempt to introduce the quantitative characteristics of the resonance "danger" $\delta^*(\mu)$ corresponding to the changing stability time in a situation when the system is asymptotically stable far from the resonance and unstable under the resonance. It has been found that by changing the resonance phases shift only, it is possible to regulate the value $\delta^*(\mu)$ making it as small as desired.

Acknowledgement. This research was supported by The Ministry of Science and Technology, State of Israel.

The paper contains the material of the report delivered at the Seminar on Differential-Functional Equations, held at the College of Judea and Samaria in Ariel, Israel.

References

1. A.M. Liapounoff, Problems général de la stability du mouvement. Princeton Univ. Press, Annals of Mathematics 17 (1949), 203-474.
2. I.G. Malkin, Theory of stability of motion. Nauka, Moscow, 1966. English translation by the Atomic Energy Association, AEC-TR-3352.
3. V.I. Arnold, Geometrical methods in the theory of ordinary differential equations. Berlin, Springer, 1983.
4. G.V. Kamenkov, Selected papers, Nauka, Moscow, 1 (1971)
5. V.M. Starjinskiy, The applied methods for nonlinear oscillation. Nauka, Moscow, 1977.

6. L. Salvadori, Sulla ricera di una Funzione di Liapounoff per un sistema differenziale interessante *la* mecanica dei sistema olonomi. Ric. Mat. 11:2 (1962), 82-94.
7. A.M. Molchanov, Stability for the case of neutral linear approximation. Doklady ANSSR 141:1 (1961), 24-27.
8. Ya.M. Goltser, Stability of neutral fixed points of autonomous mappings with internal resonances. Diff. uravnenija 15:12 (1979), 2130 -2141.
9. Ya.M. Goltser, A.L. Kunitsyn, On the stability of autonomous systems with internal resonances. Prikl. Matem. Mekhan, 39:6 (1975), 974-984.
10. A.L. Kunitsyn, A.P. Markeev, Stability in resonant cases. Moscow, Reviews in Sc. and Techn., General Mech. 4 (1979), 58-139.
11. E.E. Shnol, L.G. Khazin, Non existence of algebraic criteria for an asymptotic stability with resonances. USSR, (1977) Preprint No.45, Inst. of Appl. Problems of Math., Acad. of Sci.,
12. J.E. Marsden, M. McCracken, The Hopf Bifurcation and its Applications. Springer-Verlag, New York, 1976.
13. B.D. Hassard, N.D. Kazarinoff and Y.H. Wan, Theory and Applications of Hopf Bifurcation. Cambridge Univ. Press, 1981.
14. I.G. Malkin, Some Problems of the Theory of Nonlinear Oscillations, Nauka, Moscow, 1956.
15. J.K. Hale, Oscillations in Nonlinear Systems. McGraw- Hill, 1963.
16. V.M. Volosov, B.I. Morgunov, The Averaging Method in the Theory of Nonlinear Oscillating Systems, Moscow Univ., 1971.
17. V.A. Yakubovitsh, V.M. Starjinskiy, Linear Differential Equations with Periodical Coefficients and its Applications. Nauka, Moscow, 1972.
18. A.D. Brjuno, The Local Method of Nonlinear Analysis. Nauka, Moscow, 1979.
19. V.G. Veretennikov, The Stability and Oscillations of the Non-linear systems. Nauka, Moscow, 1984.
20. F. Takens, Singularities of vector fields. Publ. Math. THES 43 (1974), 47-100.
21. A.P. Sadovskiy, Normal forms of the system of differential equations with non-zero linear parts. Diff. uravneuija, 8:5 (1982), 827-832.
22. G.R. Belitskiy, Normal Forms, Invariants and Local Mappings. Kiev, 1979.
23. G.R. Belitskiy, The equivalence and the normal forms of smooth mapping germs. Uspechi Mat. Nauk 33:1, (199) (1978), 95-155.
24. Ya.M. Goltser, Bifurcations and stability of neutral systems in the neighbourhood of third order resonance, Prikl. Matem. and Mekhan. 43:3 (1979), 420-440.
25. Ya.M. Goltser, On the measure of closeness of a neutral system to the internal resonance. Prikl. Matem. and Mekhan 50: 6 (1986), 945-953.

THE GENERAL REPRESENTATIONS OF SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS

A.A. Khvoles and A.R. Khvoles

Dept. of Mathematics and Computer Science
Bar-Ilan University
RamaGan 52900, Israel

The theory of general representations of the solution of elliptic partial differential equations has been developed by S. Bergmann [1] and I. Vekua [2]. This theory has many applications in mathematical physics, the theory of shells, elasticity theory, etc. As usual, the problem of construction of general representations is not trivial. These representations are useful in solving boundary value problems.

In this paper, general representations for some differential equations are obtained. We consider differential equations which arise in complex analysis, elasticity theory and the theory of shells.

1. The first problem which we consider here is to find the solution of the following equation:

$$(\Delta - \lambda^2)\frac{1}{\varphi(z)}(\Delta - \lambda^2)u = f(z)\ . \tag{1.1}$$

Here $\varphi(z)$ and $f(z)$ are analytic functions in the complex plane, Δ is a Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $z = x + iy$, λ-real number.

A boundary value problem for these types of operators arises in integral geometry. Namely, a free boundary problem for a Laplace operator is related to a well-known Pompeiu problem (see survey by L. Zalcman [3]). The Pompeiu problem with two holomorphic weights leads to a boundary value problem, associated with the differential equation (1.1) [M. Agranovsky, personal communication].

The general representation for the solution of equation (1.1) is given below. This representation depends on four arbitrary holomorphic functions. In order to obtain the general representation, let us first consider the homogeneous equation:

$$(\Delta - \lambda^2)\frac{1}{\varphi(z)}(\Delta - \lambda^2)u = 0\ . \tag{1.2}$$

Introduce the complex conjugated variable $\zeta = x - iy$. Then equation (1.2) can be written in the form:

$$\left(\frac{\partial}{\partial z\partial\zeta} - \lambda^2\right)\frac{1}{\varphi(z)}\left(\frac{\partial^2 u}{\partial z\partial\zeta} - \lambda^2 u\right) = 0\ . \tag{1.3}$$

Let

$$\frac{1}{\varphi(z)}(\Delta - \lambda^2)u = V \ . \tag{1.4}$$

So problem (1.3) can be reduced to the system of A. Bizadze type [4]

$$\begin{cases} \dfrac{\partial^2 u}{\partial z \partial \zeta} - \lambda^2 u - \varphi(z)V = 0 \\ \dfrac{\partial^2 V}{\partial z \partial \zeta} - \lambda^2 V = 0 \ . \end{cases} \tag{1.5}$$

The general representation of the solution of the equation

$$\frac{\partial^2 V}{\partial z \partial \zeta} - \lambda^2 V = 0 \tag{1.6}$$

is as follows:

$$V(x,y) = \alpha_0 \mathcal{I}_0(\lambda r) + \int_0^z \phi(t)\mathcal{I}_0(i\lambda\sqrt{\zeta(z-t)})dt$$
$$+ \int_0^\zeta \phi^*(t)\mathcal{I}_0(i\lambda\sqrt{z(\zeta - t)}dt \ , \tag{1.7}$$

where $\phi(t), \phi^*(t)$ are arbitrary analytic functions, $\mathcal{I}_0(t)$ is thwe Bessel function of the first order of the imaginary argument.

One can show that the solution of equation (1.5) can be represented in the following form:

$$u(z,\zeta) = \frac{1}{\lambda^2} V_z'(z,\zeta) \int_0^z \varphi(z)\, dz \ , \tag{18}$$

where $V(z,\zeta)$ satisfies equation (1.6). So the general representation of the solution of equation (1.3) is:

$$u(z,\zeta) = u_1(z,\zeta) + \frac{1}{\lambda^2} V_z'(z,\zeta) \int_0^z \varphi(z)\, dz \ , \tag{1.9}$$

where $u_1(z,\zeta)$ is the general solution of equation (1.6) and solution (1.9) contains four arbitrary analytic functions.

The nonhomogeneous equation (1.1) has the following partial solution:

$$u_2(z,\zeta) = \frac{1}{\lambda^4} f(z)\varphi(z) \ . \tag{1.10}$$

So we have obtained the general solution of equation (1.1) which is a sum of functions (1.9) and (1.10).

2. On some elasticity theory equations with singularities.

Let us consider a prismatic shell which has a middle surface in the plane and which is bounded by straight lines $y = 0$ and $y = a$.

Values of stress on the boundary are given. Assume that the dependence of shell thickness on the y-coordinate is given by formula: $h = h_0 y^\alpha$ (h_0, α are constants).

I. Vekua's equations system has the form [2]:

$$y^2 \Delta\Delta\varphi - 2\alpha y \frac{\partial}{\partial y}\Delta\varphi - \nu\alpha(\alpha+1)\frac{\partial^2\varphi}{\partial x^2} + \alpha(\alpha+1)\frac{\partial^2\varphi}{\partial y^2} = 0 \tag{2.1}$$

$$y\Delta\Psi + \alpha\frac{\partial\Psi}{\partial y} = 0\ ,\ \nu = \frac{\lambda}{\lambda+\mu}\ , \tag{2.2}$$

when volume power is ignored, φ is the tension function and Ψ is the normal displacement.

We are looking for a solution which satisfies the following boundary conditions (I. Vekua [2]):

$$\varphi(x,0) = f_1(x)\ ,\quad \varphi_y(x,0) = f_z(x)\ ,\quad \lim_{y\to 0} y^\alpha\frac{\partial\Psi}{\partial y} = f_3(x) \tag{2.3}$$

$$\varphi(x,a) = f_1^*(x)\ ,\quad \varphi_y(x,a) = f_z^*(x)\ ,\quad \lim_{y\to a} y^\alpha\frac{\partial\Psi}{\partial y} = f_3^*(x)\ . \tag{2.4}$$

The crucial point in proving this representation is the fact that the left side in (2.1) can be decomposed in the following way ($\beta = \sqrt{(\alpha+1)(\nu\alpha-1)}$):

$$\left(y\Delta + \beta\frac{\partial}{\partial x} - (\alpha+1)\frac{\partial}{\partial y}\right)\left(y\Delta\varphi - \beta\frac{\partial\varphi}{\partial x} - (\alpha+1)\frac{\partial\varphi}{\partial y}\right) = 0\ . \tag{2.5}$$

It turns out that the general solution (2.1) can be represented as the sum (in case $\alpha \neq \frac{1}{\nu}$)

$$\varphi = \varphi_1^* + \varphi_2^* \tag{2.6}$$

where φ_1 and φ_2 are solutions of the following equations:

$$y\Delta\varphi_1^* + \beta\frac{\partial\varphi_1^*}{\partial x} - (\alpha+1)\frac{\partial\varphi_1^*}{\partial y} = 0 \tag{2.7}$$

$$y\Delta\varphi_2^* - \beta\frac{\partial\varphi_2^*}{\partial x} - (\alpha+1)\frac{\partial\varphi_1^*}{\partial Y} = 0\ . \tag{2.8}$$

Rewrite the boundary equations (2.2)-(2.4) in the form:

$$(z-\zeta)\frac{\alpha^2\Psi}{\partial z\partial\zeta} - \frac{\alpha}{2}\left(\frac{\partial\Psi}{\partial z} - \frac{\partial\Psi}{\partial\zeta}\right) = 0 \tag{2.9}$$

$$\lim_{z-\zeta\to 0}(z-\zeta)^\alpha\left(\frac{\partial\Psi}{\partial z}-\frac{\partial\Psi}{\partial\zeta}\right)=2^\alpha i^{\alpha-1}f_3^*(x) \tag{2.10}$$

$$\lim_{y\to a}(z-\zeta)^\alpha\left(\frac{\partial\Psi}{\partial z}-\frac{\partial\Psi}{\partial\zeta}\right)=2^\alpha i^{\alpha-1}f_3^*(x)\,. \tag{2.11}$$

The general representation of a real solution of equation (2.9) is (due to Darboux)

$$\begin{aligned}\Psi(z,\zeta)&=(z-\zeta)^{1-\alpha}\int_0^1\omega_1[z-t(z-\zeta)]t^{-\frac{\alpha}{2}}(1-t)^{\frac{\alpha}{2}}\,dt\\&+\int_0^1\omega_2[z+t(\zeta-z)]t^{\frac{\alpha}{2}-1}(1-t)^{\frac{\alpha}{2}-1}\,dt\\&+(\zeta-z)^{1-\alpha}\int_0^1\omega_1[\zeta+t(z-\zeta)]t^{-\frac{\alpha}{2}}(1-t)^{-\frac{\alpha}{2}}\,dt\\&+\int_0^1\omega_2[\zeta+t(z-\zeta)]t^{\frac{\alpha}{2}-1}(1-t)^{\frac{\alpha}{2}-1}\,dt\,.\end{aligned} \tag{2.12}$$

When $y=0$ we obtain the equality

$$\omega_1(x)=\frac{2^{\alpha-2}\Gamma(2-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})(1-\alpha)\sin\frac{\pi\alpha}{2}}f_3(x)\,. \tag{2.13}$$

Using (2.13) and (2.12) we obtain from equation (2.11) the following integral equation

$$\int_0^1\{\omega_2'[x+(1-2t)ai]-\omega_2'[x-(1-2t)ai]\}t^{\frac{\alpha}{2}-1}(1-2t)(1-t)^{\frac{\alpha}{2}-1}\,dt=$$

$$=iF(x)\,. \tag{2.14}$$

and we can express $F(x)$ using the function $f_3(x)$ and $f_3^*(x)$.

Now we want to find $\omega_2'(x)$. Denote $1-2t=u$. Then equation (2.14) implies the following identity:

$$\int_{-1}^1\omega_2'(x+ai)(1-u^2)^{\frac{\alpha}{2}-1}u\,du=\frac{i}{2}F(x)$$

Since

$$\omega_2(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}m(r)e^{irx}\,dr \tag{2.15}$$

we get:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}m(\tau)e^{i\tau x}\,d\tau\int_{-1}^1e^{-ua\tau}(1-u^2)^{\frac{\alpha}{2}-1}u\,du=\frac{1}{2}F(x)\,.$$

Denote

$$K(\tau)=\int_{-1}^{1}(1-u^2)^{\frac{\alpha}{2}-1}ue^{-uar}\,du\ .$$

Then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}m(\tau)K(\tau)e^{i\tau x}\,d\tau=\frac{i}{2}F(x)\,.$$

Using the inverse Fourier transform, we come to the relation:

$$m(\tau)K(\tau)=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{-i\tau x}\,dx\ . \tag{2.16}$$

Having determined $m(\tau)$ from (2.16), we can find the function $\omega_2(x)$ from (2.15). Now problem (2.9)–(2.11) is solved.

3. The stable equilibrium equations system for the prismatic shell of variable thickness.

The I. Vekua equations system has a form:

$$\mu\Delta u_1+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+a\left[(\lambda+2\mu)\frac{\partial u_1}{\partial x}+\lambda\frac{\partial u_2}{\partial y}\right]+b\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)=0$$

$$\mu\Delta u_2+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+b\left[(\lambda+2\mu)\frac{\partial u_2}{\partial y}+\lambda\frac{\partial u_1}{\partial x}\right]+a\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)=0 \tag{3.1}$$

$$\Delta u+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=0\ ,$$

where (u_1,u_2) is a shift vector in the plane Oxy , u is the shift in the normal direction, $h=h_0\ell^{ax+by}$ the variable thickness of the shell. (h_0,a,b are constants.) Set

$$u_1=\Delta w+\frac{\lambda+\mu}{\mu}\frac{\partial^2 w}{\partial y^2}+\frac{\lambda+2\mu}{\mu}b\frac{\partial w}{\partial y}+a\frac{\partial w}{\partial x}$$

$$u_2=-\left[\frac{\lambda+\mu}{\mu}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\lambda}{\mu}b\frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}\right]\ .$$

We have the following two equations for determining function u and v:

$$\Delta\Delta w+2a\frac{\partial}{\partial x}\Delta w+2b\frac{\partial}{\partial y}\Delta w-\frac{\sigma}{1-\sigma}(a^2+b^2)\Delta w$$
$$+\frac{1}{1-\sigma}\left(a^2\frac{\partial^2 w}{\partial x^2}+b^2\frac{\partial^2 w}{\partial y^2}+2ab\frac{\partial^2 w}{\partial x\partial y}\right)=0 \tag{3.2}$$

$$\Delta u+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=0\ . \tag{3.3}$$

It is known that the general integral of equation (3.3) has the form [2,4]:

$$u = \exp\left[-\frac{1}{2}(ax+by)\right] \operatorname{Re}\left\{g(z) - \int_0^z I_0(k\sqrt{\zeta(z-t)})f(t)\,dt\right\} , \qquad (3.4)$$

where I_0 is the Bessel function of the first order of the imaginary argument, $k = \frac{1}{2}\sqrt{a^2+b^2}$, $f(z)$ is an arbitrary analytic function.

Note that in the case $a^2+b^2 \neq 0$, the solution of equation (3.2) can be written as the sum

$$w = w_1 + w_2 ,$$

where the functions w_1 and w_2 satisfy the equations

$$\Delta w_1 + \left(a + b\sqrt{\frac{\sigma}{1-\sigma}}\right)\frac{\partial w_1}{\partial x} + \left(b - a\sqrt{\frac{\sigma}{1-\sigma}}\right)\frac{\partial w_1}{\partial y} = 0$$

$$\Delta w_2 + \left(a - b\sqrt{\frac{\sigma}{1-\sigma}}\right)\frac{dw_2}{dx} + \left(b + a\sqrt{\frac{\sigma}{1-\sigma}}\right)\frac{\partial w_2}{\partial y} = 0$$

Then the general solution of (3.2)

$$\begin{aligned} w = {} & \exp\left\{-\frac{1}{2}\left[\left(a + b\sqrt{\frac{\sigma}{1-\sigma}}\right)x + \left(b - a\sqrt{\frac{\sigma}{1-\sigma}}\right)y\right]\right\} \\ & \times \operatorname{Re}\left[f_1(z) - \int_0^z I_0\left(k\sqrt{\zeta(z-t)}\right) f_1(t)\,dt\right] \\ & + \exp\left\{-\frac{1}{2}\left[\left(a - b\sqrt{\frac{\sigma}{1-\sigma}}\right)x + \left(b + a\sqrt{\frac{\sigma}{1-\sigma}}\right)y\right]\right\} \\ & \times \operatorname{Re}\left[f_2(z) - \int_0^z I_0\left(k\sqrt{\zeta(z-t)}\right) f_2(t)\,dt\right] , \end{aligned}$$

where $f_1(z)$ and $f_2(z)$ are arbitrary analytic functions, and

$$k = \left[\frac{a^2+b^2}{4(1-\sigma)}\right]^{\frac{1}{2}} .$$

Case $a^2+b^2=0$ is the case of the well-known beharmonic equation.

References

1. S. Bergmann, Functions satisfying certain partial differential equations of elliptic type and their representation, Duke Math. Jour. 2 (1947), 349-366.
2. I. Vekua, New Methods for Solving Elliptic Equations, North-Holland Publ. Co., Amsterdam, 1967.
3. L. Zalcman, Offbeat integral geometry, American Mathematical Monthly 87, 3 (1980).
4. A. Bizadze, Boundary Value Problems for Second Order Elliptic Equations, North-Holland Publ. Co., Amsterdam, 1968; Izdat., Nauka, Moscow (1966), 161-175.

ON THE EXACT CONSTANT IN THE INEQUALITY FOR A NORM OF SOLUTIONS OF THE LAMÉ SYSTEM IN A HALF-SPACE

Gregory I. Kresin

The Research Institute
The College of Judea and Samaria
Ariel 44824, Israel

Abstract

The representation for the exact constant $\mathcal{K}$ in the inequality between the suprema of a norm in $\mathcal{R}_n$ of a solution of the Lamé system inside the half-space and on its boundary is obtained, where $\mathcal{R}_n$ is an n-dimensional linear normed space. It is shown that the value of constant $\mathcal{K}$ is minimal in the case of Euclidean norm in $\mathcal{R}_n$.

1. Introduction

A formula was obtained in [1] for the exact constant K in the inequality

$$\sup\left\{|\mathbf{u}(x)| : x \in E^n_+\right\} \le K \sup\left\{|\mathbf{u}(x')| : x' \in \partial E^n_+\right\}, \tag{1}$$

where $|\cdot|$ is the length of a vector in the Euclidean space E^n, $E^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$ and $\mathbf{u}$ is a solution of the n-dimensional Lamé system $\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = 0$ in the class $C(\overline{E^n_+}) \cap C^{(2)}(E^n_+)$. Here $C(\overline{E^n_+})$ is the space of bounded and continuous n- component vector-valued functions defined on $\overline{E^n_+}$. The exact constant K has the form

$$K = \frac{2\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \int_0^{\pi/2} \left[(1-\kappa)^2 + n\kappa(n\kappa - 2\kappa + 2)\cos^2\theta\right]^{1/2} \sin^{n-2}\theta\, d\theta,$$

where $\kappa = (\lambda+\mu)(\lambda+3\mu)^{-1}$.

Let $\mathcal{R}_n$ be an n-dimensional normed linear space with a norm $\|\cdot\|$ and let $\mathcal{R}^*_n$ be the space conjugate to $\mathcal{R}_n$ with the norm $\|\cdot\|^*$.

In this note we consider the exact constant $\mathcal{K}$ in the inequality

$$\sup\left\{\|\mathbf{u}(x)\| : x \in E^n_+\right\} \le \mathcal{K} \sup\left\{\|\mathbf{u}(x')\| : x' \in \partial E^n_+\right\},$$

where $\mathbf{u}$ is a solution of the Lamé system in the class $C(\overline{E^n_+}) \cap C^{(2)}(E^n_+)$.

The following statement is proved.

Proposition. *The exact constant $\mathcal{K}$ has the form*

$$\mathcal{K} = \frac{1}{\omega_n} \sup_{\|\mathbf{z}\|^*=1} \int_{S^{n-1}} \|(1-\kappa)\mathbf{z} + n\kappa \mathbf{e}_\sigma(\mathbf{e}_\sigma, \mathbf{z})\|^* \, d\sigma \,, \tag{2}$$

where ω_n is the area of the sphere $S^{n-1} = \{x \in E^n : |x| = 1\}$, $\mathbf{e}_\sigma$ is the n-dimensional unit vector joining the origin to a point $\sigma \in S^{n-1}$ and $(\cdot,\cdot)$ is the inner product in E^n.

The following equality

$$K = \min_{\|\cdot\|} \mathcal{K} \tag{3}$$

is valid, where the minimum is over all norms in the space $\mathcal{R}_n$.

It was proved in [1] that the inequality $K > 1$ holds for $\kappa \neq 0$. From this and from equality (3) it follows that the maximum norm principle (i.e., the equality $\mathcal{K} = 1$ holds) is not valid for the Lamé system in E^n_+ for $\kappa \neq 0$.

2. Proof of the Proposition

According to [2] there exists a solution of the problem

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad E^n_+ , \qquad \mathbf{u} = \mathbf{f} \quad \text{on} \quad \partial E^n_+ ,$$

that is bounded and continuous up to ∂E^n_+, where $\mathbf{f} \in C(\partial E^n_+)$, and this solution can be presented in the form

$$\mathbf{u}(x) = \int_{\partial E^n_+} M\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n} \mathbf{f}(y')\, dy' \,. \tag{4}$$

Here $y = (y', 0)$, $y' = (y_1, \ldots, y_{n-1})$, M is an $(n \times n)$-matrix-valued function on S^{n-1} with elements

$$M_{ij}(\mathbf{e}_\sigma) = \frac{2}{\omega_n} [(1-\kappa)\delta_{ij} + n\kappa(\mathbf{e}_\sigma, \mathbf{e}_i)(\mathbf{e}_\sigma, \mathbf{e}_j)] \,, \tag{5}$$

where $\kappa = (\lambda + \mu)(\lambda + 3\mu)^{-1}$.

Let $\sup \left\{\|\mathbf{f}(x')\| : x' \in \partial E^n_+\right\}$ be the norm in the space $C(\partial E^n_+)$. We fix a point $x \in E^n_+$ and find the norm $\mid \mathbf{u}(x) \mid$ of the mapping $C(\partial E^n_+) \ni \mathbf{f} \to \mathbf{u}(x) \in \mathcal{R}_n$, where $\mathbf{u}$ is defined by (4).

Using the following equalities

$$\|\mathbf{l}\| = \sup\{(\mathbf{z}, \mathbf{l}) : \|\mathbf{z}\|^* = 1\} \quad \text{and} \quad \|\mathbf{z}\|^* = \sup\{(\mathbf{z}, \mathbf{l}) : \|\mathbf{l}\| = 1\} \,,$$

the fact that the supremum operations commute, and the symmetricity of the matrix M, we find that

$$
\begin{aligned}
|\mathbf{u}(x)| &= \sup_{\|\mathbf{f}\|\le 1} \left\| \int_{\partial E_+^n} M\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n} \mathbf{f}(y')\, dy' \right\| \\
&= \sup_{\|\mathbf{f}\|\le 1} \sup_{\|\mathbf{z}\|^*=1} \left(\mathbf{z}, \int_{\partial E_+^n} M\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n} \mathbf{f}(y')\, dy'\right) \\
&= \sup_{\|\mathbf{f}\|\le 1} \sup_{\|\mathbf{z}\|^*=1} \int_{\partial E_+^n} \left(\mathbf{z}, M\left(\frac{y-x}{|y-x|}\right) \mathbf{f}(y')\right) \frac{x_n}{|y-x|^n}\, dy' \\
&= \sup_{\|\mathbf{z}\|^*=1} \sup_{\|\mathbf{f}\|\le 1} \int_{\partial E_+^n} \left(M\left(\frac{y-x}{|y-x|}\right) \mathbf{z}, \mathbf{f}(y')\right) \frac{x_n}{|y-x|^n}\, dy' \\
&= \sup_{\|\mathbf{z}\|^*=1} \int_{\partial E_+^n} \left\| M\left(\frac{y-x}{|y-x|}\right) \mathbf{z} \right\|^* \frac{x_n}{|y-x|^n}\, dy' .
\end{aligned}
$$

From this, by definition (5) of the matrix M, we get

$$
\begin{aligned}
|u(x)| &= \frac{2}{\omega_n} \sup_{\|\mathbf{z}\|^*=1} \int_{\partial E_+^n} \left\| (1-\kappa)\mathbf{z} + n\kappa \frac{(y-x)(y-x,\mathbf{z})}{|y-x|^2} \right\|^* \frac{x_n}{|y-x|^n}\, dy' \\
&= \frac{2}{\omega_n} \sup_{\|\mathbf{z}\|^*=1} \int_{S_-^{n-1}(x)} \|(1-\kappa)\mathbf{z} + n\kappa \mathbf{e}_{xy}(\mathbf{e}_{xy},\mathbf{z})\|^*\, d\sigma(y) ,
\end{aligned}
$$

where $S_-^{n-1}(x)$ is the lower half of the unit sphere in E^n centered at the point x and $\mathbf{e}_{xy} = (y-x)|y-x|^{-1}$. Since the last integral does not depend on x it follows that

$$
|u(x)| = \frac{2}{\omega_n} \sup_{\|\mathbf{z}\|^*=1} \int_{S_-^{n-1}} \|(1-\kappa)\mathbf{z} + n\kappa \mathbf{e}_\sigma(\mathbf{e}_\sigma,\mathbf{z})\|^*\, d\sigma ,
$$

where S_-^{n-1} is the lower half of the sphere S^{n-1}. Using the fact that the integrand is even with respect to $\mathbf{e}_\sigma$, we get that

$$
|u(x)| = \frac{1}{\omega_n} \sup_{\|\mathbf{z}\|^*=1} \int_{S^{n-1}} \|(1-\kappa)\mathbf{z} + n\kappa \mathbf{e}_\sigma(\mathbf{e}_\sigma,\mathbf{z})\|^*\, d\sigma ,
$$

and thus the representation (2) of the Proposition is proved.

Let $\mathbf{z}_0$ be an n-dimensional vector such that $\|\mathbf{z}\|^* = 1$ and the equality $|\mathbf{z}_0| = \max\{|\mathbf{z}| : \|\mathbf{z}\|^* = 1\}$ is valid. It is clear that for the two collinear vectors $(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma,\mathbf{z}_0)$ and $\mathbf{Z}(\sigma)$, $\|\mathbf{Z}(\sigma)\|^* = 1$, the equality

$$
\|(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma,\mathbf{z}_0)\|^* = \frac{|(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma,\mathbf{z}_0)|}{|\mathbf{Z}(\sigma)|}
$$

holds. It follows from this that

$$\|(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma, \mathbf{z}_0)\|^* \geq \frac{|(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma, \mathbf{z}_0)|}{|\mathbf{z}_0|} .$$

Then, by (2), we get

$$\begin{aligned}\mathcal{K} &\geq \frac{1}{\omega_n}\int_{S^{n-1}} \|(1-\kappa)\mathbf{z}_0 + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma, \mathbf{z}_0)\|^* \, d\sigma \\ &\geq \frac{1}{\omega_n}\int_{S^{n-1}} \left|(1-\kappa)\frac{\mathbf{z}_0}{|\mathbf{z}_0|} + n\kappa\mathbf{e}_\sigma\left(\mathbf{e}_\sigma, \frac{\mathbf{z}_0}{|\mathbf{z}_0|}\right)\right| d\sigma \\ &\geq \frac{1}{\omega_n}\int_{S^{n-1}} \left[(1-\kappa)^2 + n\kappa(n\kappa - 2\kappa + 2)\left(\mathbf{e}_\sigma, \frac{\mathbf{z}_0}{|\mathbf{z}_0|}\right)^2\right]^{1/2} d\sigma \\ &= \frac{1}{\omega_n}\int_{S^{n-1}} \left[(1-\kappa)^2 + n\kappa(n\kappa - 2\kappa + 2)\cos^2\theta_0(\sigma)\right]^{1/2} d\sigma , \qquad (6)\end{aligned}$$

where $\theta_0(\sigma)$ is the angle between the vectors $\mathbf{e}_\sigma$ and $\mathbf{z}_0|\mathbf{z}_0|^{-1}$.

By (2), the exact constant K in equality (1) is equal to

$$\frac{1}{\omega_n}\sup_{|\mathbf{z}|=1}\int_{S^{n-1}} |(1-\kappa)\mathbf{z} + n\kappa\mathbf{e}_\sigma(\mathbf{e}_\sigma, \mathbf{z})| \, d\sigma .$$

Consequently,

$$\begin{aligned}K &= \frac{1}{\omega_n}\sup_{|\mathbf{z}|=1}\int_{S^{n-1}} \left[(1-\kappa)^2 + n\kappa(n\kappa - 2\kappa + 2)(\mathbf{e}_\sigma, \mathbf{z})^2\right]^{1/2} d\sigma \\ &= \frac{1}{\omega_n}\int_{S^{n-1}} \left[(1-\kappa)^2 + n\kappa(n\kappa - 2\kappa + 2)\cos^2\theta(\sigma)\right]^{1/2} d\sigma ,\end{aligned}$$

where $\theta(\sigma)$ is the angle between the vectors $\mathbf{e}_\sigma$ and $\mathbf{z}$.

Comparing (6) with the last equality, we get that $\mathcal{K} \geq K$.

Acknowledgement. This research was supported by the Ministry of Absorption, State of Israel.

References

1. V.G. Maz'ya and G.I. Kresin, On the maximum principle for strongly elliptic and parabolic second order systems with constant coefficients. (In Russian) Matem. Sbornik 125 (167), 4 (1984), 458- 480; English transl. Math. USSR Sbornik, 53, 2 (1986) 457-479.
2. V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili and T.V. Burchuladze, Three-dimensional problems of mathematical theory of elasticity and thermoelasticity. 2nd rev. aug. ed. Nauka, Moscow (1976), English transl. North-Holland (1979).

INVARIANT LOCALLY NON-UNIQUE MANIFOLDS INDESTRUCTIBILITY AND LINEARIZATION

George Osipenko

Department of Mathematics
State Technical University
St. Petersburg, Russia

Let us consider a smooth system of differential equations

$$\dot{u} = F_0(u) , \tag{1}$$

where u lies in $\mathbf{R}^n$. We suppose the system (1) has an invariant smooth compact manifold M_0. In a space of smooth bounded vector fields $\{F\}$ we introduce a topology generated by a C^1 norm $\|F\|_1$:

$$\|F\| = \sup_u |F(u)| , \quad |\partial F(u)| = \max_{|v|=1} |\partial F(u)v| , \quad \|\partial F\| = \sup_u |\partial F(u)| ,$$

$$\|F\|_1 = \|F\| + \|\partial F\| .$$

In a space of smooth compact manifolds $\{M\}$, C^1-close to the manifold M_0 we introduce a topology in the following manner. Let $E = \{\bigcup E(x) \ , \ x \in M_0 \ , \ E(x)$ be a plane transversal to M_0 at $x\}$ be a smooth tubular neighborhood of the manifold M_0. A manifold M C^1-close to M_0 may be determined in the form of graph:

$$M = \{(x + v(x)) \ , \ x \in M_0 \ , \ v(x) \in E(x)\} ,$$

where v is a smooth vector field. The topology of space $\{M\}$ is determined in a C^1-norm of the vector field $v : \|v\|_1$. Let a distance between the manifolds M and M_0 be equal to the C^1 norm of $v : \text{dist}(M, M_0) = \|v\|_1$.

Definition 1. We say that the invariant manifold M_0 of the system (1) is indestructible if for every positive ε where there exists positive δ such that if C^1-norm of perturbation is less than δ : $\|F - F_0\|_1 < \delta$ then the perturbed system

$$\dot{u} = F(u) , \tag{2}$$

has an invariant manifold M with distance between M and M_0 less than ε : $\text{dist}(M, M_0) < \varepsilon$.

It is clear that for indestructible manifold there exists a mapping $H : V \to W$ from a C^1 neighborhood V of the field F_0 in the C^1 neighborhood W of manifold M_0 which associates to a perturbed system its invariant manifold and H is continuous at F_0. Notice that in general, the mapping H cannot be chosen continuous on the whole neighborhood of F_0.

Let $U(t, u)$ be a solution of the system (1) with initial condition $U(0, u) = u$. The invariant manifold M_0 is called normally hyperbolic if there are an invariant decomposition of tangent space into a direct sum

$$T\mathbf{R}^n|M_0 = TM_0 \oplus E^s \oplus E^u$$

and the positive constants K, λ such that

$$|\partial U^s(t,x)| < K\exp(-\lambda t) \ , \ |\partial U^s(t,x)||(\partial U_0(t,x))^{-1}| < K\exp(-\lambda t) \ , \ \text{as } t < 0 \ ,$$
$$|\partial U^u(t,x)| < K\exp(\lambda t) \ , \ |\partial U^u(t,x)||(\partial U_0(t,x))^{-1}| < K\exp(\lambda t) \ , \ \text{as } t < 0 \ ,$$

where $x \in M$, $\partial U_0, \partial U^s, \partial U^u$ are the restrictions of differential ∂U on TM_0, E^s, E^u correspondence, that is $\partial U_0 = \partial U|TM_0$, $\partial U^s = \partial U|E^s$, $\partial U^u = \partial U|E^u$. The following theorem of Sacker and Neimark is well known.

Theorem 1 [1,2,3]. *The normally hyperbolic compact invariant manifold is the indestructible one.*

In the conditions of Theorem 1, the perturbed invariant manifold M passes a property of local uniqueness: there exists a neighborhood V_0 of M_0 in R^n such that the maximal invariant set I in V of perturbed system (2) is the perturbed invariant manifold M that is $I = M$. In this case there is the unique mapping H associating to a perturbed system its invariant manifold. The indestructible invariant manifold satisfied condition of local uniqueness was called by Mane persistent manifold. He proved

Theorem 2 [4]. *The persistent manifold is normally hyperbolic.*

Thus Theorems 1 and 2 give necessary and sufficient conditions of persistent manifold. But there are many simple examples in which indestructible invariant manifolds are not persistent. For example, an equilibrium point $M_0 : x = 0$ of an equation $\dot{x} = x^3$ on the real line R is indestructible since the index of one is not 0. But this manifold is not persistent since a perturbed equation may have more than one equilibrium point near M_0. Now we consider a set of indestructible invariant manifolds wider than the set of persistent manifolds.

Definition 2. The invariant manifold M_0 of system (1) is called strongly indestructible with respect to perturbations of system (1) if there exists a C^1-neighborhood V of F_0 such that if the manifold M_0 is invariant for the system

$$\dot{u} = F_1(u) \ , \tag{3}$$

where F_1 lies in V then M_0 is indestructible with respect to perturbation of the system (3).

The importance of the strong indestructibility of invariant manifolds should be noted since practically systems of differential equations are known with certain inaccuracy. In the conditions of Theorem 1 the unique mapping H associating to a perturbed system its invariant manifold is continuous on the whole neighborhood V. Hence the persistent manifold is strongly indestructible.

Definition 3. The invariant manifold M_0 of system (1) is called weakly indestructible if it is indestructible and in every C^1 neighborhood V of F_0 is found a vector field F_1 for which M_0 is the invariant manifold of system (3) and it is not indestructible for this system.

It is not difficult to verify that the equilibrium point $M_0 : x = 0$ of the equation $\dot{x} = x^3$ gives an example of a weakly indestructible manifold.

In order to state the conditions of strong and weak indestructibility we must introduce the following notions. The stable and unstable subspaces at the point $x \in M_0$ are defined

$$E^s(x) = \{y \in T_x R^n : |\partial U(t,x)y| \to 0, |\partial U(t,x)y||(\partial U_0)^{-1}(t,x)| \to 0 \text{ as } t \to +\infty\} ,$$
$$E^u(x) = \{y \in T_x R^n : |\partial U(t,x)y| \to 0, |\partial U(t,x)y||(\partial U_0)^{-1}(t,x)| \to 0 \text{ as } t \to -\infty\} .$$

We say that the condition of transversality is valid at point $x \in M_0$ if the equality

$$T_x R^n = T_x M_0 + E^s(x) + E^u(x)$$

is valid. It should be noted that if the transversality condition is valid on the whole manifold M_0 and the sum $TM_0 + E^s + E^u$ is direct,

$$TM_0 + E^s + E^u = TM_0 \oplus E^s \oplus E^u$$

then the manifold M_0 is normally hyperbolic. It is clear that the transversality condition is wider than the normally hyperbolic condition. It may be said that the transversality condition is to the normally hyperbolic condition as the structure stability systems are to the Anosov-systems:

$$\frac{\textit{transversality}}{\textit{normally hyperbolic}} = \frac{\textit{structure stability systems}}{\textit{Anosov-systems}}$$

Strong sources and sinks. Let a point O lying on a manifold M_0 be an equilibrium one of system (1). Then the system near 0 can be represented in the form

$$\begin{aligned} \dot{x} &= Ax + Cy + f_1(x,y) , \\ \dot{y} &= \qquad By + \phi_1(x,y) , \end{aligned}$$

where f_1 and ϕ_1 are smooth mappings and vanish together with their derivatives as $x = 0$, $y = 0$; $x \in M_0$, vector y lies in the subspace transversal to M_0. We say that the condition of separation holds at the equilibrium point O if the spectrum of matrix A lies in the strip of the complex plane C $\{\mu \leq \operatorname{Re}\lambda \leq \gamma\}$ and the spectrum of matrix B lies outside this strip. If the separation conditions holds and $\gamma < 0$ then the point O is called a strong sink. If the separation condition holds and $\mu > 0$ then the point O is called a strong source. An equilibrium point O is non-degenerate if the determinant $\det \partial F_0(O) \neq 0$. Let O be an isolated equilibrium point of system (1) and S be a sphere of sufficiently small radius with center O. An index of equilibrium point O is a degree of the mapping $\Phi(u) = \frac{F_0(u)}{|F_0(u)|}$ on the sphere S.

In a similar way one can define a non-degenerate strong source and sink of periodic orbit lying on manifold M_0 and also the the index of periodic trajectory. For this one needs to consider a successive Poincaré mapping near the periodic orbit.

Let ϕ be periodic trajectory on the invariant manifold M_0 and P be its successive Poincaré mapping. Then the mapping P can be represented in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} Ax & + & Cy & + & f_1(x,y) \\ & & By & + & \phi_1(x,y) \end{pmatrix}$$

where f_1 and ϕ_1 are smooth mappings and vanish together with their derivatives as $x = 0$, $y = 0$; $O \in \phi$, $x \in M_0$, a vector y lies in the subspace transversal to M_0. We say that the condition of separation holds at the periodic trajectory ϕ if the spectrum of matrix A lies in the ring $\{\mu \geq |\lambda| \leq \gamma\}$ and the spectrum of matrix B lies outside this ring. If the separation condition holds and $\gamma < 1$ then the periodic trajectory ϕ is called a strong sink. If the separation condition holds and $\mu > 1$ then the periodic trajectory ϕ is called a strong source. The periodic trajectory ϕ is non-degenerate if a determinant $\det\big(\partial P(O) - 1\big) \neq 0$. An index of the periodic trajectory ϕ is called the index of vector field $\Phi(u) = P(u) - u$.

Theorem 3. *For the compact invariant manifold M_0 to be strongly indestructible it is necessary and sufficient that the transversality condition be valid on the whole manifold M_0 except perhaps the non-degenerate strong sources and the non-degenerate strong sinks.*

In the sufficient conditions of Theorem 3 the mapping H associating to a perturbed system its invariant manifold, in general, is not defined uniquely. It may be chosen continuous on some C^1 neighborhood V of F_0 and with the following property: if M_0 is an invariant manifold of the perturbed system (2) then $H(F) = M_0$. This means that if the perturbation does not change the invariant manifold M_0 then the mapping H gives the perturbed system the same manifold. From this equivalent definition of strong indestructible manifold follows

Definition 4. The invariant manifold M_0 of system (1) is strongly indestructible if there exists a continuous mapping $H : V \to W$ from the C^1 neighborhood of F_0 in the C^1 neighborhood of manifold M_0 which associates to a perturbed system its invariant manifold and with the following property: if M_0 is an invariant manifold of the perturbed system $\dot{u} = F(u)$ and $H(F) = M_0$.

Theorem 4. *If the transversality condition holds on the compact invariant manifold M_0 except possibly for the strong sinks and strong sources whose indices are different from zero and there exists a degenerate strong sink or source then M_0 is weakly indestructible.*

In the conditions of Theorem 4 the mapping H is not defined uniquely and it cannot be chosen continuous on any C^1 neighborhood V of F_0. While it should be remarked that there is a weakly indestructible invariant manifold with the mapping H which may be chosen continuous on same C^1 neighborhood V of F_0, it is clear that such invariant manifolds do not satisfy the conditions of Theorem 4.

Structure of space of the systems with indestructible invariant manifolds.

Consider a space of pairs $P = \{(F, M)\}$ where F is a smooth vector field and M is a smooth invariant compact manifold of the system $\dot{u} = F(u)$. Assume that space P is equipped with natural C^1 topology. Let P_i, P_s, P_w be the subsets of P consisting of pairs with the manifolds M which are indestructible, strongly indestructible, weakly indestructible respectively.

Theorem 5. i) *The set P_s is open in P.*
ii) *The sets P_s and P_w form a decomposition of P_i that is*

$$P_i = P_x \cup P_w \quad \text{and} \quad P_s \cap P_w = \emptyset$$

iii) *The set P_w is contained in a closing $\overline{P \backslash P_i}$.*

Linearization.

Recall that $E = \{\bigcup E(x)\,,\ x \in M_0\,,\ E(x) \text{ is a plane transversal to } M_0 \text{ at } x\}$ is a smooth tubular neighborhood of the compact manifold M_0. It may introduce the coordinates (x, y) where $x \in M_0$ and $y \in E(x)$. In these coordinates system (1) has a form

$$\begin{aligned} \dot{x} &= f(x) + f_1(x, y)\,, \\ \dot{y} &= \phi(x)y + \phi_1(x, y) \end{aligned} \tag{4}$$

where $f_1(x, 0) = 0$, the mapping ϕ_1 vanishes together with their derivatives as $y = 0$. We say that system (1) is linearized near manifold M_0 if there exist a neighborhood V_0 of M_0 in R^n and a homeomorphism $h : V_0 \to E$ from V_0 in the tubular neighborhood E such that the homeomorphism h makes a correspondence between the trajectory arcs of system (1) and the trajectory arcs of a linearized system

$$\dot{x} = f(x)$$
$$\dot{y} = \phi(x)y \ .$$

Theorem 6. *If the transversality condition holds on the compact invariant manifold M_0 then system (1) may be linearized near M_0.*

This theorem generalizes the well known theorem of Pugh and Shub on the linearization of a normally hyperbolic manifold [8].

Examples

Example 1. Persistent manifold – indestructible locally unique invariant manifold.

In a plane R^2 consider a system having the invariant unit circle M_0. Assume that the system has two equilibrium points disposed on M_0: a saddle A and a sink B. Let the linear part of the system at A be of the form

$$\dot{x} = x \ , \quad \dot{y} = -y$$

and at B be of the form

$$\dot{x} = x \ , \quad \dot{y} = -2y \ .$$

It can be easily verified that M_0 is normally hyperbolic. For the all points $z \in M_0$ the subspaces $E^u(z) = \{0\}$, $E^s(z)$ are straight lines transversal to M_0 at z and $E^s(A)$, $E^s(B)$ are the vertical straight lines. By Theorems 1,2 the circle M_0 is persistent. Actually this fact may be established directly.

Notice that in this case the mapping H is uniquely determined since in some neighborhood of M_0 every perturbed vector field F C^1close to the original cone has a unique invariant manifold M diffeomorphic to M_0. The manifold M consists of two equilibrium points (the saddle $A(F)$ near to A and the sink $B(F)$ near to B) and two separatrices of the saddle $A(F)$ going from $A(F)$ to $B(F)$. Hence, the mapping H must be of the form $H(F) = M$.

Example 2. Locally non-unique strongly indestructible manifold on which the transversality condition holds.

Similarly to example 1, consider a system in a plane having the invariant unit circle M_0 and two equilibrium points disposed on M_0: a source A and a sink B. Let the linear part of the system at A be of the form

$$\dot{x} = x \ , \quad \dot{y} = 2y$$

and at B be of the form

$$\dot{x} = -x \ , \quad \dot{y} = -2y \ .$$

Verify the transversality condition on the circle M_0. It is easily seen that $E^s(A) = E^u(B) = \{0\}$, $E^u(A), E^s(B)$ are the vertical straight lines. For a point $z \neq A, B$ the subspaces $E^u(z)$, $E^s(z)$ are two straight lines transversal to M_0 at z. By Theorem 3 the manifold M_0 is strongly indestructible. Actually this fact we establish directly below. Notice that in every neighborhood of M_0 there is an infinite number of smooth manifolds which are tangent to the horizontal direction at A and B. Hence the circle M_0 is not locally unique invariant manifold.

Consider a construction of the mapping H associating to a perturbed system its invariant manifold. In some neighborhood of M_0 every perturbed vector field F C^1 close to the original one has two equilibrium points (the source $A(F)$ near to A and the sink $B(F)$ near to B) and an infinite number of trajectories going from $A(F)$ to $B(F)$ and tangent to each other at these points. To construct the mapping H let us choose two points p_1, p_2 other than A, B, on the left and the right parts of circle M_0 Assume the perturbation small enough to ensure that $p_1, p_2 \neq A(F), B(F)$ and the trajectories $T(p_1), T(p_2)$ passing through p_1, p_2 tend to $A(F)$ as $t \to -\infty$ and to $B(F)$ as $t \to +\infty$. Clearly $M(p_1, p_2) = A(F) \cup B(F) \cup T(p_1) \cup T(p_2)$ is an invariant manifold of the perturbed system C^1 close to M_0. It remains to set $H(F) = M(p_1, p_2)$. It can be verified at once that the constructed mapping H satisfies Definition 4. Note that the construction of H depends on the choice of points p_1, p_2. By choosing other points we obtain a mapping different from the one constructed.

Example 3. Destruction of an invariant manifold in the case when the transversality condition does not hold at a wandering point.

In the plane R^2 let us consider a system having the invariant unit circle M_0 and two saddle points A, B. Let the linearized system at A be of the form

$$\dot{x} = x\,, \quad \dot{y} = -y$$

and at B be of the form

$$\dot{x} = -x\,, \quad \dot{y} = y\,.$$

It is easily seen that A, B are normally hyperbolic. The subspaces $E^u(A) = E^s(B) = \{0\}$, $E^s(A), E^u(B)$ are vertical straight lines. If a point $z \in M_0$, $z \neq A, B$, then $E^s(z) = E^u(z) = \{0\}$. Hence, the transversality condition is violated at z. Since the arc $AB \subset M_0$ is a separatrix joining two saddles, it can be destroyed by some perturbation, such a perturbed system having near M_0 no invariant manifold homeomorphic to M_0.

Example 4. Destruction of the C^1 structure of an invariant manifold in the case when the transversality condition does not hold at a wandering point.

As in Example 1 consider a system defined in the plane and having the invariant unit circle M_0. Assume also that $A, B \in M_0$ are their equilibrium points. Let the linear part of the system at A be the same as in Example 1 and at B of the form

$$\dot{x} = -x\ , \quad \dot{y} = -y/2\ .$$

As above the transversality condition holds at A and B is a strong sink. If a point $z \in M_0$, $z \neq A$ then $E^s(z) = E^u(z) = \{0\}$. Thus the transversality condition is violated at z. Perturbing the system in the neighborhood of some point $z \in M_0$, $z \neq A, B$, one can obtain a system whose saddle A has an unstable separatrix tangent to the vertical direction at B. Thus the perturbed system near M_0 has no invariant C^1 submanifold diffeomorphic to M_0. Observe that near M_0 there exists nevertheless an invariant C^0 manifold homeomorphic to M_0.

Example 5. The weakly indestructible manifold for which there does not exist a mapping H continuous on V.

On a real line R^1 consider a differential equation

$$\dot{x} = x^3\ .$$

The equation has an invariant manifold M_0, which is an equilibrium point $x = 0$. M_0 is indestructible since the index of this equilibrium point is distinct from zero and therefore every equation C^1 near to the initial one has at least one equilibrium point near to M_0. But M_0 is not persistent since the maximal invariant set of a perturbed equation lying in some neighborhood of M_0 may contain more than one equilibrium point. Moreover, it is not difficult to show that the invariant manifold M_0 is weakly indestructible. Clearly the mapping H is not uniquely determined. The equation also gives an example of a system for which H cannot be chosen continuous in any C^1 neighborhood V. To prove this consider the C^1 function

$$G(x) = \begin{cases} (x+d)^3\ , & x < -d \\ 0\ , & -d < x < d \\ (x-d)^3\ , & x > d\ . \end{cases}$$

where the number d is chosen, the function G is in a given neighborhood V. Show that the mapping H is not continuous at G independently of the choice of the value $H(G)$. In fact, the equation

$$\dot{x} = G(x) + \delta$$

for $\delta > 0$ has the unique equilibrium point O_1 with coordinate $\alpha_1 = -d - \delta^{1/3} < -d$ and for $\delta < 0$ the unique equilibrium point O_2 with coordinate $\alpha_2 = d - \delta^{1/3} > d$. Hence, $H(G+\delta) = \alpha_1$ for $\delta > 0$ and $H(G+\delta) = \alpha_2$ for $\delta < 0$. It follows from this that

$$\lim_{\delta \to +0} H(G+\delta) = -d\ , \qquad \lim_{\delta \to -0} H(G+\delta) = d\ .$$

Since the left and the right limits do not coincide H is not continuous at G independently of the choice value $H(G)$.

Example 6. A. Strongly indestructible manifold in the case when the transversality condition is violated at a strong source.

In a space R^3 consider a system having in a plane $X = 0$, the invariant unit circle M_0 and two equilibrium points $A(0,0,1), B(0,0,-1) \in M_0$. Suppose that in some neighborhood of A the plane $y = 0$ is invariant for the system. In this plane, let the system be of the form

$$\dot{x} = z - 1 , \qquad \dot{z} = -x ,$$

that is the equilibrium point is of the center type in this plane. In some neighborhood of A, let the linear part of the system on M_0 be of the form

$$\dot{y} = y .$$

It is clear that A is a non-degenerate strong source and the transversality condition does not hold at A. At B the linear part of the system is of the form

$$\dot{x} = -2x , \quad \dot{y} = -y , \quad \dot{z} = -2z .$$

The transversality condition is valid on $M_0 \backslash A$. In fact, for $z \in M_0 \backslash A$, $E^u(z) = \{0\}$, $E^s(z)$ is a plane transversal to M_0 at z. By Theorem 3 the invariant circle M_0 is strongly indestructible. However, we show this by direct construction of the mapping H.

The mapping H is determined, in this case, uniquely. Obviously every vector field G, C^1 close to the original one, has two equilibrium points $A(G), B(G)$ near A, B. By the Center Manifold Theorem [8] there exists a locally invariant surface $P(G)$ containing $A(G)$ and C^1 close to the plane $y = 0$. Every trajectory of G passing through a point near to $P(G)$ tends to $P(G)$ as $t \to -\infty$. On the surface $P(G)$ the equilibrium point $A(G)$ is of a center, focus or center-focus type. It is not difficult to see that in either case there are only two trajectories $T_1(G), T_2(G)$ tending to $A(G)$ as $t \to \infty$ and C^1 close to M_0. The trajectories $T_1(G), T_2(G) \to B(G)$ as $t \to +\infty$ and are tangent to each other at $A(G), B(G)$. It remains to set

$$H(G) = M = A(G) \cup B(G) \cup T_1(G) \cup T_2(G) .$$

It is easily seen that $H(G)$ depends continuously on G in C^1 topology and $H(G) = M_0$ if M_0 is invariant for G. Then the mapping H satisfies Definition 4.

Example 7. The weakly indestructible manifold for which there exists a mapping continuous on V and which does not satisfy Theorem 4's conditions.

In the plane R^2 consider a dynamic system defined in the following manner. On a rectangle $[-2,2] \times [-1,1]$ the system has a form

$$\dot{x} = y + x^2 , \quad \dot{y} = 0 .$$

The axis Ox is an invariant manifold. The trajectories over Ox are parallel to Ox and have the same direction. The lines $y = h$, $h < 0$ are invariant too. On them are equilibrium points which are defined by an equation $x^2 + y = 0$. Moreover let our system have two equilibrium points A $(-2, 2)$ and B $(2, -2)$. In some neighborhood of the point A the system has the form

$$\dot{x} = x + 2\ , \quad \dot{y} = 2(y + 2)\ .$$

In some neighborhood of point B, the system has the form

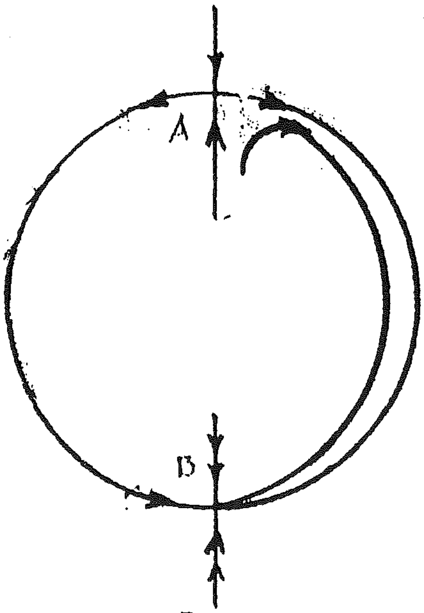
$$\dot{x} = -(x - 2)\ , \quad \dot{y} = -2(y + 2)\ .$$

Let the system be such that a trajectory T_1 beginning at point $(2, 0)$ reach equilibrium point B as $T \to +\infty$, and it is tangent to a horizontal line there. Let a trajectory T_2 beginning at point $(-2, 0)$ reach equilibrium point A as $t \to -\infty$ and be tangent to a horizontal line there. Let a horizontal segment $[A, B]$ be trajectory. Thus we have an invariant manifold M_0 consisting of the segment $[A, B]$, the trajectories T_1, T_2 and the segment $[-2, 2] \times 0$. To show that this invariant manifold is indestructible, consider a trajectory T_3 passing through point $(0, y)$, $y > 0$. If y is sufficiently small this trajectory tends to A as $t \to -\infty$ and to B as $t \to +\infty$ and is tangent to horizontal line there. Joining trajectory T_3 and segment $[A, B]$ we obtain an invariant manifold S, C^1 close to M_0. The system near manifold S is analogous to the system of Example 2. On the manifold S the transversality condition holds good. Hence there exists a mapping H_y associating to a perturbed system its invariant manifold. From the proof of Theorem 3 it follows that this mapping is continuous in y. Using mapping H_y we can construct a continuous mapping H associating to a perturbed system its invariant manifold and $H(F_0) = M_0$.

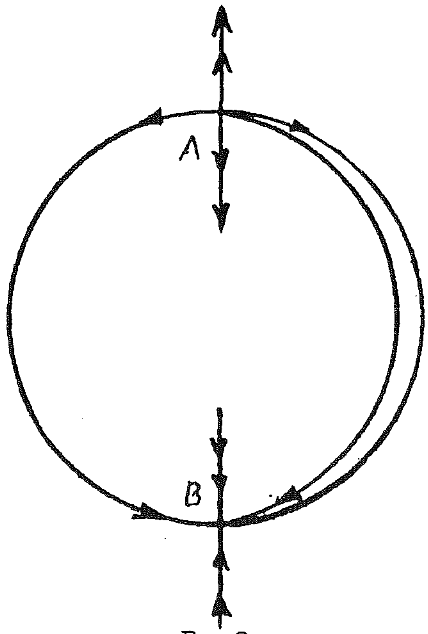
Now we show that there exists a system $\dot{u} = F_1(u)$, C^1 close to the initial one with the same invariant manifold M_0 which is not indestructible under perturbation of this system. Consider a system coinciding with the initial one outside of the rectangle $[-2, 2] \times [-1, 1]$. Inside of this rectangle there is any rectangle where a new system has a form

$$\dot{x} = -p^2 + y + x^2\ , \quad \dot{y} = -qy\ ,$$

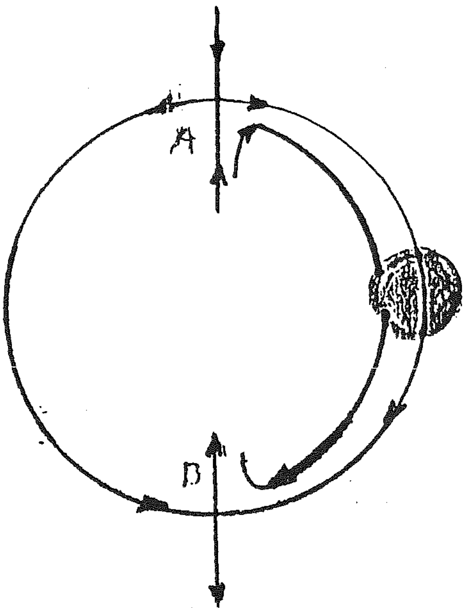
where the positive numbers p and q will be chosen later. The new system has the same invariant manifold M_0. On the axis Ox there are two equilibrium points $(-p, 0)$ and $(p, 0)$. Since $q > 0$ the equilibrium point $(-p, 0)$ is a knot and the equilibrium point $(p, 0)$ is a saddle. If $2p > q$ then at the point $(-p, 0)$ all the trajectories except the horizontal ones are tangent to a direction $(1/(2p - q), 1)$. Show that the invariant manifold M_0 is not indestructible with respect to perturbation of the new system. For this we construct a perturbation concentrated near the point $(0, 0)$ which moves an unstable manifold of the point $(p, 0)$ above the axis Ox. Then at point $(-p, 0)$ this manifold is tangent to the direction $(1/(2p - q), 1)$. Hence the perturbed system does not have an invariant manifold C^1 close to M_0.



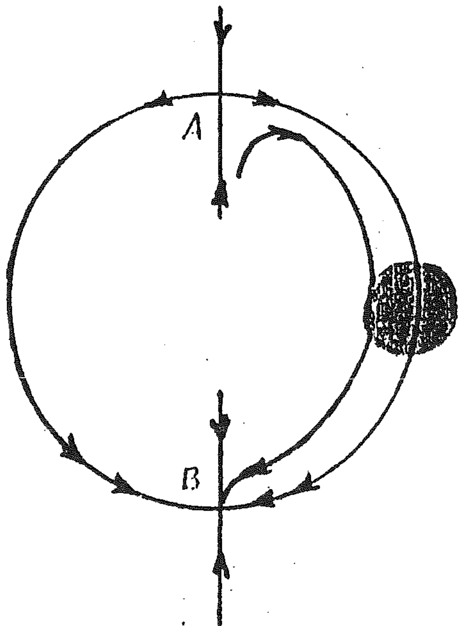
Ex.I



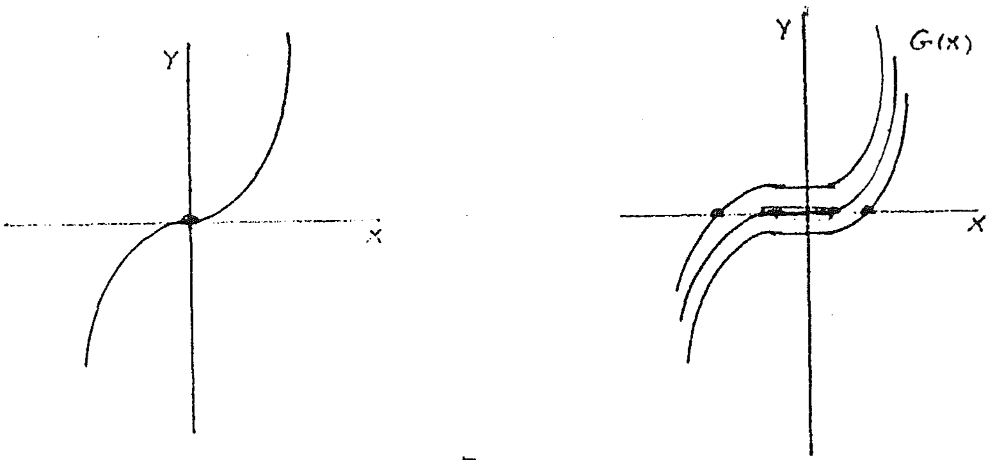
Ex.2



Ex 3

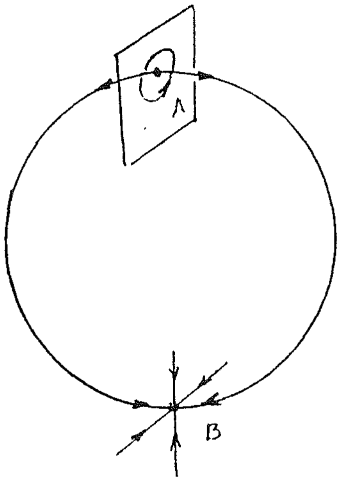


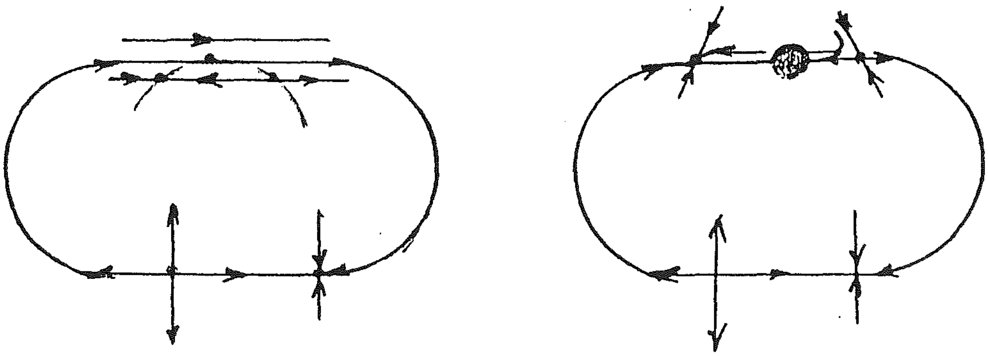
Ex.4



Ex. 5

Ex. 6





Ex. 7

References

1. R. Sacker, A perturbation theorem for invariant Riemannian manifolds, Differential Equations and Dynamical Systems, Proc. Symp. Diff. Eq. and Dyn. Syst. 1965, 43-54, Academic Press, New York, 1967,
2. Yu.I. Neimark, Integral manifolds of differential equations, Izv. Vuzov, Radiophysics 10 (1967), 321-334 (in Russian).
3. N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J. 21 (1971), 193-226.
4. R. Mañe, Persistent manifolds are normally hyperbolic, Trans. Amer. Math. Soc. 246 (1978), 261-280.
5. G.S. Osipenko, Examples of perturbations of invariant manifolds, Differential Equations 25 (1989), 675-681.
6. P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.
7. G.S. Osipenko, Coordinates in a neighborhood of invariant manifold, in "Differentsialnye and Integralnye Uraveniy'a, Gorkii, 1987, 52-56 (Russian).
8. M. Hirsch, C. Pugh, M. Shub, Invariant manifolds, Lecture Notes in Mathematics 583 (1977).
9. G.S. Osipenko, Perturbation of invariant manifolds III, Differential Equations 23 (1987), 556-561.

A UNIFIED APPROACH TO VARIOUS STOCHASTIC EQUATIONS WITH COEFFICIENTS DEPENDING ON THE PAST

Arcady Ponosov*

Fakultät für Mathematik, Ruhr-Universität Bochum
Universitätstraße, 150 W-4630 Bochum 1, Germany

Abstract

We discuss some properties of the rather general class of nolinear stochastic functional-differential equations which on one hand embraces various classes of hereditary equations which used to be studied independently, and on the other hand provides basic properties of equations; solvability, continuous dependence on initial data, etc.

1. Introduction

The central question of this paper is: what should be meant by a general stochastic functional-differential equation (SFDE)? Of course, many SFDEs can be covered by Hale-type equations, i.e., equations of the form:

$$dX_t = F(t, X_{t-})dz_t \tag{1.1}$$

where $X_t(s) = x_{t+s}$ is a stochastic process with values in the space of "initial functions", z_t is a semimartingale. Up to now the most advanced results concerning SFDEs have been obtained for equations of this type (see, e.g., [KN], [M1], [S], and references therein). Apart from these works there exist a number of papers devoted to "hereditary equations" of the form:

$$dx_t = f(t, x_{t-})dz_t \tag{1.2}$$

("Dolean-Protter" equation) with f depending on paths of solutions within the whole time interval $[0, t]$, (see, for instance, [JM], [Pr1], [WM]). Moreover, there are some sporadic SFDEs which "break down" usual frames. They are: stochastic integro-differential equations [MT], neutral equations [KN], equations with feedback in differential [Pr2], etc.

Our goal is to show how all the equations listed above can be treated within a general framework, i.e., as particular cases of a "general SFDE" for which the

* Supported by A. Von Humboldt Foundation.

basic properties hold (existence of solutions, continuous dependence on initial data, etc.). We should mention that our investigations have been strongly stimulated by the recent achievements in the deterministic theory of FDEs (see, e.g., the surveys [AM], [AMR], [R], and references therein). These achievement are assessed by this author as very fruitful. The results we are going to present here can, of course, be regarded as a generalization of deterministic ones.

2. Basic notations and preliminary results

We use mainly the terminology of the monograph [J1].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $0 \le t \le T$ be a stochastic basis with usual conditions ($(\mathcal{F}_t)$ is a right-continuous filtration; $\mathcal{F}_t, \mathcal{F}$ contain P-null sets), $z_t \equiv (z_t^j)$ be an m-dimensional semimartingale defined on $[0,T]$; (B, C, ν) be the triplet of its predictable characteristics: this means, in particular, that $B_t \equiv (B_t^j : 1 \le j \le m)$ is a predictable m-dimensional stochastic process with nondecreasing components, $C \equiv (C_t^{jk} : 1 \le j, k \le m)$ is a predictable nonnegative matrix, ν is a predictable random measure on $[0,T] \times \mathbf{R}^m$ [J2].

We also introduce a nondecreasing predictable process

$$\lambda_t \equiv \Sigma_{j \le m}(C_t^{jj} + \mathrm{Var}_{s \in [0,t]} B_s^j) + \int_{\mathbf{R}^m} \nu([a,t] \times dx) 1 \wedge |x|^2$$

($\wedge$-min, Var-variation) and Radon derivatives of (λ_t-absolutely continuous) functions B and $C + E$ w.r.t. λ (where $E_t^{jk} \equiv \int_{\mathbf{R}^m} \nu([0,t] \times dx) x^i x^j I_{\{|x| \le 1\}}$, x_i are the coordinate functions in $\mathbf{R}^m$):

$$B_t^j = \int_0^t \alpha_s^j d\lambda_s \, ; \qquad C_t^{jk} + E_t^{jk} = \int_0^t \beta_s^{jk} d\lambda_s \, . \tag{2.1}$$

We now describe the functional spaces we are going to deal with. The space k contains all $\mathcal{F}_0$-measurable random values. After identifying P-equivalent functions and endowing k with the topology of convergence in probability we get k to be a linear metric space. The space Λ_p $(1 \le p \le \infty)$ consists of row-vectors $H = (H^1, \ldots, H^m)$ with predictable components for which

$$\|H\|_{\Lambda_p}^p \equiv \int_0^T |H_s \alpha_s|^p d\lambda_s + \left(\int_0^T |H_s \beta_s H_s^T| d\lambda_s \right)^{\frac{1}{2}} < \infty \quad \text{a.s.}$$

Identifying H_1 and H_2 such that $\|H_1 - H_2\|_{\Lambda_p} = 0$ a.s. also yields a linear space with the metric $E(\|H_1 - H_2\|_{\Lambda_p} \wedge 1)$.

Using Jacod's description of z_t-integrable stochastic processes (see e.g., [J2]), one can easily see that for each $H \in \Lambda_p$ the stochastic integral $\int_0^t H_s dZ_s$ does exist, and determines the continuous operator from Λ_p to k for each t.

Now define the following set:

$$S_p \equiv \left\{ x \mid x_t - x_0 = \int_0^t H_s dz_s, \, x_0 \in k, \, H \in \Lambda_p \right\} .$$

If we identify indistinguishable stochastic processes in S_p (see e.g., [J1]) we get

2.1 Proposition. *Under the identifications just described we have the isomorphism*

$$S_p \cong \Lambda_p \times k \tag{2.2}$$

given by $x_t = \int H_s dZ_s + x_0$.

2.2 Remark. Using (2.2) we can equip the space S_p with the topology of direct product. This topology is a little bit stronger than the Emery topology of the semimartingale space S [E]. This is due to the fact that being a linear subspace of S, the space S_p is closed w.r.t. its own topology.

2.3 Remark. Using the (rather convenient) deterministic terminology (see e.g., [AMR]) we can call k a "space of initial data", S_p a "space of solutions", and Λ_p a "space of (abstract) derivatives". It will be shown later that, in fact, these spaces do play the same role as the spaces $\mathbf{R}$, L_p and W_p (absolutely continuous functions with p-summable derivatives) in the deterministic theory.

The important feature of the space W_p which leads to the basic properties of deterministic FDEs, consists of their compact imbedding in L_p (for $q < \infty$) and C (for $p > 1$). Thus, using this property, one can apply the theory of linear and nonlinear compact operators, which is rather powerful machinery.

So it would make sense to understand similar properties of the spaces S_p in order to find out which kinds of operators might be useful for the stochastic case.

First, note that we have to consider the Skorohod space D instead of C because of possible discontinuity of solutions. More precisely, we will deal with spaces $\widetilde{X}$ consisting of $(\mathcal{F}_t)$-adapted stochastic process with trajectories belonging to a given functional space X, $\widetilde{X}$ will be presumed to be equipped with the metric $E\big(||x-y||_X \wedge 1\big)$. The role of the space C will be played, in particular, by the space $\widetilde{D}$ generated by the Skorohod space of right-continuous and having left-hand limit functions with the sup-norm.

The following theorem was proved in [P5] for the case of Ito integrals. Slightly modifying the proof and using standard estimates for stochastic integrals w.r.t. semimartingales (see [J1] or [J2]) we get

2.4 Theorem A. *For* $p > 1$ *each set bounded in* S_p *is tight in* $\widetilde{D}$. **B.** *For* $p \geq 1$, $q < \infty$, *each set bounded in* S_p *is tight in* $\widetilde{L}_q$.

Recall that a set Q of random points (processes) is called tight if for any $\varepsilon > o$ there exists a compact set K such that $P\{\omega \mid x(\omega) \notin K\} < \varepsilon$ whenever $x \in Q$.

Surely this result is far from being unexpected. It is well known that, at least for equation (1.2), bounded sets of solutions are tight in $\widetilde{D}$ [JM]. On the other hand, Theorem 2.4 shows that we cannot use the theory of compact operators for our purposes and therefore we have to develop a new one which will treat operators mapping bounded sets into tights. This justifies the following

2.5 Definition. *We say that an operator h defined in a linear space of random points (processes) is tight if 1) it transforms bounded sets of its domain into tight ones, and 2) it is uniformly continuous on each tight set taking from its domain.*

One can easily observe that in case Ω shrinks into a singleton, we get the definition of compact operators which are continuous and hence uniformly continuous on compacts. It is not clear whether the condition of the uniform continuity from Definition 2.5 can be replaced (at least in what follows) by usual continuity.

Unlike the deterministic case, the tightness property of operators turns out to be too general. Thus, every nonlinear uniformly continuous operator defined on an arbitrary linear subspace of, say, k is tight. It is absolutely evident that the possible theory of such operators would be poor enough. We therefore have to introduce more hypotheses. Our point is: it would be natural to take into account the "trajectoral" nature of SFDEs. To do this we introduce the following notion (see also [Sh], [P1]).

2.6 Definition. *An operator h is said to be of local type (or simply local) if for any x, y from its domain, and for any $A \in \mathcal{F}$ the equality $x(\omega) = y(\omega)$ a.s. on A always implies that $(hx)(\omega) = (hy)(\omega)$ a.s. on A.*

Examples of local operators: 1) random operators (more precisely, superposition operators generated by random operators): $(hx)(\omega) = T(\omega, x(\omega))$; 2) stochastic integrals (also extended ones, see e.g., [NP]; 3) combinations of 1) and 2), for instance, sums, products, superpositions, etc.

Our basic observations is that there exists a nontrivial theory of local tight operators which on the one hand inherits much from the theory of compact operators, and on the other hand supplies us with a machinery which might be useful for SFDEs.

3. General SFDEs with Driving Semimartingales. Some Examples

By a general SFDE we understand the following equation:

$$dx = Fx\, dz\ , \tag{3.1}$$

where $F : S_p^n \to \Lambda_p^n$ is a local and tight operator which is additionally assumed to be independent of the future: if $x_s = y_s$ a.s. $\forall s \in [0, t]$, then $Fx_s = Fy_s$ a.s. $\forall s \in [0, t]$, where t is an arbitrary time point taking from the interval $[0, T]$. Sometimes such operators are called Volterra-type (or simply Volterra operators).

Let us now explain exactly why these conditions have been chosen. The independence of the future is related, of course, to the conditions of integrability of Fx w.r.t. z, but not only to this. It also affords us a convenient opportunity to treat solutions locally, i.e., in neighborhoods of initial points, and then to extend them as far away as possible.

The fact that we take S_p^n instead of the usual $\widetilde{D}^n$ as a domain of F is determined by examples of SFDEs. It is so for neutral equations and for some equations with hysteresis-type nonlinearities. This means that not every hereditary equation can be described by (1.1) or (1.2). On the other hand, taking S_p^n as a domain seems to be even more natural, since all solutions to (3.1) must be semimartingales of a particular form, and from this point the space $\widetilde{D}^n$ might be (and in some cases is) too wide. In fact, the space S_p^n is more convenient also from a purely theoretical point of view. It was first realized in the deterministic theory of FDEs (see, e.g., [AM], [AMR]).

Let us now turn to the properties of locality and tightness. The first means that we wish to consider only "trajectorial-type" equations, i.e., involving no averaged characteristics, expectations, etc. One might ask why we do not restrict ourselves to the case of random operators F as it is usually assumed. First, it is due to examples, neither neutral nor integro-differential equations are covered by SFDEs with random F. This was noticed also by S. Mohammed [M2, p. 6].

The most problematic assumption is probably the latter, i.e., the tightness of F. However, as it follows from examples (see below), it is actually not very restrictive. In fact, it holds for most SFDEs, except maybe some particular neutral equations. The tightness reflects the general fact that by virtue of Theorem 2.4, bounded sets of solutions of finite dimensional stochastic equations should be tight (see also Section 4).

Let us now turn to examples.

A. Ordinary equations with driving semimartingales.

$$dx_t = f(t, x_{t-})dz_t \tag{3.2}$$

Assume that

A1) $f : [0,T] \times \Omega \times \mathbf{R^n} \to \mathbf{R^{m\times n}}$ is continuous in $x \in \mathbf{R^n}$ and predictable in $(t,\omega) \in [0,T] \times \Omega$ with values in the space of $m \times n$- matrices denoted by $\mathbf{R^{m\times n}}$;

A2) for any $R > 0$ there is a stochastic process φ_t^R, summable w.r.t. λ_t and such that

$$\Gamma_p\big(t, f(t,x)\big) \le \varphi_t^R \quad \text{a.s.} \quad \big(\forall\, t \in [0,T]\,, \quad \forall\, x \in \mathbf{R}^n\,, \quad \text{s.t.}\ |x| \le R\big)\,, \tag{3.3}$$

where

$$\Gamma_p(t,u) \equiv |u\alpha_s|^p + |u\beta_s u^T|^p \tag{3.4}$$

with α, β given by (2.1).

3.1 Remark. Recall that by definition $|\alpha_s| \le 1$, $|\beta|_s \le 1$ and hence (3.3) is implied by the more simple inequality:

$$|f(t,x)|^{2p} \le \varphi_t^R \qquad (|x| \le R).$$

Note, however, that the latter estimate is sometimes too restrictive. Thus, for Ito-type equations, the drift coefficient can be p-integrable, while the diffusion one should be $2p$- integrable.

3.2 Theorem. Under Assumption A1 and A2, the operator $Fx_t \equiv f(t, x_{t-})$ is local, tight, and Volterra as the operator from S_p^n to Λ_p^n. In other words, (3.2) is a particular case of (3.1).

3.3 Remark. Clearly, the case of $p = 1$ gives the least restrictive estimate, and at the same time the widest space of solutions, namely S_1^n. The smaller the space of solutions, the stronger the estimates required.

Proof of Theorem 3.2. In our particular situation the operator F is random, $(Fx_t)(\omega) = f(t, \omega, x_t - (\omega))$, the generating function $f(t, \omega, x)$ being continuous in x. Therefore, F is local, Volterra, and uniformly continuous on tight subsets as the operator from $\widetilde{D}^n$ to Λ_p^n (the latter is ensured by A2). Clearly, the same is true for F considered in the space S_p^n endowed with the stronger topology. Moreover, since bounded subsets of the space $S_p^n (p > 1)$ are tight in $\widetilde{D}^n$, and since (evidently) F maps tight subsets of $\widetilde{D}^n$ into tight subsets of Λ_p^n, F is tight as the operator from S_p^n to $\Lambda_p^n (p > 1)$. If $p = 1$, we should first observe that for each $R > 0$ the operator $F_R x_t = f(t, \pi_R(x_{t-}))$ can be extended to the space $\widetilde{L}_2^n$ as a continuous random operator taking its values in Λ_p^n. Here π_R is just a projector of $\mathbf{R}^n$ onto the ball $B[0, R]$. Repeating the above argument and applying the second part of Theorem 2.4, we get the tightness of F_R as the operator from S_1^n to Λ_1^n. It remains to note that any bounded subset of S_1^n satisfies the condition:

$$\forall \varepsilon > 0 \; \exists R_\varepsilon \quad \text{such that} \quad P\{||x||_{D^n} > R_\varepsilon\} < \varepsilon \; ,$$

and the result follows. □

3.4 Remark. A slight modification of the above proof yields a generalization of Theorem 3.2 for the case of the Dolean-Protter equation (1.2) under conditions similar to A1 and A2. In fact, we have only to replace the domain of f ($\mathbf{R}^n$ by D^n) and to impose the Volterra condition on f. Therefore Equation (1.2) can also be put in our scheme.

The next example is actually a particular case of Equation (1.2), but it illustrates why Equation (1.1) can be involved in our consideration as well.

B. Stochastic delay equations.

$$dx_t = f(t, x_{t-}, Tx_t) dz_t \tag{3.5}$$

$$x_s = \varphi_s \, , \quad s < 0 \, , \tag{3.6}$$

where $Tx_t = \int_{(-\infty,t)} d_s R(t, s) x_s$.

Assume that

B1) $f : [0, T] \times \Omega \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{m \times n}$ is continuous in $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$ and predictable in $(\omega, t) \in [0, T] \times \Omega$;

B2) for any $R > 0$ the following estimate is fulfilled: $\Gamma_p(t, f(t, x, y)) \leq \varphi_t^R + c|y|^q$ ($c \geq 0$; $p, q \geq 1$; φ_t^R as in A2, $y \in \mathbf{R}^n$, $t \in [0, T]$, $|x| \leq \mathbf{R}^n$; Γ_p is given by (3.4));

B3) the kernel $R : [0,T] \times [-\infty, T] \times \Omega \to \mathbf{R}^{n\times n}$ is $\mathcal{B} \otimes \mathcal{P}$-measurable (where $\mathcal{B}$ and $\mathcal{P}$ stand for the σ-algebra of Borel and predictable sets respectively; we put $\mathcal{F}_s = \mathcal{F}_0$ for $s < 0$ for convenience) and satisfies the following condition:

$$\int_0^T [\mathrm{Var}_{s\in[0,T]} R^{ij}(t,s,\omega)]^{2q} d\lambda_t(\omega) < \infty \quad \text{a.s. ;}$$

B4) φ is $\mathcal{B} \otimes \mathcal{F}_0$-measurable and locally bounded on $(-\infty, 0) \times \Omega$ stochastic process.

3.5 Theorem. *Under conditions B1–B4, equation* (3.5) *coupled with Condition* (3.6) *is a particular case of* (3.1) *with* $F : S_n^p \to \Lambda_n^p$.

Proof: Note that by virtue of the properties B1–B4 the operator

$$Fx_t \equiv f\left(t, x_{t-}, \int_{[0,t)} d_s R(t,s) x_s + \psi_t\right)$$

satisfies conditions A1–A2 with the corrections mentioned in Remark 3.4. Here $\psi_t \equiv \int_{(-\infty,0)} d_s R(t,s)\psi_s$. □

3.6 Remark. It looks strange that we involve in the equation what is normally regarded as an "initial function". Let us, however, explain our approach. First of all, we should observe that the time point 0 has been excluded from the considerations. So we still need to add an initial condition to our equation. However, it is not infinite dimensional any more, and our delay equation can be treated as a finite dimensional one! Such a transformation does not seem to be anything special, but actually it changes the very nature of the equation. It was shown convincingly by recent developments in the deterministic theory of FDEs (see e.g., [AMR]). Let us mention here only one advantage of this approach. After our transformation we are not confined to a particular (sometimes rather complicated) space of initial functions. We do not need it any more! This proved to be very important for the stability theory (taken just as an example). Of course this is also important for our purposes, as it allows us to involve delay equations in the general framework.

3.7 Remark. Such a transformation can be made not only for a particular equation (3.5) but also for the general equation (1.1).

C. Integro-differential equations.

$$dx_t = f(t, x_t, Ux_t) dz_t, \tag{3.7}$$

where U is a nonlinear integral operator of the form

$$Ux_t = \psi + \int_{[0,t)} H(t,s,x_s) dz_t .$$

Such equations w.r.t. the brownian motion were studied in [MT] under Lipschitz conditions.

We assume that

C1) f satisfies Condition B1;

C2) for any $R > 0 \Gamma_p(t, f(t,x,y)) \leq \psi_t^R$ ($p \geq 1$, ψ^R as in A1 $t \in [0,T]$, $x, y \in \mathbf{R}^n$, $|x|, |y| \leq R$; Γ_p is given by (3.4));

C3) the functions $H_{ij} : [0,T] \times [0,T] \times \mathbf{R}^n \to \mathbf{R}$ ($i = 1, \ldots, n$; $j = 1, \ldots, m$) are absolutely continuous in the first variable and their derivatives w.r.t. it are $\mathcal{B} \otimes \mathcal{P}$-measurable in (t, s, ω) and continuous in x;

C4) the following estimates hold for any $R > 0$:

$$\left|H^{ij}(s,s,\omega,x)\right| \leq Q_R(t,s,\omega) \; ; \quad \left|\frac{\partial H^{ij}}{\partial t}(t,s,\omega,x)\right| \leq K_R(t,s,\omega)$$

and

$$\int_0^T d\lambda_s \left(Q_R^2(s,\omega) + \int_s^T K_R^2(\tau,s,\omega) d\tau \right) < \infty \quad \text{a.s.}$$

C5) $\psi : \Omega \times [0,T] \to \mathbf{R}^n$ is predictable and locally bounded. Some of these conditions can be relaxed if z has independent increments.

C3a) H^{ij} are $\mathcal{B} \otimes \mathcal{P}$-measurable in (t, s, ω) and continuous in x;

C4a) $\left|H^{ij}(t,s,\omega,x)\right| \leq Q_R(t,s,\omega)$ ($|x| \leq R$, $R > 0$) and

$$\int_0^T \int_0^T \left|Q_R(t,S)\right|^q d\lambda_t d\lambda_s < \infty \text{ , a.s.}$$

for some $q > 2$;

C5a) ψ is predictable and q-summable in t.

But now we have to impose an additional condition on the growth of t:

C2a) $\Gamma_p(t, f(t,x,y)) \leq \psi_t^R + c|x|^q \qquad (c \geq 0 \text{ a.s.}, |x| \leq R,\ R > 0)$.

3.8 Theorem. *Under assumptions C1–C5, or, if z_t has independent increments, under assumptions C1, C2a–C5a, equation (3.7) is a particular case of equation (3.1) with $F : S_n^p \to \Lambda_n^p$.*

Proof: Taking into acccount the proofs of the two preceding theorems, it is sufficient to establish that the integral operator

$$Ax_1 = \int_{(0,t)} H(t,s,x_s) dz_s$$

is uniformly continuous on tight subsets of the space $\widetilde{D}^n$ and takes values in $\widetilde{L}_\infty^n(\lambda)$ or in $\widetilde{L}_q^n(\lambda)$ if conditions C3 and C4 (resp. conditions C3a and C4a) are fulfilled. Both cases can be treated in a similar way. So we can restrict ourselves to the first one.

Let us represent the semimartingale z_t as a sum:

$$z_t = B_t + \beta_t + z_t I_{|\Delta z|>1}(t)\,,$$

where B_t is the first local characteristic of z_t, while the second term is a local martingale.

Now the integral operator

$$A_1 x_t = \int_{(0,t)} H(t,s,x_s)d\left(B_T + z_t I_{|\Delta z|>1}(t)\right)$$

is simply a random Stiltjes integral operator driving by the process which is absolutely continuous w.r.t. λ_t, the kernel satisfying the following estimate:

$$|H(t,s,x)| \le \max_j \left| H^j(s,s,x) + \int_s^t H'_\tau(\tau,s,x)d\tau \right| \le$$
$$\le Q_R(s) + \int_S^t K_R(t,S)d\tau \in L_1(\lambda(\omega)) \text{ a.s.}$$

Hence, the operator A_1 maps $\widetilde{D}^n$ into $\widetilde{L}^n_\infty(\lambda)$ and it is uniformly continuous on tight sets of its domain.

It remains to study the integral operator

$$A_2 x_t = \int_{[0,t)} H(t,s,x)d\beta_s$$

which can be represented (see e.g., [Pr3]) as follows

$$A_2 x_t = \int_{[0,t)} H(s,s,x_s)d\beta_s + \int_0^t d\tau \int_{[0,\tau)} H'_\tau(\tau,s,x_s)d\beta_s\,.$$

For any predictable stopping time $T_n \le T$ we have

$$E \sup_{t\le T_n} |A_2 x_t|^2 \le 2E \sup_{t\le T_n} \left(\int_{[0,t)} H(s,s,x_s)d\beta_s \right)^2 +$$
$$+2E \sup_{[0\tau)} \left(\int d\tau \int_{[0,\tau)} H'_\tau(\tau,s,x_s)d\beta_s \right)^2$$

and using the standard technique of estimating stochastic integrals w.r.t. semimartingales we obtain:

$$E \sup_{t\le T_n} |A_2 x_t|^2 \le K_1 \left(E\int_0^{T_n} Q_R^2(s)d\lambda_s + \left(\int_0^{T_n} Q_R(s)d\lambda s \right)^2 \right) +$$

$$+K_2 E \int_0^{T_n} d\lambda_s \int_s^T K_R^2(\tau, s) d\tau ,$$

so that $P\{\sup_{t\leq T} |A_2 x_t|^2 \geq K_\varepsilon\} < \varepsilon$ for abitrary $\varepsilon > 0$ and sufficiently large K_ε. On the other hand, the process $\xi \equiv \int_{[0,t)} H(t, s, x_s) d\beta_s$ admits a $\mathcal{F}_{t-} \otimes \mathcal{B}$-measurable version, hence it is equivalent to a predictable process. We have just proved that $A_2\left(\widetilde{L}_\infty^n(\lambda)\right) \subset \widetilde{L}_\infty^n(\lambda)$.

Exactly the same reasoning yields the estimate

$$E \sup_{t\leq T_n} |A_2 x_t - A_2 y_t|^2 \leq C\Bigg(E \int_0^{T_n} \left[H(s,s,x_s) - H(s,s,x_s)\right]^2 d\lambda_s +$$

$$+E \int_0^{T_n} d\lambda_s \int_s^T \left[H'_\tau(\tau, s, x_s) - H'_\tau(\tau, s, y_s)\right]^2 d\tau\Bigg) . \tag{3.8}$$

Consider the random integral operator

$$I(x_t, y_t)(\omega) \equiv \int_{[0,t)} G(s, \omega, x_s, y_s) d\lambda_s ,$$

where G equals either $[H(s,s,x) - H(s,s,y)]^2$, or $\int_s^T (H'_\tau(\tau,s,x) - H'_\tau(\tau,s,y))^2 d\tau$.

By our assumptions, $I(\omega)$ is continuous for almost all ω as an operator from $L_\infty^{2n}(\lambda(\omega))$ to $L_\infty^n(\lambda(\omega))$. Hence I considered as a superposition operator from $\widetilde{L}_\infty^{2n}(\lambda)$ to $\widetilde{L}_\infty^n(\lambda)$ is easily seen to be uniformly continuous on tight sets. In particular, given a tight set $Q \subset \widetilde{L}_\infty^n(\lambda)$ we have

$$P - \lim_{\delta\to+0} \sup \left| \int_{[0,t)} G(s, x_s, y_s) d\lambda_s \right| = 0 ,$$

where supremum is taken over $t \in T$, $x, y \in Q$, $E(\|x-y\|_{D^n} \wedge 1) \leq \delta$. In other words, for some exhaustive sequence of predictable stopping times

$$\lim_{\delta\to+0} \sup \left| \int_{(0,t\wedge T_n)} G(s, x_s, y_s) d\lambda_s \right| = 0 .$$

Making use of estimate (3.11) we obtain that

$$\lim_{\delta\to+0} \sup |A_2 x_{t\wedge T_n} - A_2 y_{t\wedge T_n}|^2 = 0$$

This means that A_2 is continuous on Q.

Let us complete the proof of Theorem 3.8. If $p > 1$, then the result immediately follows from Theorem 2.4. If $p = 1$, we have first to replace our operator A by a "truncated" one, as was done once in the course of the proof of Theorem 3.2. □

D. Neutral stochastic equations. We are not going to study here all possible classes of neutral equations. It would extend our paper enormously. Instead, we will consider only one particular type of neutral equation which is rather illustrative, but comparatively simple technically:

$$dx_t = f(t, Tx_t, Sx_t)dz_t \tag{3.9}$$

where

$$Tx_t = \int_{(-\infty,t)} d_s R(t,s)x_s\,, \qquad Sx_t = \int_{(-\infty,t)} Q(t,s)dx_s\,.$$

Of course, we have to supply this equation by a "prehistory":

$$x_s = \xi_s\,, \quad s < 0\,. \tag{3.10}$$

Now introduce the following assumptions:

D1) f satisfies B1;

D2) $\forall R > 0\ \Gamma_p(t, f(t,x,y)) \le \varphi_t^R + |x|^q$ $(p > 1,\ q \ge 1;\ t \in [0,T];\ x, y \in \mathbf{R}^n;$ $|y| \le R$; Γ_p is given by (3.4));

D3) R satisfies B3;

D4) Q is absolutely continuous in t, while its derivative Q'_t satisfies the same measurablity condition as R and additionally the following estimate:

$$\int_0^T \left(\int_s^T |Q'_\tau(\tau,s)|^2 d\tau \right)^r d\lambda_s + \int_0^T |Q(s,s)|^{2r} d\lambda_s < \infty \text{ a.s.}$$

D5) $\int_{-\infty,0)} Q(t,s)d\varphi_s$ exists and is locally bounded in t.

3.9 Theorem. *Under assumptions D1–D5, equation* (3.9) *with the "prehistory"* (3.10) *is a particular case of the general equation* (3.1) *where* $F : S_p^n \to \Lambda_p^n$.

Sketch of the proof. The crucial point is to verify the tightness of the operator $S_0 x_t \equiv \int_{[0,y)} Q(t,s)dx_s$ regarded as a mapping from S_p^n to $\widetilde{D}^n$. The following estimates can be derived easily from the proof of the preceding theorem:

$$E \sup_{0\le t\le T_n} |S_0 x_t|^2 \le E\left(\int_0^{T_n} \left(\int_s^b |Q'_\tau(\tau,s)|^2 d\tau \right)^r d\lambda_s \right)^{\frac{1}{r}} +$$

$$+E\left(\int_0^{T_n} |Q(s,s)|^{2r} d\lambda_s \right)^{\frac{1}{r}} \tag{3.11}$$

for some exhaustive sequence $\{T_n\}$ of predictable stopping times. This means that for any $\varepsilon > 0$ there is a number N for which $P\{T_n < T\} < \varepsilon$ for all $n \ge N$. So S_0 is a bounded linear operator from S_p^n to $\widetilde{D}^n$. Now approximating the kernel Q by degenerated kernels Q^n and using the similar estimate as (3.11), we obtain that S_0 can be uniformly approximated by finite dimensional random (and hence tight) operators acting from S_p^n to $\widetilde{D}^n$. Therefore, S_0 should be tight as well. Now combining the proof of Theorem 3.2 with the fact just established we get the required result. □

4. Basic properties of Equation (3.1).

As we have already seen, a large number of SFDEs can be regarded as particular cases of equation (3.1) under rather weak assumptions. In this section we would like to state some general results concerning solvability, continuous dependence on initial data, etc. of equation (3.1). To do this we use some rather profound facts from the theory of local-type operators. However, let us first formulate the results we have been keeping in mind throughout the preceding sections.

1. For any initial conditions $x_0 = \kappa \in k^n$ there is at least one weak solution to equation (3.1) defined on a random interval $[a, \tau]$ where $\tau > a$ a.s. is a stopping time.
2. Any weak solution of the Cauchy problem $x_0 = \kappa$ for equation (3.1) can be extended up to either the time-point b, or an explosion time.

It is assumed in the next two properties that all solutions of (3.1) reach the time-point b.

3. Any set of solutions of (3.1) bounded in S_p^n is tight in S_p^n (and hence in $\widetilde{D}^n$).
4. If all solutions to (3.1) with all initial data $\kappa \in k^n$ satisfy the property of the pathwise uniqueness, then they are all strong (i.e., they are defined on the initial stochastic basis) and continuously (in S_p^n-topology) depend on $\kappa \in k^n$.

These properties were proved in [P7] for the case of Ito-type SFDEs. So it would probably make sense not to repeat it in detail, but only to highlight main points and to consider new difficulties more thoroughly.

First, however, let us give a more accurate definition of a weak solution to equation (3.1).

4.1 Definition. A stochastic basis $(\Omega^*, \mathcal{F}^*, P^*)$ is called a (regular) splitting of the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ if there exists a $(\mathcal{F}^*, \mathcal{F})$-measurable surjective mapping $c : \Omega^* \to \Omega$ such that:

1) $P^* c^{-1} = P$;
2) $c^{-1}(\mathcal{F}_t) \subset \mathcal{F}_t^* \quad (\forall t)$;
3) $z_t c$ is again a semimartingale on $(\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)$ with the same local characteristics as z_t.

The third property implies, in particular, that brownian motion still preserves its properties after regular splitting.

4.2 Definition. A weak solution to equation (3.1) is a stochastic process which is defined on some regular splitting of the initial stochastic basis, which belongs to the space $S_p^{n*} \supset S_p^n$ (the space S_p^{n*} as well as the space Λ_p^{n*} are constructed similarly to S_p^n and Λ_p^n w.r.t. the new stochastic process) and which satisfies an equation

$$dx_t = F^* x_t d(z_t c)$$

on a random interval $[0, \tau]$, where $F^* : S_p^{n*} \to \Lambda_p^{n*}$ is the (unique) local continuous operator extending F.

In fact, F^* is also tight.

Let us observe that such an extension of F does exist. This statement is far from being trivial in general (see [P2] for the proof), but it is rather evident in particular examples.

Thus, in Examples A and B the operator F is random, i.e., $(Fx)(\omega) = P(\omega, x(\omega))$ for appropriate P, so that $(F^*x)(\omega^*) = P(\omega, x(\omega^*))$, $\omega^* \in \Omega^*$. In Example C we have $Fx = f(t, x, Ux_t)$, $Ux_t = \int_{[0.t)} H(t, s, x_s)dz_s$, so that $F^*x = f(t, x, U^*x_t)$, where $U^*x_t = \int_{[0,t)} H(t, s, x_s)d(z_sc)$.

Finally, in Example D

$$Fx = f(t, T_0x, S_0x) \ , \quad S_0x = \int_{[0,t)} Q(t, s)dx_s$$

and

$$F^*x = f(t, T_0x, S_0^*x) \ , \quad S_0^*x = \int_{[0,t)} Q(t, s)dx_s \quad (x \in S_p^{n*}) \ .$$

Proof of the stated results. We consider the following operator equation in Λ_p^n:

$$H = \Phi H \ , \tag{4.1}$$

where $\Phi H \equiv F(\kappa + \int_{[0,\cdot)} H_s dz_s)$. By assumptions, Φ is local, tight and Volterra. So we might apply the fixed point theorem for local tight operators which states that if such an operator h has an invariant ball in a space of random points (processes) which satisfies the so-called "Π-property" (see below), then h has at least one weak fixed point (see [P6] for the proof, or [P3] for the formulation).

It is easy to check that the space Λ_p^n satisfies the "Π-property": there exists a sequence of random finite dimensional Volterra projections $P^m : \Lambda_p^n \to \Lambda_p^n$, strongly convergent to the identity (see also [P4] for the proof). The only thing we have to verify, therefore, is the existence of an invariant ball. Of course this property fails in general, but fortunately we have the Volterra property of F and the technique proposed in [P7]. To exploit this we should find a random Volterra projection $\pi : \Lambda_p^n \to B^n$ where B is given by

$$B = \left\{ x \in \Lambda_p : \int_0^T \Gamma(t, x_t)d\lambda_t \le 1 \right\} .$$

and Γ_p is defined in (3.4). Without loss of generality we can assume that $n = 1$.

Put

$$\pi x_t = x_t((I_{\{\Gamma_p(t,x_t)<1\}} + \gamma_x I_{[\tau]}) \ ,$$

where $\tau \equiv \inf\{t : \int_0^t \Gamma_p(s, x_s) \ge 1\}$, and $\gamma_x \ge 0$ is chosen in such a way that $\gamma_x = 0$ if $\tau = +\infty$ and

$$\int_0^\tau \Gamma_p(s, \pi x_s)d\lambda_s \le 1 \tag{4.2}$$

Let us show the existence of such γ_x for the case of $\tau < +\infty$. Consider an equation

$$Au^2 + Bu + C = 1$$

where $A = |x\beta_t x|^p \Delta\lambda_\tau$, $B = |x\alpha_t|^p \Delta\lambda_\tau$, $C = \int_{[0,\tau)} \Gamma_p(s, x_s) d\lambda_s$.

Since $A, B, C \geq 0$ and $C \leq 1$ there is only one positive solution u_x to this equation. Put $\gamma_x = \sqrt[p]{u_x}$. We have then

$$\int_0^\tau \Gamma_p(s, \pi x_s) d\lambda_s = \int_{[0,\tau)} \Gamma_p(s, x_s) d\lambda_s + \Gamma_p(\tau, \gamma_x x_\tau)\Delta\lambda_\tau = \gamma_x^p B + \gamma_x^{2p} A + C = 1$$

and therefore

$$\int_0^T \Gamma_p(s, \pi x_s) d\lambda_s = 1\,, \qquad \text{if} \quad \tau < 0$$

and

$$\int_0^T \Gamma_p(s, \pi x_s) d\lambda_s < 1\,, \qquad \text{if} \quad \tau = \infty$$

(of course, all here depends on ω).

Clearly, γ_x is a continuous function of $x \in \Lambda_p$. We claim that (i) $\pi(\Lambda_p) \subset B$, (ii) $\pi x = x$ if $x \in B$, (iii) $x^n \to x$ implies $\pi x^n \to \pi x$. To see this, note that (i) follows directly from (4.2), (ii) can be deduced from the following: $\int_0^T \Gamma_p(s, x_s) d\lambda_s < 1\,, \Rightarrow \tau = +\infty \Rightarrow \gamma_x = 0$ & $[0,\tau) \cap [0,T] = [0,T] \Rightarrow \pi x_t = x_t$. Finally, in order to prove continuity of π we have to use continuity of the integral and the continuity of γ_x.

Let us come back to the properties of equation (3.1) and consider an operator equation

$$H = \pi\Phi H$$

in the space Λ_p^n.

By the fixed point theorem for local tight operators [P3] there is at least one weak solution to this equation, belonging, in general, to the enlarged space Λ_p^{n*}. Put $\tau = \inf\{t : ||(\pi\Phi)^* H^* I[0,t]||_{\Lambda_p^{n*}} \geq 1\}$. Then τ is a predictable stopping time and, moreover, since

$$\lim_{t \to +0} ||(\pi\Phi)^* H^* I_{[0,t]}||_{\Lambda_p^{n*}} = 0$$

we have $P^*\{\tau > 0\} = 1$.

As τ is predictable, there exists a stopping time η, $0 < \eta < \tau$. We have therefore $\pi(\Phi^* H^*) I_{[0,\eta]} = F^* H^* I_{[0,\eta]}$. Now taking into account that Φ is local and Volterra we deduce

$$\Phi^*(H^* I_{[0,\eta]}) = (\Phi^* H^*) I_{[0,\eta]} = \pi(\Phi^* H^*) I_{0,\eta]} = H^* I_{[0,\eta]}\,,$$

so that $x_t^* \equiv \kappa + \int H_s^* d(z_s c)$ is a weak solution to (3.1) on the random segment $[0,\eta]$, and the first property follows.

To prove the second one we inductively construct a sequence of weak solutions $x_t^m \equiv x_0 + \int_0^t H_s^m d(z_s c^m)$ defined on some $[0, \eta_m]$, where $c^m : \Omega_m \to \Omega$ stands for the m-th splitting mapping and H_t^m is a weak solution to the equation

$$H = \Phi_m H \equiv \pi_m \Phi(H I_{[\eta_{m-1},T]} + H^{m-1} I_{[0,\eta_{m-1})}) , \tag{4.3}$$

π_m being a Volterra projection onto the set $\{\Gamma_p(t, H_t) \leq m\}$. We also put $\Omega^0 = \Omega$, $\mathcal{F}^0 = \mathcal{F}$, $\mathcal{F}_t^0 = \mathcal{F}_t$, $P^0 = P$ for convenience. At least one weak solution H^m to equation (4.3) does exist due to the same fixed point theorem for local tight operators, and as before we can define the stopping time η^m by

$$\eta^m \equiv \inf\{t : ||\Phi_m H^*|| \geq m\} .$$

Now using the definition of Φ_m, we have:

$$H^m|_{[0,\eta^{m-1})} = H^{m-1} c^{m,m-1} ,$$

where $c^{m,m-1} : \Omega^m \to \Omega^{m-1}$ connects two splittings and $(\Omega^m, \mathcal{F}^m, \mathcal{F}_t^m, P^m)$ is exactly the splitting on which H^m is defined.

These splittings form a projective family of probability spaces and a simple computation shows that its projective limit $\{\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P}\}$ does exist and inherits the property of regularity. Putting $\eta = \sup_m\{\eta^m\}$, $\bar{H}|_{[0,\eta^m)} \equiv H^m \bar{c}^m$ and $\bar{x}_t \equiv \kappa + \int_0^t \bar{H}_s d(z_s \bar{c}^0)$, where $\bar{c}^m : \bar{\Omega} \to \Omega_m$ $(m \geq 0)$ are projections generated by the projective limit, we get the predictable stopping time and the solutions to equations (4.1) and (3.1) respectively. Moreover, as follows immediately from the definition of Φ_m,

$$\bar{P}\{||\bar{H} I_{[0,\eta)}||_{\Lambda_p^n} = +\infty\} + \bar{P}\{\eta = T\} = 1$$

and therefore

$$\bar{P}\{||\bar{x} I_{[0,\eta)}||_{\bar{D}_p^n} = +\infty\} + \bar{P}\{\eta = T\} = 1$$

This ends the proof of the second property.

The proofs of the third and fourth properties are similar to those presented in [P7] and hence can be omitted.

5. Conclusion.

We have shown that there exists a quite general class of SFDEs which on the one hand embraces many particular examples of SFDEs, and on the other hand admits basic properties of differential equations: existence, continuous dependence on initial data, etc. We should however remark that there are some classes of SFDEs which cannot be put into this framework, e.g., neutral equations involving "delayed derivatives". Such equations arise naturally as randomly perturbed deterministic neutral equations of the form $\dot{x} = f(t, Tx, S\dot{x})$ where T and S are delay operators (for example, T can be defined as in equation (3.5) and $S\dot{x} = \dot{x}(t-h)$). Some results concerning such equations can be found in [P7]. The crucial point is that they can be transformed to equation (3.1) for which a local and tight operator F is defined implicitly.

References

[AM] N.V. Azbelev, V.P. Maksimov, Equations with delayed argument. Differential Equations 18:12 (1982), 1419-1441.

[AMR] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional-Differental Equations. Nauka, Moscow 1991 (In Russian).

[E] M. Emery, Une topologie sur l'espace des semimartingales. Sèminaire de Proab. XIII, Cet. Notes in Mathematics, 721 (1979), Springer-Verlag, 180-260.

J1] J. Jacod, Calcul stochastique et problèmes de martingales. Lect. Notes in Mathematics, 714 (1979), Springer-Verlag.

[J2] J. Jacod, Weak and strong solutions of stochastic differential equations. Stochastics 3 (1980), 171-191.

[JM] J. Jacod, J. Memin, Existence of weak solutions for stochastic differential equations with driving semimartingales. Stochastics 4 (1981), 317-337.

[KN] V.B. Kolmanovskii, V.R. Nosov, Stability of Functional Differential Equations. Academic Press, London (1986).

[M1] S.E.A. Mohammed, Stochastic Functional Differential Equations. Research notes in Mathematics No. 99, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984.

[M2] S.E.A. Mohammed, Lyapunov exponents and stochastic flows of linear and affine hereditary systems: a survey. Research Report No. 91-98, Dept. of Mathematics, Carnegie Mellon Univ., January 1991.

[MT] V.J. Mizel, V. Trutzer, Stochastic hereditary equations: existence and asymptotic stability. Journal of Integral Equations 7 (1984), 1-72.

[NM] D. Nualart, E. Pardoux, Stochastic calculus with anticipating integrands. Probability Theory and Related Fields 78 (1988), 535-581.

[P1] A.V. Ponosov, On the Nemytskii conjecture. Soviet Math. Doklady 34:1 (1987), 231-233.

[P2] A.V. Ponosov, On the extensions of local-type operators. Boundary Value Problems (1987), 92-96 (In Russian).

[P3] A.V. Ponosov, Fixed point method in the theory of stochastic differential equations. Soviet Math. Doklady 37:2 (1988), 426-429.

[P4] A.V. Ponosov, Approximations of Volterra operators. Boundary Value Problems (1988), 77-82 (In Russian).

[P5] A.V. Ponosov, Tightness property for operators associated with the Wiener process. Statistical Methods (1988), 148-154 (In Russian).

[P6] A.V. Ponosov, Schauder-type Theorem for locval operators. Functional-Differential Equations (1988), 40-49 (In Russian).

[P7] A.V. Ponosov, On the theory of reducible Ito equations. Differential Equations 25:2 (1990), 1347-1355.

[Pr1] Ph.E. Protter, Right-continuous solutions of stochastic integral equations. J. Multivariate Anal. 7 (1977), 204-214.

[Pr2] Ph.E. Protter, Stochastic differential equations with feedback in the differentials. Lect. Notes in Mathematics, 920 (1982), 459-468.

[Pr3] Ph.E. Protter, Volterra equations driven by semimartingales. Ann. Probab. 13:2 (1985), 519-530.

[R] L.F. Rakhmatullina, On the definition of solutions to equations with deviating argument. Functional-Differential Equations (1985), 13-18 (In Russian).

[S] M.K.R. Scheutzow, Qualitative behavior of stochastic delay equations with a bounded memory. Stochastics 12 (1984), 41- 80.

[Sh] I.V. Shragin, Abstract Nemytskii operators are locally defined operators. Soviet Math. Doklady 17:2 (1976), 354-357.

[WM] R.Q. Wu, X. Mao, Existence and Uniqueness of solutions of stochastic differential equations. Stochastics 12 (1984), 41-80.

STEADY MODES IN A DISCONTINUOUS CONTROL SYSTEM WITH TIME DELAY

E.I. Shustin
School of Math. Sci.
Tel Aviv University
Tel Aviv, Israel

E.M. Fridman
Dept. of Math.
Railroad Engineering Inst.
Samara, Russia

L.M. Fridman
Dept. of Math.
Architecture & Building Inst.
Samara, Russia

Abstract

We describe the solutions space for the scalar system with a discontinuous delay control element

$$\dot{x}(t) = -\text{sign } x(t-1) + F(x(t),\ t)\ .$$

We show that the time delay does not allow to realize an ideal sliding mode, but implies oscillations, whose stability is determined by one discrete parameter – oscillation frequency.

Introduction

It is fruitful for many control problems to use relay control algorithms providing sliding modes, i.e. special kinds of motions on a discontinuity surface [1,2,6]. One of the unavoidable difficulties in realizing such algorithms is the time delay, always existing in real systems. It implies auto-oscillations and does not allow to realize an ideal sliding mode [4].

In this paper a description of a solution space for the equation

$$\dot{x}(t) = -\text{sign } x(t-1) + F(x(t),t), \quad t \geq 0 \tag{0.1}$$

with

$$|F(x,t)| \leq p < 1, \quad F \in C^1(\mathbf{R}^2)\ , \tag{0.2}$$

$$x(t) = \varphi(t),\ t \in [-1;0],\ \varphi \in C[-1,0] \tag{0.3}$$

is presented. Obviously, under condition (0.2), for any $\varphi \in C[-1;0]$, there exists a unique continuous solution $x_\varphi(t)$, $t \in [-1;\infty)$, of the problem (0.1), (0.3). We consider further only such solutions. Two solutions are called *equivalent* if they coincide after a relevant time moment. The main result of this study is that (section 1.1) each solution of the equation (0.1) is equivalent to a steady mode (SM), which is a solution with a constant frequency. That means we have *finite time* of entrance to steady mode. Moreover, in the autonomous case, $F(x,t) \equiv F(x)$, there exists a countable set of periodic SM generating all other SM by translations in t. Another important result consists in a description of classes of stable and unstable SM (section 1.2). Also we give sufficient conditions for an existance of stable bounded stady modes in the case of an unbounded function F (section 1.3).

The following example underlines the meaning of results. The simplest equation

$$\dot{x}(t) = -\text{sign } x(t-1) \tag{0.4}$$

has the 4-periodic SM

$$g_0(t) = \begin{cases} t, & -1 \le t \le 1 , \\ 2-t, & 1 \le t \le 3, \end{cases} \quad g_0(t+4k) = g_0(t) , \quad k \in \mathbb{Z}$$

It generates all other SM: namely, it is easy to verify that the $4/(4n+1)$-periodic function

$$g_n(t) = \frac{1}{4n+1} g_0((4n+1)t) , \quad t \in \mathbf{R} ,$$

is a solution of (0.4) for each integer $n \ge 1$. This example is essential one because, as will be shown, each solution $x(t) \not\equiv 0$ of (0.4) is equivalent to $g_n(t+\alpha)$ for some $n \ge 0$, $\alpha \in \mathbf{R}$; moreover, a solution $g_n(t)$ is stable for $n = 0$, and unstable for $n \ge 1$.

1. Formulation of Results

1.1 Steady Modes. The main object of this section is a special characteristic of solution, its *frequency*. Our main result (Theorem 1.1.7) states that, for any solution, its frequency becomes constant after a suitable time moment. In fact, that means each solution is equivalent to some SM – a solution with a constant frequency.

We formulate further all necessary steps leading to the main result. The proofs are presented in section 2.

Let Z_φ denote a set of zeros of $x_\varphi(t)$. Put $Z_\varphi^+ = Z_\varphi \cap [0; +\infty)$.

Lemma 1.1.1. *For any $\varphi \in C[-1;0]$ the set Z_φ is non-empty and unbounded.*

So we can define the frequency function $\nu_\varphi : Z_\varphi^+ \longrightarrow \mathbb{N} \cup \{0\} \cup \{\infty\}$ by

$$\nu_\varphi(t) = \text{card } (Z_\varphi \cap (t; t-1)) , \quad t \in Z_\varphi^+ .$$

Lemma 1.1.2. *For any $\varphi \in C[-1;0]$ the function ν_φ does not increase.*

This implies that there exists a limit

$$N_\varphi \stackrel{\text{def}}{=} \lim_{\substack{t \to \infty \\ t \in Z_\varphi^+}} \nu_\varphi(t) .$$

Lemma 1.1.3. *If $N_\varphi < \infty$ then N_φ is even, and $C[-1;0]$ is divided into sets*

$$U_\infty = \{\varphi \in C[-1;0] | N_\varphi = \infty\} ,$$
$$U_n = \{\varphi \in C[-1;0] | N_\varphi = 2n\} , \quad n \geq 0 .$$

Introduce the following subset of $C[-1;0]$:

$$\mathcal{F} = \{\varphi \in C[-1;0] | \varphi^{-1}(0) \text{ is finite}\}$$

It follows immediately from lemma 1.1.2 that

$$\mathcal{F} \subset \bigcup_{0 \leq n < \infty} U_n$$

Definition 1.1.4. A solution $x_\varphi(t)$ with $\nu_\varphi \equiv$ const is called *steady mode* (SM).

The set of SM is represented naturally as a union of disjoint sets $S_n = \{x_\varphi(t) | \nu_\varphi \equiv 2n\}$, $n \geq 0$, $S_\infty = \{x_\varphi(t) | \nu_\varphi \equiv \infty\}$.

Lemma 1.1.5. *For any integer $n \geq 0$ and real $T \geq 0$ there exists $g(t) \in S_n$ such that*

$$g(T) = 0 , \quad \dot{g}(T) > 0 . \tag{1.1.6}$$

If $n = 0$ then such SM is unique.

As a consequence of above statements we obtain

Theorem 1.1.7. *Any solution $x_\varphi(t)$ of the (0.1), (0.3) is equivalent to a suitable SM.*

In the autonomous case, we give a more precise description of the SM set:

Theorem 1.1.8. *In the autonomous case*
(i) $S_\infty = \{0\}$ if $F(0) = 0$, and $S_\infty = \emptyset$ if $F(0) \neq 0$.
(ii) there are periodic steady modes $g_0, g_1, \ldots, g_n, \ldots$ such that

$$S_n = \{g_n(t + \alpha) \mid \alpha \in \mathbf{R}\} , \quad n \geq 0 ,$$

and their periods satisfy inequalities

$$\tau_0 > 2 , \quad n^{-1} > \tau_n > (n+1)^{-1} , \quad n \geq 1 . \tag{1.1.9}$$

This means that a moduli space of solutions (up to equivalence) is a union of a countable set of disjoint circles, and, maybe, also a point.

1.2. Stability.

In this section the characteristics of SM stability are presented.

Lemma 1.2.1. *The set U_0 has nonempty interior. Moreover,* Int U_0 *contains the non-empty set*

$$\widetilde{U}_0 = U_0 \cap \{\varphi \in C[-1;0] \mid \text{mes } \varphi^{-1}(0) = 0\} .$$

In particular, we get that the property $N_\varphi = 0$ is stable if mes $\varphi^{-1}(0) = 0$.

Theorem 1.2.2. *If*

$$\int_0^\infty \max_x \left|\frac{\partial F(x,t)}{\partial t}\right| dt < \infty \tag{1.2.3}$$

then all solutions $x_\varphi(t)$, $\varphi \in \widetilde{U}_0$, are non-asymptotically stable.

We should underline that there are unstable solutions $x_\varphi(t)$ with $\varphi \in U_0$. For example, let $\psi \in U_n$, $n \geq 1$, then $\varphi(t) = \max\{0; \psi(t)\} \in U_0$, but $\varphi_\tau(t) = \varphi(t) + \tau\psi(t) \in U_n$, for any $\tau > 0$.

Theorem 1.2.4. *If*

$$\sup\left|\frac{\partial F}{\partial x}\right| = M_x < 2(1-p)^2(1+p)^{-3} \tag{1.2.5}$$

or

$$\sup\left|\frac{\partial F}{\partial t}\right| = M_t < 2(1-p)^2(1+p)^{-2} \tag{1.2.6}$$

then all solutions $x_\varphi(t)$, $\varphi \in \bigcup_{1 \leq n \leq \infty} U_n$, are unstable.

Note that conditions of theorems 1.2.2 and 1.2.4 are fulfilled in the autonomous case.

1.3 The Case of Unbounded Function F. Consider the equation

$$\begin{aligned} \dot{x}(t) &= -\text{sign} x(t-1) + F(x(t)) \\ F \in C^1(\mathbf{R})\,, &\quad F(0) = p \in (-1;1) \end{aligned} \tag{1.3.1}$$

without restriction (0.2). The most interesting example is

$$F(x) = kx\,, \quad k = \text{const}\ . \tag{1.3.2}$$

In general, a solution of (1.3.1) might be unbounded or inextensible on the infinite interval. We shall indicate conditions, when solutions are extensible, bounded and stable.

Introduce the numbers from $\mathbf{R} \cup \{-\infty; +\infty\}$:

$$x_1^+ = \inf\{x > 0 | F(x) = 1\} \ , \quad x_{-1}^+ = \inf\{x > 0 | F(x) = -1\}$$

$$x_1^- = \sup\{x < 0 | F(x) = 1\} \ , \quad x_{-1}^- = \sup\{x < 0 | F(x) = -1\} \ .$$

Theorem 1.3.3. *If*

(i)
$$x_{-1}^+ \le x_1^+ \ , \quad or \quad \int_0^{x_1^+} \frac{dx}{1 + F(x)} > 1 \ ,$$

and

(ii)
$$x_1^- \ge x_{-1}^- \ , \quad or \quad \int_{x_{-1}^-}^0 \frac{dx}{1 - F(x)} > 1 \ ,$$

then there is $\delta > 0$ such that all solutions $x_\varphi(t)$ of the equation (1.3.1), where $\varphi \in C[-1;0]$, $\|\varphi\| < \delta$, are extensible on $[-1;\infty)$, uniformly bounded, and possess all properties, mentioned in sections 1.1, 1.2. (Here $\|\varphi\|$ means $\max|\varphi(t)|$.)

From this theorem, it is not difficult to deduce:

Corollary 1.3.4. *Under conditions (1.3.2)*
(i) if $k \le 0$ then all solutions are bounded,
(ii) if $0 < k < ln 2$ then all solutions $x_\varphi(t)$, $\|\varphi\| < (2 \cdot \exp(-k) - 1)/k$, are uniformly bounded;
(iii) if $k \ge ln 2$ then there is no a stable bounded solution.

2. Proofs

Lemma 1.1.1. is obvious.

Proof of Lemma 1.1.2. If $t_1 < t_2$, $t_1, t_2 \in Z_\varphi^+$, then, according to Rolle's theorem and (0.1), (0.2), there exists $\xi \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$. Therefore

$$\text{card}\,(Z_\varphi \cap (t_1 - 1; t_2 - 1)) \ge \text{card}\,\left(Z_\varphi^+ \cap (t_1; t_2)\right) + 1 \ ,$$

hence

$$\nu_\varphi(t_1) = \text{card}\,(Z_\varphi \cap (t_1 - 1; t_1)) \ge$$
$$\ge \text{card}\,(Z_\varphi \cap (t_2 - 1; t_2)) = \nu_\varphi(t_2) \ .$$

Proof of Lemma 1.1.3. Let $\nu_\varphi(t) = N_\varphi < \infty$, when $t \geq T$. Then $x_\varphi(t)$ changes its sign at every point $t \in Z_\varphi \cap [T; +\infty)$. Indeed, if $t_1 < t_2$ are neighbour points from $Z_\varphi \cap [T+1; \infty)$ then, according to above assumption, there is a unique $z \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$, and hence $x_\varphi(t)$ changes its sign at z. Now it is easy to see that N_φ is even.

Proof of Lemma 1.1.5. In the case $N = 0$ the desired statement is obvious. Fix even $N > 0$. Put

$$\Sigma = \{(a_0, \dots, a_N) \in \mathbf{R}^{N+1} \mid a_0 \geq 0, \dots, a_N \geq 0 \ , a_0 + \dots + a_N = 1\} \ .$$

Let $Z_\varphi \cap [T; +\infty)$ be locally finite, and

$$T = t_1 < t_2 < t_3 < \cdots$$

be all zeros of $x_\varphi(t)$ in $[T; +\infty)$. Let $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$. Define the following vectors of sign changes: $\overline{a} = (a_0, \dots, a_N)$, $\overline{b} = (b_0, \dots, b_N) \in \Sigma$, where

$$a_0 = t_k - t_{k-1}, a_1 = t_{k-1} - t_{k-2}, \dots, a_{N-1} = t_{k-N+1} - t_{k-N} \ ,$$
$$a_N = t_{k-N} - (t_k - 1) \ ,$$
$$b_0 = t_{k+1} - t_k, b_1 = t_k - t_{k-1}, \dots, b_{N-1} = t_{k-N+2} - t_{k-N+1} \ ,$$
$$b_N = t_{k-N+1} - (t_{k+1} - 1) \ .$$

Thus we obtain a correspondence

$$(\overline{a}, \alpha, \varepsilon) \mapsto (\overline{b}, \beta, -\varepsilon) \ , \tag{2.2}$$

where $\alpha = t_k$, $\beta = t_{k+1}$, $\varepsilon = \text{sign } \dot{x}_\varphi(t_k)$.

Proposition 2.3. *For a fixed ε, the correspondence inverse to (2.2), is a smooth map*

$$M_\varepsilon : \Sigma \times \mathbf{R} \to \Sigma \times \mathbf{R} \ .$$

Proof: Denote by $x_\varepsilon(t_0, x_0, a)$, $\varepsilon = \pm 1$, the solution of the Cauchy problem

$$\frac{dx}{da} = \varepsilon + F(x, t_0 + a), \ x(0) = x_0 \ .$$

Define functions $T = \lambda_\varepsilon(t, a), \varepsilon = \pm 1$, by equations

$$\begin{cases} x_{-\varepsilon}(t + a, \ x_\varepsilon(t, 0, a), \ b) = 0 \ , \\ T = t + a + b \end{cases} \ . \tag{2.4}$$

It is easy to see that for a fixed t_0, functions $\lambda_\pm(t_0, a)$ increases strictly, and $\lambda_\pm(t_0, a) > a$ if $a > 0$. Therefore, for a fixed t_0, we can define positive functions of $b > 0$:

(i) $\rho_\varepsilon(t_0, b)$ inverse to $b = \lambda_\varepsilon(t_0, \rho_\varepsilon)$

(ii) $\sigma_\varepsilon(t_0, b) = b - \rho_\varepsilon(t_0, b)$. Hence $(\overline{a}, \alpha) = M_\varepsilon(\overline{b}, \beta)$ can be defined as

$$\begin{cases} a_0 = b_1, \ a_1 = b_2, \dots, a_{N-2} = b_{N-1} \ , \\ a_{N-1} = b_N + \sigma_\varepsilon(\beta - b_0, \ b_0), \ a_N = \rho_\varepsilon(\beta - b_0, b_0) \\ \alpha = \beta - b_0 \end{cases} \ . \tag{2.5}$$

□

So, from a given triple $(\overline{a}, \alpha, \varepsilon)$, using (0.1) we can construct a solution of (0.1) for $t \geq \alpha$, and using maps $M_\pm$ we can extend this solution on the interval $(-\infty, \alpha)$ with a constant frequency function. Now let us introduce the decreasing sequence of closed connected sets

$$\Pi_0 = \Sigma \times \mathbf{R}, \ \Pi_{n+1} = (M_- M_+)(\Pi_n), \ n \geq 0 .$$

Let $\Pi = \Pi_0 \cap \Pi_1 \cap \Pi_2 \cap \ldots$ The statement of lemma 1.1.5 means that $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$ for any $\alpha \in \mathbf{R}$. It is obvious that, for any $k > 0$,

$$\Pi_k \cap (\Sigma \times \{\alpha\}) \neq \emptyset , \tag{2.6}$$

if α is big enough or small enough. Then (2.6) is fulfilled for any $k \geq 0, \alpha \in \mathbf{R}$ because of connectedness of Π_k, $k \geq 0$. Thus $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$, because $\Pi_k \cap (\Sigma \times \{\alpha\}) \neq \emptyset$, $k \geq 0$, are compact.

Proof of Theorem 1.1.7. It is easy to deduce from the proof of Proposition 2.3 that every solution $g(t)$, $t \geq T$, of (0.1) with a constant finite frequency can be extended on $[-1; \infty)$ with the same frequency. That finishes the proof according to lemmas 1.1.1, 1.1.3, 1.1.5.

Proof of Theorem 1.1.8. Here we omit the proof of statement (i), which is explaned in [5]. It is rather tedious, and is proposed to be published in another paper.

Now we will prove that, for any $n \geq 1$ and a fixed $T \in \mathbf{R}$, there is an unique $g_{n,T} \in S_n$ with property (1.1.6). Since M_ε, defined by (2.5), doesn't depend on β we get a map $M_\varepsilon : \Sigma \to \Sigma$ such that

$$\begin{cases} \overline{a} = M_\varepsilon(\overline{b}); \overline{a}, \overline{b} \in \Sigma \\ a_0 = b_1, \ a_1 = b_2, \ldots, a_{N-2} = b_{N-1} , \\ a_{N-1} = b_{N-2} + \sigma_\varepsilon(b_0), \ a_N = \rho_\varepsilon(b_0) , \end{cases} \tag{2.7}$$

where $N = 2n$ and according to the definition of $\rho_\varepsilon, \sigma_\varepsilon$ (see Proposition 2.3) and (0.2)

$$\frac{1-p}{2} \leq \rho'_\varepsilon(b) \leq \frac{1+p}{2} , \ \frac{1-p}{2} \leq \sigma'_\varepsilon(b) \leq \frac{1+p}{2} . \tag{2.8}$$

We have to show that the intersection of a decreasing sequence of compacts

$$(M_- \circ M_+)^k(\Sigma) , \quad k \geq 0 ,$$

is one point. That follows from

Proposition 2.9. *For the metric*

$$\|\overline{a}-\overline{b}\| = \sum_{i=0}^{N} |a_i - b_i| \tag{2.10}$$

the operator

$$M = (M_- \circ M_+)^{N(N+1)} : \Sigma \longrightarrow \Sigma$$

is a contraction with a coefficient $1-\gamma$, *where*

$$\gamma = \left(\frac{1-p}{2}\right)^{N(N+1)}$$

Proof: If $\overline{a}, \overline{b} \in \Sigma$ then the vector $\overline{a}-\overline{b}$ has at least one pair of coordinates with different signs. Let

$$a_j - b_j = \max_i\{a_i - b_i\} > 0 \ , \quad a_k - b_k = \min\{a_i - b_i\} < 0 \ .$$

It is easy to see that

$$\begin{aligned} a_j - b_j &\geq \|\overline{a}-\overline{b}\|/(2N) \ , \\ b_k - a_k &\geq \|\overline{a}-\overline{b}\|/(2N) \ . \end{aligned} \tag{2.11}$$

According to (2.6) $\overline{c} = M_\varepsilon(\overline{a}) - M_\varepsilon(\overline{b})$ can be defined by

$$\begin{aligned} c_0 &= \rho_\varepsilon'(\theta)\cdot(a_0 - b_0),\ c_1 = a_1 - b_1, \ldots, c_{N-1} = a_{N-1} - b_{N-1} \ , \\ c_N &= a_N - b_N + \sigma_\varepsilon'(\theta)\cdot(a_0 - b_0) \ . \end{aligned}$$

Thus the transformation $\overline{a}-\overline{b} \mapsto \overline{c}$ can be described as a multiplication by a matrix $\{\alpha_{ij}\}$ (depending on $\overline{a}, \overline{b}$), where according to (2.8)

$$\begin{aligned} &\alpha_{ij} \geq 0 \ , \quad 0 \leq i,j \leq N \ , \\ &\sum_{i=0}^{N} \alpha_{ij} = 1 \ , \quad j = 0, \ldots, N \\ &\min\{\alpha_{ij} \mid \alpha_{ij} > 0\} \geq (1-p)/2 \ . \end{aligned}$$

Hence the transformation $\overline{a}-\overline{b} \mapsto M(\overline{a}) - M(\overline{b})$ can be described as a multiplication by a matrix $M = \{m_{ij}\}$, where

$$\begin{aligned} &m_{ij} > 0 \ , \quad 0 \leq i,j \leq N \ , \\ &\sum_{i=0}^{N} m_{ij} = 1 \ , \quad j = 0, \ldots, N \ , \\ &\min\{m_{ij}\} \geq \gamma \ . \end{aligned}$$

Then, according to (2.10), (2.11),

$$\|M(\overline{a}) - M(\overline{b})\| = \sum_{i=0}^{N} |\sum_{q=0}^{N} m_{iq}(a_q - b_q)| \leq$$

$$\leq \sum_{i=0}^{N} \left(\sum_{q=0}^{N} m_{iq}|a_q - b_q| - 2\gamma \cdot \|\overline{a} - \overline{b}\|/(2N) \right) <$$

$$< \|\overline{a} - \overline{b}\| - \gamma \cdot \|\overline{a} - \overline{b}\| = (1-\gamma)\|\overline{a} - \overline{b}\| \ . \qquad \square$$

This uniqueness and the autonomy imply the equality $g_{n,T}(t) = g_{n,0}(t-T)$, $t, T \in \mathbf{R}$, as well as the periodicity of $g_{n,0}$. Inequalities (1.1.9) follow from that the frequency of $g_{n,0}$ is equal to $2n$.

Proof of Lemma 1.2.1. The set $\widetilde{U}_0$ is non-empty because it contains $S_0 \neq \emptyset$. Now let $\varphi \in U_0$, and $\mathrm{mes}\varphi^{-1}(0) = 0$. Then $x_\varphi(t) = g_{0,T}(t)$, $t \geq T$, for a relevant $T \in \mathbf{R}$. That means

$$x_\varphi(T) = 0, \quad \dot{x}_\varphi(t) > 0, \quad t \in \left(T; T + \frac{2}{1+p}\right) \ .$$

If $\psi \in C[-1;0]$ is close to φ, then $\psi^{-1}(0)$ is contained in a sufficiently small neighbourhood of $\varphi^{-1}(0)$, and

$$\mathrm{mes}\ (\{\varphi > 0\} \circ \{\psi > 0\}), \quad \mathrm{mes}\ (\{\varphi < 0\} \circ \{\psi < 0\})$$

are small enough, where $A \circ B$ denotes $(A \backslash B) \cup (B \backslash A)$. Hence $Z_\psi \cap [0; T+2]$ is contained in a sufficiently small neighbourhood of $Z_\varphi \cap [0; T+2]$. Therefore

$$x_\psi(t) > 0 \ , \quad t \in \left(T + \delta; T + \frac{2}{1+p} - \delta\right) \ ,$$

$$2\delta < \frac{2}{1+p} - 1 \ ,$$

that implies $\psi \in U_0$.

Proof of Theorem 1.2.2. Let $\varphi \in \widetilde{U}_0$, and $x_\varphi(t) = g_{0\alpha}(t)$, $t \geq T$. We have just showed that if ψ is sufficiently close to φ then $x_\psi(t) = g_{0\beta}(t)$, $t \geq T$, where $|\beta - \alpha|$ is small enough. Let

$$\alpha = t_1 < t_2 < \dots \ , \quad \beta = t_1' < t_2' < \dots$$

be all zeros of functions $g_{0\alpha}, g_{0\beta}$ respectively in the interval $[T; \infty)$. It is enough to prove that

$$C_1 \cdot |\beta - \alpha| < |t_k - t_k'| < C_2 \cdot |\beta - \alpha| \ , \quad k = 1, 2, \dots$$
$$C_1, C_2 = \mathrm{const} \ .$$

According to the definition of functions $\lambda_\pm(t_0, a)$

$$t_{k+1} = \lambda_\pm(t_k, 1) \ , \quad t'_{k+1} = \lambda_\pm(t'_k, 1) \ ,$$

hence

$$t'_{k+1} - t_{k+1} = \frac{\partial \lambda_\pm(\theta_k, 1)}{\partial t} \cdot (t'_k - t_k) \ , \quad |\theta_k - t_k| < |t'_k - t_k| \ , \quad k \geq 1 \qquad (2.12)$$

$$t'_n - t_n = \Pi_{k=1}^{n-1} \frac{\partial \lambda_\pm(\theta_k, 1)}{\partial t} \cdot (\beta - \alpha)$$

The statement follows from

Proposition 2.13. *Under condition (1.2.3), the product*

$$\Pi_{k=1}^{\infty} \frac{\partial \lambda_\pm}{\partial t}(\theta_k, 1)$$

converges uniformly when

$$\theta_{k+1} \geq \theta_k + 1 \ , \quad k = 1, 2, 3, \ldots \qquad (2.14)$$

Proof: We will show that the series

$$\sum_{k=1}^{\infty} \left(\frac{\partial \lambda_\pm}{\partial t}(\theta_k, 1) - 1 \right)$$

converges uniformly. Put

$$\mu(t) = \max_x \left| \frac{\partial F}{\partial t}(x, t) \right| \ , \quad t \geq 0 \ .$$

It follows from (2.4) and well-known formulae for derivatives of solutions with respect to initial data [3], that

$$\frac{\partial \lambda_\varepsilon}{\partial t}(t, a) = 1 - (-\varepsilon + F(0, T))^{-1} \cdot \exp \int_{t+a}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt \times$$
$$\times \left(\int_{t+a}^{\tau} \frac{\partial F}{\partial t}(x_{-\varepsilon}, t) dt + \int_t^{t+a} \frac{\partial F}{\partial t}(x_\varepsilon, t) dt \cdot \exp \int_t^{t+a} \frac{\partial F}{\partial x}(x_\varepsilon, t) dt \right) \ ,$$

where $\tau = \lambda_\varepsilon(t, a)$, hence

$$\left| \frac{\partial \lambda_\varepsilon}{\partial t}(\theta, 1) - 1 \right| \leq \frac{1}{1-p} \exp \int_{\theta+1}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt \times$$
$$\times \left(\int_{\theta+1}^{\tau} \mu(t) dt + \int_\theta^{\theta+1} \mu(t) dt \cdot \exp \int_0^{\theta+1} \frac{\partial F}{\partial x}(x_\varepsilon, t) dt \right) \qquad (2.15)$$

According to (2.14) one may admit

$$\theta \gg 0\ , \quad \int_\theta^\infty \mu(t)dt \le 1\ .$$

Then

$$\int_{\theta+1}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t)dt = \int_{\theta+1}^{\tau} \frac{\partial F}{\partial t} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1}dt -$$

$$- \int_{\theta+1}^{\tau} \frac{\partial F}{\partial t} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1}dt \le 2p +$$

$$+ \frac{1}{1-p} \cdot \int_{\theta+1}^{\tau} \mu(t)dt \le 2p + \frac{1}{1-p}\ ,$$

$$\int_\theta^{\theta+1} \frac{\partial F}{\partial x}(x_\varepsilon, t)dt \le 2p + \frac{1}{1-p}\ .$$

Put $q = \exp(2p + 1/(1-p))$, $N = [(1+p)/(1-p)] + 1$. Then (2.15) implies

$$\left|\frac{\partial \lambda_\varepsilon}{\partial t}(\theta, 1) - 1\right| \le \frac{q^2}{1-p} \int_\theta^\tau \mu(t)dt\ ,$$

$$\sum_{\theta_i > \theta} \left|\frac{\partial \lambda_\pm}{\partial t}(\theta_i, 1) - 1\right| \le \frac{q^2 N}{1-p} \int_\theta^\infty \mu(t)dt \underset{\theta\to\infty}{\longrightarrow} 0\ ,$$

because $\tau \le \theta + (1+p)/(1-p)$ according to (0.2), that completes the proof. □

For the proof of Theorem 1.2.4 we need two following propositions.

Proposition 2.16. *If*

$$a \le (1+p)/2 \tag{2.17}$$

and one of (1.2.5), (1.2.6) is fulfilled, then

$$\frac{\partial \lambda_\varepsilon}{\partial a}(t, a) \ge q > 1\ , \quad \varepsilon = \pm 1\ . \tag{2.18}$$

Proof: It is not difficult to conclude (see [3]) that

$$\frac{\partial \lambda_\varepsilon}{\partial a}(t, a) = 1 + (1 - \varepsilon F(0, T))^{-1} \exp \int_{t+a}^{T} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t)dt \times$$

$$\times \left(1 + \varepsilon F(x_\varepsilon(t, 0, t+a), t+a) + \varepsilon \int_{t+a}^{T} \frac{\partial F}{\partial t}(x_{-\varepsilon}, t)dt\right)\ ,$$

where $T = \lambda_\varepsilon(t,a)$. Therefore (1.2.6) implies

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt$$
$$>1-p-(T-t-a)M_t \geq 1-p-aM_t(1+p)/(1-p)>0\ ,$$
$$\int_{t+a}^{T}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt=\int_{t+a}^{T}\left(\frac{dF}{dt}-\frac{\partial F}{\partial t}\right)\cdot(\dot{x}_{-\varepsilon})^{-1}dt=$$
$$=\int_{t+a}^{T}\frac{dF}{dt}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt-\int_{t+a}^{T}\frac{\partial F}{\partial t}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt\geq$$
$$\geq -2p-M_t\cdot(T-t-a)/(1-p)\geq -2p-M_ta(1+p)(1-p)^{-2}\geq$$
$$\geq -2p-M_t\cdot(1+p)^2(1-p)^{-2}/2\ ,$$

that implies (2.18). Analogously (1.2.5) implies

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt=$$
$$=1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{dF}{dt}dt-$$
$$-\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial x}\cdot\dot{x}_{-\varepsilon}dt\geq 1+\varepsilon\cdot F(0,T)-$$
$$-M_x(1+p)(T-t-a)\geq 1-p-M_xa(1+p)^2/(1-p)>0\ ,$$
$$\int_{t_0}^{t_0+a}\frac{\partial F}{\partial x}(x_\varepsilon,t)dt\geq -M_xa\geq -M_x(1+p)/2\ ,$$

that implies (2.18).

Proposition 2.19. *Under conditions of theorem 1.2.4 the measure of the set* Π *from the proof of lemma 1.1.5 is zero.*

Proof: First we show that any $\overline{a}=(a_0,\ldots,a_N)=M_\varepsilon(\overline{b}),\overline{b}\in\Sigma$, satisfies $a_N\leq(1+p)/2$. Indeed, we have $a_N\leq a_{N-1}(1+p)/(1-p)$, that implies the above inequality.

Now from (2.5) the Jacobian $|M_\varepsilon'|$ of the map M_ε is equal to

$$\frac{\partial\rho_\varepsilon}{\partial b}(t,b)\Big|_{t=\alpha,b=b_0}=\left(\frac{\partial\lambda_\varepsilon}{\partial a}(t,a)\Big|_{t=\alpha,a=a_N}\right)^{-1}\leq\frac{1}{q}<1$$

according to proposition 2.16. Then

$$|(M_-\circ M_+)'|\leq q^{-2}<1\ . \tag{2.20}$$

Fix $A \in \mathbf{R}$ and $T > A$. Then

$$\Pi \cap (\Sigma \times (-\infty; A]) \subset \bigcup_{k \geq n} (M_- \circ M_+)^k (\Sigma \times [T; T+1]) ,$$

where n might be chosen big enough because $T > A$ is arbitrary. Thus we obtain from (2.20)

$$\operatorname{mes}(\Pi \cap (\Sigma \times (-\infty; A])) \leq q^{-2(n-1)} \frac{\operatorname{mes}(\Sigma)}{q^2 - 1} \underset{n \to \infty}{\longrightarrow} 0 ,$$

that completes the proof.

Proof of Theorem 1.2.4. Now fix $\varphi \in U_n$ and a neighbourhood V of φ in $C[-1; 0]$. The set $\mathcal{F}$ is dense in $C[-1; 0]$, evidently. Put

$$m = \min\{k \mid \mathcal{F} \cap U_k \cap V \neq \emptyset\} .$$

Assume $m \geq 1$, and $\psi \in \mathcal{F} \cap U_m \cap V$. Then there is $\xi \in S_m$ such that $x_\psi(t) = \xi(t)$, $t \geq T$, $\xi(T) = 0$. Let $2k$ be a number of sign changes of ψ in $[-1; 0]$, and $\overline{a} \in \Sigma_k \subset \mathbf{R}^{2k+1}$ be a vector of sign changes of ψ, constructed as in the proof of lemma 1.1.5, as well as $\overline{b} \in \Sigma_m \subset \mathbf{R}^{2m+1}$ be a vector of sign changes of ξ in $(T-1; T)$. Suppose $\overline{c} \in \Sigma_t, \overline{d} \in \Sigma_s$ are vectors of sign changes of $x_\psi(t)$ in intervals $(t_n - 1; t_n)$ and $(t_{n+1} - 1; t_{n+1})$ respectively. If $r = s$ then, according to the proof of lemma 1.1.5, the equation (0.1) generates a diffeomorphism of neighbourhoods of $(\overline{c}, t_n)$, $(\overline{d}, t_{n+1})$ in $\Sigma_r \times \mathbf{R}$. If $r < s$ then it is possible to deduce, following arguments from the proof of lemma 1.1.5,

$$\begin{aligned} &c_0 = d_1, \ldots, c_{2s-1} = d_{2s},\ c_{2r} = \Lambda(d_0, c_{2s}, \ldots, c_{2r-2}, t_{n+1}) , \\ &c_{2r-1} = 1 - c_0 - \cdots - c_{2r-2} - c_{2r},\ t_n = t_{n+1} - d_0 , \end{aligned}$$

where Λ is some smooth function. Hence an inverse image of $(\overline{d}, t_{n+1})$ in a neighbourhood of $(\overline{c}, t_n)$ in $\Sigma_r \times \mathbf{R}$ has the codimension $2s + 1$. That implies the measure of an inverse image of $\Pi \cap (\Sigma_m \times \mathbf{R})$ in $\Sigma_k \times \mathbf{R}$ is zero. Therefore, after a suitable small variation of $(\overline{a}, 0)$ in $\Sigma_k \times \mathbf{R}$ an image of $(\overline{a}, 0)$ in $\Sigma_m \times \mathbf{R}$ leaves Π, i.e. a limit frequency of the changed solution is less than $2m$, what contradicts to definition of m, and hence to our assumption $m > 0$.

Thus we get that $U_0 \cap \mathcal{F}$ is dense in $\mathcal{F}$, and also in $C[-1; 0]$, because $\mathcal{F}$ is dense in $C[-1; 0]$. According to theorem 1.2.2, it means that $U_\infty \cup \bigcup_{k \geq 1} U_k$ is dense nowhere in $C[-1; 0]$.

Proof of Theorem 1.3.3. It is easy to see that there are $x_1 \in (0; \min\{x_1^+; x_{-1}^+\})$, $x_2 \in (\max\{x_1^-; x_{-1}^-\}; 0)$ such that

$$\int_0^{x_1} \frac{dx}{1+F(x)} = \int_{x_2}^{0} \frac{dx}{1-F(x)} = 1 .$$

The statement of the theorem follows from

Proposition 2.2.1. *If* $\varphi \in C[-1;0]$, $\varphi(0) = 0$ *then*

$$x_2 \leq x_\varphi(t) \leq x_1 , \quad t \geq 0 .$$

Proof: Assume $x_\varphi(t_0) > x_1$, $t_0 > 0$. Let $t_1 = \max\{t < t_0 \mid x_\varphi(t) = 0\}$. Without any loss of generality one may suppose $\dot{x}_\varphi(t) = 1 + F(x_\varphi(t)) > 0$, $t \in (t_1; t_0)$. That means $t_0 - t_1 < 1$. From the other side

$$t_0 - t_1 = \int_0^{x_\varphi(t_0)} \frac{dx}{1+F(x)} > \int_0^{x_1} \frac{dx}{1+F(x)} = 1 .$$

Therefore $x_\varphi(t) \leq x_1$, $t \geq 0$. Analogously, $x_\varphi(t) \geq x_2$.

References

1. J. Andre, P. Seibert, Über die Stuckweise lineare Differentialgleichungen, die bei Regelungsproblem auftreten. Arch. Math. 7:3 (1956), 148-164.
2. A.F. Filippov, Differential equations with the discontinuous right hand side. Dordrecht, Kluwer Acad. Press, 1988.
3. Hartman Ph. Ordinary differential equations. 2nd ed., Boston, Birkhauser, 1982.
4. V.B. Kolmanovsky, V.R. Nosov, Stability and periodic modes of controllable systems with time delay. Moscow, Nauka, 1981 (Russian).
5. E.I. Shustin, Super-high-frequency modes in a discontinuous dynamic delay system. Preprint, Tel Aviv University, 1992.
6. V.I. Utkin, Sliding modes in control optimization. Berlin etc., Springer, 1991.

APPLICATION OF LAGRANGE'S APPROACH FOR THE INVESTIGATION OF ONE LINEAR OPTIMAL CONTROL PROBLEM WITH MIXED CONSTRAINTS IN DISCRETE TIME

Ioshua Sobolevsky

Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem, Israel

Abstract

Linear optimal control problem is studied by Lagrange's approach. Optimality principle for this problem is established. Finite search algorithm is presented. Several examples are consider.

Paper is devoted to the investigation of one linear optimal problem in discrete time. The peculiarity of this problem is that the control domain at next stage depends on the state of the system at the present stage. The problems of such type was considered firstly by R.Bellman ([1], Dynamic Programming, Princeton University Press, 1957). They were called by the name of bottleneck problems. R.Bellman completely investigated the simplest problem of such type in continuous time. There is an extensive bibliography in the book ([2], E.J.Anderson, P.Nash, Linear programming in infinite dimensional spaces, 1987) and in the paper ([3], R.T.Rockafellar, "Linear-quadratic programming and optimal control", SIAM Journal on Control and Optimization 25 (1987), 781–814).

In my paper I shall use by Lagrange's method for extremal problems. It will permit to establish a necessary and sufficient condition of optimality and (in decomposable case) to find the algorithm of search of optimal solution. I shall give also an example of an application of this algorithm.

Consider the problem

$$x(t) = x(t-1) + Au(t)\,; \qquad t = 1, \ldots, N\,; \qquad x(0) = x^0\,;$$

$$Bu(t) \leq x(t-1)\,; \qquad u(t) \geq 0\,; \qquad t = 1, \ldots, N\,;$$

$$\max\,(\,c\,,\,x(N)\,)$$

Here $u(t)\ (t = 1, \ldots, N)$ is a control and $x(t)\ (t = 0, \ldots, N)$ is a trajectory (or state).

Pair $(\,u(\cdot)\,,\,x(\cdot)\,)$ is called a feasible process if it satisfies conditions written in the first three lines. This pair is called an optimal process if it maximizes the linear form $(\,c\,,\,x(N)\,)$.

Suppose that the matrices A and B have a positive (or nonnegative) elements, $x(t) \in R^m$, $u(t) \in R^n$ (m, n — are natural numbers), $c > 0$, $x^0 > 0$, and all inequalities are understood in the sense of the cone of the vectors with nonnegative coordinates.

Let $(u(\cdot), x(\cdot))$ is an arbitrary feasible process. Then we have $x(0) = x^0$; $x(1) = x(0)+Au(1) = x^0+Au(1);\ldots;x(t) = x^0+A(u(1)+\cdots+u(t));\ldots;x(N) = x^0 + A(u(1) + \cdots + u(N))$. Let now $h(t)$ $(t = 1,\ldots,N)$ is an arbitrary m-dimensional vector-function (direction in the control space) and let $\epsilon > 0$ is an arbitrary positive number. Put $u_{\epsilon,h}(t) = u(t) + \epsilon h(t)$ $(t = 1,\ldots,N)$; $x_{\epsilon,h}(0) = x^0$; $x_{\epsilon,h}(t) = x_{\epsilon,h}(t-1) + Au_{\epsilon,h}(t)$; $(t = 1,\ldots,N)$. Then we have $x_{\epsilon,h}(0) = x^0$; $x_{\epsilon,h}(1) = x^0 + Au_{\epsilon,h}(1) = x^0 + Au(1) + \epsilon Ah(1) = x(1) + \epsilon Ah(1)$; $x_{\epsilon,h}(2) = x^0 + A(u_{\epsilon,h}(1) + u_{\epsilon,h}(2)) = x^0 + A(u(1) + u(2)) + \epsilon A(h(1) + h(2)) = x(2) + \epsilon A(h(1) + h(2));\ldots; x_{\epsilon,h}(t) = x^0 + A(u_{\epsilon,h}(1) + \cdots + u_{\epsilon,h}(t)) = x^0 + A(u(1) + \cdots + u(t)) + \epsilon A(h(1) + \cdots + h(t)) = x(t) + \epsilon A(h(1) + \cdots + h(t));\ldots; x_{\epsilon,h}(N) = x^0 + A(u_{\epsilon,h}(1) + \cdots + u_{\epsilon,h}(N)) = x^0 + A(u(1) + \cdots + u(N)) + \epsilon A(h(1) + \cdots + h(N)) = x(N) + \epsilon A(h(1) + \cdots + h(N))$. We suppose here that $u(\cdot)$ is a feasible control and $x(\cdot)$ is a corresponding trajectory. The perturbation (or direction) $h(\cdot)$ we want to choose by such a manner, that the displacement of this direction on small value wouldn't break only one restriction. At the beginning we shall record our conditions in the coordinate form

$$x_i(0) = x_i^0, \; i = 1,\ldots,m \tag{1}$$

$$x_i(t) = x_i(t-1) + \sum_{j=1}^{n} a_{ij}u_j(t); \; i = 1,\ldots,m; \; t = 1,\ldots,N \tag{2}$$

$$\sum_{j=1}^{n} b_{ij}u_j(t) \le x_i(t-1); \; i = 1,\ldots,m; \; t = 1,\ldots,N \tag{3}$$

$$u_j(t) \ge 0; \; j = 1,\ldots,n; \; t = 1,\ldots,N \tag{4}$$

$$\max \sum_{i=1}^{m} c_i x_i(N)$$

The first two equalities are fulfilled evidently also for $u_{\epsilon,h}(\cdot)$ and $x_{\epsilon,h}(\cdot)$. Therefore we must check of validity of the conditions (or inequalities) (3) and (4). They will be looked so

$$\sum_{j=1}^{n} b_{ij}u_{\epsilon,h,j}(t) \le x_{\epsilon,h,i}(t-1); \; i = 1,\ldots,m; \; t = 1,\ldots,N \tag{5}$$

$$u_{\epsilon,h,j}(t) \ge 0; \; j = 1,\ldots,n; \; t = 1,\ldots,N \tag{6}$$

Taking into account that $u_{\epsilon,h,j}(t) = u_j(t) + \epsilon h_j(t)$; $x_{\epsilon,h,i}(t) = x_i(t-1) + \epsilon(A(h(1) + \cdots + h(t-1)))_i = x_i(t-1) + \epsilon \sum_{j=1}^{n} a_{ij}(h_j(1) + \cdots + h_j(t-1))$, we obtain (substituting in (5) and (6))

$$\sum_{j=1}^{n} b_{ij}h_j(t) \leq x_i(t-1) + \epsilon \sum_{j=1}^{n} a_{ij}(h_j(1) + \cdots + h_j(t-1))\,;\ i = 1,\ldots,m\,;\ t = 1,\ldots,N \tag{7}$$

$$u_j(t) + \epsilon h_j(t) \geq 0\,;\ j = 1,\ldots,n\,;\ t = 1,\ldots,N \tag{8}$$

Define now two sets of indices $I(t)$ and $J(t)$, $(t = 1,\ldots,N)$:

$$I(t) = \{\ i \mid \sum_{j=1}^{n} b_{ij}u_j(t) = x_i(t-1)\ \} \tag{9}$$

$$J(t) = \{\ j \mid u_j(t) = 0\ \} \tag{10}$$

Let $i \notin I(t)$. Then from (3) it follows, that for such i and for sufficiently small $\epsilon > 0$ the inequality (7) is fulfilled. Analogously, if $j \notin J(t)$, then for sufficiently small $\epsilon > 0$ the inequality (8) is fulfilled. Therefore we must check only the cases of $i \in I(t)$ or $j \in J(t)$. So, let $i \in I(t)$. Then

$$\sum_{j=1}^{n} b_{ij}u_j(t) = x_i(t-1) \tag{11}$$

And therefore for the validity of the inequality (7) we must demand

$$\sum_{j=1}^{n} b_{ij}h_j(t) \leq \sum_{j=1}^{n} a_{ij}(h_j(1) + \cdots + h_j(t-1)) \tag{12}$$

Further, let $j \in J(t)$. Then

$$u_j(t) = 0 \tag{13}$$

And therefore for the validity of (8) we must suppose, that

$$h_j(t) \geq 0 \tag{14}$$

Thus for feasibility of the process $(u_{\epsilon,h}(\cdot), x_{\epsilon,h}(\cdot))$ we must demand, that

$$\sum_{j=1}^{n} b_{ij}h_j(t) \leq \sum_{j=1}^{n} a_{ij}(h_j(1) + \cdots + h_j(t-1))\,;\ i \in I(t)\,;\ t = 1,\ldots,N \tag{15}$$

and

$$h_j(t) \geq 0\,;\ j \in J(t)\,;\ t = 1,\ldots,N \tag{16}$$

Now we suppose, that $(u(\cdot),\ x(\cdot))$ is an optimal process and $h(\cdot)$ is a feasible perturbation (or direction). Then the inequalities (15), (16) are fulfilled and thanks to the optimality we obtain the decreasing of an objective functional, i.e.

$$(c,\, x_{\epsilon,h}(N)) \le (c,\, x(N)) \tag{17}$$

for small ϵ. But we proved early, that

$$x_{\epsilon,h}(N) = x(N) + \epsilon\, A\,(h(1) + \cdots + h(N)) \tag{18}$$

Therefore from (17) we obtain $(c,\, A\,(h(1)+\cdots+h(N))) \le 0$ or in coordinate form

$$\sum_{i=1}^{m} c_i \sum_{j=1}^{n} a_{ij}\,(h_j(1) + \cdots + h_j(N)) \le 0 \tag{19}$$

or

$$\sum_{\tau=1}^{N} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_i a_{ij} \right) h_j(\tau) \le 0 \tag{20}$$

So, from the inequalities (15) and (16) it follows (20). It is necessary and sufficient condition for local and (thanks to linearity) global maximum. Now we shall try to record the inequalities (15), (16) and (20) in the comfortable form. For this purpose we introduce the linear functionals, acting in finite linear dimensional space of vectors with coordinates $\{h_j(\tau) \mid j = 1,\ldots,n;\ \tau = 1,\ldots,N\}$. The first functional we define by the formula

$$F\{h_j(\tau)\} = \sum_{\tau=1}^{N} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_i a_{ij} \right) h_j(\tau)\,. \tag{21}$$

Then inequality (20) we can record in the form

$$F\{h_j(\tau)\} \le 0 \tag{22}$$

Further, we define the linear functionals Φ^{it}, $i \in I(t)$, $t = 1,\ldots,N$ putting

$$\Phi^{it}\{h_j(\tau)\} = \sum_{\tau=1}^{t-1} \sum_{j=1}^{n} a_{ij} h_j(\tau) - \sum_{j=1}^{n} b_{ij} h_j(t) \tag{23}$$

Then inequality (15) is rewritten in the form

$$\Phi^{it}\{h_j(\tau)\} \ge 0 \tag{24}$$

At last we put

$$\Psi^{ks}\{h_j(\tau)\} = h_k(s);\ k \in J(s);\ s = 1,\ldots,N \tag{25}$$

(16) is rewritten so

$$\Psi^{ks}\{h_j(\tau)\} \geq 0;\; k \in J(s);\; s = 1,\ldots,N \tag{26}$$

Thus, we know that from inequalities

$$\Phi^{it}\{h_j(\tau)\} \geq 0;\; i \in I(t);\; t = 1,\ldots,N \tag{27}$$

$$\Psi^{ks}\{h_j(\tau)\} \geq 0;\; k \in J(s);\; s = 1,\ldots,N \tag{28}$$

it follows inequality

$$F\{h_j(\tau)\} \leq 0 \tag{29}$$

It means, that linear functional $-F$ belongs to the cline, generated by the functionals Φ^{it}, Ψ^{ks}, i.e. there exist nonnegative numbers

$$\alpha^{it} \geq 0;\; i \in I(t);\; t = 1,\ldots,N \tag{30}$$

$$\beta^{ks} \geq 0;\; k \in J(s);\; s = 1,\ldots,N \tag{31}$$

such that

$$F + \sum_{t=1}^{N} \sum_{i \in I(t)} \alpha^{it}\Phi^{it} + \sum_{s=1}^{N} \sum_{k \in J(s)} \beta^{ks}\Psi^{ks} = 0 \tag{32}$$

(We uses here by the following theory from the separation theory: Let $f_1,\ldots,f_m,f_{m+1}$ be linear functionals on the finite dimensional space E and let the following implication is true

$$(\, f_1(x) \geq 0,\ldots, f_m(x) \geq 0 \,) \implies (\, f_{m+1}(x) \geq 0 \,)$$

Then f_{m+1} is a conic linear combination of the functionals $f_1,\ldots,f_m$, i.e. there exist nonnegative numbers $\alpha_1,\ldots,\alpha_m \geq 0$ such that $f_{m+1} = \alpha_1 f_1 + \cdots + \alpha_m f_m$. By the other words f_{m+1} belongs to the cone (cline) generated by the functionals $f_1,\ldots,f_m$).

Now we shall write the last equality in the coordinate form with help of (21), (23) and (25). I.e. we must record (32) for every coefficient of $h_j(\tau)$

$$\sum_{i=1}^{m} c_i a_{ij} + \sum_{t=\tau+1}^{N} \left(\sum_{i \in I(t)} a_{ij}\alpha^{it} \right) - \sum_{i \in I(\tau)} b_{ij}\alpha^{i\tau} + \beta^{j\tau} = 0 \tag{32}$$

We can record (32) without sets of indices I(t) and J(t)

$$\alpha^{it} \geq 0;\; i = 1,\ldots,m;\; t = 1,\ldots,N \tag{33}$$

$$\beta^{ks} \geq 0;\; k = 1,\ldots,n;\; s = 1,\ldots,N \tag{34}$$

and

$$\alpha^{it}\left(x_i(t-1) - \sum_{j=1}^{n} b_{ij}u_j(t)\right) = 0\ ;\ i = 1,\ldots,m\ ;\ t = 1,\ldots,N \tag{35}$$

$$\beta^{ks}u_k(s) = 0\ ;\ k = 1,\ldots,N \tag{36}$$

Then (32) is rewritten in the form

$$F + \sum_{t=1}^{N}\sum_{i=1}^{m}\alpha^{it}\Phi^{it} + \sum_{s=1}^{N}\sum_{k=1}^{n}\beta^{ks}\Psi^{ks} = 0 \tag{37}$$

It means that all coordinates of the functional, i.e. all coefficients of $h_j(\tau)$ are equal to zero. So, we apply the functional (37) to an arbitrary vector $\{h_j(\tau)\,;\, j = 1,\ldots,n\,;\ \tau = 1,\ldots,N\}$. Then we obtain

$$F\{h_j(\tau)\} + \sum_{t=1}^{N}\sum_{i=1}^{m}\alpha^{it}\Phi^{it}\{h_j(\tau)\} + \sum_{s=1}^{N}\sum_{k=1}^{m}\beta^{ks}\Psi^{ks}\{h_j(\tau)\} \tag{38}$$

Now we calculate an each term separately

$$F\{h_j(\tau)\} = \sum_{\tau=1}^{N}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}c_i a_{ij}\right)h_j(\tau) \tag{39}$$

$$\Phi^{it}\{h_(\tau)\} = \sum_{\tau=1}^{t-1}\sum_{j=1}^{n}a_{ij}h_j(\tau) - \sum_{j=1}^{n}b_{ij}h_j(t) \tag{40}$$

$$\Psi^{ks}\{h_j(\tau)\} = h_k(s)\ . \tag{41}$$

Therefore from (37) and (38) we obtain

$$\left.\begin{aligned}&\sum_{\tau=1}^{N}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}c_i a_{ij}\right)h_j(\tau)+\sum_{t=1}^{N}\sum_{i=1}^{m}\alpha^{it}\left(\sum_{\tau=1}^{t-1}\sum_{j=1}^{n}a_{ij}h_j(\tau)-\sum_{j=1}^{n}b_{ij}h_j(t)\right)\\&+\sum_{s=1}^{N}\sum_{k=1}^{n}\beta^{ks}h_k(s)=\sum_{\tau=1}^{N}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}c_i a_{ij}\right)h_j(\tau)+\sum_{\tau=1}^{N-1}\sum_{j=1}^{n}\left(\sum_{t=\tau+1}^{N}\sum_{i=1}^{m}\alpha^{it}a_{ij}\right)h_j(\tau)\\&-\sum_{\tau=1}^{N}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\alpha^{i\tau}b_{ij}\right)h_j(\tau)+\sum_{\tau=1}^{N}\sum_{j=1}^{n}\beta^{j\tau}h_j(\tau)=0\end{aligned}\right\} \tag{42}$$

It is true for every real numbers $h_j(\tau)$. Therefore all coefficients here are equal to zero. Let at the beginning $\tau = 1, \ldots, N-1$. Then

$$\sum_{i=1}^{m} c_i a_{ij} + \sum_{t=\tau+1}^{N} \sum_{i=1}^{m} \alpha^{it} a_{ij} \sum_{i=1}^{m} \alpha^{i\tau} b_{ij} + \beta^{i\tau} = 0 \, ; \; j = 1, \ldots, n \, ; \; \tau = 1, \ldots, N-1 \tag{43}$$

At last for $\tau = N$ we have

$$\sum_{i=1}^{m} c_i a_{ij} - \sum_{i=1}^{m} \alpha^{iN} b_{ij} + \beta^{jN} = 0 \, ; \; j = 1, \ldots, n \tag{44}$$

Besides this we have

$$x_i(t) = x_i(t-1) + \sum_{j=1}^{n} a_{ij} u_(t) \, ; \; i = 1, \ldots, m \;\; t = 1, \ldots, N$$

$$x_i(0) = x_i^0 \, ; \; i = 1, \ldots, m$$

$$\sum_{i=1}^{n} b_{ij} u_j(t) \leq x_i(t-1) \, ; \; i = 1, \ldots, m \;\; t = 1, \ldots, N$$

$$u_j(t) \geq 0 \, ; \; j = 1, \ldots, n \, ; \; t = 1, \ldots, N$$

There are $2(m+n)N$ unknowns and $2(m+n)N$ equations here. Now we write all correlations once again

$$\alpha^{it} \geq 0 \, ; \; i = 1, \ldots, m \, ; \; t = 1, \ldots, N \tag{45}$$

$$\beta^{ks} \geq 0 \, ; \; k = 1, \ldots, n \, ; \; s = 1, \ldots, N \tag{46}$$

$$x_i(t-1) - \sum_{j=1}^{n} b_{ij} u_j(t) \geq 0 \, ; \; i = 1, \ldots, ; \; t = 1, \ldots, N \tag{47}$$

$$u_k(s) \geq 0 \, ; \; k = 1, \ldots, n \;\; s = 1, \ldots, N \tag{48}$$

$$\alpha^{it} \left(x_i(t-1) - \sum_{j=1}^{n} b_{ij} u_j(t) \right) = 0 \, ; \; i = 1, \ldots, m \, ; \; i = 1, \ldots, N \tag{49}$$

$$\beta^{ks} u_k(s) = 0 \, ; \; k = 1, \ldots, n \, ; \; s = 1, \ldots, N \tag{50}$$

$$x_i(t) = x_i(t-1) + \sum_{j=1}^{n} a_{ij} u_j(t) \, ; \; i = 1, \ldots, m \, ; \; t = 1, \ldots, N \tag{51}$$

$$x_i(0) = x_i^0 \, ; \; i = 1, \ldots, m$$

$$\sum_{i=1}^{m} c_i a_{ij} + \sum_{t=\tau+1}^{N} \sum_{i=1}^{m} \alpha^{it} a_{ij} - \sum_{i=1}^{m} \alpha^{i\tau} b_{ij} + \beta^{j\tau} = 0 \, ;$$

$$j = 1, \dots, n; \; \tau = 1, \dots, N-1 \tag{52}$$

$$\sum_{i=1}^{m} c_i a_{ij} - \sum_{i=1}^{m} \alpha^{iN} b_{ij} + \beta^{jN} = 0; \; j = 1, \dots, n; \; \tau = N \tag{53}$$

Now we shall transform this system to the form that can be obtained from the Boltyanskii's optimality principle. We put

$$\lambda_i(t) = -\alpha^{it}; \; i = 1, \dots, m; \; t = 1, \dots, N \tag{54}$$

$$\mu_j(t) = -\beta^{jt}; \; j = 1, \dots, n; \; t = 1, \dots, N \tag{55}$$

Then

$$\lambda_i(t) \leq 0; \; i = 1, \dots, m; \; t = 1, \dots, N \tag{56}$$

$$\mu_j(t) \leq 0; \; i = 1, \dots, n; \; t = 1, \dots, N \tag{57}$$

$$\lambda_i(t) \left(\sum_{j=1}^{n} b_{ij} u_j(t) - x_i(t-1) \right) = 0; \; i = 1, \dots, m; \; t = 1, \dots, N \tag{58}$$

$$\mu_j(t) \left(- u_j(t) \right) = 0; \; j = 1, \dots, n; \; t = 1, \dots, N \tag{59}$$

Equalities (52) and (53) are rewritten in the form

$$\sum_{i=1}^{m} \left(\left(c_i + \sum_{t=\tau+1}^{N} \alpha^{it} \right) a_{ij} - \alpha^{i\tau} b_{ij} \right) + \beta^{j\tau} = 0; \quad j = 1, \dots, n; \quad \tau = 1, \dots, N \tag{60}$$

$$\sum_{i=1}^{m} (c_i a_{ij} - \alpha^{iN} b_{ij}) + \beta^{jN} = 0; \quad j = 1, \dots, n; \quad \tau = N \tag{61}$$

We put now

$$\Psi_i(\tau) = c_i + \sum_{t=\tau+1}^{N} \alpha^{it}; \quad i = 1, \dots, m; \quad \tau = 1, \dots, N-1 \tag{62}$$

$$\Psi_i(N) = 0; \quad i = 1, \dots, m; \quad \tau = N \tag{63}$$

Then taking into account (55) and (62) and also (60) and (61) we can write

$$\sum_{i=1}^{m} (a_{ij} \Psi_i(\tau) + b_{ij} \lambda_i(\tau)) = \mu_j(\tau); \quad j = 1, \dots, n; \quad \tau = 1, \dots, N \tag{64}$$

$$\sum_{i=1}^{m} (a_{ij} c_i + b_{ij} \lambda_i(N)) = \mu_j(N); \quad j = 1, \dots, n; \quad \tau = N \tag{65}$$

Taking (54) into account we can rewrite (62) in the form

$$\Psi_i(\tau) = c_i - \sum_{t=\tau+1}^{N} \lambda_i(t); \quad i = 1, \dots, m \quad \tau = 1, \dots, N-1 \tag{66}$$

or in the vector form

$$\Psi(\tau) = c - \sum_{t=\tau+1}^{N} \lambda(t); \quad \tau = 1, \dots, N-1 \quad \Psi(N) = 0 \tag{67}$$

In particularly for $\tau = N - 1$ we obtain from (67) that

$$\Psi(N-1) = c - \lambda(N) \tag{68}$$

Further for $1 \leq \tau \leq N - 2$ from (67) we have

$$\Psi(\tau+1) = c - \sum_{t=\tau+2}^{N} \lambda(t) \tag{69}$$

Therefore from (67) and (69) we have

$$-\Psi(\tau) + \Psi(\tau+1) = \lambda(\tau+1) \tag{70}$$

So

$$-\Psi(\tau) + \Psi(\tau+1) - \lambda(\tau+1) = 0; \quad \tau = 1, \dots, N-2 \tag{71}$$

$$c - \Psi(N-1) - \lambda(N) = 0 \tag{72}$$

$$\Psi(N) = 0 \tag{73}$$

Thus we obtain all the correlations that gives Boltyanskii's optimality principle. Now we shall write all these condition in one place. Thus we proved the following theorem;

For optimality of the process $(u(\cdot), x(\cdot))$ it is necessary and sufficient the existence of vectors

$$\Psi(t) = \{\Psi_1(t), \dots, \Psi_m(t); \quad t = 1 = 1, \dots, N \tag{74}$$

$$\lambda(t) = \{\lambda_1(t), \dots, \lambda_m(t)\}; \quad t = 1, \dots, N \tag{75}$$

$$\mu(t) = \{\mu_1(t), \dots, \mu_n(t)\}; \quad t = 1, \dots, N \tag{76}$$

such, that the following conditions are satisfied:

$$(A) \quad -\Psi(t) + \Psi(t+1) - \lambda(t+1) = 0; \quad t = 1, \dots, N-2 \tag{77}$$

$$c - \Psi(N-1) - \lambda(N) = 0; \tag{78}$$

$$\Psi(N) = 0 \tag{79}$$

$$(B) \quad A^*\Psi(t) + B^*\lambda(t) = \mu(t) \tag{80}$$

$$A^*c + B^*\lambda(N) = \mu(N) \tag{81}$$

(Here A^* and B^* are matrices adjoint to A and B respectively.)

$$(C) \quad \lambda_i(t) \leq 0; \quad \mu_j(t) \leq 0; \quad i = 1, \ldots, m; \quad j = 1, \ldots, n; \quad t = 1, \ldots, N \quad (82)$$

$$\lambda_i(t) \left(\sum_{j=1}^{n} b_{ij} u_j(t) - x_i(t-1) \right) = 0; \quad i = 1, \ldots, m; \quad t = 1, \ldots, N \quad (83)$$

$$\mu_j(t)(-u_j(t)) = 0; \quad j = 1, \ldots, n; \quad t = 1, \ldots, N \quad (84)$$

Taking into account (80) and (81) we can write

$$\sum_{i=1}^{m} (a_{ij} \Psi_i(t) + b_{ij} \lambda_i(t))(-u_j(t)) = 0; \quad j = 1, \ldots, n; \quad t = 1, \ldots, N-1 \quad (84)$$

$$\sum_{i=1}^{m} (a_{ij} c_i + b_{ij} \lambda_i(N))(-u_j(N)) = 0; \quad j = 1, \ldots, n \quad (86)$$

From (80), (81) and (82) we obtain

$$\sum_{i=1}^{m} (a_{ij} \Psi_i(t) + b_{ij} \lambda_i(t)) \leq 0; \quad j = 1, \ldots, n; \quad t = 1, \ldots, N \quad (87)$$

$$\sum_{i=1}^{m} (a_{ij} c_i + b_{ij} \lambda_i(N)) \leq 0; \quad j = 1, \ldots, n \quad (88)$$

Consider now the case, when for control we have some independent inequalities

$$u_1(t) + \ldots + u_{k_1}(t) \leq x_1(t-1) \quad (89)$$

$$u_{k_1+1}(t) + \ldots + u_{k_2}(t) \leq x_2(t-1) \quad (90)$$

....

....

$$u_{k_{m-1}+1}(t) + \ldots + u_{k_m}(t) \leq x_m(t-1) \quad (91)$$

In this case matrix $B = (b_{ij})$ has a form

$$B = \begin{pmatrix} 1 \cdots 1 \, 0 \cdots 0 \cdots\cdots 0 \cdots 0 \\ 0 \cdots 0 \, 1 \cdots 1 \cdots\cdots 0 \cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \underbrace{\underbrace{\underbrace{0 \cdots 0}_{k_1} \, 0 \cdots 0}_{k_2} \cdots\cdots 1 \cdots 1}_{k_m = n} \end{pmatrix} \quad (92)$$

Then the equalities (85) and (86) take the form

$$(a_{11}\Psi_1(t) + \cdots + a_{m1}\Psi_m(t) + \lambda_1(t))(-u_1(t)) = 0 \tag{93}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1k_1}\Psi_1(t) + \cdots + a_{mk_1}\Psi_m(t) + \lambda_1(t))(-u_{k_1}(t)) = 0 \tag{94}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1,k_{m-1}+1}\Psi_1(t) + \cdots + a_{m,k_{m-1}+1}\Psi_m(t) + \lambda_m(t))(-u_{k_{m-1}+1}(t)) = 0 \tag{95}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1,k_m}\Psi_1(t) + \cdots + a_{m,k_m}\Psi_m(t) + \lambda_m(t))(-u_{k_m}(t)) = 0 \tag{96}$$

for $t = 1, \ldots, N-1$ and for $t = N$ we have

$$(a_{11}c_1 + \cdots + a_{m1}c_m + \lambda_1(N))(-u_1(N)) = 0 \tag{97}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1,k_1}c_1 + \cdots + a_{m,k_1}c_m + \lambda_1(N))(-u_{k1}(N)) = 0 \tag{98}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1,k_{m-1}+1}c_1 + \cdots + a_{m,k_{m-1}+1}c_m + \lambda_m(N))(-u_{k_{m-1}+1}(N)) = 0 \tag{99}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1,k_m}c_1 + \cdots + a_{m,k_m}c_m + \lambda_m(N))(-u_{k_m}(N)) = 0 \tag{100}$$

Besides this, we can write (87) and (88) in the form

$$a_{11}\Psi_1(t) + \cdots + a_{m1}\Psi_m(t) + \lambda_1(t) \leq 0 \tag{101}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$a_{1,k_1}\Psi_1(t) + \cdots + a_{m,k_1}\Psi_m(t) + \lambda_1(t) \leq 0 \tag{102}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$a_{1,k_{m-1}+1}\Psi_1(t) + \cdots + a_{m,k_{m-1}+1}\Psi_m(t) + \lambda_m(t) \leq 0 \tag{103}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$a_{1,k_m}\Psi_1(t) + \cdots + a_{m,k_m}\Psi_m(t) + \lambda_m(t) \leq 0 \tag{104}$$

for $t = 1, \cdots, N-1$ and for $t = N$ we have

$$a_{11}c_1 + \cdots + a_{m1}c_m + \lambda_1(N) \leq 0 \tag{105}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1,k_1}c_1 + \cdots + a_{m,k_1}c_m + \lambda_1(N) \leq 0 \tag{106}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1,k_{m-1}+1}c_1 + \cdots + a_{m,k_{m-1}+1}c_m + \lambda_m(N) \leq 0 \tag{107}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1,k_m}c_1 + \cdots + a_{m,k_m}c_m + \lambda_m(N) \leq 0 \tag{108}$$

Further, the equalities (83) have a form

$$\lambda_1(t)(u_1(t) + \cdots + u_{k_1}(t) - x_1(t-1)) = 0 \tag{109}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_1(t)(u_{k_{m-1}+1}(t) + \cdots + u_{k_m}(t) - x_m(t-1)) = 0 \tag{110}$$

Here

$$\lambda_i(t) \leq 0 \tag{111}$$

Since matrix A and B have positive elements, then from (51) it follows that

$$x_i(t) > 0; \quad i = 1, \ldots, m; \quad t = 1, \ldots, N \tag{112}$$

Furter, from (105), (106) we obtain that

$$\lambda_1(N) < 0 \tag{113}$$

Analogously,

$$\lambda_i(N) < 0; \quad i = 1, \ldots, n \tag{114}$$

Then from (119)–(110) we obtain

$$u_1(N) + \cdots + u_{k_1}(N) = x_1(N-1) \tag{115}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_{k_{m-1}+1}(N) + \cdots + u_{k_m}(N) = x_m(N-1) \tag{116}$$

From (115) and (112) we have, that at least one of $u_1(N), \dots, u_{k_1}(N)$ is not equal to zero. Analogously, one of $u_{k_{m-1}+1}(N), \dots, u_{k_m}(N)$ is not equal to zero. Therefore one of the parenthesis(97)–(98) is equal to zero. In order to find the index $J_1(N) \in [1, k_1]$, for which corresponding parenthesis is equal to zero, we must calculate

$$\max_{1 \le j \le k_1} (a_{1j}c_1 + \cdots + a_{mj}c_m) = a_{1J_1(N)}c_1 + \cdots + a_{mJ_1(N)}c_m \qquad (117)$$

It may be that such index $J_1(N)$ is not unique. Then we can take one of them. Now we can find $\lambda_1(N)$ from the equality

$$a_{1J_1(N)}c_1 + \cdots + a_{mJ_1(N)}c_m + \lambda_1(N) = 0 \qquad (118)$$

Analogously, index $J_2(N) \in [k_1 + 1, k_2]$ we find from the correlation

$$\max_{k_1+1 \le j \le k_2} (a_{1j}c_1 + \cdots + a_{mj}c_m) = a1J_2(N)c_1 + \cdots + a_{mJ_2(N)}c_m \qquad (119)$$

and for $\lambda_2(N)$ we have

$$a_{1J_2(N)}c_1 + \cdots + a_{mJ2(N)}c_m + \lambda_2(N) = 0 \qquad (120)$$

Further we find $J_3(N), \dots, J_m(N)$ and $\lambda_3(N), \dots, \lambda_m(N)$.Thus, we found vector $\lambda(N)$. Then from (78) we find

$$\Psi(N-1) = c - \lambda(N) \qquad (121)$$

Record correlations (93)–(96) for $t = N - 1$. Then we obtain

$$(a_{11}\Psi_1(N-1) + \cdots + a_{m1}\Psi_m(N-1) + \lambda_1(N-1))(-u_1(N-1)) = 0 \qquad (122)$$

....

....

$$(a_{1,k_1}\Psi_1(N-1) + \cdots + a_{m,k_1}\Psi_m(N-1) + \lambda_1(N-1))(-u_{k_1}(N-1)) = 0 \qquad (123)$$

....

....

$$(a_{1,k_{m-1}+1}\Psi_1(N-1) + \cdots + a_{m,k_{m-1}+1}\Psi_m(N-1) + \lambda_1(N-1))(-u_{k_{m-1}+1}(N-1)) = 0 \qquad (124)$$

....

....

$$(a_{1,k_m}\Psi_1(N-1) + \cdots + a_{m,k_m}\Psi_m(N-1) + \lambda_1(N-1))(-u_{k_m}(N-1)) = 0 \qquad (125)$$

As above we find indices $J_1(N-1),\dots,J_m(N-1)$ from the correlations

$$\max_{1\le j\le k_1}(a_{1j}\Psi_1(N-1)+\cdots+a_{mj}\Psi_m(N-1)) = a_{1J_1(N-1)}\Psi_1(N-1) + \cdots + a_{mJ_(N-1)}\Psi_m(N-1) \tag{126}$$

We obtain $\lambda_1(n-1)$ from the equality

$$a_{1J_(N-1)}\Psi_1(N-1)+\cdots+a_{mJ_1(N-1)}\Psi_m(N-1)+\lambda(N-1)=0 \tag{127}$$

Analogously, we find $J_2(N-1),\dots,J_m(N-1)$ and $\lambda_2(N-1),\dots,\lambda_m(N-1)$. Thus we found the vector $\lambda(N-1)$. Then from (77) for $t=N-2$ we find that

$$\Psi(N-2)=\Psi(N-1)-\lambda(N-1) \tag{128}$$

Further, record correlations (93)–(96) for $t=N-2$ and so on. By the same method we find $J_1(k),\dots,J_m(k)$, $\lambda_1(k),\dots,\lambda_m(k)$, $\Psi_1(k),\dots,\Psi_m(k)$ for every $k\in[0,\,N]$. From (77) we have

$$\Psi(t)=\Psi(t+1)-\lambda(t+1) \tag{129}$$

From this

$$\Psi(t)\ge\Psi(t+1) \tag{130}$$

Since $\Psi(N-1)=c-\lambda(N)\ge c>0$, then

$$\Psi(t)\ge c>0 \tag{131}$$

Therefore from (101)–(102) we have

$$\lambda_1(t),0;\quad t=1,\dots,N \tag{132}$$

Analogously,

$$\lambda_i(t)<0;\quad i=1,\dots,m;\quad t=1,\dots,N \tag{133}$$

Then from (109)–(110) we have

$$u_1(t)+\cdots+u_{k1}(t)=x_1(t-1) \tag{134}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$u_{k_{m-1}+1}(t)+\cdots+u_{k_m}(t)=x_m(t-1) \tag{135}$$

We see from this that one of $u_1(t),\dots,u_{k_1}(t)$ is not equal to zero. Analogously, one of $u_{k_{m-1}+1}(t),\dots,u_{k_m}(t)$ is not equal to zero.

Now we shall find the optimal control and the corresponding trajectory. Find at first $u(1)$. Substitute $t = 1$ in (93)–(94) and put

$$\left.\begin{array}{r} u_{J_1(1)}(1) = x_1(1-1) = x_1(0) = x_1^0; \\ u_{J_2(1)}(1) = x_2(1-1) = x_2(0) = x_2^0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{J_m(1)}(1) = x_m(1-1) = x_m(0) = x_m^0; \\ u_j(1) = 0; \quad j \neq J_1(1), \ldots, J_m(1) \end{array}\right\} \tag{136}$$

Then (109)–(110) are satisfied. Further, (93)–(96) are satisfied also. Validity of the inequalities (101)–(104) follows from the choice of the indices $J_1(1), \ldots, J_m(1)$ and numbers $\lambda_1(1), \ldots, \lambda_m(1)$. Further, we put

$$x(1) = x(0) + Au(1) \tag{137}$$

where $x(0) = x^0$ and $u(1)$ was found. Further, we put

$$u_{J_1(2)}(2) = x_1(1); \cdots; u_{J_m(2)}(2) = x_m(1); \quad u_j(2) = 0; \quad j \neq J_1(2), \ldots, J_m(2) \tag{138}$$

It is easy to see that also in this case the correlations (93)–(96) are satisfied for $t = 2$. Then we find

$$x(2) = x(1) + Au(2) \tag{139}$$

Analogously, we find

$$u(1), \ldots, u(N) \tag{140}$$

and

$$x(0), x(1), \ldots, x(N) \tag{141}$$

.

So, we found the control (140) and the trajectory (141). We also found $\Psi(t)$ and $\lambda(t)$. Functions $u(t), x(t), \Psi(t), \lambda(t)$ satisfy the correlations (93)–(111) and (82), (84), (77)–(79), (51). Therefore $u(t)$ is an optimal control and $x(t)$ is an optimal trajectory.

Now we can describe the problem and the algorithm of search of optimal process.

Consider the problem

$$x_i(t) = x_i(t-1) + \sum_{j=1}^{n} a_{ij} u_j(t); \quad i = 1, \ldots, m; \quad t = 1, \ldots, N \tag{142}$$

$$x_i(0) = x_i^0; \quad i = 1 \ldots, m \tag{143}$$

$$u_j(t) \geq 0; \quad j = 1, \ldots, n; \quad t = 1, \ldots, N \tag{144}$$
$$u_1(t) + \cdots + u_{k_1}(t) \leq x_1(t-1) \tag{145}$$
$$u_{k_1+1}(t) + \cdots + u_{k_2}(t) \leq x_2(t-1) \tag{146}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_{k_{m-1}+1}(t) + \ldots + u_{k_m}(t) \leq x_m(t-1) \tag{147}$$
$$\max \sum_{i=1}^{m} c_i x_i(N) \tag{148}$$

We suppose, that

$$a_{ij} \geq 0; \quad i = 1, \ldots, m; \quad j = 1, \ldots, n \tag{150}$$

Now we describe an algorithm. Index $J_1(N)$ and number $\lambda_1(N)$ are defined by the equalities

$$max_{1 \leq j \leq k_1}(a_{1j}c_1 + \cdots + a_{mj}c_m) = a_{1J_1(N)}c_1 + \ldots + a_{mJ_1(N)}c_m = -\lambda_1(N) \tag{151}$$

Analogously, for $p = 2, \ldots, m$ we have

$$\max_{k_{p-1}+1 \leq j \leq k_p}(a_{1j}c_1 + \ldots + a_{mj}c_m) = a_{1J_p(N)}c_1 + \ldots + a_{mJ_p(N)}c_m = -\lambda_p(N) \tag{152}$$

So, we find $J_1(N), \ldots, J_m(N) \quad \lambda_1(N), \ldots, \lambda_m(N)$. Further

$$\Psi(N-1) = c - \lambda(N) \tag{153}$$

Here $c = (c_1, \ldots, c_m), \quad \lambda(N) = (\lambda_1(N), \ldots, \lambda_m(N))$ Now we obtain the numbers $J_p(N-1), \lambda_p(N-1)$ from the equalities

$$\max_{kp-1+1 \leq j \leq k_p}(a_{1J}\Psi_1(N-1) + \ldots + a_{mj}\Psi_m(N-1))$$
$$= a_{1J_p(N-1)}\Psi_1(N-1) + \ldots + a_{mJ_p(N-1)}\Psi_m(N-1) = -\lambda_p(N-1) \tag{154}$$
$$(k_0 = 0), \quad p = 1, \ldots, m \quad (k_m = n)$$

Then we find $\Psi(N-2)$

$$\Psi(N-2) = \Psi(N-1) - \lambda(N-1) \tag{155}$$

and so on. $(\Psi(t) = \Psi(t+1) - \lambda(t+1))$. By this method we find

$$J(t), \lambda(t), \quad t = 1, \ldots, N; \quad \Psi(t), \quad t = 1, \ldots, N-1 \tag{156}$$

Now we find optimal control and trajectory

$$u(1): \quad u_{J_1(1)}(1) = x_1^0; \quad u_{J_2(1)}(1) = x_2^0; \ldots$$

$$u_{J_m(1)}(1) = x_m^0; \quad u_j(1) = 0; \quad j = J_1(1), \ldots, J_m(1); \tag{157}$$

$$x(1) = x(0) + Au(1) \tag{158}$$

$$u(2): \quad u_{J_1(2)}(2) = x_1(1); \quad u_{J_2(2)}(2) = x_2(1); \ldots$$

$$u_{J_m(2)}(2) = x_m(1); \quad u_j(2) = 0; \quad j = J_1(2), \ldots, J_m(2); \tag{159}$$

$$x(2) = x(1) + Au(2) \tag{160}$$

By the same method we find

$$u(1), \ldots, u(N) \tag{161}$$

$$x(0), x(1), \ldots, x(N) \tag{162}$$

Now consider an example

$$x_1(t) = x_1(t-1) + u_1(t) + 2u_2(t) + 3u_3(t) + 4u_4(t)$$
$$x_2(t) = x_2(t-1) + 4u_1(t) + 3u_2(t) + 2u_3(t) + u_4(t)$$
$$x_1(0) = 1, \quad x_2(0) = 1; \quad N = 2;$$
$$u_1(t) + u_2(t) \leq x_1(t-1);$$
$$u_3(t) + u_4(t) \leq x_2(t-1);$$
$$\max\{x_1(2) + x_2(2)\}; \quad c_1 = c_2 = 1$$
$$\max(1+4, 2+3) = 5 = -\lambda_1(2); \quad J_1(2) = 1;$$

Here there are two opportunities for the choice of index $J_1(2) = 1$ or $J_1(2) = 2$. We prefer here the first opportunity. However, we could choose the other opportunity. So,

$$\lambda_1(2) = -5; \quad J_1(2) = 1; \quad max(3+2, 4+1) = 5 = -\lambda_2(2); \quad J_2(2) = 3;$$

Here also there are two opportunities for choice and we prefer the first of them. So,

$$\lambda_2(2) = -5; \quad J_2(2) = 3; \quad \Psi(1) = c - \lambda(2) = (c_1 - \lambda_1(2), c_2 - \lambda_2(2)) = (1+5, 1+5) = ($$

Thus, we have

$$J_1(1) = 1; \quad J_2(1) = 3; \quad \lambda_1(1) = -30; \quad \lambda_2(1) = -30;$$
$$J_1(2) = 1; \quad J_2(2) = 3; \quad \lambda_1(2) = -5; \quad \lambda_2(2) = -5;$$
$$\Psi_1(1) = 6; \quad \Psi_2(1) = 6.$$
$$u_1(1) = x_1^0 = 1; \quad u_3(1) = x_2^0 = 1; \quad u_2(1) = u_4(1) = 0$$
$$x_1(1) = 1 + 1 + 3 = 5; \quad x_2(1) = 1 + 4 + 2 = 7;$$
$$u_1(2) = x_1(1) = 5; \quad u_3(2) = x_2(1) = 7; \quad u_2(2) = u_4(2) = 0.$$
$$x_1(2) = 5 + 5 + 21 = 31 \quad x_2(2) = 7 + 20 + 14 = 41$$
$$x(0) = (1, 1); \quad u(1) = (1, 0, 1, 0); \quad x(1) = (5, 7);$$
$$u(2) = (5, 0, 7, 0); \quad x(2) = (31, 41); \quad max = x_1(2) + x_2(2) = 31 + 41 = 72;$$

Now we make an another choice on the first stage: $J_1(2) = 2$. So,

$J_1(2) = 2; \quad J_2(2) = 3; \quad \lambda_1(2) = -5; \quad \lambda_2(2) = -5;$
$\Psi(1) = (6, 6); \quad J_1(1) = 1; \quad J_2(1) = 3; \quad \lambda_1(1) = \lambda_2(1) = -30;$
$u_1(1) = u_3(1) = 1; \quad u_2(1) = u_4(1) = 0; \quad x_1(1) = 5; \quad x_2(1) = 7;$
$u_2(2) = 5; \quad u_3(2) = 7; \quad x_1(2) = 5 + 10 + 21 = 36; \quad x_2(2) = 7 + 15 + 14 = 36;$
$max = x_1(2) + x_2(2) = 36 + 36 = 72.$

We obtained the same result as in the first case. Suppose now, that the second choice is other

$J_1(1) = 2; \quad J_2(1) = 3; \quad u_2(1) = 1; \quad u_3(1) = 1; \quad u_1(1) = u_4(1) = 0;$
$x_1(1) = 1 + 2 + 3 = 6; \quad x_2(1) = 1 + 3 + 2 = 6; \quad J_1(2) = 1; \quad J_2(2) = 3;$
$u_1(2) = 6; \quad u_3(2) = 6; \quad u_2(2) = u_4(2) = 0; \quad x_1(2) = 6 + 6 + 18 = 30;$
$x_2(2) = 6 + 24 + 12 = 42; \quad max = x_1(2) + x_2(2) = 72$

And again we obtained the same result.

Consider now the case of $N = 3$ (three stages).

$\max(1 + 4, 2 + 3) = 5 = -\lambda_1(3); \quad J_1(3) = 1; \quad J_2(3) = 1; \quad \lambda_1(3) = -5;$
$\max(2 + 3, 4 + 1) = 5 = -\lambda_2(3); \quad J_2(3) = 3; \quad J_2(3) = 3; \quad \lambda_2(3) = -5;$
$\Psi(2) = c - \lambda(3) = (c_1 - \lambda_1(3), c_2 - \lambda_2(3)) = (6, 6);$
$J_1(2) = 1; \quad J_2(2) = 3; \quad \lambda_1(2) = \lambda_2(2) = -30;$
$\Psi(1) = \Psi(2) - \lambda(2) = (6 - \lambda_1(2), 6 - \lambda_2(2)) = (36, 36);$
$J_1(1) = 1; \quad J_2(1) = 3; \quad \lambda_1(1) = \lambda_2(1) = -180.$
$u_1(1) = 1; \quad u_3(1) = 1; \quad u_3(1) = 1; \quad u_2(1) = u_4(1) = 0;$
$x_1(1) = 1 + 1 + 3 = 5; \quad x_2(1) = 1 + 4 + 2 = 7;$
$u_1(2) = 5; \quad u_3(2) = 7; \quad u_2(2) = u_4(2) = 0;$
$x_1(2) = 5 + 5 + 21 = 31; \quad x_2(2) = 7 + 20 + 14 = 41;$
$u_1(3) = 31; \quad u_3(3) = 41; \quad u_2(3) = u_4(3) = 0;$
$x_1(3) = 31 + 31 + 123 = 185; \quad x_2(3) = 41 + 124 + 82 = 247;$
$max = x_1(3) + x_2(3) = 432$

Now consider the other opportunity when

$J_1(3) = 2; \quad J_2(3) = 3; \quad u_2(3) = 31; \quad u_1(3) = u_4(3) = 0;$
$x_1(3) = 31 + 62 + 123 = 216; \quad x_2(3) = 41 + 93 + 82 = 216;$
$max = x_1(3) + x_2(3) = 432.$

Consider now the other opportunity

$$J_1(1) = 2; \quad J_2(1) = 3; \quad u_2(1) = 1; \quad u_3(1) = 1; \quad u_1(1) = u_4(1) = 0;$$
$$x_1(1) = 1 + 2 + 3 = 6; \quad x_2(1) = 1 + 2 + 3 = 6; \quad x_2(1) = 1 + 3 + 2 = 6;$$
$$u_1(2) = 6; \quad u_3(2) = 6; \quad u_2(2) = u_4(2) = 0;$$
$$x_1(2) = 6 + 6 + 18 = 30; \quad x_2(2) = 6 + 24 + 12 = 42;$$
$$u_1(3) = 30; \quad u_3(3) = 42; \quad u_2(3) = u_4(3) = 0;$$
$$x_1(3) = 30 + 30 + 126 = 186; \quad x_2(3) = 42 + 120 + 84 = 246;$$
$$max = x_1(3) + x_2(3) = 432.$$

Thus, we obtained the same result.

References

[1]. R. Bellman, *Dynamic Programming*, Princeton University Press, 1957.

[2]. E.J. Anderson, P.Nash, *Linear programming in infinite dimensional spaces*, 1987.

[3]. R.T. Rockafellar, *Linear-quadratic programming and optimal control*, SIAM Journal on control and optimization 25 (1987), 781–814.

BEHAVIOUR OF ITERATION PROCESSES NEAR THE BOUNDARY OF STABILITY DOMAIN

with applications to the Socio-Spatial Relative Dynamics

M. Sonis

Bar-Ilan University
Ramat Gan, 52900 Israel

Abstract

The objective of this research is the elaboration of an algorithm for the complete description of the qualitative properties of the discrete iteration processes. The field of iteration processes is mature enough for the construction of Calculus of Iteration Processes: the fragments of such a calculus are currently appearing in numerous studies. The first basic element of a calculus is the transformation of classical Routhian formalism, which gives a complete description of the behaviour of the iteration processes near the boundaries of the Stability Domains of equilibria. The use of Routhian formalism leads to the superposition of the space of eigenvalues and the phase space.

The study of behaviour of the iteration processes near the boundaries of Stability Domains can be achieved by the travels of equilibria in the phase space. Crossing the boundaries of the Stability Domain reveals the plethora of possible ways from stability, periodicity, Arnold mode-locking tongues and quasi-periodicity to chaos. The numerical procedure of the description of such phenomena includes spatial bifurcation diagrams in which the bifurcation parameter is the equilibrium itself. The second basic element of the Calculus of Iteration Processes is the universality property of the iteration processes. The content of such universality lies in the fact that for each iteration process with a big enough number of external parameters it is possible to construct the only one realization of this iteration process with a preset combination of qualitative properties of equilibria. The essential part of the proposed research is connected to the application of the Calculus of Iteration Processes to the newly developed class of the discrete relative m-population/n-location Socio-Spatial dynamics. The computer realization of the proposed calculus of iterations is presented in detail with the application to the one-population/three location log-linear relative dynamics.

Introduction

In recent decades a new paradigm of Deterministic Chaos for understanding complicated dynamic behaviour appeared in the form of the scientific approach and the methods to deal with manifestations of chaos and turbulence in different sciences. At present the essence of scientific efforts has shifted to the further elaboration of the conceptual framework, to the standardization of numerical methods and to the detailed description of the important new domains of application.

The main objective of this research is twofold: to elaborate the standardization of the analysis of behaviour of autonomous finite-dimensional discrete iteration processes in the form of the elements of **Calculus of Iteration Processes** and to apply this calculus to the study of a new branch of Chaos studies: the Discrete Relative m-population/n-location Socio-Spatial Dynamics elaborated by D.S. Dendrinos and M. Sonis starting in 1983 (see the book: D.S. Dendrinos, M. Sonis, **Chaos and Socio-Spatial Dynamics**, Springer Verlag Series of Applied Mathematic, Vol. 86, 1990).

Thus, this research presents an attempt to use the computer as a theoretical tool.

1. Elements of the Calculus of Autonomous Iteration Processes

Let us start from the explicit form of the n-dimensional discrete time iteration processes (other explicit and implicit forms of the iteration processes can also be considered):

$$x_i(t+1) = F_i(\mathbf{A}; \mathbf{x}(t)), \quad i = 1, 2, \ldots, n, \quad t = 0, 1, 2 \ldots \tag{1.1}$$

where the vectors

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

represent the states of the iteration process in the time points $t = 0, 1, 2, \ldots,$ $\mathbf{A}$ is the set of external constants (external bifurcation parameters), and the functions $F_i(\mathbf{A}; \mathbf{y})$ $i = 1, 2, \ldots, n$, are the differentiable functions of all their components $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

The elements of the calculus of iteration processes include, first of all, the description of possible equilibria $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ of the iteration process (1.1) given by the system of equations

$$x_i^* = F_i(\mathbf{A}; \mathbf{x}^*), \qquad i = 1, 2, \ldots, n\ . \tag{1.2}$$

The algebraic equations (1.2) in many cases allow the representation of the part $\mathbf{A}_1$ of the external bifurcation parameters from the set $\mathbf{A}$ with the help of the components of the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ and the remaining part $\mathbf{A}_2 = \mathbf{A} \backslash \mathbf{A}_1$. As a result, if the parameters from $\mathbf{A}_2$ are stable, then the coordinates of the equilibrium play the role of true internal bifurcation parameters.

As will be explained further, the remaining part of parameters $\mathbf{A}_2 = \mathbf{A} \backslash \mathbf{A}_1$ is used for the description of the boundaries of the domain of stability of equilibria within the phase space. Thus, the change of the coordinates of fixed points and the stability of the parameters from $\mathbf{A}_2$ generate the movement of the equilibrium in the phase space and the immovability of the boundaries of the stability domain.

The movement of equilibrium points can be placed on the segments of straight lines. This allows the complete computerized description of the appearance of different bifurcation phenomena in the phase space.

The first basic element of the Calculus of Iterations includes the travels of equilibria in the phase space which reveal the qualitative features of the behaviour of the trajectories of the iteration process near the boundaries of the domain of stability of equilibria.

The standard computational procedure can be applied, including the numerical realization of the following analytical constructions:

1. The matrix of the linear approximation of the iteration process - the Jacobi matrix -

$$J_{t:t+1} = \|s_{ij}(t+1,t)\| \tag{1.3}$$

where

$$s_{ij}(t+1,t) = \frac{\partial x_i(t+1)}{\partial x_j(t)}\,, \quad i,j = 1,2,\ldots,n\,. \tag{1.4}$$

2. The value of the Jacobi matrix $J^* = \|s^*_{ij}\|$ at the equilibrium $\mathbf{x}^* = (x^*_1, x^*_2, \ldots, x^*_n)$.
3. The characteristic polynomial of the Jacobi matrix J^*:

$$P(\mu) = \mu^n + a_1\mu^{n-1} + \ldots + a_{n-1}\mu + a_n\,.$$

As well known, the construction of the analytical forms of the coefficients of the characteristic polynomial $P(\mu)$ can be done with the help of the principal minors of the Jacobi matrix J^*. Thus, the following analytical objects should be computed:

4. Principal minors of the Jacobi matrix J^*.

By the well-known von Neumann theorem the equilibrium $\mathbf{x}^*$ is asymptotically stable iff for all its eigenvalues μ the following condition holds:

$$|\mu| < 1\,. \tag{1.5}$$

Consider the space of all coefficients of the characteristic polynomials of the order n - **the space of eigenvalues**. Condition (1.5) defines in this space the geometrical domain of asymptotic stability. The analytical description of this stability domain can be constructed with the help of the classic Routh-Gurvitz-Samuelson procedure in the form of non-linear inequalities.

This procedure can be described as follows: (see: Samuelson, 1983, pp. 435–

437). First of all, construct the parameters

$$b_0 = \sum_{i=0}^{n} a_i\,;$$

$$b_1 = \sum_{i=0}^{n} a_i(n-2i)\,, \quad \text{where} \quad a_0 = 1\,;$$

$$\vdots$$

$$b_r = \sum_{i=0}^{n} a_i \sum_{k=0}^{n} (-1)^k \binom{n-1}{r-k}\binom{i}{k}\,, \quad \text{where} \quad \binom{i}{k} = \begin{cases} \frac{i!}{k!(i-k)!} & i \geq k\,; k \geq 0 \\ 0 & i < k \\ 0 & k < 0 \end{cases}\;;$$

$$\vdots$$

$$b_n = 1 - a_1 + a_2 - \ldots + (-1)^{n-1}a_{n-1} + (-1)^n a_n\,. \tag{1.6}$$

Further, construct the matrix

$$\begin{pmatrix} b_1 & b_3 & b_5 & \ldots \\ b_0 & b_2 & b_4 & \ldots \\ 0 & b_1 & b_3 & \ldots \\ 0 & b_0 & b_2 & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{pmatrix} \tag{1.7}$$

and its principal minors $\Delta_1, \Delta_2, \ldots, \Delta_n$.

The conditions of asymptotic stability are:

$$b_0 > 0\,; \quad \Delta_r > 0\,, \quad r = 1, 2, \ldots, n\,. \tag{1.8}$$

Remark The reader may verify that for $n = 2$

$$b_0 = 1 + a_1 + a_2\,; \quad b_1 = 2 - 2a_2\,; \quad b_2 = 1 - a_1 + a_2\,; \tag{1.9}$$

and the stability domain in the space of eigenvalues is defined by the linear inequalities:

$$-1 \pm a_1 < a_2 < 1\,. \tag{1.10}$$

For $n = 3$

$$\begin{aligned} b_0 &= 1 + a_1 + a_2 + a_3\,; \\ b_1 &= 3 + a_1 - a_2 - 3a_3\,; \\ b_2 &= 3 - a_1 - a_2 + 3a_3\,; \\ b_3 &= 1 - a_1 + a_2 - a_3\,; \end{aligned} \tag{1.11}$$

and the stability domain is defined by the linear and quadratic inequalities:

$$\begin{aligned}
&1 + a_1 + a_2 + a_3 > 0\,;\\
&3 + a_1 - a_2 - 3a_3 > 0\,;\\
&1 - a_1 + a_2 - a_3 > 0\,;\\
&1 - a_2 + a_1 a_3 - a_3^2 > 0\,.
\end{aligned} \tag{1.12}$$

5. The boundaries of the stability domain in the space of eigenvalues are determined with the help of the Routhian procedure described above by the non-linear equalities:

$$b_0 = 0\,; \quad \Delta_r = 0\,, \quad r = 1.2, \ldots, n\,. \tag{1.13}$$

Next, because the components of the Jacobi matrix J^* are the functions of the coordinates of the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, it is possible to construct:

6. Analytical and geometric images of the boundaries of the domain of stability in the phase space.

It is important to underline that because the parameters from $\mathbf{A_1}$ can be analytically presented with the help of the coordinates of the fixed points, then the boundaries of the domain of stability in the phase space depend only on the parameters from $\mathbf{A}_2$.

7. Travels of equilibria in the phase space.

The superposition of the space of eigenvalues and the phase space together with the immovability of the boundaries of the domain of stability in the phase space represent the possibility of describing all admissible qualitative features of the behaviour of the iteration process near the boundaries of the stability domain. The travels of the equilibrium in the phase space on the segments of straight lines and crossing the boundaries of the stability domain reveal the plethora of possible ways from stability, periodicity, Arnold horns and quasi-periodicity to chaos. The numerical procedure of the description of such phenomena includes the construction of

8. Spatial bifurcation diagrams in which the bifurcation parameter is the equilibrium itself.

The organization of the travels of equilibria in the phase space on the segments of straight lines can be done in the following way: it is possible to parameterize the segment of the straight line between the equilibria $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x}(j) = \mathbf{x}\left(1 - \frac{j}{T}\right) + \mathbf{y}\frac{j}{T}\,, \quad j = 0, 1, \ldots, T\,, \tag{1.14}$$

where j is a bifurcation parameter and T is a number of bifurcation steps. The usual bifurcation diagram can be obtained from (1.14) by fixing some coordinate of the vectors $\mathbf{x}(j)$.

The second element of the Calculus is the universality properties of the iteration processes. In this paper we present only a specific type of universality; the content of such universality lies in the fact that for each iteration process with a big enough number of external bifurcation parameters it is possible to construct the realization of this iteration process with a preset combination of qualitative properties of equilibria.

Remark. It is important to note that there are different ways of describing the universality properties of iteration processes which justify the elaboration of the Calculus of Iteration Processes. One such universality property is presented in the **Entropy Principle of Extremality** (Gontar, 1981; Dendrinos and Sonis, 1986; Sonis and Gontar, 1992):

Consider a functional

$$E = -\sum_{j=0}^{n} p_j(t+1)\ln p_j(t+1) + \\ + \sum_{j=0}^{n} p_j(t+1) F_j\big(\mathrm{A}\,;\mathrm{x}(0),\mathrm{x}(1),\mathrm{x}(2),\dots,\mathrm{x}(t)\big)\,, \qquad (1.15)$$

where the probabilistic vector

$$\mathbf{p}(t+1) = \big(p_0(t+1), p_1(t+1), p_2(t+1), \dots, p_n(t+1)\big) \qquad (1.16)$$

has the components

$$p_0(t+1) = 1/[1+\sum_{s=1}^{n} \exp x_s(t+1)]\,;$$

$$p_j(t+1) = \exp x_j(t+1)/[1+\sum_{s=1}^{n} \exp x_s(t+1)] \\ = \exp x_j(t+1) p_0(t+1)\,; \quad j=1,2,\dots,n\,; \qquad (1.17) \\ 0 < p_0(t+1)\,, \quad p_j(t+1) < 1\,, \quad j=1,2,\dots,n\,;$$

$$p_0(t+1) + \sum_{j=1}^{n} p_j(t+1) = 1\,.$$

Then the system

$$\frac{\partial E}{\partial x_r(t+1)} = 0\,, \qquad r=1,2,\dots,n \qquad (1.18)$$

is equivalent to the system of difference equations (1.1) defining the iteration process and the maximum of the functional (1.15) equals

$$E_{\max} = -\ln p_0(t+1)\,. \qquad (1.19)$$

This entropy principle presents the unification of two different branches of non-linear discrete dynamics: relative discrete Socio- Spatial dynamics and Iterative Physico-Chemical Reaction processes (Sonis and Gontar, 1992).

2. Calculus of Two-dimensional Iteration Processes

In this part we present a brief realization of the Calculus of Iterations for two-dimensional iterations of the type (see, for example, Lauwerier, 1986):

$$\begin{aligned} x(t+1) &= G\big(x(t), y(t)\big) \\ y(t+1) &= H\big(x(t), y(t)\big) \end{aligned} \tag{2.1}$$

2.1 Stability. The standard linear stability analysis of the general two-dimensional discrete map (2.1) is based on the consideration of the general Jacobi matrix

$$J(t+1,t) = \begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{bmatrix} \tag{2.2}$$

and its value J^* on the fixed point x^*, y^*

$$J^* = \begin{bmatrix} \frac{\partial G^*}{\partial x^*} & \frac{\partial G^*}{\partial y^*} \\ \frac{\partial H^*}{\partial x^*} & \frac{\partial H^*}{\partial y^*} \end{bmatrix} \tag{2.3}$$

where $G^* = G(x^*, y^*)$, $H^* = H(x^*, y^*)$.

The eigenvalues of the Jacobi matrix J^* are the solutions of the quadratic equation

$$\mu^2 - TrJ^*\mu + \Delta^* = 0 \ , \tag{2.4}$$

where

$$\begin{aligned} \operatorname{Tr} J^* &= \frac{\partial g^*}{\partial x^*} + \frac{\partial H^*}{\partial y^*} \, , \\ \Delta^* &= \det J^* \ . \end{aligned} \tag{2.5}$$

The eigenvalues μ_1, μ_2 of the Jacobi matrix J^* are

$$\mu_{1;2} = \operatorname{Tr} J^* \pm \sqrt{(\operatorname{Tr} J^*)^2 - 4\Delta^*} \tag{2.6}$$

where the Vieta conditions hold:

$$\mu_1 + \mu_2 = \operatorname{Tr} J^*; \mu_1\mu_2 = \Delta^* \, . \tag{2.7}$$

Next we will summarize the qualitative properties of the behaviour of discrete maps which are the result of the standard linear stability analysis (see, for example, Hsu, 1977; Thompson and Stewart, 1986, pp. 150–161; Sonis, 1990).

It is possible to consider the eigenvalues as points in the complex plane. Then the conditions of stability (1.5) mean that the eigenvalues lie within the unit circle which presents the stability boundary on the complex plane. If the eigenvalues are both inside the unit circle the dynamics is asymptotically stable; the convergence towards the fixed point is nodal or alternate nodal in the case of real eigenvalues, and spiral or oscillatory focal in the case of conjugate eigenvalues. On the unit circle the situation is as follows: if $|\mu_1| < 1$ and $\mu_2 = 1$, then the dynamics shows the incipient divergence; if $\mu_1 = -1$ and $|\mu_2| < 1$, then the dynamics shows incipient flip. If at least one of the eigenvalues is outside the circle, the dynamics is unstable.

The various routes crossing the stability boundary correspond to three qualitatively different phenomena of behaviour of the orbits: divergence, flip and flutter.

If both eigenvalues are real, the following bifurcation events occur:

if the smaller eigenvalue μ_1 is positive and the bigger, μ_2, crosses the unit circle at point 1, then the dynamics show the transfer from monotonic convergence to the attracting node to direct saddle monotonic divergence, through the incipient divergence phenomenon;

if both eigenvalues are bigger than 1, we have a repelling node;

if the smaller eigenvalue lies in the interval $-1 < \mu_1 < 0$, then the convergence and divergence are oscillatory;

if the smaller eigenvalue μ_1 crosses the unit circle at point -1 and the bigger eigenvalue stays within the unit circle, then the dynamics show the transfer from alternate convergence to the attracting node to oscillatory flip dynamics with increasing amplitudes through the incipient neutral flip.

In the case of complex conjugate eigenvalues the behaviour of discrete is much more complicated:

in the case of the positive real part of the eigenvalues, the dynamics show spiral convergence to the attracting focus if the eigenvalues lie within the unit circle, or spiral divergence from the repelling focus if the eigenvalues lie outside the unit circle.

if the real part of the eigenvalues is negative, the dynamics spiral and oscillate at the same time - the flutter phenomenon - going from convergence to flutter divergence through neutral oscillations of constant amplitude when the conjugate eigenvalues cross the unit circle.

2.2 The space of eigenvalues. A better understanding can be achieved by considering the construction of the two-dimensional space of the eigenvalues of discrete dynamics.

Figure 1 presents the complete description of the linear stability analysis for two-dimensional discrete dynamics (2.1) in terms of the invariants $\operatorname{Tr} J^*$, Δ^* of the Jacobi matrix J^*. This presentation extends and makes more precise the known features of the linear stability analysis. This close look is taken to set up the framework for obtaining a full view of the dynamic events and the various bifurcation they entail.

Outcome of the general Routh-Hurvitz conditions: the polynomial $x^2 + a_1 x +$

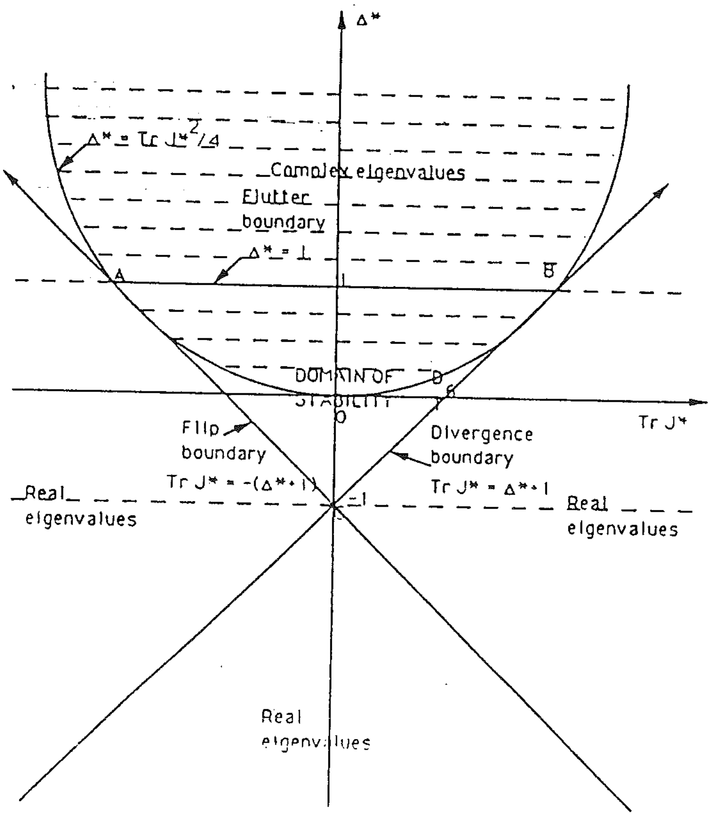


Figure 1. Space of eigenvalues: domains of stability of eigenvalues properties.

a_2 has roots less than 1 in absolute value if and only if $-1 \pm a_1 < a_2 < 1$ (see Samuelson, 1983, p. 436, or Dendrinos, Sonis, 1990, p. 79); thus, the conditions of stability of the fixed points imply that

$$-1 \pm \operatorname{Tr} J^* < \Delta^* < 1 . \tag{2.8}$$

The two-dimensional plane with coordinate axes $\operatorname{Tr} J^*$ and Δ^* is the space of eigenvalues (see Figure 1). One can distinguish the following major categories of stability of qualitative properties of eigenvalues and their geometrical location (domains, intervals of straight lines, segments of parabolas and specific points) in the space of eigenvalues.

The parabola $\Delta^* = \frac{1}{4} \operatorname{Tr} J^{*2}$ divides the plane of eigenvalues into two major domains: the eigenvalues are real outside the parabola and are complex conjugate inside this parabola. On the parabola itself the eigenvalues are equal: $\mu_1 = \mu_2 = \frac{1}{2} \operatorname{Tr} J^*$.

In the plane of eigenvalues the domain of stability defined by conditions (2.8) is the triangle ABC with the vertices:

$$A=\begin{bmatrix}-2\\1\end{bmatrix},B=\begin{bmatrix}2\\1\end{bmatrix}, \quad C=\begin{bmatrix}0\\-1\end{bmatrix}.$$

The sides of the triangle of stability are generated by the following straight lines:
– the divergence boundary

$$\mathrm{Tr}\,J^* = \Delta^* + 1; \tag{2.9}$$

– the flip boundary

$$\mathrm{Tr}\,J^* = -(\Delta^* + 1); \tag{2.10}$$

– the flutter boundary

$$\Delta^* = 1. \tag{2.11}$$

The Vieta conditions (2.7) imply that
– on the divergence boundary $(\mu_1 - 1)(\mu_2 - 1) = 0$; i.e., at least one of the eigenvalues is equal to 1.

The crossing of this boundary gives the orbits approaching infinity. Such divergence starts from the points within the domain of stability; this domain is the infinity-locking domain.
– on the flip boundary $(\mu_1 + 1)(\mu_2 + 1) = 0$; i.e., at least one of the eigenvalues is equal to -1.

Each point on the flip boundary corresponds to the two-periodic cycle, and the movement outside the domain of stability generates the Feigenbaum type periodic doubling sequence, leading to chaos (Feigenbaum, 1978).
- on the flutter boundary $|\mu_1| = |\mu_2| = 1$.

It is easy to describe all points on the segment AB on the flutter boundary. The condition $|\mu_1| = |\mu_2| = 1$ means that $\mu_1 = e^{i2\pi\Omega}$, $\mu_2 = ei2\pi\Omega$, $0 \le \Omega \le 1$, and therefore $\mathrm{Tr}\,J^* = \mu_1 + \mu_2 = 2\cos 2\pi\Omega$. Thus, all points of the segment AB have the form $\begin{bmatrix}2\cos 2\pi\Omega\\1\end{bmatrix}$. If Ω is a rational fraction: $\Omega = \frac{p}{q}$, then we have q-periodic (resonance) fixed points: between them the fixed points of strong resonance with $\Omega = \frac{1}{3}$, $\Omega = \frac{1}{4}$, correspond to the points $\begin{bmatrix}-1\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$. Other ration fractions $\Omega = \frac{p}{q}$ represent the points of weak resonance. (Table 1 represents all resonances for $q \le 12$.)

The same periodic behaviour is also observed in a small domain near resonance. This domain - the mode-locking domain - is the image of the Arnold tongue from the corresponding domain of change of eigenvalues in complex plane (Arnold, 1977). For strong resonance the mode-locking domain is started within the domain of stability (Kogan, 1991).

In conclusion, there are sixteen distinct domains of the stability of qualitative properties of the eigenvalues defining sixteen major structurally stable classes of dynamic behaviour of trajectories near fixed points.

Table 1. *Resonances of two-dimensional iteration processes.*

Type of resonance periodicity q		Arguments of eigenvalues $\Omega = \frac{2\pi p}{q}$	Values of $TrJ^* = 2\cos\frac{2\pi p}{q}$
3	-	$\frac{2\pi}{3}$	- 1
4	-	$\frac{\pi}{2}$	0
5	-	$\frac{2\pi}{5}$	0.61803
	-	$\frac{4\pi}{5}$	-1.61803
6	-	$\frac{\pi}{3}$	1
7	-	$\frac{2\pi}{7}$	1.24698
	-	$\frac{4\pi}{7}$	- 0.44504
	-	$\frac{6\pi}{7}$	- 1.80194
8	-	$\frac{\pi}{4}$	1.41421
9	-	$\frac{2\pi}{9}$	1.53209
	-	$\frac{4\pi}{9}$	0.34730
	-	$\frac{8\pi}{9}$	-1.87939
10	-	$\frac{\pi}{5}$	1.61803
	-	$\frac{3\pi}{5}$	-0.61803
11	-	$\frac{2\pi}{11}$	1.68251
	-	$\frac{4\pi}{11}$	0.83083
	-	$\frac{6\pi}{11}$	- 0.28463
	-	$\frac{8\pi}{11}$	- 1.30972
	-	$\frac{10\pi}{11}$	- 1.91899
12	-	$\frac{\pi}{6}$	1.73205

3. Discrete Relative One Population/n-Location Socio-Spatial Dynamics

3.1 General properties. In this section the ideas of Calculus of iterations will be applied for the specific cases of a new general model of discrete relative one population/multiple location socio-spatial dynamics (see Dendrinos and Sonis, 1990; the case of multiple population/multiple location discrete dynamics can be treated analogously).

Let the vector

$$\mathbf{x}(t) = \left(x_1(t), x_2(t), \ldots, x_n(t)\right), \quad t = 0, 1, 2, \ldots$$

be the relative population size distribution at time t at location i in the environment of n locations. Such a formulation could be specified for any socio-economic quantity, normalized over a regional or national total.

The one population/multiple location relative discrete socio-spatial dynamics is then given by:

$$x_i(t+1) = f_i\left(\mathbf{x}(t)\right) \Big/ \sum_{j=1}^{n} f_j\left(\mathbf{x}(t)\right), \quad i = 1, 2, \ldots, n; \quad t = 0, 1, 2, \ldots; \tag{3.1}$$

$$f_i\left(\mathbf{x}(t)\right) > 0, \quad i = 1, 2, \ldots, n;$$

$$0 < x_i(0) < 1, \quad i = 1, 2, \ldots, n; \quad \sum_j x_j(0) = 1. \tag{3.2}$$

The expression $f_i\left(\mathbf{x}(t)\right)$ is the locational comparative advantages enjoyed by the population at (i, t). Functions f_i depend on the relative distribution of the population in all locations, and on other environmental parameters.

A specific log-linear formulation for the functions f_i with the universality properties may be represented by the following:

$$F_i\left(\mathbf{x}(t)\right) = A_i \prod_j x_j(t)^{a}ij; \tag{3.3}$$

$$-\infty < a_{ij} < \infty; \qquad A_i > 0, \qquad i = 1, 2, \ldots, n;$$

where $A_1, A_2, \ldots, A_n$ are the composite locational advantages of the locations $1, 2, \ldots, n$, and the matrix $\|a_{ij}\|$ is the matrix of the composite elasticities of relative population growth. The universality of this dynamics means that under different parameter specifications this iteration process can reproduce each preset dynamic behaviour including stability, periodic motion, quasi-periodicity and various forms of chaotic movement.

It is important to stress that the relative dynamics (3.1) can be generated by the following extreme principle (cf. Gontar, 1981; Sonis and Gontar, 1992):

the relative Socio-Spatial dynamics proceed in such a way that in the transfer from time t to time $t+1$ the entropy functional

$$E(t,t+1) = \sum_{i=1}^{n} x_i(t+1)\left[lnx_i(t+1) - lnf_i(\mathbf{x}(t) - 1\right] \tag{3.4}$$

reaches its minimum in the space of vectors $\mathbf{x}(t+1)$ subject to the conservation condition:

$$\sum_{i=1}^{n} x_i(t+1) = 1\,.$$

This extreme principle defines a new law of **collective** non-local population redistribution behaviour which is a meso-level counterpart of the utility optimization individual behaviour!.

For the realization of the scheme of Calculus of Iteration processes we will use the following form of one population/three locations log-linear map (see Dendrinos, Sonis, 1990, p. 85):

$$\begin{aligned} x_1(t+1) &= 1/[1+f_2(t)+f_3(t)]: \\ x_2(t+1) &= f_2(t)/[1+f_2(t)+f_3(t)]\,; \\ x_3(t+1) &= f_3(t)/[1+f_2(t)+f_3(t)]\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} f_2(t) &= A_2 x_1(t)^{a_{21}} x_2(t)^{a_{22}} x_3(t)^{a_{23}}\;; \\ f_3(t) &= A_3 x_1(t)^{a_{31}} x_2(t)^{a_{32}} x_3(t)^{a_{33}}\;; \\ &A_2, A_3 > 0\;; \end{aligned} \tag{3.6}$$

$$0 < x_1(t), x_2(t), x_3(t) < 1; \qquad \sum_{j=1}^{3} x_j(t) = 1; \quad t = 0,1,2,\ldots; \tag{3.7}$$

and the matrix of elasticities has a form

$$\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad -\infty < a_{ij} < +\infty\,, \quad i = 2,3\;; \quad j = 1,2,3. \tag{3.8}$$

3.2 A Moebius plane as a phase space. Moebius plane is the two-dimensional space (plane) defined by three barycentric coordinates x_1, x_2, x_3, $x_1 + x_2 + x_3 = 1$, of each point within it. The scale element of this plane is the Moebius equilateral triangle with the unit scale on its sides. This triangle is generated by three coordinate axes (Figure 2). It is possible to measure the barycentric coordinates of each point in Moebius plane by projecting it (parallel to the sides) onto the sides of the Moebius triangle. If the point P lies within the Moebius triangle, then its barycentric coordinates x_1, $x_2 . x_3$ must be between 0 and 1:

$$x_1 + x_2 + x_3 = 1 ; \qquad 0 \le x_1, x_2, x_3 \le 1 . \tag{3.9}$$

If the point Q lies outside the Moebius triangle, then one of the barycentric coordinates must be negative, and another greater than 1, but the condition $x_1 + x_2 + x_3 = 1$ always holds. The vertices of the Moebius triangle are

$$\begin{aligned} X &: x_1 = 1 ; \quad x_2 = 0 ; \quad x_3 = 0 , \\ Y &: x_1 = 0 ; \quad x_2 = 1 ; \quad x_3 = 0 , \\ Z &: x_1 = 0 ; \quad x_2 = 0 ; \quad x_3 = 1 . \end{aligned} \tag{3.10}$$

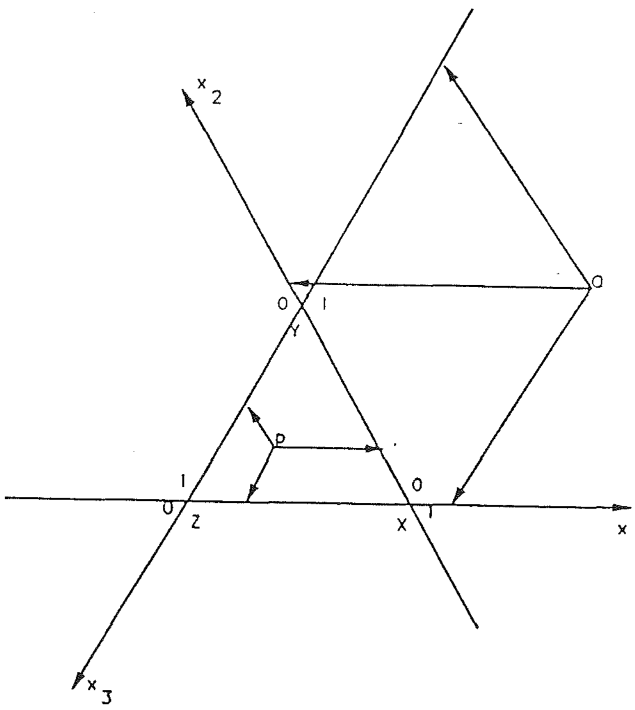


Figure 2. Barycentric coordinates in Moebius plane.

3.3 Straight lines in the Moebius space. The equation of the straight line in the Moebius plane has the form

$$ax_1 + bx_2 + cx_3 = 0\,. \tag{3.11}$$

The elementary analysis shows that if $a \neq b$; $a \neq c$; $b \neq c$ then the straight line (39) intersects the x_1-axis in the point

$$P_1 : x_1 = c/(c-a)\,; \quad x_2 = 0\,; \quad x_3 = a/(a-c)\,; \tag{3.12}$$

the intersection with the x_2-axis will be the point

$$P_2 : x_1 = b/(b-a)\,; \quad x_2 = a/(a-b)\,; \quad x_3 = 0\,; \tag{3.13}$$

the intersection with the x_3-axis will be the point

$$P_3 : x_1 = 0\,; \quad x_2 = c/(c-b)\,; \quad x_3 = b/(b-c)\,. \tag{3.14}$$

Moreover, the point P_1 lies on the x_1-side ZX of the Moebius triangle if and only if sign $a \neq$ sign c; the point P_2 lies on the x_2-side XY of the Moebius triangle if and only if sign $a \neq$ sign b; and the point P_3 lies on the x_3-side YZ of the Moebius triangle if and only if sign $b \neq$ sign c. Thus, the straight line (3.11) intersects the Moebius triangle if and only if sign $a \neq$ sign b or sign $a \neq$ sign c, or sign $b \neq$ sign c. Furthermore, if sign $a =$ sign $b =$ sign c, then the Moebius triangle lies on one side of the straight line (3.11). For the one population/three location relative dynamics the Moebius triangle gives the natural way to present the trajectories of dynamics and their fixed points. Moreover, because of conditions (3.9) the trajectories of the relative dynamics occur within the Moebius triangle itself.

3.4 Fixed points of one population/three location relative log-linear dynamics. Now we will concentrate on the graphical representation of the behaviour of the non-periodic fixed point x_1^*, x_2^*, x_3^* of the dynamics (3.6)–(3.8) within the Moebius triangle under the arbitrary change in the composite comparative advantages $A_2, A_3 > 0$ and the arbitrary change of the matrix of elasticities

$$\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\;; \quad -\infty < a_{ij} < +\infty\,, \quad i = 2,3,\,; \quad j = 1,2,3\,.$$

It is possible to prove (see Sonis, 1990) that the coordinates x_1^*, x_2^*, x_3^* of the fixed point satisfy the equations:

$$\begin{aligned} x_1^{*-(a_{21}+1)} x_2^{*1-a_{22}} x_3^{*-a_{23}} &= A_2\;; \\ x_1^{*-(a_{31}+1)} x_2^{*-a_{32}} x_3^{*1-a_{33}} &= A_3\;. \end{aligned} \tag{3.15}$$

Moreover, the system (3.15) is equivalent to

$$\begin{aligned} &x_1^* + C_2 x_1^{*a} + C_3 x_1^{*b} = 1 \; ; \\ &x_2^* = C_2 x_1^{*a} \; , \\ &x_3^* = C_3 x_1^{*b} \; , \end{aligned} \tag{3.16}$$

where

$$\Delta_{12} = \begin{vmatrix} a_{21}+1 & a_{22}-1 \\ a_{31}+1 & a_{32} \end{vmatrix} ; \quad \Delta_{23} = \begin{vmatrix} a_{22}-1 & a_{23} \\ a_{32} & a_{33}-1 \end{vmatrix} ; \quad \Delta_{31} = \begin{vmatrix} a_{23} & a_{21}+1 \\ a_{33}-1 & a_{31}+1 \end{vmatrix} \tag{3.17}$$

$$a = \Delta_{31}/\Delta_{23} \; ; \qquad b = \Delta_{12}/\Delta_{23} \; , \tag{3.18}$$

and

$$C_2 = \left[A_2^{1-a_{33}} A_3^{a_{23}}\right]^{1/\Delta_{23}} \; ; \quad C_3 = \left[A_2^{a_{32}} A_3^{1-a_{22}}\right]^{1/\Delta_{23}} \; . \tag{3.19}$$

Further, the following proposition holds, describing the conditions of existence and the number of solutions for system (3.16):

Proposition. *In the case* $\Delta_{23} \neq 0$

1. *If* $\text{Sign}\,\Delta_{23} = \text{Sign}\,\Delta_{31} = \text{Sign}\,\Delta_{12}$ (*i.e.*, $a, b \geq 0$), *then the first equation of* (3.16) *always has only one solution* x_1^*; *this means that a unique non-periodic fixed point*

$$x_1^*, \; x_2^* = C_2 x_1^{*a} \, , \qquad x_3^* = C_3 x_1^{*b}$$

   always exists.
2. *If* $\text{Sign}\,\Delta_{23} \neq \text{Sign}\,\Delta_{31}$ *or* $\text{Sign}\,\Delta_{23} \neq \text{Sign}\,\Delta_{12}$, *then equations* (3.16) *have no more than two solutions; the unique solution appears only if*

$$\Delta_{23} x_1^* + \Delta_{31} x_2^* + \Delta_{12} x_3^* = 0 \; .$$

It is easy to calculate the non-periodic fixed points from equations (3.6)–(3.8) with the help of the computation of the values of the left parts of the first from equations (3.16) in two points of the x_1^*-axis. Refinement of the mesh size near a suspected fixed point by dividing it in two makes it possible to pin down the location of any fixed point. At the beginning let us consider all log-linear models (3.6)–(3.8) with the fixed matrix of elasticities and the changeable composite advantages A_2, A_3. By choosing the appropriate parameters A_2, A_3 from equations (3.15) one can put the non-periodic fixed point into an arbitrary place within the Moebius triangle. Thus equations (3.15) allow the conversion of the fixed points of the dynamics (3.6)–(3.8) into the true internal bifurcation parameters.

Now consider the straight line

$$\Delta_{23} x_1^* + \Delta_{31} x_2^* + \Delta_{12} x_3^* = 0 \tag{3.20}$$

which we will call the **bifurcation line.** The position of the bifurcation line depends only on the elasticities a_{ij}, and does not depend on the parameters A_2, A_3. If $\operatorname{Sign}\Delta_{23} \neq \operatorname{Sign}\Delta_{31}$ or $\operatorname{Sign}\Delta_{23} \neq \operatorname{Sign}\Delta_{12}$ or $\operatorname{Sign}\Delta_{31} \neq \operatorname{Sign}\Delta_{12}$ then the bifurcation line intersects the Moebius triangle, and the choice of the fixed point from one side of the bifurcation line immediately implies the existence of another fixed point from the other side of the straight line; this fixed point corresponds to the same choice of elasticities and advantages parameters. These fixed points merge on the bifurcation line.

If parameters A_2, A_3 are moving on a curve in the space of composite locational advantages, then the corresponding non-periodic fixed points are moving on two different curves within the Moebius triangle; these two curves intersect only on the bifurcation line. In the same way, the choice of some domain in the space of composite locational advantages generates two domains of fixed points within the Moebius triangle such that their intersection includes some interval from the bifurcation line.

If $\operatorname{Sign}\Delta_{23} = \operatorname{Sign}\Delta_{31} = \operatorname{Sign}\Delta_{12}$, then the Moebius triangle lies on one side of the bifurcation line, and for each choice of A_2, A_3 there is only one fixed point. In this case the movement on the curve or in some domain in the space of composite locational advantages generates only one curve or one domain of non-periodic fixed points within the Moebius triangle.

Thus, the change in elasticities will change only the position of the bifurcation curve, but the qualitative properties of the existence of non-periodic fixed points will be the same.

4. The Elements of the Calculus of Iterations for One Population/Three Location Log-linear Relative Dynamics

4.1 The Jacobi matrix. Consider the slope-response functions

$$s_{ij}(t+1,t) = \frac{\partial x_i(t+1)}{\partial x_j(t)}, \qquad i,j = 1,2,3,$$

which are the entries of the Jacobi slope-matrix

$$J(t+1,t) = \|s_{ij}(t+1,t)\|.$$

The direct calculation gives

$$s_{1j}(t+1,t) = -\frac{x_1(t+1)}{x_j(t)}\left[a_{2j}x_2(t+1) + a_{3j}x_3(t+1)\right], \quad j = 1,2,3;$$

$$s_{ij}(t+1,t) = \frac{x_i(t+1)}{x_j(t)}\left[a_{ij} - a_{2j}x_2(t+1) - a_{3j}x_3(t+1)\right], \quad i = 2,3; \quad j = 1,2,3. \tag{4.1}$$

The conservation conditions

$$\sum_{j=1}^{3} x_j(t) = 1 ; \quad t = 0, 1, 2, \ldots$$

imply that the determinant of the Jacobi matrix (Jacobian) is equal to zero:

$$\det J(t+1, t) = 0$$

At the fixed point x_1^*, x_2^*, x_3^* the Jacobi slope-matrix $J^* = \|s_{ij}^*\|$ has the entries

$$\begin{aligned} s_{11}^* &= -[a_{21}x_2^* + a_{31}x_3^*] ; \quad s_{1j}^* = -\frac{x_1^*}{x_j^*}[a_{2j}x_2^* + a_{3j}x_3^*] , \quad j = 2, 3 ; \\ s_{ij}^* &= \frac{x_i^*}{x_j^*}[a_{ij} - a_{2j}x_2^* - a_{3j}x_3^*], , \quad i = 2, 3 ; \quad j = 2, 3 , \end{aligned} \tag{4.2}$$

such that at the fixed point $\det J^* = 0$.

The characteristic equation of the Jacobi matrix J^*:

$$\mu^3 - \operatorname{Tr} J^* \mu^2 + \Delta^* \mu - \det J^* = 0 \tag{4.3}$$

where

$$\operatorname{Tr} J^* = \sum_{i=1}^{3} s_{ij}^* = a_{22} + a_{33} - x_2^*(a_{21} + a_{22} + a_{23}) - x_3^*(a_{31} + a_{32} + a_{33}) ; \tag{4.4}$$

and

$$\Delta^* = \begin{vmatrix} s_{11}^* & s_{12}^* \\ s_{21}^* & s_{22}^* \end{vmatrix} + \begin{vmatrix} s_{11}^* & s_{13}^* \\ s_{31}^* & s_{33}^* \end{vmatrix} + \begin{vmatrix} s_{22}^* & s_{23}^* \\ s_{32}^* & s_{33}^* \end{vmatrix}$$

It is possible to prove (see Dendrinos, Sonis, 1990, p. 87) that

$$\Delta^* = \begin{vmatrix} x_1^* & x_2^* & x_3^* \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} . \tag{4.5}$$

Since $\det J^* = 0$, the non-zero eigenvalues of the Jacobi matrix J^* are the solutions of the quadratic equation

$$\mu^2 - \operatorname{Tr} J^* \mu + \Delta^* = 0 .$$

4.2 Linear superposition of the Moebius plane and the space of eigenvalues. A linear dependence between the Moebius plane and the two-dimensional (trace-determinant) space of eigenvalues for log-linear dynamics is defined with the help of linear transformation given by formulas (4.4) and (4.5):

$$\begin{aligned} \operatorname{Tr} J^* &= s x_1^* + (s - s_2) x_2^* + (s - s_3) \\ \Delta^* &= \Delta_{23}^* x_1^* + \Delta_{31}^* x_2^* + \Delta_{12}^* x_3^* \end{aligned} \tag{4.6}$$

where

$$s = a_{22} + a_{33}\,; \quad s_2 = a_{21} + a_{22} + a_{23}\,; \quad s_3 = a_{31} + a_{32} + a_{33}\,; \tag{4.7}$$

$$\Delta_{12}^* = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\,; \quad \Delta_{23}^* = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\,; \quad \Delta_{31}^* = \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}\,. \tag{4.8}$$

This linear transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} \to \begin{bmatrix} \operatorname{Tr} J^* \\ \Delta^* \end{bmatrix}$$

preserves straight lines and half-planes. Therefore the Moebius triangle XYZ from the Moebius plane has an image in space of eigenvalues which is also triangle $X^*Y^*Z^*$: the vertices of the Moebius triangle X, Y, Z are transformed to the following vertices X^*, Y^*, Z^* in the eigenvalues space:

$$\begin{aligned} X &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} s \\ \Delta_{23}^* \end{bmatrix} = X^*\,; \\ Y &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} s - s_2 \\ \Delta_{31}^* \end{bmatrix} = Y^*\,; \\ Z &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} s - s_3 \\ \Delta_{12}^* \end{bmatrix} = Z^*\,. \end{aligned} \tag{4.9}$$

This correspondence permits the allocation of the eigenvalues space by the structure of the Moebius plane, defined with the help of the non-equilateral Moebius triangle $X^*Y^*Z^*$.

It is also possible to transfer the Euclidean structure of the eigenvalues space to the Moebius plane. For this purpose present the linear dependencies (4.6) between the barycentric coordinates x_1^*, x_2^*, x_3^* and $\operatorname{Tr} J^*$, Δ^* in the following form:

$$s x_1^* + (s - s_2) x_2^* + (s - s_3) x_3^* = \operatorname{Tr} J^*$$

$$\Delta_{23}^* x_1^* + \Delta_{31}^* x_2^* + \Delta_{12}^* x_3^* = \Delta^* \tag{4.10}$$
$$x_1^* + x_2^* + x_3^* = 1\,.$$

Thus the matrix

$$A = \begin{bmatrix} s & s - s_2 & s - s_3 \\ \Delta_{23}^* & \Delta_{31}^* & \Delta_{12}^* \\ 1 & 1 & 1 \end{bmatrix}$$

transforms the Moebius plane into the eigenvalues space:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} \overset{A}{\to} \begin{bmatrix} \mathrm{Tr}\, J^* \\ \Delta^* \\ 1 \end{bmatrix} \sim \begin{bmatrix} \mathrm{Tr}\, J^* \\ \Delta^* \end{bmatrix}$$

where the sign $\sim$ means equivalence.

The determinant of the matrix A is equal to

$$\begin{aligned} \det A &= s_2\,(\Delta_{23}^* - \Delta_{12}^*) + s_3\,(\Delta_{31}^* - \Delta_{23}^*) = \\ &= (s_2 - s_3)\Delta_{23}^* - s_2\Delta_{12}^* + s_3\Delta_{31}^*\,. \end{aligned} \tag{4.11}$$

If $\det A \neq 0$ then the inverse transformation A^{-1} exists and transforms the eigenvalues space into the Moebius plane:

$$\begin{bmatrix} \mathrm{Tr}\, J^* \\ \Delta^* \end{bmatrix} \sim \begin{bmatrix} \mathrm{Tr}\, J^* \\ \Delta^* \\ 1 \end{bmatrix} \overset{A^{-1}}{\to} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix}.$$

It is possible to check that

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \Delta_{31}^* - \Delta_{12}^* & s_2 - s_3 & (s - s_2)\Delta_{12}^* - (s - s_3)\Delta_{31}^* \\ -\Delta_{23}^* + \Delta_{12}^* & s_3 & -s\Delta_{12}^* + (s - s_3)\Delta_{23}^* \\ \Delta_{23}^* - \Delta_{31}^* & -s_2 & s\Delta_{31}^* + (s - s_2)\Delta_{23}^* \end{bmatrix}. \tag{4.12}$$

Now consider the images of the divergence, flip and flutter boundaries in the Moebius plane.

The equation of the divergence boundary $\mathrm{Tr}\, J^* = \Delta^* + 1$ after the substitutions (4.6) become

$$(\Delta_{23}^* - s + 1)x_1^* + (\Delta_{31}^* - s + s_2 + 1)x_2^* + (\Delta_{12}^* - s + s_3 + 1)x_3^* = 0 \tag{4.14}$$

It is easy to prove that

$$\Delta_{23}^* - s + 1 = \Delta_{23}\,;$$
$$\Delta_{31}^* - s + s_2 + 1 = \Delta_{31}\,; \tag{4.15}$$

$$\Delta_{12}^* - s + s_3 + 1 = \Delta_{23} .$$

Therefore the equation of the divergence boundary (4.14) coincides with the equation of the bifurcation straight line (see 3.20)):

$$\Delta_{23} x_1^* + \Delta_{31} x_2^* + \Delta_{12} x_3^* = 0 ,$$

on which the unique non-periodic fixed point first appears, and bifurcates further into two fixed points under the change in the locational advantages parameters A_2, A_3.

The equation of the flip boundary $\mathrm{Tr}\, J^* = -(\Delta^* + 1)$ after the substitutions (4.6) become

$$(\Delta_{23}^* + s + 1)x_1^* + (\Delta_{31}^* + s - s_2 + 1)x_2^* + (\Delta_{12}^* + s - s_3 + 1)x_3^* = 0 . \quad (4.16)$$

The equation of the flutter boundary $\Delta^* = 1$ after the substitutions (4.6) become

$$\Delta_{23}^* x_1^* + \Delta_{31}^* x_2^* + \Delta_{12}^* x_3^* = 1 \quad (4.17)$$

The location of these boundaries in the Moebius plane can be determined with the help of points A^*, B^*, C^* with coordinates (see Figure 3):

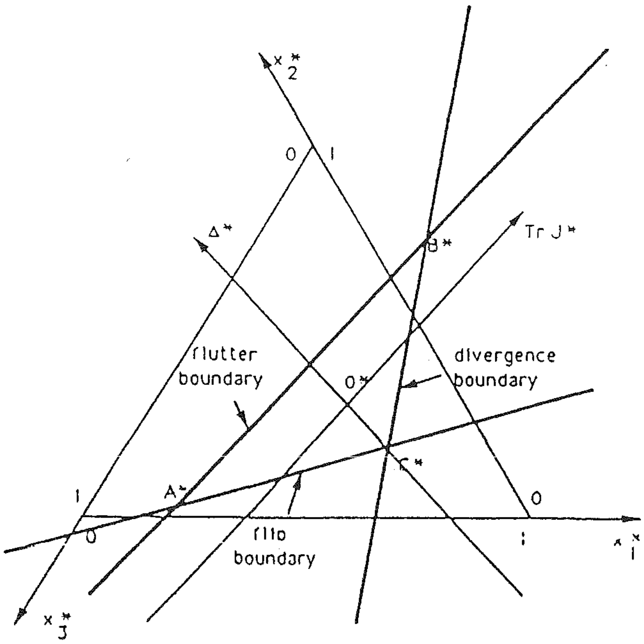


Figure 3. Superposition of the space of eigenvalues and the Moebius plane for one population/three location relative dynamics with changeable locational advantages A_2,A_3 and constant elasticities a_{ij}.

$$A = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{A^{-1}} \tag{4.18}$$

$$A^* \sim \begin{cases} x_1^* = \frac{1}{\det A}\left[(s_2 - s_3) + (s - s_2 + 2)\Delta_{12}^* - (s - s_3 + 2)\Delta_{31}^*\right] \\ x_2^* = \frac{1}{\det A}\left[s_3 - (s + 2)\Delta_{12}^* + (s - s_3 + 2)\Delta_{23}^*\right] \\ x_3^* = \frac{1}{\det A}\left[-s_2 + (s + 2)\Delta_{31}^* - (s - s_2 + 2)\Delta_{23}^*\right] . \end{cases}$$

$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{A^{-1}} \tag{4.19}$$

$$B^* \sim \begin{cases} x_1^* = \frac{1}{\det A}\left[(s_2 - s_3) + (s - s_2 - 2)\Delta_{12}^* - (s - s_3 - 2)\Delta_{31}^*\right] \\ x_2^* = \frac{1}{\det A}\left[s_3 - (s - 2)\Delta_{12}^* + (s - s_3 - 2)\Delta_{23}^*\right] \\ x_3^* = \frac{1}{\det A}\left[-s_2 + (s - 2)\Delta_{31}^* - (s - s_2 - 2)\Delta_{23}^*\right] . \end{cases}$$

$$C = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \xrightarrow{A^{-1}} \tag{4.20}$$

$$C^* \sim \begin{cases} x_1^* = \frac{1}{\det A}\left[(s_2 - s_3) + (s - s_2)\Delta_{12}^* - (s - s_3)\Delta_{31}^*\right] \\ x_2^* = \frac{1}{\det A}\left[-s_3 - s\Delta_{12}^* + (s - s_3)\Delta_{23}^*\right] \\ x_3^* = \frac{1}{\det A}\left[s_2 + s\Delta_{31}^* - (s - s_2)\Delta_{23}^*\right] . \end{cases}$$

These points define the place of images of the coordinate axes $\operatorname{Tr} J^*$ and Δ^* in the Moebius plane (see Figure 3).

It is evident that the image A^*B^* of the flutter segment AB has the form:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = A^{-1} \begin{bmatrix} 2\cos 2\pi\Omega \\ 1 \\ 1 \end{bmatrix}, \quad 0 \le \Omega \le \frac{1}{2} . \tag{4.21}$$

Therefore, the $p:q$ resonances have a form:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = A^{-1} \begin{bmatrix} 2\cos 2\pi\frac{p}{q} \\ 1 \\ 1 \end{bmatrix} \tag{4.22}$$

for the rational fractions $\frac{p}{q}$, $0 \le \frac{p}{q} \le 1$; for example, the points of strong resonances $1:3$ and $1:4$ are correspondingly

$$A^{-1} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} . \tag{4.23}$$

Thus, the superposition of the space of eigenvalues and the Moebius plane clarifies the qualitative description of the local features or relative dynamics within a vicinity of fixed points, and also the global features of dynamics connected with the "competition" between possible attractors and repellers.

5. The Qualitative Universality of Log-linear Relative Dynamics

The qualitative universality of the log-linear relative dynamics means that this dynamics can replicate all possible qualitative properties of behaviour of an arbitrary discrete dynamics near the boundaries of their asymptotic stability. Moreover, for each preset qualitative property the specific combination of elasticities and locational advantages exists, i.e., the specific log-linear model exists which reproduces the given qualitative property. Furthermore, the universality within universality phenomenon holds: it is possible to choose a model which itself can reproduce all possible qualitative properties of discrete dynamics under the change in the locational advantages parameters only, without a change in elasticities. In detail, the fundamental property of qualitative universality follows from the possibility of placing the image of the Moebius triangle at each place in the space of eigenvalues, and from the possibility to place a fixed point at each point in the Moebius triangle.

The technical details of these impositions will be elaborated below for one population/three location relative log-linear dynamics.

Consider three arbitrary points in the phase space

$$A^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B^* = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad C^* = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

which can play the role of the images of the stability triangle ABC

$$A = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}; \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

in the Moebius plane. The condition of the construction of a non-degenerated triangle is that the matrix

$$Z = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \tag{5.1}$$

is invertible: $\det Z \neq 0$. For the construction of the log-linear model corresponding to the stability triangle $A^*B^*C^*$ we need the evaluation of the matrix of elasticities

$$\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

First of all let us find the components of the matrix

$$A = \begin{bmatrix} s & s - s_2 & s - s_3 \\ \Delta^*_{23} & \Delta^*_{31} & \Delta^*_{12} \\ 1 & 1 & 1 \end{bmatrix}$$

satisfying the condition

$$\begin{bmatrix} s & s-s_2 & s-s_3 \\ \Delta_{23}^* & \Delta_{31}^* & \Delta_{12}^* \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \tag{5.2}$$

or in the matrix form:

$$AZ = B$$

where

$$B = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} .$$

Therefore

$$A = BZ^{-1} \tag{5.3}$$

Thus the right parts of the following non-linear algebraic equations (see (4.7) and (4.8)) are known:

$$\begin{aligned} &a_{22} + a_{33} = s \ ; \\ &a_{21} + a_{22} + a_{23} = s - s_2 \ ; \\ &a_{31} + a_{32} + a_{33} = s - s_3 \ ; \\ &\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \Delta_{23}^* \ ; \\ &\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} = \Delta_{31}^* \ ; \\ &\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \Delta_{12}^* \ . \end{aligned} \tag{5.4}$$

The last three equations give

$$a_{21}\Delta_{23}^* + a_{22}\Delta_{31}^* + a_{23}\Delta_{12}^* = a_{31}\Delta_{23}^* + a_{32}\Delta_{31}^* + a_{33}\Delta_{12}^* = 0$$

and

$$(s_3 - s_2)a_{21} + (s - s_2)a_{31} = \Delta_{31}^* - \Delta_{12}^*$$

$$(s_3 - s)a_{22} + (s - s_2)a_{32} = \Delta_{12}^* - \Delta_{23}^*$$

$$(s_3 - s)a_{23} + (s - s_2)a_{33} = \Delta_{23}^* - \Delta_{31}^* \ .$$

Therefore the elasticities a_{ij} can be constructed with the help of formulas:

$$\begin{aligned}
a_{23} &= \left(s + \frac{\Delta_{31}^* - \Delta_{23}^*}{s - s_2} + \Delta_{23}^* \frac{s - s_2}{\Delta_{31}^* - \Delta_{23}^*}\right) \Big/ \left(\frac{s - s_3}{s - s_2} + \frac{\Delta_{23}^* - \Delta_{12}^*}{\Delta_{31}^* - \Delta_{23}^*}\right) , \\
a_{33} &= \frac{s - s_3}{s - s_2}\left(a_{23} + \frac{\Delta_{23}^* - \Delta_{31}^*}{s - s_3}\right) , \\
a_{22} &= s - a_{33} \\
a_{21} &= s - s_2 - a_{22} - a_{23} , \\
a_{31} &= \frac{s - s_3}{s - s_2}\left(a_{21} + \frac{\Delta_{31}^* - \Delta_{12}^*}{s - s_3}\right) , \\
a_{32} &= \frac{s - s_3}{s - s_2}\left(a_{22} + \frac{\Delta_{12}^* - \Delta_{23}^*}{s - s_3}\right) .
\end{aligned} \qquad (5.5)$$

In conclusion, the fundamental property of qualitative universality of the relative log-linear dynamics is the result of the possibility of placing (with the help of formulas (5.3) and (5.5), defining the elasticities a_{ij}) the triangle of stability in a preset position in the phase space, and of executing (with the help of formulas (3.15), defining the comparative advantages A_2, A_3) the travels of equilibria in the preset direction in the phase space.

6. An Example of Computer Realization of the Calculus of Iterations

In this part we will present an example which visionalizes the proposed calculus of iterations with the help of a specially constructed log-linear model. The parameters of this model are chosen in such a way that the image $A^*B^*C^*$ of the triangle of stability ABC will have the following location on the phase space (see Figure 4):

$$A^* = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} , \quad B^* = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} , \quad C^* = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} .$$

Therefore formulas (5.3) and (5.5) give the following elasticities for such a log-linear model:

$$\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0.25 & 0.5 & 2.25 \\ 2.5 & 1 & 2.5 \end{bmatrix} .$$

The sides A^*C^*, A^*B^*, B^*C^* of the triangle of stability $A^*B^*C^*$ represent the flip, flutter and divergence boundaries for the chosen model.

The intersection of the flip boundary A^*C^* will give the Feigenbaum double periodic way to chaos. As an example the movement of the fixed point $\mathbf{x}$ on the

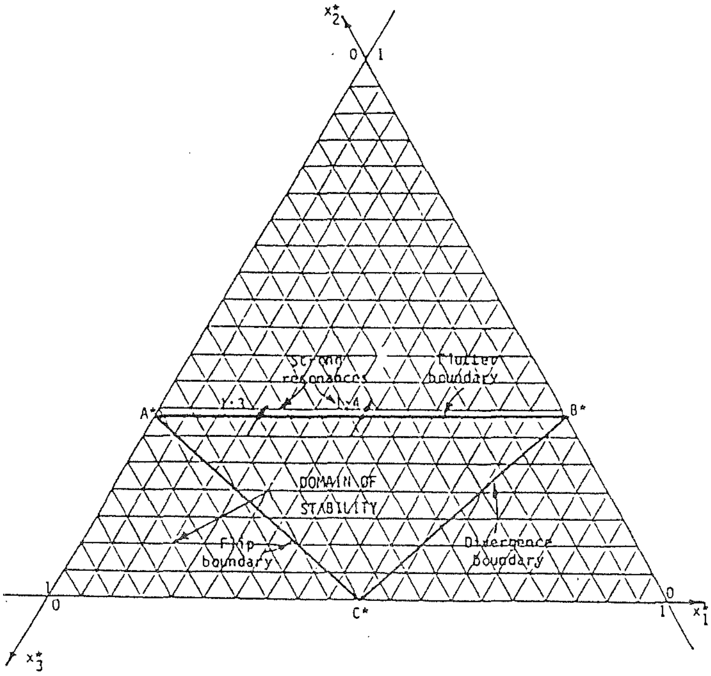


Figure 4. Domain of stability and movement of equilibria for the log-linear relative dynamics with the elasticities

$$\begin{bmatrix} 0 & 0 & 0 \\ 0.25 & 0.5 & 2.25 \\ 2.5 & 1 & 2.5 \end{bmatrix}.$$

segment of a straight line between the points $\begin{bmatrix} 0.25 \\ 0.2 \\ 0.55 \end{bmatrix}$ and $\begin{bmatrix} 0.15 \\ 0.1 \\ 0.75 \end{bmatrix}$ is chosen (see Figure 4). The corresponding spatial bifurcation diagram is presented on Figure 5 where the double periodic bifurcation development is shown; Figure 6 presents the usual bifurcation diagram for the first coordinates $x_1(t)$ of the orbits.

The intersection of the flutter boundaries leads through the resonances and their Arnold tongues to the Hopf-Neimark bifurcations. Formulas (4.23) give the coordinates of the strong (three-periodic and four-periodic) resonances. The three-periodic resonance sits in the point $\begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}$. This resonance is surrounded by the mode-locking Arnold tongue which covers a part of the triangle of stability.

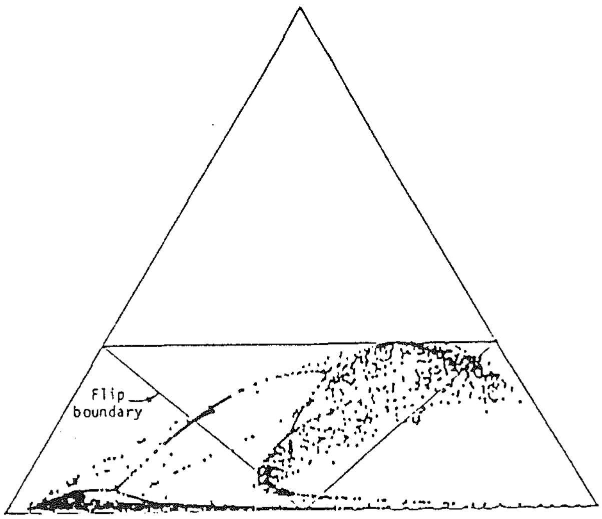


Figure 5. The Feigenbaum way to chaos.

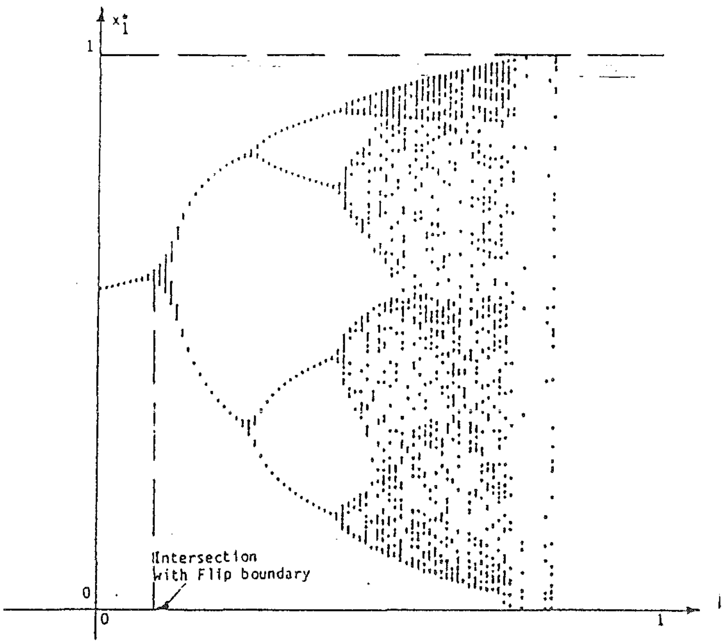


Figure 6. Bifurcation diagram for periodic doubling.

The movement of the fixed point x on the segment of a straight line between the points $\begin{bmatrix} 0.167 \\ 0.313 \\ 0.52 \end{bmatrix}$ and $\begin{bmatrix} 0.167 \\ 0.35 \\ 0.483 \end{bmatrix}$ (see Figure 4) gives on the bifurcation diagram (Figure 7) the jump from the stable equilibrium to the three-periodic cycle and, further, to its doubling. The spatial bifurcation diagram (Figure 8) presents this jump in the phase space; it is also possible to see that the three-periodic cycle is started within the triangle of stability.

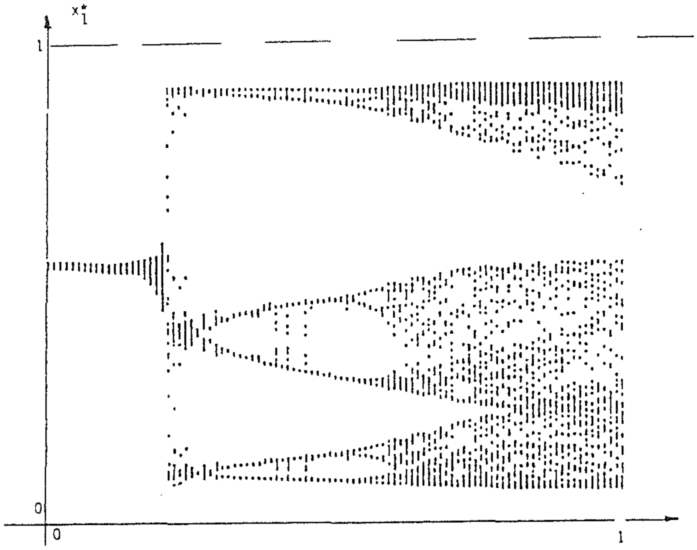


Figure 7. Bifurcation diagram: stable fixed points, three-period cycles and their doubling.

The four-periodic resonance is located on the point $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$. The movement of the fixed point $\mathbf{x}$ on the segment of a straight line between the points $\begin{bmatrix} 0.333 \\ 0.313 \\ 0.354 \end{bmatrix}$ and $\begin{bmatrix} 0.333 \\ 0.367 \\ 0.3 \end{bmatrix}$ (see Figure 4) gives on the spatial bifurcation diagram (Figure 9) the rotating four-periodic cycle and its period- doubling.

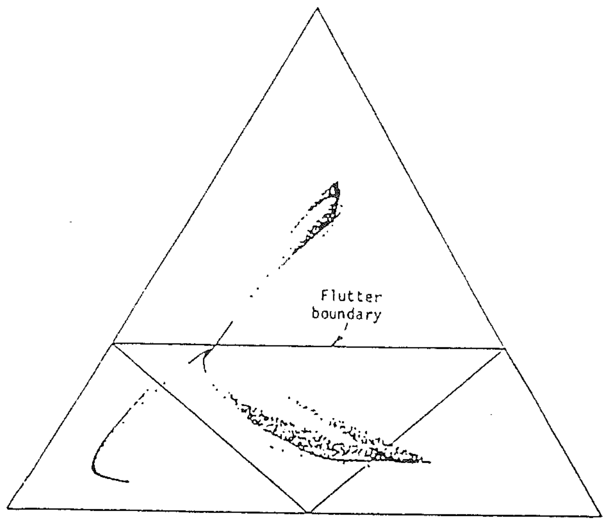


Figure 8. Strong 1:3 resonance and its doubling.

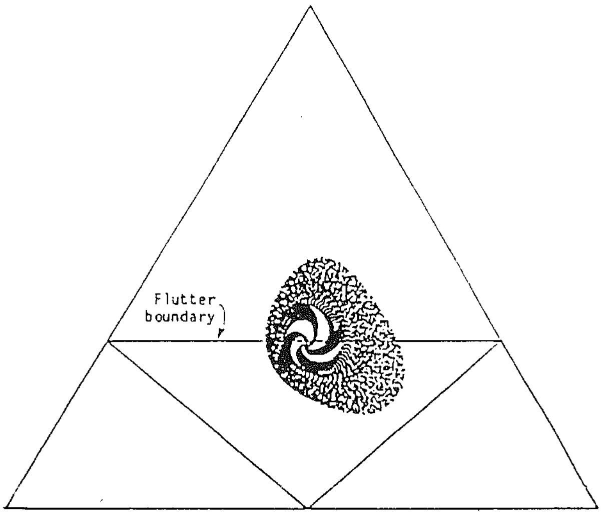


Figure 9. Strong 1:4 resonance: rotating four-period cycles and their doubling.

References

V.I. Arnold, Loss of stability of self-oscillations close to resonance and versal deformations of equivalent vector fields, Functional Analysis and Applications II (1977), 85-92.

D.S. Dendrinos, M. Sonis, Variational principles and conservation conditions in Volterra's ecology and in urban/relative dynamics. Journal of Regional Science 26:2 (1986), 359-377.

D.S. Dendrinos, M. Sonis, Chaos and Socio-Spatial dynamics, Applied Mathematical Sciences 86 (1990), Springer- Verlag; New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong.

M. Feigenbaum, Qualitative Universality for a Class of Nonlinear Transformations, Journal of Statistical Physics 19 (1978), 25-52.

V.G. Gontar, New principles and a new mathematical model for formal chemical kinetics, Russian Journal of Physical Chemistry LV:9 (1981),2301-2304.

C.S. Hsu, On non-linear parametric exitation problem, Advances in Applied Mechanics 17 (1977), 245–301.

V. Kogan, Loss of stability of auto-oscillations in the vicinity of resonances of order 3 and 4, In "Systems Dynamics", Gorky University Press: Gorky, Russia, 1991.

H.A. Lauwerier, Two-dimensional iterative maps, In "Chaos" (A.V. Holden, ed.), Princeton University Press: Princeton, New Jersey, 1986, 58-95.

P.A. Samuelson, Foundations of Economic Analysis, Enlarged Edition, Harvard Economic Studies, Vol. 80, 1983; Harvard University Press, Cambridge, Massachusetts, and London, England.

M. Sonis, Universality of Relative Socio-Spatial Dynamics: the case of one population/three locations log-linear dynamics, Series on Socio-Spatial Dynamics, Univ. of Kansas, 1:3 (1990), 179-204.

M. Sonis, V. Gontar, Entropy principle of extremality, Socio-Spatial Dynamics 3:2 (1992), 61-73.

J.M.T. Thompson, H.B. Stewart, Nonlinear Dynamics and Chaos, Geometrical Methods for Engineers and Scientists, John Wiley: Chichester,. New York, Brisbane, Totonto, Singapore 1986.