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SECOND ORDER ACCURACY DIFFERENCE SCHEME FOR APPROXIMATE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS.

A. ASHYRALYEV*, H. AKÇA† AND U. GURAY‡

Abstract. We describe a second order accuracy difference scheme for the approximate solutions of linear delay differential equations. A sufficient condition for the stability of this difference scheme is given. A convergence estimate for the difference scheme is obtained.

AMS(MOS) subject classification. 34K40, 34K20, 65L05, 65L20

1. Introduction and preliminary results. Numerical solutions of the delay differential equations have been studied extensively by many researchers (cf.,e.g. [1-2], [4-7] and the references therein) and developed over the last two decades. In the literature mostly the condition $|b(t)| \leq \operatorname{Re} a(t)$ is considered. The subject of this paper is the stability analysis of high order accuracy approximate solutions of variable coefficients delay differential equations. In addition, we introduced the second order accuracy difference schemes for the approximate solutions of the initial value problem for linear delay equations. A sufficient condition for the stability of this difference scheme is given. Convergence estimates for second order accuracy difference schemes are also obtained.

Usually delay equations can be solved by adapting standard numerical methods for differential equations without delay. But it is difficult to generalize for any numerical method to obtain for instance high order accuracy algorithms, because high order methods may not give good results. Even if $a(t), c(t)$ and $g(t)$ are arbitrary differentiable functions, but $u(t)$ may not have

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the required number of derivatives for a sufficiently large t. This property can be examined on the following test equation:

$$\begin{aligned} u'(t) &= -u(t) + u(t-1), \qquad t \geq 0, \\ u(t) &= t+1, \qquad -1 \leq t \leq 0, \end{aligned} \tag{1.1}$$

For $t \in [0,1]$, $u(t-1) = t$ is obvious. Therefore on the interval $[0,1]$ the test problem can be considered as an ordinary initial value problem (IVP) such as

$$u_1'(t) = -u_1(t) + t, \qquad u_1(0) = 1. \tag{1.2}$$

Having solved this simple initial value problem, one can compute $u(t)$ on the interval $[1,2]$ by solving the delay differential equation

$$u_2'(t) = -u_2(t) + u_1(t-1), \qquad u_2(1) = u_1(1) \tag{1.3}$$

and, in general,

$$u_i'(t) = -u_i(t) + u_{i-1}(t-1), u_i(i-1) = u_{i-1}(i-1), i = 1,2,\ldots \tag{1.4}$$

By mathematical induction, approximate solutions of the problem (1.1) can be easily obtained. Using the recursive formula (1.4) the first few terms of the sequence of the solutions can be found as follows:

$$\begin{aligned} u_1(t) &= 2e^{-t} + t - 1, \\ u_2(t) &= 2e^{-t} + 2te^{-(t-1)} + t - 3, \\ u_3(t) &= 2e^{-t} + (t^2+2)e^{-(t-1)} + t - 5. \end{aligned} \tag{1.5}$$

The second derivatives of the solutions sequence $u_1(t)$ and $u_2(t)$ have different values at the point $t = 1$. That means $u''(t)$ does not exist.

2. Statement of the problem. Consider the linear delay differential equation

$$\begin{aligned} u'(t) &= -a(t)u(t) + b(t)u(t-\omega), \quad t \geq 0, \\ u(t) &= g(t), \qquad -\omega \leq t \leq 0, \end{aligned} \tag{2.1}$$

where $a(t), c(t) \in C^2([0,\infty), \mathbf{R})$ and $g(t) \in C^2([-\omega, 0], \mathbf{R})$, $\omega \geq 0$ is a constant delay. For existence and uniqueness of the solutions of the initial value problem (2.1) see [4,5,6]. We are interested to introduce an asymptotically stable high order accuracy difference scheme for the approximate solutions of (2.1). It has been already proved that if $|b(t)| \le a(t)$ for every $t \ge 0$, then the solutions of (2.1) are asymptotically stable [4,5,7]. A solution $u(t)$ of (2.1) is said to be *asymptotically stable* if

$$|u(t)| \le \max_{-\omega \le t \le 0} |g(t)| \text{ for every } t, \omega \ge 0. \tag{2.2}$$

Using the second order accuracy implicit difference scheme for differential equations without delay (see, for example, [3]), we have the following approximate solutions for the IVP (2.1)

$$\begin{aligned} &u_0 = g(0), \quad t_k = k\tau, \quad N\tau = \omega, \quad 0 \le k \le N, \text{and} \\ &a_k = a(t_k - \tfrac{1}{2}\tau), \qquad b_k = b(t_k - \tfrac{1}{2}\tau), \quad k \ge 1, \\ &\tfrac{1}{\tau}(u_k - u_{k-1}) + (a_k + \tfrac{1}{2}\tau a_k^2)u_k = \tfrac{1}{2}a_k b_k \tau g(t_k - \omega) \\ &\quad + \tfrac{1}{2}b_k[g(t_k - \omega) + g(t_{k-1} - \omega)], \ 1 \le k \le N, \\ &\tfrac{1}{\tau}(u_k - u_{k-1}) + (a_k + \tfrac{1}{2}\tau a_k^2)u_k = \tfrac{1}{2}(\tau a_k + 1)b_k u_{k-N} \\ &\quad + \tfrac{1}{2}b_k u_{k-N-1}, \ N + 1 \le k. \end{aligned} \tag{2.3}$$

The difference scheme (2.3) approximates the solutions of (2.1) on the entire interval. In addition, if the condition $|b(t)| \le a(t)$ for every $t \ge 0$ is satisfied, then the difference scheme (2.3) defines an asymptotically stable solution for the IVP (2.1).

Denote

$$D_k = \left\{1 + \tau a_k + \frac{1}{2}(\tau a_k)^2\right\}^{-1} \quad \text{for} \quad k > 0, \tag{2.4}$$

$$u(k,j) = \begin{cases} D_k \cdots D_{j+1}, & k > j, \\ 1, & k = j. \end{cases} \tag{2.5}$$

LEMMA 1. *If D_k is defined by* (2.4), *then the following identity*

$$I - D_j = \left\{\tau a_j + \frac{1}{2}(\tau a_j)^2\right\} D_j \tag{2.6}$$

holds for $j = 1, 2, 3 \cdots$ The proof of the lemma is obvious.

LEMMA 2. *An approximate solution of the IVP* (2.1) *can be defined as follows:*

$$\begin{aligned} u_k = u(k,0)g(0) + \sum_{j=1}^{k} \tfrac{1}{2}\tau u(k, j-1)[a_j b_j \tau g(t_j - \omega) \\ + b_j(g(t_j - \omega) + g(t_{j-1} - \omega))] \end{aligned} \tag{2.7}$$

where $1 \le k \le N$, *and*

$$u_k = u(k,N)u_N + \sum_{j=N+1}^{k} \tfrac{1}{2}\tau u(k,j-1)[(a_j\tau+1)b_j u_{j-N} + b_j u_{j-N-1}] \tag{2.8}$$

where $k \ge N+1$. The proof of the lemma is obvious.

THEOREM 1. *Assume that the condition* $|b(t)| \le a(t)$ *(for every* $t \ge 0$*) holds and* u_k *is defined by* (2.7) – (2.8)*. Then for every* $k \ge 0$ *and for every step size* τ *we have*

$$|u_k| \le \max_{-N \le j \le 0} |g(t_j)|. \tag{2.9}$$

Proof. First consider the case $1 \le k \le N$. From the hypothesis and using the formula (2.7), we have

$$|u_k| \le u(k,0)|g(0)| + \sum_{j=1}^{k} \frac{1}{2}\tau u(k,j) D_j[(|a_j||b_j|\tau + |a_j|)$$

$$(|g(t_j-\omega)|) + |b_j||g(t_{j-1}-\omega)|]$$

$$\le u(k,0)|g(0)| + \sum_{j=1}^{k} \frac{1}{2}\tau u(k,j) D_j \left\{[a_j^2|\tau + |a_j|][|g(t_j-\omega)|] + |a_j||g(t_{j-1}-\omega)|\right\}$$

$$\le u(k,0)|g(0)| + \max_{-N\le j\le 0} |g(t_j)| \sum_{j=1}^{k} \tau u(k,j) D_j \left\{\frac{1}{2}a_j^2\tau + a_j\right\}$$

and by the Lemma 1

$$\sum_{j=1}^{k} u(k,j) D_j[a_j\tau + \tfrac{1}{2}(a_j\tau)^2] = \sum_{j=1}^{k} [u(k,j) - u(k,j-1)] = 1 - u(k,0) \tag{2.10}$$

It follows that

$$|u_k| \le u(k,0)|g(0)| + \max_{-N\le j\le 0} |g(t_j)|(1 - u(k,0)),$$

consequently we obtain the result

$$|u_k| \le \max_{-N\le j\le 0} |g(t_j)| \tag{2.11}$$

for $1 \leq k \leq N$. Applying the mathematical induction, one can easily show that it is true for every k. Namely, assume that the inequality (2.9) is true for $N(n-1) \leq k \leq nN,\ n = 1, 2, 3, \cdots$ Thus

$$|u_k| \leq u(k, nN)|u_{nN}| + \sum_{j=nN+1}^{k} u(k, j-1)\left\{[\frac{(a_j\tau)^2}{2} + a_j\tau]|u_{j-N}| + \frac{a_j\tau}{2}|u_{j-N-1}|\right\}$$

$$\leq u(k, N)|u_{nN}| + \sum_{j=nN+1}^{k} u(k, j)D_j\left\{\frac{(a_j\tau)^2}{2} + a_j\tau\right\} \max_{nN+1 \leq j \leq k} |u_j|$$

and by Lemma 2 we have

$$\sum_{j=nN+1}^{k} u(k, j)D_j\left\{\frac{(a_j\tau)^2}{2} + a_j\tau\right\} = 1 - u(k, nN)$$

and it follows that

$$|u_k| \leq u(k, N)|u_{nN}| + \max_{nN+1 \leq j \leq k} |u_j| \{1 - u(k, N)\}$$

$$\leq \max_{(n-1)N \leq j \leq nN} |u_j| \leq \max_{-N \leq j \leq 0} |g(t_j)|$$

for every $k,\ nN \leq k \leq (n+1)N$.
This result completes the proof of the theorem .

3. Convergence estimate of the difference scheme. We have already shown in Section 2 that approximate solutions of (1.2) defined by (2.3) are bounded for every k. Before going to discuss convergence properties of the solutions, let us introduce the following notations.

$e_k = u(t_k) - u_k$ represents the component of the error vector e^τ, and A_k denotes the component of the approximation vector A^τ where

$$\text{(3.1)} \qquad A_k = \frac{1}{\tau}(e_k - e_{k-1}) + (a_k + \frac{\tau a_k^2}{2})e_k, \quad e_0 = 0, \quad 1 \leq k \leq N,$$

$$\text{(3.2)} \qquad \begin{aligned} A_k &= \frac{1}{\tau}(e_k - e_{k-1}) + (a_k + \frac{\tau a_k^2}{2})e_k \\ &- \frac{1}{2}(\tau a_k + 1)b_k e_{k-N} - \frac{1}{2}b_k e_{k-N-1}, N + 1 \leq k. \end{aligned}$$

The following equations can be obtained in an obvious way:

$$u(t_k - \tfrac{\tau}{2}) - \tfrac{1}{2}(u(t_k) + u(t_{k-1}))$$

$$= \tfrac{1}{2} \int_{t_k-\tau/2}^{t_k} \int_{s}^{s-\tau/2} u''(\lambda)\, d\lambda\, ds, \quad t_{k-1} \le s \le t_k \tag{3.3}$$

and

$$u(s) - u(t_k - \tfrac{\tau}{2}) = (s - t_k - \tfrac{\tau}{2})u'(t_k - \tfrac{\tau}{2})$$

$$+ \int_{t_k-\tau/2}^{s} (s-\lambda)u''(\lambda)\, d\lambda, \quad t_{k-1} \le s \le t_k. \tag{3.4}$$

THEOREM 2. *The estimate*

$$\sum_{i=1}^{k} \tau |A_i| \le M\tau^2, \qquad \textit{for all} \quad k \ge 1 \tag{3.5}$$

holds, where M is a real number independent of τ.

Proof. First consider the case $1 \le k \le N$. Using the definition of A_k, we may write from (2.1)

$$A_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} (-a(s)u(s) + b(s)g(s-\omega))ds + (a_k + \frac{\tau}{2}a_k^2)u(t_k)$$

$$+g(t_{k-1} - \omega)) - a_k b_k \frac{\tau}{2} g(t_k - \omega) - \frac{1}{2} b_k (g(t_k - \omega)$$

$$= \frac{1}{\tau} \int_{t_{k-1}}^{t_k} (b(s)g(s-\omega) - b_k g(t_k - \frac{\tau}{2}))\, ds + b_k [g(t_k - \frac{\tau}{2})$$

$$[-\frac{1}{2}(g(t_k - \omega) + g(t_{k-1} - \omega))] + \frac{1}{\tau} \int_{t_{k-1}}^{t_k} (a_k u(t_k - \frac{\tau}{2}) - a(s)u(s))\, ds$$

$$+a_k \left\{ \int_{t_k-\tau/2}^{t_k} (a_k u(t_k) - a(s)u(s))\, ds + \int_{t_k-\tau/2}^{t_k} (b(s)g(s-\omega) - b_k g(t_k - \omega))\, ds \right\}.$$

Analogously, the definition of A_k for the case $N+1 \leq k$ is

$$A_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} (-a(s)u(s) + b(s)g(s-\omega))ds + (a_k + \frac{\tau}{2}a_k^2)u(t_k)$$

$$-\frac{1}{2}(\tau a_k + 1)b_k u(t_{k-N}) - \frac{1}{2}b_k u(t_{k-N-1})$$

$$= \frac{1}{\tau}\int_{t_{k-1}}^{t_k} (b(s)u(s-\omega) - b_k u(t_k - \frac{\tau}{2}))\,ds + b_k[u(t_k - \frac{\tau}{2})$$

$$-\frac{1}{2}(u(t_{k-N}) + u(t_{k-1-N}))]\,ds + \frac{1}{\tau}\int_{t_{k-1}}^{t_k} (a_k u(t_k - \frac{\tau}{2}) - a(s)u(s))\,ds$$

$$+a_k\left\{\int_{t_k-\tau/2}^{t_k} (a_k u(t_k) - a(s)u(s))\,ds + \int_{t_k-\tau/2}^{t_k} (b(s)u(s-\omega) - b_k u(t_{k-N}))\,ds\right\}.$$

Thus we can rewrite A_k using the formulas (3.3)–(3.4) for the case $1 \leq k \leq N$:

$$A_k = \frac{1}{\tau}\left\{\int_{t_{k-1}}^{t_k}\int_{t_k-\tau/2}^{s} (s-\lambda)(b(\lambda)g(\lambda-\omega))''\,d\lambda\,ds + \frac{1}{2}\int_{t_k-\tau/2}^{t_k}\int_{s}^{s-\tau} g''(\lambda-\omega)\,d\lambda\,ds\right\}$$

$$+\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_{s}^{t_k-\tau/2} (s-\lambda)[(a''(\lambda) - a(\lambda)a'(\lambda))u(\lambda) + (2a'(\lambda) - a^2(\lambda))u'(\lambda)$$

$$+a(\lambda)[b'(\lambda)g(\lambda-\omega) + b(\lambda)g'(\lambda-\omega)]]\,d\lambda\,ds$$

$$+a_k\left\{\int_{t_k-\tau/2}^{t_k}\left[\int_{s}^{t_k-\tau/2} a'(\lambda)u(s)\,d\lambda + \int_{s}^{t_k} a_k u'(\lambda)\,d\lambda\right] ds\right.$$

$$+\int_{t_k-\tau/2}^{t_k}\left[\int_s^{t_k-\tau/2} b'(\lambda)g(s-\omega)\,d\lambda+\int_s^{t_k} b_k g'(\lambda-\omega)\,d\lambda\right]ds\Bigg\}.$$

It follows that

$$\sum_{i=1}^{k}\tau|A_i|\le M\tau^2. \tag{3.6}$$

Finally, for the case $k\ge N+1$, the same estimate as before may be obtained by a similar type of calculations. The proof of the theorem is complete.

THEOREM 3. *If $u(t)$ is a smooth solution of the IVP (2.1) and the approximate solutions u_k are defined by (2.3), then for every $k\ge 1$ the convergence estimate*

$$|u(t_k)-u_k|\le M\tau^2 \tag{3.7}$$

holds, where M is a real number independent of τ.

Proof. The proof of the theorem is based on the estimate (3.6) and

$$|e_k|\le\sum_{j=1}^{k}\tau|A_j| \qquad \text{for all} \quad k\ge 1. \tag{3.8}$$

First let us consider the case $1\le k\le N$. Using (2.4)–(2.5), we can obtain

$$|e_k|\le\sum_{j=1}^{k}\tau u(k,j-1)|A_j|\le\sum_{j=1}^{k}\tau|A_j|. \tag{3.9}$$

By mathematical induction it can be shown that the estimate (3.9) is also true for any k. Namely, assume the inequality (3.8) is true for $(n-1)N\le k\le nN$. Then it follows that

$$|e_k|\le u(k,nN)|e_{nN})|$$

$$+\sum_{j=nN+1}^{k}u(k,j-1)\left\{[\frac{(a_j\tau)^2}{2}+a_j\tau]|e_{j-N}|+\frac{a_j\tau}{2}|e_{j-N-1}|\right\}$$

$$+\sum_{j=nN+1}^{k}\tau u(k,j)|A_j|$$

$$\leq u(k,nN)|e_N| + \sum_{j=nN+1}^{k} u(k,j)D_j \left\{ \frac{(a_j\tau)^2}{2} + a_j\tau \right\} \max_{(n-1)N\leq j\leq nN} |e_j|$$

$$+ \sum_{j=nN+1}^{k} \tau|A_j|$$

$$\leq [u(k,nN) + (I - u(k,nN))] \max_{(n-1)N\leq j\leq nN} |e_j|$$

$$+ \sum_{j=nN+1}^{k} \tau|A_j| \leq \sum_{j=1}^{k} \tau|A_j|$$

which completes the proof.

Note that the better convergence of difference scheme (2.3) in comparison with the convergence of Euler's difference scheme for the nonsmooth solution of the problem is supported by the results of numerical experiments.

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NONOSCILLATION OF A SECOND ORDER LINEAR DELAY DIFFERENTIAL EQUATION WITH A MIDDLE TERM *

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Abstract. For a delay differential equation

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t)) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = 0, \; h_k(t) \le t, \; g_k(t) \le t,$$

a generalized Riccati inequality is constructed which implies nonoscillation of the differential equation.

Comparison theorems and an explicit nonoscillation condition are presented.

Key Words. Oscillation, second order delay equation, comparison theorems.

AMS(MOS) subject classification. 34K15, 34K25.

1. Introduction. This paper deals with oscillation problems for a scalar linear delay differential equation of the second order. Such equations attract attention of many mathematicians due to their significance in applications. We mention here the monographs of A. D. Myshkis [1], S.B. NORKIN [2], G.S. LADDE, V. LAKSHMIKANTHAM and B.G. ZHANG [3], I. GYORI and G. LADAS [4], L.N. ERBE, Q. KONG and B.G. ZHANG [5] and references therein. The monographs contain examples of physical models leading to equations of the type

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t)) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = f(t), \; h_k(t) \le t, \; g_k(t) \le t.$$

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The term with the first derivative is usually called "a damping term". For example, in the usual equation of oscillations (without delay) such a term corresponds to the media resistance. The greater part of the literature deals with equations not containing the term with the first derivative. For these equations if the coefficients and a solution are positive on the semiaxis then its derivative is nonnegative. This fact is very important; it is employed in most investigations on second order delay differential equations. If the first derivative is included in the equation explicitly (the equation contains the middle term), then a sign of a solution does not uniquely define the sign of its derivative. Therefore the study of oscillation properties of the equations with the middle term is more complicated. This is the reason why such equations are much less studied than the equations without the middle term. The following particular cases were considered: the middle term is not delayed (see, for example, papers [6,7]) and the delay is constant [8,9].

In this paper we consider the general class of equations containing the middle term with deviating argument and study properties of these equations concerned with nonoscillation. The main result is the following: if a generalized Riccati inequality (which is constructed here) has a nonnegative solution for $t \geq t_0$, then the differential equation for $t \geq t_0$ has a positive solution with a nonnegative derivative and the fundamental function of this equation is positive. If the middle term is not delayed, this immediately yields that the following four properties are equivalent: nonoscillation of solutions of this equation and the corresponding differential inequality, positiveness of the fundamental function and existence of a nonnegative solution of a generalized Riccati inequality.

We employ a generalized Riccati inequality to compare oscillation properties of two equations without comparing their solutions. One can treat these results as a natural generalization of the well-known Sturm comparison theorem for a second order ordinary differential equation.

By applying the positiveness of the fundamental function we compare positive solutions of two nonoscillation equations.

The paper also contains explicit nonoscillation conditions obtained by construction solutions of generalized Riccati inequality.

The paper is organized as follows. Section 2 contains relevant definitions and notations. In section 3 we obtain the main result. Section 4 deals with comparison results. The last section includes some explicit nonoscillation conditions.

In this paper we follow the method employed in [10] for the second order delay differential equation without middle term.

2. Preliminaries. We consider a scalar delay differential equation of the second order

(1) $$\ddot{x}(t)+\sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t))+\sum_{k=1}^{m} b_k(t)x(g_k(t))=0,$$

under the following assumptions:

(a1) a_k, b_k are Lebesgue measurable and locally essentially bounded functions;

(a2) h_k, g_k are Lebesgue measurable functions,

$$h_k(t)\le t,\ \lim_{t\to\infty} h_k(t)=\infty,\ g_k(t)\le t,\ \lim_{t\to\infty} g_k(t)=\infty.$$

Together with (1) consider for each $t_0 \ge 0$ an initial value problem with a right hand side

(2) $$\ddot{x}(t)+\sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t))+\sum_{k=1}^{m} b_k(t)x(g_k(t))=f(t),\ t\ge t_0,$$

(3) $$x(t)=\varphi(t),\dot{x}(t)=\psi(t),\ t<t_0;\ x(t_0)=x_0,\ \dot{x}(t_0)=x_0'.$$

We also assume that the following hypothesis holds

(a3) $f:[t_0,\infty)\to R$ is a Lebesgue measurable locally essentially bounded function, φ, $\psi:(-\infty,t_0)\to R$ are Borel measurable bounded functions.

DEFINITION 1. *Suppose a function $x:[t_0,\infty)\to R$ is differentiable and $\dot{x}$ is locally absolutely continuous function. Extend the functions x and $\dot{x}$ for $t\le t_0$ by the help of equalities (3). We say that extended so function x is a* a solution *of problem (2), (3) if it satisfies equation (2) for almost every $t\in[t_0,\infty)$* .

DEFINITION 2. *For each $s\ge 0$ the solution $X(t,s)$ of the problem*

(4) $$\begin{gathered}\ddot{x}(t)+\sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t))+\sum_{k=1}^{m} b_k(t)x(g_k(t))=0,\ t\ge s,\\ x(t)=0,\ \dot{x}(t)=0,\ t<s;\ x(s)=0,\ \dot{x}(s)=1,\end{gathered}$$

is called a fundamental function *of equation (1).*

Remark. In literature [16] the fundamental function is also called *the Cauchy function.*

We assume $X(t,s)=0,\ 0\le t<s$.

Let functions x_1 and x_2 be the solutions of the following problems

$$\ddot{x}(t)+\sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t))+\sum_{k=1}^{m} b_k(t)x(g_k(t))=0,\ t\ge t_0;$$

$$x(t)=0,\ \dot{x}(t)=0,\ t<t_0,$$

with initial values $x(t_0)=1,\ \dot{x}(t_0)=0$ for x_1 and $x(t_0)=0,\ \dot{x}(t_0)=1$ for x_2, respectively.

By definition $x_2(t)=X(t,t_0)$.

LEMMA 1. [11] *Let (a1)-(a3) hold. Then there exists one and only one solution of problem (2), (3) that can be presented in the form*

$$x(t)=x_1(t)x_0+x_2(t)x_0^{'}+\int_{t_0}^{t}X(t,s)f(s)ds-\sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)b_k(s)\varphi(g_k(s))ds-$$

$$\sum_{k=1}^{r}\int_{t_0}^{t}X(t,s)a_k(s)\psi(h_k(s))ds+\int_{t_0}^{t}X(t,s)f(s)ds \tag{5}$$

(The functions $\varphi(t)$ and $\psi(t)$ which describe "the prehistory" of the process are not defined for $t\ge t_0$. In the latter equality we assume $\varphi(s)=0$, if $s>t_0$ and $\psi(s)=0$, if $s>t_0$.)

3. Nonoscillation Criteria. Denote

$$a^+=\max\{a,0\},\ a^-=\max\{-a,0\}.$$

The following theorem establishes a sufficient condition of existence of a nonoscillatory solution.

THEOREM 1. *Suppose there exist $t_0\ge 0$ and a nonnegative locally absolutely continuous function u satisfying the following conditions:*

$$\text{(b1)}\quad \dot{u}(t)+u^2(t)+\sum_{k=1}^{r\ \prime}a_k^+(t)u(h_k(t))\exp\left\{-\int_{h_k(t)}^{t}u(s)ds\right\}+$$

$$\sum_{k=1}^{m\ \prime}b_k^+(t)\exp\left\{-\int_{g_k(t)}^{t}u(s)ds\right\}\le 0 \tag{6}$$

where the sums contain only those terms for which $h_k(t)\ge t_0$ or, respectively, $g_k(t)\ge t_0$.

(b2) The equation

$$\dot{z}(t)+u(t)z(t)+\sum_{k=1}^{r}a_k(t)z(h_k(t))=0 \tag{7}$$

has a positive fundamental function $Z(t,s)>0$ for $t>s>t_0$.

Then

1) the fundamental function of (1) and its derivative in t X_t' *are positive for* $t > s > t_0$: $X(t,s) > 0,\ X_t'(t,s) \geq 0$;

2) there exists a solution $x(t)$ *of (1) such that* $x(t) > 0,\ \dot{x}(t) \geq 0,\ t > t_0$.

Proof. 1) Consider an initial value problem

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}(h_k(t)) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = f(t),\ t \geq t_0, \tag{8}$$

$$x(t) = \dot{x}(t) = 0,\ t \leq t_0.$$

Denote

$$z(t) = \dot{x}(t) - u(t)x(t), \tag{9}$$

where x is the solution of (8) and u is a nonnegative solution of (6). From (9) we obtain

$$x(t) = \int_{t_0}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds, \tag{10}$$

$$\dot{x} = z + ux, \quad \ddot{x} = \dot{z} + \dot{u}x + uz + u^2x = \dot{z} + uz + (\dot{u} + u^2)x.$$

Substituting $\dot{x}, \ddot{x}$ into (8) we obtain

$$\dot{z}(t) + u(t)z(t) + \sum_{k=1}^{r} a_k(t)z(h_k(t)) = \tag{11}$$

$$-(\dot{u}(t) + u^2(t))x(t) - \sum_{k=1}^{r} a_k(t)u(h_k(t))x(h_k(t)) - \sum_{k=1}^{m} b_k(t)x(g_k(t)) + f(t).$$

Equalities (8) and (9) imply $z(t_0) = 0$. Using (10) we can rewrite equation (11) in the form

$$\dot{z}(t) + u(t)z(t) + \sum_{k=1}^{r}{}' a_k(t)z(h_k(t)) =$$

$$-\left(\dot{u}(t) + u^2(t) + \sum_{k=1}^{r}{}' a_k^+(t)u(h_k(t)) \exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\} + \right.$$

$$\sum_{k=1}^{m}{}' b_k^+(t)\exp\left\{-\int_{g_k(t)}^t u(s)ds\right\}\Big)\int_{t_0}^t \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds+$$

$$\sum_{k=1}^{r}{}' a_k^+(t)u(h_k(t)\int_{h_k(t)}^t \exp\left\{\int_s^{h_k(t)} u(\tau)d\tau\right\} z(s)ds+$$

$$\sum_{k=1}^{m}{}' b_k^+(t)\int_{g_k(t)}^t \exp\left\{\int_s^{g_k(t)} u(\tau)d\tau\right\} z(s)ds+$$

$$\sum_{k=1}^{r}{}' a_k^-(t)u(h_k(t))\int_{t_0}^{h_k(t)} \exp\left\{\int_s^{h_k(t)} u(\tau)d\tau\right\} z(s)ds+$$

$$\sum_{k=1}^{m}{}' b_k^-(t)\int_{t_0}^{g_k(t))} \exp\left\{\int_s^{g_k(t)} u(\tau)d\tau\right\} z(s)ds+f(t), \quad z(t_0)=0. \tag{12}$$

Denote by $Z(t,s)$ the fundamental function of (7) and by $Fz+f$ the right-hand side of the equation (12). Then equation (12) is equivalent to the following equation

$$z = Hz + p, \tag{13}$$

where

$$(Hz)(t) = \int_{t_0}^t Z(t,s)(Fz)(s)ds, \quad p(t) = \int_{t_0}^t Z(t,s)f(s)ds. \tag{14}$$

Inequalities (6) and $Z(t,s) > 0$ yield that if $z(t) \geq 0$ then $(Hz)(t) \geq 0$ (i.e operator H is positive).

Denote

$$c(t) = \dot{u}(t) + u^2(t) + \sum_{k=1}^{r}{}' a_k^+(t)u(h_k(t))\exp\left\{-\int_{h_k(t)}^t u(s)ds\right\}+$$

$$\sum_{k=1}^{m}{}' b_k^+(t)\exp\left\{-\int_{g_k(t)}^t u(s)ds\right\}.$$

Since u is locally absolutely continuous, $c \in L_{[t_0,b]}$ for every $b > t_0$, where $L_{[a,b]}$ is a space of all Lebesgue integrable on $[a,b]$ functions with the usual norm.

Function $Z(t,s)$ is bounded [12] in any square $[t_0,b]\times[t_0,b]$, hence for a certain $K>0$, $|Z(t,s)|\le K,\ b\ge t\ge s\ge t_0$.

Then we have for $t\in[t_0,b]$

$$|(Hz)(t)|\le K\exp\left\{\int_{t_0}^{b}u(\tau)d\tau\right\}\int_{t_0}^{t}(|c(s)|+$$

$$\sum_{k=1}^{r}|a_k(s)||u(h_k(s))|+\sum_{k=1}^{m}|b_k(s)|)\int_{t_0}^{s}|z(\tau)|d\tau ds=K\exp\left\{\int_{t_0}^{b}u(\tau)d\tau\right\}\times$$

$$\times\int_{t_0}^{t}\left(\int_{\tau}^{t}\left[|c(s)|+\sum_{k=1}^{r}|a_k(s)||u(h_k(s))|+\sum_{k=1}^{m}|b_k(s)|\right]ds\right)|z(\tau)|d\tau.$$

The kernel of Volterra integral operator H is bounded in each square $[t_0,b]\times[t_0,b]$, hence [13, p.519] $H: L_{[t_0,b]}\to L_{[t_0,b]}$ is a weakly compact operator and his square is compact operator. The spectral radius of a compact Volterra integral operator in the space $L_{[a,b]}$ is equal to zero [14, Th.6.2, p.143].

Therefore the spectral radius of operator H: $r(H)=0$.

Thus if in (13) $p(t)\ge 0$ then

$$z(t)=p(t)+(Hp)(t)+(H^2p)(t)+\ldots\ge 0.$$

If $f(t)\ge 0$ then by (14) $p(t)\ge 0$. Hence for equation (11) we have the following: if $f(t)\ge 0$ then the solution of this equation $z(t)\ge 0$.

Therefore equality (10) implies that the solution of (8) and its derivative are nonnegative for any nonnegative right-hand side.

The solution of this equation can be presented in the form (5), consequently

$$x(t)=\int_{t_0}^{t}X(t,s)f(s)ds,\quad \dot{x}(t)=\int_{t_0}^{t}X_t'(t,s)f(s)ds. \tag{15}$$

As it was shown $f(t)\ge 0$ implies $x(t)\ge 0$ *and* $\dot{x}(t)\ge 0$. Consequently, the kernels of the integral operators (15) are nonnegative. Therefore $X(t,s)\ge 0$ and $X_t'(t,s)\ge 0$.

Since $X_t'(s,s)=1$ implies $X_t'(t,s)>0$ on some interval $[s,s+\sigma]$ for a certain $\sigma>0$, the strict inequality $X(t,s)>0, t>s\ge t_0$ holds.

2) A function $x(t)=X(t,t_0)$ is a positive solution of the equation (1) with nonnegative derivative.

The proof is complete. □

COROLLARY 1. *Suppose $a_k(t) \le 0$ and there exist $t_0 \ge 0$ and a nonnegative locally absolutely continuous function u such that the following condition holds*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m}{}' b_k^+(t) \exp\left\{-\int_{g_k(t)}^t u(s)ds\right\} \le 0$$

where sums contain only terms for which $g_k(t) \ge t_0$. Then

1) $X(t,s) > 0,\ X_t'(t,s) \ge 0,\ t > s > t_0$;

2) there exists a solution $x(t)$ of (1) such that $x(t) > 0,\ \dot{x}(t) \ge 0,\ t > t_0$.

Proof. We have to prove only positiveness of the fundamental function of (7). The ordinary differential equation

$$\dot{z}(t) + u(t)z(t) = 0$$

has a positive fundamental function. Then inequality $a_k(t) \le 0$ and Theorem 2 of [15] imply that the fundamental function of (7) is positive. □

We will demonstrate that condition (b1) in Theorem 1 is a necessary one for nonoscillation of equation (1) with nonnegative coefficients. To this end consider the delay differential inequality

$$\ddot{y}(t) + \sum_{k=1}^{r} a_k(t)\dot{y}(h_k(t)) + \sum_{k=1}^{m} b_k(t)y(g_k(t)) \le 0,\ t \ge 0. \tag{16}$$

THEOREM 2. *Suppose $a_k(t) \ge 0,\ b_k(t) \ge 0$. If there exists $t_0 \ge 0$ such that inequality (16) has a positive solution with a nonnegative derivative for $t > t_0$, then there exists $t_1 \ge t_0$ such that inequality (6) has a nonnegative solution for $t \ge t_1$.*

Proof. Let $y(t)$ be a positive solution of inequality (16) for $t > t_0$ with nonnegative derivative. Then there exists a point t_1 such that $h_k(t) \ge t_0, g_k(t) \ge t_0$, if $t \ge t_1$. We can assume without loss of generality that $y(t_1) = 1$.

Denote $u(t) = \frac{\dot{y}(t)}{y(t)}$, if $t \ge t_1$ and $u(t) = 0$, if $t < t_1$. Then u is a nonnegative locally absolutely continuous on $[t_1, \infty)$ function. The equalities $\dot{y}(t) - u(t)y(t) = 0,\ y(t_1) = 1$ imply that

$$y(t) = \exp\left\{\int_{t_1}^t u(s)ds\right\},\ \dot{y}(t) = u(t)\exp\left\{\int_{t_1}^t u(s)ds\right\},$$

$$\ddot{y}(t) = \dot{u}(t)\exp\left\{\int_{t_1}^t u(s)ds\right\} + u^2(t)\exp\left\{\int_{t_1}^t u(s)ds\right\}. \tag{17}$$

Substitute (17) into (16) we obtain

$$\exp\left\{\int_{t_1}^{t} u(s)ds\right\}\left[\dot u(t)+u^2(t)+\sum_{k=1}^{r}{}' a_k(t)u(h_k(t))\exp\left\{-\int_{h_k(t)}^{t}u(s)ds\right\}+\right.$$
$$\left.\sum_{k=1}^{m}{}' b_k(t)\exp\left\{-\int_{g_k(t)}^{t}u(s)ds\right\}\right]+\sum_{k=1}^{r}{}'' a_k(t)\dot y(h_k(t))+ \tag{18}$$
$$\sum_{k=1}^{m}{}'' b_k(t)y(g_k(t))\le 0,$$

where the sums $\sum''$ contain such terms that $t_0 \le h_k(t) < t_1$ or $t_0 \le g_k(t) < t_1$, respectively. For $t \ge t_0$ $y(t) \ge 0$, $\dot y(t) \ge 0$, $a_k(t) \ge 0$, $b_k(t) \ge 0$, consequently, the last two terms in (18) are positive. Therefore (18) implies inequality (6). □

In the case $h_k(t) \equiv t$ as a corollary of Theorems 1 and 2 we may obtain necessary and sufficient nonoscillation conditions. To this end consider the following equation

$$\ddot x(t)+a(t)\dot x(t)+\sum_{k=1}^{m} b_k(t)x(g_k(t))=0,\ t\ge 0, \tag{19}$$

and delay differential inequality

$$\ddot y(t)+a(t)\dot y(t)+\sum_{k=1}^{m} b_k(t)y(g_k(t))\le 0,\ t\ge 0. \tag{20}$$

COROLLARY 2. *Suppose* $a(t) \ge 0$, $b_k(t) \ge 0$, $k = 1,\dots,m$. *Then for equation (19) the following statements are equivalent:*

1) There exists $t_1 \ge 0$ *such that inequality (20) has a positive solution with a nonnegative derivative for* $t > t_1$.

2) There exists $t_2 \ge 0$ *such that the inequality*

$$\dot u(t)+u^2(t)+a(t)u(t)+\sum_{k=1}^{m}{}' b_k(t)\exp\left\{-\int_{g_k(t)}^{t}u(s)ds\right\}\le 0 \tag{21}$$

has a nonnegative locally absolutely continuous $[t_2,\infty)$ *solution, where the sum* $\sum'$ *contains only the terms for which* $g_k(t) \ge t_2$.

3) There exists $t_3 \ge 0$ *such that* $X(t,s) > 0$, $X'_t(t,s) \ge 0$, $t > s \ge t_3$.

4) There exists $t_4 \ge 0$ *such that equation (19) has a positive solution with nonnegative derivative for* $t > t_4$.

Remark. For equations without middle term ($a(t) \equiv 0$) this result was obtained in [10].

4. Comparison Theorems. Theorem 1 can be employed for comparison of oscillation properties. To this end consider together with equation (1) the following equation

$$\ddot{x}(t) + \sum_{k=1}^{r} \bar{a}_k(t)\dot{x}(h_k(t)) + \sum_{k=1}^{m} \bar{b}_k(t)x(\bar{g}_k(t)) = 0, \; t \geq 0. \tag{22}$$

Suppose (a1) and (a2) hold for equation (22) and denote by $Y(t,s)$ a fundamental function of this equation.

THEOREM 3. *Suppose $a_k(t) \geq 0$, $b_k(t) \geq 0$, and the conditions of Theorem 1 hold for some $t_0 \geq 0$. If*

$$\bar{a}_k(t) \leq a_k(t), \; \bar{b}_k(t) \leq b_k(t), \bar{g}_k(t) \leq g_k(t), \; t \geq t_0, \tag{23}$$

then equation (22) has a positive solution with a nonnegative derivative for $t > t_0$ and $Y(t,s) > 0$, $Y_t^{'}(t,s) \geq 0$, $t > s > t_0$.

Proof. By the assumptions of Theorem 1 there exists a nonnegative solution u of the inequality (6) for $t \geq t_0$. Inequalities (23) yield that u is also a solution of the inequality

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{r}{}' \bar{a}_k^{+}(t)u(h_k(t)) \exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\} +$$

$$\sum_{k=1}^{m}{}' \bar{b}_k^{+}(t) \exp\left\{-\int_{\bar{g}_k(t)}^{t} u(s)ds\right\} \leq 0, \; t \geq t_0,$$

where the sums contain only terms for which $h_k(t) \geq t_0$ or $\bar{g}_k(t) \geq t_0$, correspondingly. By a comparison theorem (Theorem 2 in [14]), applied to equation (7) and the equation

$$\dot{z}(t) + u(t)z(t) + \sum_{k=1}^{r} \bar{a}_k(t)z(h_k(t)) = 0,$$

the fundamental function of the last equation is positive.

Hence Theorem 1 implies all the statements of this theorem. □

COROLLARY 3. *If $a(t) \geq 0, b_k(t) \geq 0$ and an ordinary differential equation*

$$\ddot{y}(t) + a(t)\dot{y}(t) + \sum_{k=1}^{m} b_k(t)y(t) = 0$$

is nonoscillating, then equation (19) is also nonoscillating for any $g_k(t) \leq t$.

Let us compare now the solutions of problem (2), (3) and the following one

$$\ddot{y}(t)+\sum_{k=1}^{r}\bar{a}_k(t)\dot{y}(h_k(t))+\sum_{k=1}^{m}\bar{b}_k(t)y(g_k(t))=r(t),\ t\geq t_0, \tag{24}$$

$$y(t)=\bar{\varphi}(t),\ \dot{y}(t)=\bar{\psi}(t),\ t<t_0,\ y(t_0)=y_0,\ \dot{y}(t_0)=y_0'. \tag{25}$$

Denote by $x(t)$ the solution of (2), (3), by $y(t)$ and $Y(t,s)$ the solution and the fundamental function, respectively, of (24), (25).

THEOREM 4. *Suppose all the conditions of Theorem 1 hold ,* $x(t)>0$, $\dot{x}(t)\geq 0,\ t>t_0$ *and* $a_k(t)\geq\bar{a}_k(t)\geq 0,\ b_k(t)\geq\bar{b}_k(t)\geq 0,\ r(t)\geq f(t)$; $\varphi(t)\geq\bar{\varphi}(t),\ \psi(t)\geq\bar{\psi}(t),\ t<t_0;\ y_0=x_0,\ y_0'\geq x_0'$. *Then* $y(t)\geq x(t),\ t\geq t_0$ *and* $Y(t,s)\geq X(t,s)>0,\ t>s>t_0$.

Proof. Denote by u a nonnegative solution of inequality (6). The inequalities $a_k(t)\geq\bar{a}_k(t), b_k(t)\geq\bar{b}_k(t)$ yield that the function u is also a solution of the inequality, corresponding to (6) for equation (24) and the fundamental function of the equation corresponding to (7) is positive [14]. Hence by Theorem 1 $Y(t,s)>0,\ t>s>t_0$.

Rewrite (2) in the form

$$\ddot{x}(t)+\sum_{k=1}^{r}\bar{a}_k(t)\dot{x}(h_k(t))+\sum_{k=1}^{m}\bar{b}_k(t)x(g_k(t))=$$

$$-\sum_{k=1}^{r}[a_k(t)-\bar{a}_k(t)]\dot{x}(h_k(t))-\sum_{k=1}^{m}[b_k(t)-\bar{b}_k(t)]x(g_k(t))+f(t).$$

Hence (see (5)) for the solutions of (2), (3) and (24), (25) we have

$$x(t)=y_1(t)x_0+Y(t,t_0)x_0'-\sum_{k=1}^{r}\int_{t_0}^{t}Y(t,s)[a_k(s)-\bar{a}_k(s)]\dot{x}(h_k(s))ds-$$

$$\sum_{k=1}^{m}\int_{t_0}^{t}Y(t,s)[b_k(s)-\bar{b}_k(s)]x(g_k(s))ds-\sum_{k=1}^{r}\int_{t_0}^{t}Y(t,s)\bar{a}_k(s)\psi(h_k(s))ds-$$

$$\sum_{k=1}^{m}\int_{t_0}^{t}Y(t,s)\bar{b}_k(s)\varphi(g_k(s))ds+\int_{t_0}^{t}Y(t,s)f(s)ds,$$

$$y(t) = y_1(t)y_0 + Y(t,t_0)y_0' - \sum_{k=1}^{r} \int_{t_0}^{t} Y(t,s)\bar{a}_k(s)\bar{\psi}(h_k(s))ds -$$

$$\sum_{k=1}^{m} \int_{t_0}^{t} Y(t,s)\bar{b}_k(s)\bar{\varphi}(g_k(s))ds + \int_{t_0}^{t} Y(t,s)r(s)ds,$$

where $\varphi(g_k(s)) = \psi(h_k(s)) = \bar{\varphi}(g_k(s)) = \bar{\psi}(h_k(s)) = 0$, if $h_k(s) > t_0, g_k(s)) > t_0$ and $x(g_k(s)) = 0, \dot{x}(h_k(s)) = 0$ if $g_k(t) < t_0$ or $h_k(s) < t_0$.

Therefore $y(t) \geq x(t) > 0,\ t > t_0$.

Comparing the solutions of equation (4) and the corresponding equation for $Y(t,s)$ we see that $Y(t,s) \geq X(t,s),\ t > s > t_0$, which completes the proof. □

If in (2), (3) $b_k(t) \leq 0, h_k(t) \equiv t$ we can obtain a stronger comparison result. To this end consider the following two equations

$$\ddot{x}(t) + a(t)\dot{x}(t) - \sum_{k=1}^{m} b_k(t)x(g_k(t)) = f(t),\ t \geq t_0,$$

(26) $$x(t) = \varphi(t), t < t_0; x(t_0) = x_0, \dot{x}(t_0) = x_0',$$

$$\ddot{y}(t) + \bar{a}(t)\dot{y}(t) - \sum_{k=1}^{m} \bar{b}_k(t)y(g_k(t)) = r(t),\ t \geq t_0,$$

(27) $$y(t) = \bar{\varphi}(t), t < t_0; y(t_0) = y_0, \dot{y}(t_0) = y_0'.$$

Denote by $x(t), y(t)$ and $X(t,s), Y(t,s)$ solutions and fundamental functions of (26) and (27), respectively.

THEOREM 5. *Suppose*

$$a(t) \geq \bar{a}(t), \bar{b}_k(t) \geq b_k(t) \geq 0,$$

$$\bar{\varphi}(t) \geq \varphi(t) \geq 0, r(t) \geq f(t) \geq 0, y_0 \geq x_0 > 0, y_0' \geq x_0' \geq 0.$$

Then $y(t) \geq x(t) > 0, t \geq t_0$ *and* $Y(t,s) \geq X(t,s) > 0,\ t > s > t_0$.

Proof. Inequality (6) for equation (26) is

(28) $$\dot{u}(t) + u^2(t) + a^+(t)u(t) \leq 0.$$

Straightforward calculations imply that the function

$$u(t)=\frac{\exp\{-\int_{t_0}^{t}a^+(s)ds\}}{\int_{t_0}^{t}\exp\{-\int_{t_0}^{s}a^+(\tau)d\tau\}ds}$$

is a nonnegative solution of the equation corresponding to inequality (28).

Then for equation (26) and similar for equation (27) we have $X(t,s)>0, Y(t,s)>0,\ t>s>t_0$.

Compare now solution x of (26) and solution z of an equation

$$\ddot{z}(t)+a(t)\dot{z}(t)=f(t),\quad z(t_0)=x_0,\quad \dot{z}(t_0)=x_0'. \tag{29}$$

By straightforward calculations we have that $z(t)>0, t\geq t_0$.

Rewrite equation (29) in the form

$$\ddot{z}(t)+a(t)\dot{z}(t)-\sum_{k=1}^{m}b_k(t)z(g_k(t))=-\sum_{k=1}^{m}b_k(t)z(g_k(t))+f(t).$$

Then for solutions of equations (26) and (29) we have

$$x(t)=x_1(t)x_0+X(t,t_0)x_0'+\sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)b_k(s)\varphi(g_k(s))ds+\int_{t_0}^{t}X(t,s)f(s)ds,$$

$$z(t)=x_1(t)x_0+X(t,t_0)x_0'+\sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)b_k(s)\varphi(g_k(s))ds$$

$$-\sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)b_k(s)z(g_k(s))ds+\int_{t_0}^{t}X(t,s)f(s)ds.$$

Hence $x(t)\geq z(t)>0, t>t_0$ and as a consequence $x_1(t)>0$. Similarly, $y(t)>0,\ y_1(t)>0$.

Now the same computation as in the previous theorem imply the assertion of the theorem. □

COROLLARY 4. *Suppose $b_k(t)\leq 0$, x and y are solutions of equation (19) and inequality (20), respectively, such that $x(t)=y(t),\ t\leq t_0; \dot{x}(t_0)=\dot{y}(t_0)$. Then $x(t)\geq y(t), t\geq t_0$.*

The proof is based on solution representation (5) and inequality $X(t,s)>0, t>s>0$.

5. Explicit Nonoscillation Conditions. We will employ Theorem 1 for obtaining explicit sufficient conditions of nonoscillation.

THEOREM 6. *Suppose the following conditions hold*

$$\frac{1}{2}\sum_{k=1}^{r} a_k^+(t)\sqrt{\frac{t^3}{h_k(t)}} + \sum_{k=1}^{m} b_k^+(t)\sqrt{t^3 g_k(t)} \le 1/4, \ t \ge t_0, \tag{30}$$

$$\sum_{k=1}^{m} \int_{h(t)}^{t} a_k^+(\tau)\sqrt{\frac{\tau}{h_k(\tau)}}d\tau \le 1/e, \ t \ge t_0, \tag{31}$$

where $h(t) = \min_k\{h_k(t)\}$.

Then equation (1) has a positive solution with a nonnegative derivative for $t > t_0$.

Proof. Let $u = \frac{1}{2t}$. Then inequality (6) takes a form

$$-\frac{1}{4t^2} + \frac{1}{2}\sum_{k=1}^{r} \frac{a_k^+(t)}{h_k(t)}\sqrt{\frac{h_k(t)}{t}} + \sum_{k=1}^{m} b_k^+(t)\sqrt{\frac{g_k(t)}{t}} \le 0,$$

and is equivalent to an inequality (30).

Equation (7) with $u = \frac{1}{2t}$ is

$$\dot{z}(t) + \frac{1}{2t}z(t) + \sum_{k=1}^{r} a_k^+(t)z(h_k(t)) = 0. \tag{32}$$

Substitution $z(t) = \frac{v(t)}{\sqrt{t}}$ in equation (32) we obtain

$$\dot{v}(t) + \sum_{k=1}^{r} a_k(t)\sqrt{\frac{t}{h_k(t)}}v(h_k(t)) = 0. \tag{33}$$

Condition (31) yields [15] that the fundamental function of (33), and therefore of (32), is positive. Theorem 1 implies the statement of the theorem. □

COROLLARY 5. *Suppose the following condition holds*

$$\frac{1}{2}a^+(t)t + \sum_{k=1}^{m} b_k^+(t)\sqrt{t^3 g_k(t)} \le 1/4, \ t \ge t_0.$$

Then equation (19) has a positive solution with a nonnegative derivative for $t > t_0$.

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NONANTICIPATING SELECTORS OF SET-VALUED MAPPINGS AND ITERATED PROCEDURES*

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Abstract. An analog of the known method of programmed iterations (MPI) [1]–[10] is considered. This method is used [1]–[6],[8, 9] both for solving differential games (DG) (see monographs [11]–[14] and for an investigation of other problems [7, 10, 15, 16]. In particular, by MPI constructing solving quasistrategies [1]–[6], [8],[17]–[19] was realized (in this connection, see generalized constructions [1]–[6]). But, the above-mentioned construction of solving quasistrategies requires the preliminary determination of some (very important) auxiliary objects; of course, these objects (the value of DG, the stable bridge) represent a very essential interest for solving the corresponding DG. Now, it is important to note the following fact: under the employment of constructions [1]–[6],[8] the solving quasistrategy is defined in terms of the above-mentioned (auxiliary) objects. In contrast to the above-mentioned approach in the given paper a "direct" method of constructing set-valued "quasistrategies" on the basis of iterations is considered.

Key Words. Nonanticipating selector, set-valued mapping, iterated procedure, quasistrategy.

1. Introduction. In many concrete problems it is important to construct an useful control in the form of the nonanticipating response to the realization of a priori indefinite factors. Sometimes, it is advisable to consider the motion of a system in the form of the above-mentioned response. We consider the unit "pointer" $I \triangleq [0,1[$ and the set $\mathbf{C}$ of all piecewise constant and continuous from the right (real-valued) mappings $c(\cdot): I \longrightarrow [-1,1]$. We call an operator $\alpha: \mathbf{C} \longrightarrow \mathbf{C}$ nonanticipating mapping in the case, when

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$\forall v_1(\cdot) \in \mathbf{C}\ \forall v_2(\cdot) \in \mathbf{C}\ \forall t \in]0,1[:$

$$((v_1(\cdot) \mid [0,t[) = (v_2(\cdot) \mid [0,t[)) \Longrightarrow$$

$$((\alpha(v_1(\cdot)) \mid [0,t[) = (\alpha(v_2(\cdot)) \mid [0,t[). \tag{1.1}$$

We use in (1.1) the natural notion of a restriction of a function to a nonempty subset of domain of definition. The definition (1.1) is analogous to the corresponding one of [17] and many other investigations. It is natural to introduce a set-valued variant of (1.1). In this case we consider in the capacity of α a mapping from $\mathbf{C}$ into the family of all subsets of $\mathbf{C}$. We obtain a set-valued variant of pseudo-strategies of [20]. In addition, for (1.1) it is possible to choose a set-valued analog. Namely, under conditions of (1.1) it is possible to introduce the following requirement on the choice of the considered set-valued mapping α:

$$\begin{aligned}((v_1(\cdot) \mid [0,t[) = (v_2(\cdot) \mid [0,t[)) \Longrightarrow (\{(u(\cdot) \mid [0,t[) :\\ u(\cdot) \in \alpha(v_1(\cdot))\} = \{(u(\cdot) \mid [0,t[) : u(\cdot) \in \alpha(v_2(\cdot))\}).\end{aligned} \tag{1.2}$$

Of course, here $\alpha(v(\cdot))$ is a subset of $\mathbf{C}$ under $v(\cdot) \in \mathbf{C}$. The property (1.2) defines an important case of the above-mentioned set-valued "pseudo-strategies" (see [20]). Here we have a set-valued quasistrategy (we use the terminology of [17]). In many problems it is possible to consider an "arbitrary" set-valued mapping $\mathcal{C}$ operating on $\mathbf{C}$; namely, $\mathcal{C}(v(\cdot)) \subset \mathbf{C}$ under $v(\cdot) \in \mathbf{C}$. The mapping $\mathcal{C}$ corresponds to a goal of the investigated problem. In addition, $v(\cdot)$ is an infinite-dimensional parameter influencing on the attainment of the above-mentioned goal. We interpret $\mathcal{C}(v(\cdot))$ as the set of controls $u(\cdot)$ resolving the problem for the parameter $v(\cdot)$. But, it is possible that a priory $v(\cdot)$ is not known. The values $v(t)$ become known only with respect to an development of $v(\cdot)$ as a time function. Therefore, the control $u(\cdot) \in \mathcal{C}(v(\cdot))$ is formed as a reaction on $v(\cdot)$ with the validity of requirements analogous to (refl.1) and (refl.2). As a result, we obtain the problem of constructing nonanticipating (set-valued) selectors of the given set-valued mapping.

2. General definitions. We use the set-theoretic symbolics including quantors, propositional connectives and other current stipulations; denote by $\triangleq$ the equality by definition and use the natural abbreviation *def*. If A and B are sets, denote by B^A the set of all mappings from A into B; for each choice of a mapping $f \in B^A$ and a set C, $C \subset A$, we denote by $(f \mid C)$ the natural restriction of the mapping f to the set C (see [21, Ch. I]). For the designation of superposition (of mappings) we use the natural symbol $\circ$. It is advisable to remind some notions of general topology. Namely, we use the usual notion of a subspace of a topological space (TS). Moreover, below we use the natural notions of compact, countably compact and sequentially compact subsets in arbitrary TS; in these definitions we operate on the corresponding subspace of the given TS (see [22]–[24]; in particular, see [24, p.239]). We use nets and the Moore-Smith convergence (see [22]–[24]). It is convenient to accept the following stipulation in the connection with the designations for nets. Namely, we denote each net as the triplet for which the first elements compose a nonempty directed set and the third element is a mapping defined on the above-mentioned (directed) set; if the last mapping has own values in the given set H, we call the considered net as a net in H. Finally, if $(D, \preceq, h)$ is a net in the set H and H is equipped with the topology $\mathcal{T}$, then we call $(D, \preceq, h)$ as a net in TS $(H, \mathcal{T})$. If $(D, \preceq, h)$ is a net in TS $(H, \mathcal{T})$ and $x \in H$, then we denote by $(D, \preceq, h) \xrightarrow{\mathcal{T}} x$ the Moore-Smith convergence [22, 23] of $(D, \preceq, h)$ to x in the sense of topology $\mathcal{T}$. For the convergence of a sequence $(x_i)_{i \in \mathcal{N}}$ in TS $(H, \mathcal{T})$ to a point $x \in H$ we use the more traditional designation $(x_i)_{i \in \mathcal{N}} \xrightarrow{\mathcal{T}} x$. Of course, in the last case we have the following particular case: in the capacity of $(D, \preceq)$ we use the positive integers $\mathcal{N} \triangleq \{1; 2; ...\}$ with the ordinary order $\leq$ of $\mathcal{N}$ (clearly, that $\leq$ is a direction of $\mathcal{N}$). If $(\Delta, \ll)$ and $(D, \preceq)$ are nonempty directed sets, then we denote by $(Isot)[\Delta; \ll; D; \preceq]$ the set of all operators $l \in D^{\Delta}$ such that

$$(2.1)\qquad \begin{gathered}(\forall d \in D\ \exists \delta \in \Delta :\ d \preceq l(\delta))\&(\forall \delta_1 \in \Delta\ \forall \delta_2 \in \Delta :\\ (\delta_1 \ll \delta_2) \Longrightarrow (l(\delta_1) \preceq l(\delta_2))).\end{gathered}$$

In (2.1) the isotone mappings of directed sets are defined. This construction is coordinated with the general notion of a compactness (see [22, 23]). It is important to note the following particular case of (2.1): $(D, \preceq) = (\mathcal{N}, \leq)$. In this connection we suppose for each directed set $(D, \preceq)$, $D \neq \emptyset$, that $(isot)[D; \preceq] \triangleq (Isot)[D; \preceq; \mathcal{N}; \leq]$. The last definition is useful under operations with countable compact sets. The corresponding property is realized by the procedure that is analogous to the construction of the passage to a subnet by an isotone mapping considered in [22, Ch. 2]. But, in the case of the consideration of countable compact sets we apply elements of $(isot)[D; \preceq]$ in the capacity of isotone mappings of [22]. Below we use the following designation $\mathcal{N}_0 \triangleq \mathcal{N} \cup \{0\}$. So, $\mathcal{N}_0 = \{0; 1; 2; ...\}$. In the following the natural notion of the power of a mapping is used. If H is a nonempty set, then we denote by $\mathbb{I}_H$ the identical operator from H^H: $\mathbb{I}_H \in H^H$ and $\mathbb{I}_H(h) \triangleq h$ under $h \in H$. If H is a nonempty set and $\mathbf{T} \in H^H$, then we define the sequence

$$(\mathbf{T}^k)_{k\in\mathcal{N}_0} : \mathcal{N}_0 \longrightarrow H^H \tag{2.2}$$

by the following natural conditions: $(\mathbf{T}^0 \triangleq \mathbb{I}_H)\&(\forall s \in \mathcal{N} : \mathbf{T}^s = \mathbf{T} \circ \mathbf{T}^{s-1})$. In terms of (2.2) it is possible to realize the basic constructions of MPI [1]–[8]. In the given investigation we operate by powers of mappings defined in terms of (2.2). For some specific kinds of such mappings we introduce below the infinite power. Namely, we use the above-mentioned operation of the raising to the infinite power for set-valued mappings.

3. Nonanticipating set-valued mappings and the problem of a selection of arbitrary set-valued mappings. We consider an abstract construction of the determination of nonanticipating set-valued mappings. The natural concrete variant of this procedure is the iterated construction of [1]–[8]. But, it is possible to represent other concrete variants of the realization of MPI.

We fix: 1) nonempty sets X and Υ; 2) a nonempty family $\mathcal{X}$ of nonempty subsets of X; 3) TS (Y,τ), $Y \neq \emptyset$; 4) a nonempty set Z, $Z \subset Y^X$; 5) a nonempty set Ω, $\Omega \subset \Upsilon^X$. Suppose that $\otimes^X(\tau)$ is the natural topology of the set Y^X corresponding to the Tychonoff product of samples of TS (Y,τ) with the index set X. Consider the set Z as a subspace of the Tychonoff product $(Y^X, \otimes^X(\tau))$. We equip the set Z with the topology θ induced [22, Ch. 1] from $(Y^X, \otimes^X(\tau))$. Denote by $\mathbf{X}$ (by $\mathbb{Z}$) the family of all subsets of X (of Z). Let $X \triangleq \mathbf{X} \setminus \{\emptyset\}$. Then $\mathcal{X} \subset X$. In addition, $\mathbb{Z}^\Omega$ defines the basic kind of set-valued mappings considered below. Moreover, we use "partial" set-valued mappings. Let Σ be the family of all subsets of Ω. Denote by Σ_0 the family of all nonempty sets of Σ : $\Sigma_0 \triangleq \Sigma \setminus \{\emptyset\}$. We consider elements of $\mathbb{Z}^T$ (where $T \in \Sigma_0$) as partial set-valued mappings. We use operators $\mathcal{H} \in \mathbb{Z}^T$ under $T \neq \Omega$ for constructing a parallel version of MPI.

Example 3.1. Consider (see section 1) the known concrete variant of the general setting. This variant corresponds to the case of control problems. Namely, in this example suppose that $X \triangleq I = [0,1[$ and $\mathcal{X} \triangleq \{[0,t[:\ t \in]0,1[\}$. So, $(X,\mathcal{X})$ is realized. Suppose that (in this example) $\Upsilon \triangleq [-1,1]$ and $\Omega \triangleq \mathbf{C}$ (see section 1). We interprete functions $\omega \in \Omega$ as unknown controls or controls of "opponent". Moreover, we suppose that in the given case (Y,τ) is the real line $\mathbb{R}$ with the "ordinary" $|\cdot|$-topology $\tau_{\mathbb{R}}$. In the capacity of Z we consider (in this example) the set $\mathbf{C}_0(I)$ of all uniformly continuous functions from X into $\mathbb{R}$. We introduce the set $\mathcal{U}$ of all Borel functions U from X into $\mathbb{R}$ with the following property: for each $t \in X = I$ the inequality $| U(t) | \leq 1$ takes place. Then, we suppose that principle of constructing mappings from Ω into $\mathbb{Z}$ is defined by integration of sum of controls $U \in \mathcal{U}$ and $V \in \Omega$. This corresponds to the simplest version of DG. So, we define under $V \in \Omega$ the set $\mathbb{X}(V)$ of all functions

$$t \longmapsto \int\limits_{[0,t[} U(\xi)\lambda(d\xi) + \int\limits_{0}^{t} V(\xi)d\xi : \ [0,1[\longrightarrow \mathbb{R}$$

(λ is the restriction of Lebesque measure to σ-algebra of Borel subsets of $X = [0,1[)$, where $U \in \mathcal{U}$. Of course, $\mathbb{X}(V) \subset Z = \mathbf{C}_0(I)$. Fix functions $\alpha_* : I \longrightarrow \mathbb{R}$ and $\beta_* : I \longrightarrow \mathbb{R}$. If $V \in \Omega$, then we suppose that

$$\mathbb{X}_0(V) \triangleq \{x(\cdot) \in \mathbb{X}(V) \mid \forall \xi \in I : \ \alpha_*(\xi) \leq x(\xi) \leq \beta_*(\xi)\}.$$

We obtain a compactum in the sense of the metric of the uniform convergence in Z. So, in the form of $\mathbb{X}_0(\cdot)$ we have a compact-valued mapping on Ω.

Returning to the general case we note that in the following we consider two nonempty functional sets Ω and Z. In addition, X is the common domain of all functions of Ω and Z. Sets of $\mathcal{X}$ are used for the determination of nonanticipating reactions of type $z = z(\omega)$, where $\omega \in \Omega$ and $z \in Z$. The corresponding strict definition is realized in terms of restrictions of the above-mentioned functions to sets of $\mathcal{X}$.

Remark 3.1. Consider the case connected with (1.1),(1.2). Namely, in this case it is possible to use the following (natural for control problems) stipulation: $X = I$; $\Upsilon = Y = [-1,1]$; τ is (in this example) the natural $| \cdot |$-topology of $[-1,1]$; $\Omega = Z = \mathbf{C}$; $\mathcal{X} = \{[0,t[: \ t \in]0,1[\}$. Of course, in the given case all following constructions are coordinated with (1.1) and (1.2); moreover, see [17, 20].

Now we return to the general case. Introduce new definitions. If $T \in \Sigma_0$, $\omega \in \Omega$ and $A \in X$, then we suppose

$$(3.1) \qquad (Ge)[T;\omega \mid A] \triangleq \{t \in T \mid (\omega \mid A) = (t \mid A)\};$$

in the case $\omega \in T$ we have in (3.1) the nonempty set. Moreover, $\forall \omega \in \Omega \ \forall A \in X$:

$$(3.2) \quad \Omega_0(\omega \mid A) \triangleq (Ge)[\Omega;\omega \mid A] = \{\tilde{\omega} \in \Omega \mid (\omega \mid A) = (\tilde{\omega} \mid A)\} \in \Sigma_0.$$

If the triplet (T, ω, A) corresponds to the conditions defining (3.1), then

$$(3.3) \qquad (Ge)[T;\omega \mid A] = T \cap \Omega_0(\omega \mid A).$$

In (3.1) - (3.3) we have germs of elements of Ω; by these germs we introduce nonanticipating mappings and a specific operator acting in the set $\mathbb{Z}^T$, where $T \in \Sigma_0$. The case $T = \Omega$ is basic. If $T \in \Sigma_0$, then we suppose that

$$\gamma_T : \mathbb{Z}^T \longrightarrow \mathbb{Z}^T \tag{3.4}$$

is defined by the following condition; namely, $\forall C \in \mathbb{Z}^T \; \forall \omega \in T$:

$$\gamma_T(C)(\omega) \triangleq \{f \in C(\omega) \mid \forall A \in \mathcal{X} \; \forall \tilde{\omega} \in (Ge)[T; \omega \mid A] \; \exists \tilde{f} \in C(\tilde{\omega}) : \; (f \mid A) = (\tilde{f} \mid A)\}. \tag{3.5}$$

In terms of (3.4),(3.5) we introduce the general notion of a nonanticipating mapping. If $T \in \Sigma_0$, then $N[T] \triangleq \{C \in \mathbb{Z}^T \mid C = \gamma_T(C)\}$ is the set of all nonanticipating mappings with the domain T. Suppose $\forall T \in \Sigma_0 \; \forall \alpha \in \mathbb{Z}^T$:

$$N_0[T; \alpha] \triangleq \{C \in N[T] \mid \forall \omega \in T : \; C(\omega) \subset \alpha(\omega)\}. \tag{3.6}$$

In (3.6) we consider the set of all nonanticipating mappings that are set-valued selectors of the given (set-valued) mapping α. We have in the form $\Gamma \triangleq \gamma_\Omega$ an element of the set $\mathcal{Z}$ of all operators acting in the set $\mathbb{Z}^\Omega$. So, for $C \in \mathbb{Z}^\Omega$ and $\omega \in \Omega$ we obtain in the form of $\Gamma(C)(\omega)$ the set of all $f \in C(\omega)$ such that

$$\forall A \in \mathcal{X} \; \forall \tilde{\omega} \in \Omega_0(\omega \mid A) \; \exists \tilde{f} \in C(\tilde{\omega}) : \; (f \mid A) = (\tilde{f} \mid A). \tag{3.7}$$

The property (3.7) is used in the basic definition of nonanticipating set-valued global mappings. The mapping $\Gamma : \mathbb{Z}^\Omega \longrightarrow \mathbb{Z}^\Omega$ we use for the iterated determination of elements of the set

$$\mathbb{N} \triangleq \{C \in \mathbb{Z}^\Omega \mid \Gamma(C) = C\}. \tag{3.8}$$

As a corollary, we have the following basic representation for nonanticipating selectors of a set-valued mapping. Namely, $\forall \alpha \in \mathbb{Z}^\Omega : \; \mathbb{N}_0[\alpha] \triangleq N_0[\Omega; \alpha]$. But, now it is advisable to introduce basic structures of $\mathbb{Z}^\Omega$ used in the following. If $H_1 \in \mathbb{Z}^\Omega$ and $H_2 \in \mathbb{Z}^\Omega$, then suppose *def*:

$$(H_1 \sqsubseteq H_2) \Longleftrightarrow (\forall \omega \in \Omega : \; H_1(\omega) \subset H_2(\omega)). \tag{3.9}$$

We use in the following the notion of a monotonocity of Γ only in the sense of (3.9). Moreover, in the sequel we use the "ordinary" set-theoretical convergence of sets [25, Ch. I]: if H is a set, $(A_i)_{i \in \mathcal{N}}$ is a sequence of subsets of H and A is a subset of H, then $(A_i)_{i \in \mathcal{N}} \downarrow A$ denotes that A is the intersection of all sets A_i, $i \in \mathcal{N}$, and (moreover) $\forall k \in \mathcal{N}: \ A_{k+1} \subset A_k$. If $(\mathcal{C}_i)_{i\in\mathcal{N}}$ is a sequence in $\mathbb{Z}^\Omega$ and $\mathcal{C} \in \mathbb{Z}^\Omega$, then *def*:

$$(3.10) \qquad ((\mathcal{C}_i)_{i\in\mathcal{N}} \Downarrow \mathcal{C}) \Longleftrightarrow (\forall\omega \in \Omega: \ (\mathcal{C}_i(\omega))_{i\in\mathcal{N}} \downarrow \mathcal{C}(\omega)).$$

We equip $\mathbb{Z}^\Omega$ with the order (3.9) and with the convergence (3.10). So, we have the basic space $(\mathbb{Z}^\Omega, \sqsubseteq, \Downarrow)$. Returning to (3.8), we obtain that $\mathrm{N}_0[\alpha] = \{\mathcal{C} \in \mathrm{N} \mid \mathcal{C} \sqsubseteq \alpha\}$ under $\alpha \in \mathbb{Z}^\Omega$. By (3.3) the following useful property is used. Namely, for $\mathcal{C} \in \mathbb{Z}^\Omega$, $T \in \Sigma_0$ and $\omega \in T$ we have

$$(3.11) \qquad \Gamma(\mathcal{C})(\omega) \subset \gamma_T((\mathcal{C} \mid T))(\omega) \subset \mathcal{C}(\omega).$$

From (3.11) we obtain $\forall \mathcal{C} \in \mathrm{N} \ \forall T \in \Sigma_0: \ (\mathcal{C} \mid T) \in N[T]$. Moreover, we have $\forall a \in \mathbb{Z}^\Omega \ \forall \alpha \in \mathrm{N}_0[a] \ \forall T \in \Sigma_0$:

$$(3.12) \qquad (\alpha \mid T) \in N_0[T; (a \mid T)].$$

In the connection with (3.12) we note the important property: under $a \in \mathbb{Z}^\Omega$ and $T \in \Sigma_0$ it is possible that in the set $N_0[T; (a \mid T)]$ singular elements are contained. It is impossible to consider these elements as mappings on the left-hand side of (3.12).

Example 3.2. Return to the particular case connected with (1.1), (1.2). We use the concrete definition of Remark 3.1. Consider $a \in \mathbb{Z}^\Omega = \mathbb{Z}^{\mathbf{C}}$ such that under $v_*(\cdot) \in \mathbf{C}$ the $v_*(\cdot)$-image $a(v_*(\cdot))$ is the set of all constant controls $u_*(\cdot) \in \mathbf{C}$, $u_*(t) \equiv u^* \in [-1, 1]$, for which

$$\int_0^1 u_*(t)dt + \int_0^1 v_*(t)dt = 0.$$

Then $a(v(\cdot)) \neq \emptyset$ under $v(\cdot) \in \mathbf{C}$. If $\alpha \in \mathbb{Z}^\Omega = \mathbb{Z}^{\mathbf{C}}$ has the property (1.2) and (moreover) $\alpha(v(\cdot)) \subset a(v(\cdot))$ under each $v(\cdot) \in \mathbf{C}$, then α is the "empty-valued" mapping. Namely, under the above-mentioned conditions the property $\alpha(v(\cdot)) \equiv \emptyset$ takes place. Fix $v_0(\cdot) \in \mathbf{C}$ and introduce one-element set $T_0 \triangleq \{v_0(\cdot)\}$. Then $a_0 \triangleq (a \mid T_0)$ is the trivial nonanticipating (on T_0) mapping with nonempty values $a_0(t)$, $t \in T_0$. Consequently, it is impossible to obtain a_0 in the form (3.12).

4. Some general properties. We consider the basic space $(\mathbb{Z}^\Omega, \sqsubseteq, \Downarrow)$ and the mapping $\Gamma \in \mathcal{Z}$. Denote by Z the family of all sets $\mathbf{H}$, $\mathbf{H} \subset \mathbb{Z}^\Omega$, such that $\forall \mathcal{H} \in \mathbf{H}: \ \Gamma(\mathcal{H}) \in \mathbf{H}$. Of course, elements of Z are Γ-invariant subspaces of $\mathbb{Z}^\Omega$ and only they. If R is a subfamily of Z, then the union of all sets $R \in R$ is an element of Z; in the case $R \neq \emptyset$ the intersection of all sets $R \in R$ in an element of Z. Finally, $\emptyset \in Z$ and $\mathbb{Z}^\Omega \in Z$.

Denote by $\mathbf{Z}$ the family of all sets H, $H \subset \mathbb{Z}^\Omega$, such that for each sequence $(H_i)_{i \in \mathcal{N}}$ in H and for each mapping $H \in \mathbb{Z}^\Omega$ the implication

$$((H_i)_{i \in \mathcal{N}} \Downarrow H) \Longrightarrow (H \in H)$$

takes place. Then it is possible to call $H \in \mathbf{Z}$ a closed set in the space $(\mathbb{Z}^\Omega, \Downarrow)$; the last space is the structure equipped with the convergence of sequences. Of course, $\emptyset \in \mathbf{Z}$, $\mathbb{Z}^\Omega \in \mathbf{Z}$; if R is a nonempty subfamily of $\mathbf{Z}$, then the intersection of all sets $R \in R$ is an element of $\mathbf{Z}$. For an investigation of Z and $\mathbf{Z}$ in terms of the mapping Γ we consider a topological construction connected with (Z, θ).

So, let: $\mathbb{F}$ be the family of all closed in (Z, θ) subsets of Z (of course, $\mathbb{F} \subset \mathbb{Z}$); $\mathcal{F}$ be the family of all sequentially closed [26] in (Z, θ) subsets of Z; $\mathbb{K}$ be the family of all compact [23, Ch. 3] in (Z, θ) subsets of Z; $\mathcal{K}$ be the family of all sequentially compact [24, p. 239] in (Z, θ) subsets of Z; $\mathbb{C}$ be the family of all countably compact [24, p. 239] in (Z, θ) subsets of Z. Moreover, we suppose $\mathbb{T} \triangleq \mathbb{C} \cap \mathbb{F}$, obtaining the family of all closed countable compact in (Z, θ) subsets of Z. Then $\mathbb{K} \cup \mathcal{K} \subset \mathbb{C}$. If (Y, τ) is a Hausdorff space, then (Z, θ) is a Hausdorff space and $\mathbb{K} \subset \mathbb{T}$; moreover, in this case (of a Hausdorff space (Y, τ)) the inclusion $\mathcal{K} \subset \mathcal{F}$ takes place.

PROPOSITION 4.1. *If (Y, τ) is a Hausdorff space, then $\mathbb{K}^\Omega \in Z$ and $\mathcal{K}^\Omega \in Z$.*

Proof. Let $U \in \mathbb{K}^\Omega$ and $\omega \in \Omega$. It is sufficient to establish the property $\Gamma(U)(\omega) \in \mathbb{F}$ (see (3.5),(3.7)). We use the known Birkhoff theorem about the representation of the closure operator (see [21]–[24]). Let $(D, \preceq, \varphi)$ be a net in $\Gamma(U)(\omega)$ and $f \in Z$. Moreover, let $(D, \preceq, \varphi) \xrightarrow{\theta} f$. Of course, $f \in U(\omega)$. Here we use two following properties: 1) $U(\omega) \in \mathbb{F}$; 2)$\varphi(d) \in U(\omega)$ under $d \in D$. Let $A \in \mathcal{X}$ and $\tilde{\omega} \in \Omega_0(\omega \mid A)$. We use the axiom of choice and (3.7). Let $\tilde{\varphi}$ be a mapping from D into $U(\tilde{\omega})$ for which $\forall d \in D$: $(\varphi(d) \mid A) = (\tilde{\varphi}(d) \mid A))$. But, $U(\tilde{\omega}) \in \mathbb{K}$. Therefore (see [22, Ch. 2]), it is possible to choose $\tilde{f} \in U(\tilde{\omega})$, a nonempty directed set $(\Delta, \ll)$ and a mapping $l \in (Isot)[\Delta; \ll; D; \preceq]$ for which the convergence $(\Delta, \ll, \tilde{\varphi} \circ l) \xrightarrow{\theta} \tilde{f}$ takes place. Then by known [22, 23] properties of Tychonoff products and relative topologies we have $\forall x \in X$:

$$(4.1)\quad ((\Delta, \ll, (\tilde{\varphi} \circ l)(\cdot)(x)) \xrightarrow{\tau} \tilde{f}(x))\&((\Delta, \ll, (\varphi \circ l)(\cdot)(x)) \xrightarrow{\tau} f(x)).$$

In (4.1) and below we use "ordinary" stipulations: $(\varphi o l)(\cdot)(x)$ and $(\tilde{\varphi} o l)(\cdot)(x)$ are mappings from D into $U(\omega)$ and $\tilde{U}(\omega)$ with the values $(\varphi \circ l)(\delta)(x)$ and

$(\tilde{\varphi} \circ l)(\delta)(x)$ respectively. We use (4.1) in the case $x \in A$. Then by the choice of $\tilde{\varphi}$ we have $(f \mid A) = (\tilde{f} \mid A)$, since (Y, τ) is a Hausdorff space. So, $\tilde{f} \in U(\tilde{\omega}) : \ (f \mid A) = (\tilde{f} \mid A)$. But, the choice of A and $\tilde{\omega}$ was arbitrary. Therefore, from (3.7) we have $f \in \Gamma(U)(\omega)$. Since the choice of $(D, \preceq, \varphi)$ and f also was arbitrary, we have $\Gamma(U)(\omega) \in \mathbb{F}$. The last property reduces to the property of a compactness of $\Gamma(U)(\omega)$. The statement $\mathbb{K}^\Omega \in Z$ has been established. The proof of the property $\mathcal{K}^\Omega \in Z$ is analogous (here the more simple "sequential" procedure is used that is in essence analogous to [4]–[6]).

It is possible easily to show that $\mathbb{T}^\Omega \in \mathbf{Z}$.

PROPOSITION 4.2. *If (Y, τ) is a T_1- space [22]–[24], $(\mathcal{C}_i)_{i \in \mathcal{N}}$ is a sequence in $\mathbb{T}^\Omega$ and $\mathcal{C} \in \mathbb{Z}^\Omega$, then*

$$((\mathcal{C}_i)_{i \in \mathcal{N}} \Downarrow \mathcal{C}) \Longrightarrow ((\Gamma(\mathcal{C}_i))_{i \in \mathcal{N}} \Downarrow \Gamma(\mathcal{C})).$$

Proof. Let $(\mathcal{C}_i)_{i \in \mathcal{N}}$ be a convergent (to $\mathcal{C}$) sequence in $\mathbb{T}^\Omega$ (see (3.10)). Since Γ is the monotone (in the sense of (3.9)) mapping, it is sufficient to establish that under $\omega \in \Omega$ the intersection of all sets $\Gamma(\mathcal{C}_i)(\omega)$, $i \in \mathcal{N}$, is a subset of $\Gamma(\mathcal{C})(\omega)$. Fix $\omega \in \Omega$. Let φ be an element of the intersection of all sets $\Gamma(\mathcal{C}_i)(\omega)$, $i \in \mathcal{N}$. Then $\varphi \in \mathcal{C}(\omega)$, since $\mathcal{C}(\omega)$ is the intersection of all sets $\mathcal{C}_i(\omega)$, $i \in \mathcal{N}$; see (3.10). Fix $A^* \in \mathcal{X}$ and $\omega^* \in \Omega_0(\omega \mid A^*)$. Choose $(\varphi_i^*)_{i \in \mathcal{N}}$ from the product of all sets $\mathcal{C}_i(\omega^*)$, $i \in \mathcal{N}$, with the property $\forall j \in \mathcal{N} : \ (\varphi \mid A^*) = (\varphi_j^* \mid A^*)$; in this connection see (3.7). In addition, $(\mathcal{C}_i(\omega^*))_{i \in \mathcal{N}}$ converges monotonically to $\mathcal{C}(\omega^*)$. As a corollary, $(\varphi_i^*)_{i \in \mathcal{N}}$ is the sequence in $\mathcal{C}_1(\omega^*) \in \mathbb{C}$. Therefore, it is possible to choose $\varphi^* \in \mathcal{C}_1(\omega^*)$, a nonempty directed set $(D, \preceq)$ and a mapping $l \in (isot)[D; \preceq]$ for which

$$(D, \preceq, (\varphi^*_{l(d)})_{d \in D}) \xrightarrow{\theta} \varphi^*. \tag{4.2}$$

Of course, from (4.2) the following property follows. Namely, $\forall x \in X$:

$$(D, \preceq, (\varphi^*_{l(d)}(x))_{d \in D}) \xrightarrow{\tau} \varphi^*(x).$$

By the basic property of T_1-space [22]–[24] we have the equality for restrictions: $(\varphi \mid A^*) = (\varphi^* \mid A^*)$. By properties of l we have $\varphi^* \in \mathcal{C}_n(\omega^*)$ under $n \in \mathcal{N}$ (here we use the following property: $\mathcal{C}_k(\omega^*) \in \mathbb{F}$, $k \in \mathcal{N}$). As a corollary, we obtain $\varphi^* \in \mathcal{C}(\omega^*)$, $(\varphi \mid A^*) = (\varphi^* \mid A^*)$. Since the choice of A^* and ω^* was arbitrary, we have $\varphi \in \Gamma(\mathcal{C})(\omega)$.

Note that in the case when (Y, τ) is a Hausdorff space, the property $\mathbb{K}^\Omega \in \mathbf{Z}$ takes place. From Proposition 4.2 the useful statement follows. Namely, we have

PROPOSITION 4.3 ([27]). *Let (Y, τ) be a Hausdorff space, $(C_i)_{i \in \mathcal{N}}$ be a sequence in $\mathbb{K}^\Omega$ and $C \in \mathbb{Z}^\Omega$. Then*

$$((C_i)_{i \in \mathcal{N}} \Downarrow C) \Longrightarrow ((\Gamma(C_i))_{i \in \mathcal{N}} \Downarrow \Gamma(C)). \quad (4.3)$$

If (Y, τ) is a Hausdorff space, then $\mathcal{K}^\Omega \in \mathbf{Z}$. The following statement takes place.

PROPOSITION 4.4 ([27]). *If (Y, τ) is a Hausdorff space, $(C_i)_{i \in \mathcal{N}}$ is a sequsence in $\mathcal{K}^\Omega$ and $C \in \mathbb{Z}^\Omega$, then the implication (4.3) is true.*

The proof is a highly obvious "sequential version" of the corresponding proof of Proposition 4.2.

PROPOSITION 4.5. *Let (Y, τ) be a T_1- space [22]-[24]. Then $\mathrm{N} \cap \mathbb{T}^\Omega \in \mathbf{Z}$.*

Proof. Let $(C_i)_{i \in \mathcal{N}}$ be a sequence in $\mathrm{N} \cap \mathbb{T}^\Omega$ and $C \in \mathbb{Z}^\Omega$. Moreover, let $(C_i)_{i \in \mathcal{N}} \Downarrow C$. Then $C \in \mathbb{T}^\Omega$ (we know that $\mathbb{T}^\Omega \in \mathbf{Z}$). If $j \in \mathcal{N}$, then $C_j = \Gamma(C_j)$ (by definitions of section 3). But, from Proposition 4.2 we have the convergence $(\Gamma(C_i))_{i \in \mathcal{N}} \Downarrow \Gamma(C)$. As a corollary, $\Gamma(C) = C$. So, $C \in \mathrm{N}$. The proof is completed.

PROPOSITION 4.6. *If (Y, τ) is a Hausdorff space, then $\mathrm{N} \cap \mathbb{K}^\Omega \in \mathbf{Z}$.*

The proof follows from Proposition 4.3.

PROPOSITION 4.7. *If (Y, τ) is a Hausdorff space, then $\mathrm{N} \cap \mathcal{K}^\Omega \in \mathbf{Z}$.*

The proof obviously follows from Proposition 4.4. In conclusion of the present section we note a useful corollary. Namely, if (Y, τ) is a Hausdorff space, then $\mathbb{K}^\Omega \in Z \cap \mathbf{Z}$ and $\mathcal{K}^\Omega \in Z \cap \mathbf{Z}$; moreover, in this case the basic operator Γ has the property of the sequential continuity on $\mathbb{K}^\Omega$ (see Proposition 4.3) and the analogous property on $\mathcal{K}^\Omega$ (see Proposition 4.4). So, for natural case of a Hausdorff space (Y, τ) the spaces $\mathbb{K}^\Omega$ and $\mathcal{K}^\Omega$ are very "good".

5. The basic iterations method. We consider iterated procedures in $\mathbb{Z}^\Omega$ and in $\mathbb{Z}^T$, where $T \in \Sigma_0$. In this section the first case is discussed. First we will do several observations. If M is a subset of N (see (3.8)), then the mapping

$$\omega \longmapsto \bigcup_{C \in M} C(\omega) : \Omega \longrightarrow \mathbb{Z} \quad (5.1)$$

is an element of N. As a corollary, we have the following property: if $\alpha \in \mathbb{Z}^\Omega$ and M is a subset of $\mathrm{N}_0[\alpha]$, then (5.1) is an element of $\mathrm{N}_0[\alpha]$ (note that $\mathrm{N}_0[\alpha] \neq \emptyset$; the "empty-valued" mapping $\alpha_\emptyset$, for which $\alpha_\emptyset(\omega) = \emptyset$ under $\omega \in \Omega$, is an element of $\mathrm{N}_0[\alpha]$). In particular, the last statement takes place in the case $M = \mathbb{N}_0[\alpha]$. So, for $\alpha \in \mathbb{Z}^{\Omega}$ in the form of the mapping $(na)[\alpha]$ of the kind

$$\omega \longmapsto \bigcup_{\mathcal{C} \in \mathbb{N}_0[\alpha]} \mathcal{C}(\omega) : \Omega \longrightarrow \mathbb{Z} \tag{5.2}$$

we have the greatest in $(\mathbb{Z}^{\Omega}, \sqsubseteq)$ element of $\mathbb{N}_0[\alpha]$. The determination (under $\alpha \in \mathbb{Z}^{\Omega}$) of the mapping $(na)[\alpha]$ (5.2) is our basic problem. So, $\forall \alpha \in \mathbb{Z}^{\Omega}$:

$$((na)[\alpha] \in \mathbb{N}_0[\alpha])\&(\forall \mathcal{C} \in \mathbb{N}_0[\alpha] : \mathcal{C} \sqsubseteq (na)[\alpha]). \tag{5.3}$$

For the attainment of the above-mentioned goal we use the iterated procedure on the basis of Γ (see section 3). Moreover, below we consider analogous procedures on the basis of the operators (3.4),(3.5). Therefore, we introduce general definitions in terms of the mappings (3.4). In addition, we use the natural construction (2.2). Then for $T \in \Sigma_0$ we have in the form of γ_T^0 the identical mapping acting in $\mathbb{Z}^T$ and (as a corollary) $\gamma_T^0 = \mathbb{I}_H \mid_{H=\mathbb{Z}^T}$; under $k \in \mathcal{N}$ the equality $\gamma_T^k = \gamma_T \circ \gamma_T^{k-1}$ takes place. In particular, $(\Gamma^k)_{k \in \mathcal{N}_0}$ is a sequence in $\mathcal{Z}$; $\Gamma^0 = \gamma_\Omega^0$ and $\forall k \in \mathcal{N} : \Gamma^k = \Gamma \circ \Gamma^{k-1}$. In terms of the last sequence we introduce the basic kind of an iterated process. Under $\mathcal{C} \in \mathbb{Z}^{\Omega}$ we have $\Gamma^0(\mathcal{C}) = \mathcal{C}$ and the following property: if $m \in \mathcal{N}_0$, $n \in \mathcal{N}_0$ and $m \leq n$, then $\Gamma^n(\mathcal{C}) \sqsubseteq \Gamma^m(\mathcal{C})$. By Proposition 4.1 we obtain that in the case, when (Y, τ) is a Hausdorff space, $\forall n \in \mathcal{N}_0$:

$$(\forall \mathcal{U} \in \mathbb{K}^{\Omega} : \Gamma^n(\mathcal{U}) \in \mathbb{K}^{\Omega})\&(\forall \mathcal{V} \in \mathcal{K}^{\Omega} : \Gamma^n(\mathcal{V}) \in \mathcal{K}^{\Omega}). \tag{5.4}$$

Now, we introduce in the consideration an infinite power of set-valued mappings. If $T \in \Sigma_0$, then we define the mapping γ_T^{∞} operating from $\mathbb{Z}^T$ into $\mathbb{Z}^T$ by the following rule; namely, $\forall \mathcal{C} \in \mathbb{Z}^T \; \forall t \in T$:

$$\gamma_T^{\infty}(\mathcal{C})(t) \triangleq \bigcap_{k \in \mathcal{N}_0} \gamma_T^k(\mathcal{C})(t). \tag{5.5}$$

Of course, we have $\Gamma^{\infty} = \gamma_\Omega^{\infty}$; in addition, for $\mathcal{C} \in \mathbb{Z}^{\Omega}$, $\omega \in \Omega$ and $m \in \mathcal{N}_0$ we have in the form of $\Gamma^{\infty}(\mathcal{C})(\omega)$ the intersection of all sets $\Gamma^k(\mathcal{C})(\omega)$, $k \in \mathcal{N}_0$, $m \leq k$. Note that $\forall \mathcal{C} \in \mathbb{Z}^{\Omega}$:

$$(\Gamma^k(\mathcal{C}))_{k \in \mathcal{N}} \Downarrow \Gamma^{\infty}(\mathcal{C}). \tag{5.6}$$

The relation (5.6) is a simple corollary of the definition (5.5). The following statement is true.

PROPOSITION 5.1. *Let (Y, τ) be a Hausdorff space. Then*

$$(\forall \mathcal{C} \in \mathbb{K}^{\Omega} : \Gamma^{\infty}(\mathcal{C}) \in \mathbb{K}^{\Omega})\&(\forall \tilde{\mathcal{C}} \in \mathcal{K}^{\Omega} : \Gamma^{\infty}(\tilde{\mathcal{C}}) \in \mathcal{K}^{\Omega}).$$

The proof follows from properties established in section 3 and from the relations (5.4),(5.6). We have the following

PROPOSITION 5.2. *If* (Y,τ) *is a Hausdorff space, then*

$$(\forall \mathcal{C} \in \mathbb{K}^{\Omega} : (\Gamma \circ \Gamma^{\infty})(\mathcal{C}) = \Gamma^{\infty}(\mathcal{C}))\&(\forall \tilde{\mathcal{C}} \in \mathcal{K}^{\Omega} : (\Gamma \circ \Gamma^{\infty})(\tilde{\mathcal{C}}) = \Gamma^{\infty}(\tilde{\mathcal{C}})).$$

The proof follows from Propositions 4.3, 4.4 and from (5.6).

From the monotonocity of Γ in the sense of (3.9) we have $\forall \alpha \in \mathbb{Z}^{\Omega}\ \forall \mathcal{H} \in \mathrm{N}_0[\alpha]$:

$$\mathcal{H} \sqsubseteq \Gamma^{\infty}(\alpha). \tag{5.7}$$

Along with (5.7) it is advisable to note the obvious monotonocity of Γ^{∞} in the sense of (3.9) and such simple fact that $\forall \mathcal{C} \in \mathbb{Z}^{\Omega} : \ \Gamma^{\infty}(\mathcal{C}) \sqsubseteq \mathcal{C}$. The following important statement is true.

THEOREM 5.1. *If* (Y,τ) *is a Hausdorff space, then* $\forall \alpha \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$: $\Gamma^{\infty}(\alpha) = (na)[\alpha]$.

The proof of Theorem 5.1 is the obvious combination of Proposition 5.2 and (5.7).

COROLLARY 1. *Let* (Y,τ) *be a Hausdorff space. Then*

$$(\forall \alpha \in \mathbb{K}^{\Omega} : (na)[\alpha] \in \mathbb{K}^{\Omega})\&(\forall \beta \in \mathcal{K}^{\Omega} : (na)[\beta] \in \mathcal{K}^{\Omega}).$$

The proof is the natural combination of Proposition 5.1 and Theorem 5.1. The last corollary is a highly general fact meaning the validity of important hereditary properties of the greatest nonanticipating set-valued selector with respect to the initial set-valued mapping.

Note that $\mathrm{N} = \{\mathcal{C} \in \mathbb{Z}^{\Omega} \mid \mathcal{C} = \Gamma^{\infty}(\mathcal{C})\}$. We suppose $\forall T \in \Sigma_0\ \forall \mathcal{C} \in \mathbb{Z}^{T}$:

$$(DOM)[\mathcal{C}] \triangleq \{\omega \in T \mid \mathcal{C}(\omega) \neq \emptyset\}. \tag{5.8}$$

We have in (5.8) the effective domain of set-valued mapping. Now we consider the basic case $T = \Omega$ (see (5.8)). The following statement is true.

THEOREM 5.2 ([27]). *If* (Y,τ) *is a Hausdorff space and* $\mathcal{C} \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$, *then*

$$((DOM)[\Gamma^{k}(\mathcal{C})])_{k \in \mathcal{N}} \downarrow (DOM)[\Gamma^{\infty}(\mathcal{C})].$$

The proof follows (in the case $\mathcal{C} \in \mathbb{K}^{\Omega}$) from the known property of centered systems of closed sets in compact spaces [21]–[24]. In "sequentially compact" case the standard procedure of the choice of an convergent subsequence is used. From Theorem 5.2 in the case of a Hausdorff space (Y, τ) we have $\forall \mathcal{C} \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega} \forall \omega \in \Omega$:

$$(\Gamma^{\infty}(\mathcal{C})(\omega) = \emptyset) \Longleftrightarrow (\exists n \in \mathcal{N} : \Gamma^{n}(\mathcal{C})(\omega) = \emptyset).$$

Now we consider the important question about the existence of nontrivial nonanticipating set-valued mappings being selectors of the initial set-valued mapping. The above-mentioned nontriviality is identified with the property $(DOM)[\mathcal{C}] = \Omega$ for the considered mapping $\mathcal{C} \in \mathbb{Z}^{\Omega}$. Suppose $\forall a \in \mathbb{Z}^{\Omega}$:

$$\mathbb{N}^{0}[a] \triangleq \{\alpha \in \mathbb{N}_0[a] \mid (DOM)[\alpha] = \Omega\}. \tag{5.9}$$

It is possible to consider elements of the set (5.9) as analogs of set-valued quasistrategies of [1]–[6],[8, 18, 19]. Of course (see (5.7)), $\forall a \in \mathbb{Z}^{\Omega}: (\mathbb{N}^{0}[a] \neq \emptyset) \Longrightarrow ((DOM)[\Gamma^{\infty}(a)] = \Omega)$. If (Y, τ) is a Hausdorff space, then $\forall a \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$:

$$(\mathbb{N}^{0}[a] \neq \emptyset) \Longleftrightarrow ((DOM)[\Gamma^{\infty}(a)] = \Omega). \tag{5.10}$$

Under the proof of (5.10) Theorem 5.1 and (5.7) are used. It is useful to introduce in the consideration the analogs of (one-valued) quasistrategies of [17, 20]. Of course, we consider the "quasistrategies" solving some problem which by initial set-valued mapping is defined. Namely, if $\mathcal{C} \in \mathbb{Z}^{\Omega}$, then we denote by $\mathbf{n}^{0}[\mathcal{C}]$ the set of all

$$h \in \prod_{\omega \in \Omega} \mathcal{C}(\omega)$$

such that

$$\forall \omega \in \Omega \, \forall A \in \mathcal{X} \, \forall \tilde{\omega} \in \Omega_0(\omega \mid A) : \ (h(\omega) \mid A) = (h(\tilde{\omega}) \mid A).$$

In this connection we recall (1.1) and constructions of [17, 20]. If $\mathcal{C} \in \mathbb{Z}^{\Omega}$, then from $\mathbf{n}^{0}[\mathcal{C}] \neq \emptyset$ the statement $(DOM)[\Gamma^{\infty}(\mathcal{C})] = \Omega$ follows. In the connection with the problem of the existence of nonanticipating one-valued selector of nonanticipating set-valued mapping we note [28]. It is advisable to consider "partial" nonanticipating set-valued mappings. If $\mathcal{C} \in \mathbb{Z}^{\Omega}$, then we denote by $\mathbb{N}^{0}_{\neq\emptyset}[\mathcal{C}]$ the set of all mappings $\mathcal{H} \in \mathbb{N}_0[\mathcal{C}]$ such that $(DOM)[\mathcal{H}] \neq \emptyset$. It is obvious that under $\mathcal{C} \in \mathbb{Z}^{\Omega}$ from $\mathbb{N}^{0}_{\neq\emptyset}[\mathcal{C}] \neq \emptyset$ the statement $(DOM)[\Gamma^{\infty}(\mathcal{C})] \neq$

$\emptyset$ follows. If (Y,τ) is a Hausdorff space and $\mathcal{C} \in \mathbb{K}^\Omega \cup \mathcal{K}^\Omega$, then the statements $\mathbb{N}^0_{\neq\emptyset}[\mathcal{C}] \neq \emptyset$ and $(DOM)[\Gamma^\infty(\mathcal{C})] \neq \emptyset$ are equivalent. Note that in Example 3.2 under $a \in \mathbb{Z}^\Omega$ with the property $a(v(\cdot)) \neq \emptyset$, $v(\cdot) \in \Omega = \mathbf{C}$, we have $\mathbb{N}^0_{\neq\emptyset}[a] = \emptyset$ (we leave the given obvious proof to the reader). From Propositions 4.6 and 4.7 the following statement follows.

PROPOSITION 5.3. *Let (Y,τ) be a Hausdorff space, $\mathcal{C} \in \mathbb{Z}^\Omega$ and $(\mathcal{C}_i)_{i\in\mathcal{N}}$ is a sequence in $\mathbb{Z}^\Omega$. Moreover, let: $(\mathcal{C}_i)_{i\in\mathcal{N}}$ be the sequence in $\mathbb{K}^\Omega$ or $(\mathcal{C}_i)_{i\in\mathcal{N}}$ be the sequence in $\mathcal{K}^\Omega$. Then*

$$((\mathcal{C}_i)_{i\in\mathcal{N}} \Downarrow \mathcal{C}) \Longrightarrow ((\Gamma^\infty(\mathcal{C}_i))_{i\in\mathcal{N}} \Downarrow \Gamma^\infty(\mathcal{C})).$$

In the conclusion of this section we consider briefly a construction connected with the notion of "universal" fixed points. In this part we use a "stratification" of the operator Γ. If $A \in X$ (see section 3), then we introduce the operator $\Gamma_A \in \mathcal{Z}$ for which $\forall \mathcal{C} \in \mathbb{Z}^\Omega\ \forall \omega \in \Omega$:

$$\Gamma_A(\mathcal{C})(\omega) \triangleq \{f \in \mathcal{C}(\omega) \mid \forall \tilde{\omega} \in \Omega_0(\omega \mid A)\ \exists \tilde{f} \in \mathcal{C}(\tilde{\omega}):$$

$$(f \mid A) = (\tilde{f} \mid A)\}. \tag{5.11}$$

From (3.7) and (5.11) we have the obvious property: if $\mathcal{C} \in \mathbb{Z}^\Omega$ and $\omega \in \Omega$, then $\Gamma(\mathcal{C})(\omega)$ is the intersection of all sets $\Gamma_A(\mathcal{C})(\omega)$, $A \in \mathcal{X}$. As a corollary, we obtain the equality $\mathbb{N} = \{\mathcal{C} \in \mathbb{Z}^\Omega \mid \forall A \in \mathcal{X} : \Gamma_A(\mathcal{C}) = \mathcal{C}\}$. But, the basic property is defined by the following practically obvious statement.

PROPOSITION 5.4. *If $A \in X$, then $\Gamma_A \circ \Gamma_A = \Gamma_A$.*

The proof follows from (5.11). We obtain the important property of idempotency of Γ_A, where $A \in X$. As a corollary, we have $\forall A \in X$:

$$(f.p.)[A] \triangleq \{\mathcal{C} \in \mathbb{Z}^\Omega \mid \Gamma_A(\mathcal{C}) = \mathcal{C}\} = \{\Gamma_A(\mathcal{H}) : \mathcal{H} \in \mathbb{Z}^\Omega\}. \tag{5.12}$$

From (5.12) many useful corollaries follow. It is obvious that $\mathbb{N}$ is the intersection of all sets $(f.p.)[A]$, $A \in \mathcal{X}$. So, nonanticipating set-valued mapping are "universal" fixed points and only they. It is useful to supplement this statement by some properties connected with the basic problem of the (nonanticipating) selection. Proposition 5.4 is the basis of following constructions; the corresponding proofs are very simple. If $\alpha \in \mathbb{Z}^\Omega$ and $A \in X$, then suppose $(f.p. - \alpha)[A] \triangleq \{\mathcal{C} \in (f.p.)[A] \mid \mathcal{C} \sqsubseteq \alpha\}$. From definitions of section 3 the obvious statement follows: if $a \in \mathbb{Z}^\Omega$, then $\mathbb{N}_0[a]$ is the intersection of all the sets $(f.p. - a)[A]$, $A \in \mathcal{X}$. Under $\alpha \in \mathbb{Z}^\Omega$ it is useful to introduce $\mathbb{S}[\alpha] \triangleq \{\mathcal{H} \in \mathbb{Z}^\Omega \mid \mathcal{H} \sqsubseteq \alpha\}$; moreover, if $A \in X$, then we have $(f.p. - \alpha)[A] = \{\Gamma_A(\mathcal{C}) : \mathcal{C} \in \mathbb{S}[\alpha]\}$ (see Proposition 5.4).

Until the end of this section we suppose that (Y, τ) is a Hausdorff space. By analogy with Proposition 4.1 the following property is proved: if $A \in X$, then

$$(\forall \mathcal{C} \in \mathbb{K}^\Omega : \Gamma_A(\mathcal{C}) \in \mathbb{K}^\Omega)\&(\forall \mathcal{D} \in \mathcal{K}^\Omega : \Gamma_A(\mathcal{D}) \in \mathcal{K}^\Omega).$$

On this basis it is possible essentially to supplement the above-mentioned representation of $\mathbb{N}$: the set $\mathbb{N} \cap \mathbb{K}^\Omega$ (the set $\mathbb{N} \cap \mathcal{K}^\Omega$) is the intersection of all the sets $\{\mathcal{C} \in \mathbb{K}^\Omega \mid \Gamma_A(\mathcal{C}) = \mathcal{C}\}$ (of all the sets $\{\mathcal{C} \in \mathcal{K}^\Omega \mid \Gamma_A(\mathcal{C}) = \mathcal{C}\}$), $A \in \mathcal{X}$. These last sets assume a more simple representation connected with Proposition 5.4. Namely, $\forall A \in X$:

$$(\{\mathcal{C} \in \mathbb{K}^\Omega \mid \Gamma_A(\mathcal{C}) = \mathcal{C}\} = (f.p.)[A] \cap \mathbb{K}^\Omega = \{\Gamma_A(\mathcal{H}) : \mathcal{H} \in \mathbb{K}^\Omega\})\&$$
$$(\{\mathcal{C} \in \mathcal{K}^\Omega \mid \Gamma_A(\mathcal{C}) = \mathcal{C}\} = (f.p.)[A] \cap \mathcal{K}^\Omega = \{\Gamma_A(\mathcal{H}) : \mathcal{H} \in \mathcal{K}^\Omega\}).$$

For completion of these representations we consider again the problem of the nonanticipating selection of set-valued mappings. If $M \in \mathbb{Z}^\Omega$, then we introduce in the consideration two following families: $\mathbb{S}_{\mathbb{K}}[M] \triangleq \mathbb{S}[M] \cap \mathbb{K}^\Omega = \{\mathcal{C} \in \mathbb{K}^\Omega \mid \mathcal{C} \sqsubseteq M\}$ and $\mathbb{S}_{\mathcal{K}}[M] \triangleq \mathbb{S}[M] \cap \mathcal{K}^\Omega = \{\mathcal{C} \in \mathcal{K}^\Omega \mid \mathcal{C} \sqsubseteq M\}$. Moreover, we have $\forall A \in X\ \forall M \in \mathbb{Z}^\Omega$:

$$\begin{aligned}&((\mathbb{K} - f.p. - M)[A] \triangleq (f.p. - M)[A] \cap \mathbb{K}^\Omega = \{\Gamma_A(\mathcal{C}) :\\ &\mathcal{C} \in \mathbb{S}_{\mathbb{K}}[M]\})\&((\mathcal{K} - f.p. - M)[A] \triangleq (f.p. - M)[A] \cap \mathcal{K}^\Omega =\\ &= \{\Gamma_A(\mathcal{C}) : \mathcal{C} \in \mathbb{S}_{\mathcal{K}}[M]\}).\end{aligned} \tag{5.13}$$

We omit obvious proofs connected with Proposition 5.4. Return to the important case corresponding to the given compact-valued initial set-valued mapping and the analogous mapping with sequentially compact values. In addition, $\forall M \in \mathbb{Z}^\Omega$:

$$(\mathbb{N}_0[M] \cap \mathbb{K}^\Omega = \bigcap_{A \in \mathcal{X}} (\mathbb{K} - f.p. - M)[A])\&(\mathbb{N}_0[M] \cap \mathcal{K}^\Omega = \bigcap_{A \in \mathcal{X}} (\mathcal{K} - f.p. - M)[A]). \tag{5.14}$$

It is advisable to consider (5.13) and (5.14) in a natural combination. We restrict ourselves to the consideration of the case of a compact-valued initial mapping. Note that (5.14) is especially important by virtue of Theorem 5.1. So, under $M \in \mathbb{K}^\Omega$ we have $\Gamma^\infty(M) = (na)[M] \in \mathbb{N}_0[M] \cap \mathbb{K}^\Omega$. In addition, in this case we have (see (5.14)) in the form of $\Gamma^\infty(M)$ the greatest in $(\mathbb{Z}^\Omega, \sqsubseteq)$ element of intersection $M_{\mathbb{K}} \triangleq \mathbb{N}_0[M] \cap \mathbb{K}^\Omega$ of all sets $(\mathbb{K} - f.p. - M)[A]$, $A \in \mathcal{X}$;

namely, $(na)[M] = \Gamma^\infty(M) \in M_{\mathbb{K}}$ and $\forall \mathcal{H} \in M_{\mathbb{K}}: \ \mathcal{H} \sqsubseteq \Gamma^\infty(M)$. As a corollary, we have the following scheme of the determination of $(na)[M] = \Gamma^\infty(M)$, where $M \in \mathbb{K}^\Omega$. Namely, under $A \in \mathcal{X}$ it should be constructed the set $\{\Gamma_A(\mathcal{C}) : \ \mathcal{C} \in \mathbb{S}_{\mathbb{K}}[M]\}$ (or $(\mathbb{K} - f.p. - M)[A]$). Later we consider all common elements of the families $\{\Gamma_A(\mathcal{C}) : \ \mathcal{C} \in \mathbb{S}_{\mathbb{K}}[M]\}$, $A \in \mathcal{X}$, obtaining (as a result) $M_{\mathbb{K}}$. Among them we choose the greatest element determined by $\Gamma^\infty(M)$; so, $(na)[M] = \Gamma^\infty(M)$ is the greatest element of $M_{\mathbb{K}}$. Now we note a natural sequential version. Namely, let (until the end of the given section) $M \in \mathcal{K}^\Omega$; it is possible to consider the intersection $M_{\mathcal{K}}$ of all sets $(\mathcal{K}-f.p.-M)[A] = \{\Gamma_A(\mathcal{C}) : \ \mathcal{C} \in \mathbb{S}_{\mathcal{K}}[M]\}$, $A \in \mathcal{X}$. Then $(na)[M] = \Gamma^\infty(M)$ is the greatest element of $M_{\mathcal{K}}$; so, $\Gamma^\infty(M) \in M_{\mathcal{K}}$ and $\forall \mathcal{H} \in M_{\mathcal{K}} : \ \mathcal{H} \sqsubseteq \Gamma^\infty(M)$. So, for two above-mentioned cases ($M \in \mathbb{K}^\Omega$ and $M \in \mathcal{K}^\Omega$) we obtain a distinctive "algorithm on functional level". For this "algorithm" the operation connected with Proposition 5.4 is characteristic: the image of a set in the space $\mathbb{Z}^\Omega$ under the operation of the mapping Γ_A should be constructed.

6. The local analysis of nonanticipating set-valued mappings. In this section we consider questions connected with a natural localization of "global" nonanticipating mappings. In particular, we investigate the following important question: when the initial set-valued mapping does contain a nonanticipating set-valued selector with nonempty values? First we do several absolutely simple observations. Denote by $\mathbb{H}$ the family of all sets $H \in \Sigma$ such that

$$\forall \omega \in H: \ \bigcup_{A \in \mathcal{X}} \Omega_0(\omega \mid A) \subset H. \tag{6.1}$$

It is possible (see (3.7),(3.8)) to verify the following obvious property: if $\mathcal{C} \in \mathbb{N}$, then $(DOM)[\mathcal{C}] \in \mathbb{H}$ and $\Omega \setminus (DOM)[\mathcal{C}] \in \mathbb{H}$. Below we consider other corollaries of definitions on the basis of (6.1). Suppose $\mathbb{N}_0 \triangleq \{\mathcal{C} \in \mathbb{N} \mid (DOM)[\mathcal{C}] \neq \emptyset\}$ and $\forall T \in \Sigma_0 : \ N_T^0 \triangleq \{\mathcal{C} \in N[T] \mid (DOM)[\mathcal{C}] = T\}$. If $\mathcal{C} \in \mathbb{N}_0$ and $C \triangleq (DOM)[\mathcal{C}]$, then $(\mathcal{C} \mid C) \in N_C^0$. Moreover, we note that $\forall \alpha \in \mathbb{Z}^\Omega: \ \mathbb{N}_{\neq\emptyset}^0[\alpha] = \{\mathcal{H} \in \mathbb{N}_0 \mid \mathcal{H} \sqsubseteq \alpha\}$. Let $\forall T \in \Sigma_0 \ \forall \alpha \in \mathbb{Z}^T$:

$$\tilde{N}_T^0[\alpha] \triangleq \{\mathcal{C} \in N_0[T;\alpha] \mid (DOM)[\mathcal{C}] = T\}. \tag{6.2}$$

Elements of (6.2) are nontrivial nonanticipating set-valued selectors of the ("partial" set-valued) mapping α and only they. By properties of $\mathbb{H}$ we obviously have the following useful property: if $\alpha \in \mathbb{Z}^\Omega$, $\mathcal{H} \in \mathbb{N}_{\neq\emptyset}^0[\alpha]$ and

$\mathbb{A} \triangleq (DOM)[\mathcal{H}]$, then $(\mathcal{H} \mid \mathbb{A}) \in \tilde{N}^0_{\mathbb{A}}[(\alpha \mid \mathbb{A})]$. Finally, as a corollary of (5.7), we note that $\forall \mathcal{C} \in \mathbb{Z}^\Omega\ \forall \mathcal{H} \in \mathbb{N}_0[\mathcal{C}] : (DOM)[\mathcal{H}] \subset (DOM)[\Gamma^\infty(\mathcal{C})]$. On this basis it is possible to obtain an useful estimate of effective domains of nonanticipating selectors. If $T \in \Sigma_0$ and $\alpha \in \mathbb{Z}^T$, then (3.6) is the nonempty set, since "empty-valued" mapping on T is the element of (3.6). As a corollary, $\forall \alpha \in \mathbb{Z}^\Omega : \mathbb{N}_0[\alpha] \neq \emptyset$. Therefore, if $\alpha \in \mathbb{Z}^\Omega$, then $(\mathcal{DOM})[\alpha] \triangleq \{(DOM)[\mathcal{H}] : \mathcal{H} \in \mathbb{N}_0[\alpha]\} \neq \emptyset$, $\emptyset \in (\mathcal{DOM})[\alpha]$.

PROPOSITION 6.1. *If (Y, τ) is a Hausdorff space and $\mathcal{C} \in \mathbb{K}^\Omega \cup \mathcal{K}^\Omega$, then $\mathbf{D} \triangleq (DOM)[\Gamma^\infty(\mathcal{C})] = (DOM)[(na)[\mathcal{C}]]$ is the greatest (in the sense of inclusions) element of $(\mathcal{DOM})[\mathcal{C}]$: $(\mathbf{D} \in (\mathcal{DOM})[\mathcal{C}])\&(\forall H \in (\mathcal{DOM})[\mathcal{C}] : H \subset \mathbf{D})$.*

The proof follows from statements of section 5. We introduce a trivial extension of partial set-valued mappings. If $T \in \Sigma_0$ and $\alpha \in \mathbb{Z}^T$, then $(\emptyset - ext)[\alpha] \in \mathbb{Z}^\Omega$ is *def* the mapping for which

$$(\forall t \in T : (\emptyset - ext)[\alpha](t) \triangleq \alpha(t))\&(\forall \omega \in \Omega \setminus T :$$

$$(\emptyset - ext)[\alpha](\omega) \triangleq \emptyset). \tag{6.3}$$

Of course, it is possible to use (6.3) for the extension of nonanticipating mapping. In this connection we recall the construction of $\mathbb{H}$; we note that $\mathbb{H}$ is simultaneously a Boolean algebra [25, Ch. I] and a topology of Ω; in particular, we have $\forall H \in \mathbb{H} : \Omega \setminus H \in \mathbb{H}$. These properties follow from (3.2); moreover, we note that $\forall \omega \in \Omega\ \forall A \in X\ \forall \tilde{\omega} \in \Omega_0(\omega \mid A) : \Omega_0(\omega \mid A) = \Omega_0(\tilde{\omega} \mid A)$. For $\mathbb{H}_0 \triangleq \mathbb{H}\setminus\{\emptyset\}$, $\mathbb{H}_0 \subset \Sigma_0$, we have (in the connection with (6.3)) the property: if $T \in \mathbb{H}_0$ and $\mathcal{H} \in N[T]$, then $(\emptyset - ext)[\mathcal{H}] \in \mathbb{N}$. As a corollary, $\forall \alpha \in \mathbb{Z}^\Omega\ \forall T \in \mathbb{H}_0\ \forall \mathcal{H} \in N_0[T; (\alpha \mid T)] : (\emptyset - ext)[\mathcal{H}] \in \mathbb{N}_0[\alpha]$. From this property we have (see section 5) $\forall a \in \mathbb{Z}^\Omega\ \forall H \in \mathbb{H}_0\ \forall \alpha \in N_0[H; (a \mid H)]\ \forall h \in H : \alpha(h) \subset \Gamma^\infty(a)(h)$. As a corollary, $\forall a \in \mathbb{Z}^\Omega\ \forall H \in \mathbb{H}_0\ \forall \alpha \in N_0[H; (a \mid H)] : (DOM)[\alpha] \subset (DOM)[\Gamma^\infty(a)]$.

Until the end of the present section we suppose that the following natural condition is correct.

CONDITION 6.1. *The family $\mathcal{X}$ of section 3 is a basis of a filter of the set X, i.e. $\forall A \in \mathcal{X}\ \forall B \in \mathcal{X}\ \exists C \in \mathcal{X} : C \subset A \cap B$.*

PROPOSITION 6.2. *If $\omega \in \Omega$, then the union of all sets $\Omega_0(\omega \mid E)$, $E \in \mathcal{X}$, is an element of $\mathbb{H}_0$.*

The proof is the obvious corollary of the definition of $\mathbb{H}$ and Condition 6.1. Therefore, we omit the corresponding reasoning.

Note that Proposition 6.2 generate a new approach to the investigation of nonanticipating set-valued mappings. Namely, introduce the following partition

$$\mathcal{G} \triangleq \{\bigcup_{E \in \mathcal{X}} \Omega_0(\omega \mid E) : \omega \in \Omega\} \tag{6.4}$$

of Ω in the disjoint union of nonempty subsets; namely,

$$\begin{gathered}(\mathcal{G} \subset \mathbb{H}_0)\&(\Omega = \bigcup_{G \in \mathcal{G}} G)\&(\forall G_1 \in \mathcal{G}\, \forall G_2 \in \mathcal{G} : \\ (G_1 \cap G_2 \neq \emptyset) \Longrightarrow (G_1 = G_2)).\end{gathered} \tag{6.5}$$

Of course, in terms of (6.4),(6.5) it is possible to introduce an equivalence relation. Namely, let def $\forall \omega_1 \in \Omega\, \forall \omega_2 \in \Omega$:

$$(\omega_1 \sim \omega_2) \Longleftrightarrow (\exists G \in \mathcal{G} : (\omega_1 \in G)\&(\omega_2 \in G)). \tag{6.6}$$

It is obvious that under $\omega_1 \in \Omega$ and $\omega_2 \in \Omega$ the equivalence

$$(\omega_1 \sim \omega_2) \Longleftrightarrow (\exists A \in \mathcal{X} : (\omega_1 \mid A) = (\omega_2 \mid A)) \tag{6.7}$$

takes place. In terms of (6.6) and (6.7) it is possible to consider $\mathcal{G}$ as the natural factor space. If $(\alpha_G)_{G \in \mathcal{G}} \in \prod_{G \in \mathcal{G}} \mathbb{Z}^G$, then

$$\square_{G \in \mathcal{G}}\, \alpha_G : \Omega \longrightarrow \mathbb{Z} \tag{6.8}$$

is the mapping, for which $\forall P \in \mathcal{G}\, \forall \omega \in P$:

$$(\square_{G \in \mathcal{G}}\, \alpha_G)(\omega) \triangleq \alpha_P(\omega). \tag{6.9}$$

By (6.8),(6.9) the natural operation of the glueing of local mappings is defined. If $(\alpha_G)_{G \in \mathcal{G}}$ is an element of the product of all sets $N[G]$, $G \in \mathcal{G}$, then (6.8) is an element of $\mathbb{N}$. Moreover, if $\mathcal{C} \in \mathbb{Z}^\Omega$ and $(\alpha_G)_{G \in \mathcal{G}}$ is an element of the product of all sets $N_0[G; (\mathcal{C} \mid G)]$ (of all sets $\tilde{N}^0_G[(\mathcal{C} \mid G)]$), $G \in \mathcal{G}$, then (6.8) is an element of $\mathbb{N}_0[\mathcal{C}]$ (of $\mathbb{N}^0[\mathcal{C}]$). If $T \in \Sigma_0$ and $a \in \mathbb{Z}^T$, then we suppose that $N^0_{\neq\emptyset}[T; a]$ is the set of all $\mathcal{H} \in N_0[T; a]$ such that $(DOM)[\mathcal{H}] \neq \emptyset$. It is clear that under $\mathcal{C} \in \mathbb{Z}^\Omega$ and $(\mathcal{U}_G)_{G \in \mathcal{G}} \in \prod_{G \in \mathcal{G}} N_0[G; (\mathcal{C} \mid G)]$:

$$(\square_{G \in \mathcal{G}}\, \mathcal{U}_G \in \mathbb{N}^0_{\neq\emptyset}[\mathcal{C}]) \Longleftrightarrow (\exists P \in \mathcal{G} : \mathcal{U}_P \in N^0_{\neq\emptyset}[P; (\mathcal{C} \mid P)]). \tag{6.10}$$

We note that under $\mathcal{C} \in \mathbb{Z}^\Omega$ the product of all the sets $N_0[G; (\mathcal{C} \mid G)]$, $G \in \mathcal{G}$, is a nonempty set (we use axiom of choice).

PROPOSITION 6.3. *Let $\mathcal{C} \in \mathbb{Z}^{\Omega}$. Then the mapping φ defined as*

$$(6.11) \qquad (\mathcal{U}_G)_{G\in\mathcal{G}} \longmapsto \square_{G\in\mathcal{G}}\,\mathcal{U}_G : \prod_{G\in\mathcal{G}} N_0[G;(\mathcal{C} \mid G)] \longrightarrow \mathbb{N}_0[\mathcal{C}]$$

is bijective; moreover, $\mathbb{N}^0[\mathcal{C}]$ is the image of the set $\prod\limits_{G\in\mathcal{G}} \tilde{N}^0_G[(\mathcal{C} \mid G)]$ under the mapping φ.

The proof is very simple corollary of the definitions (6.8),(6.9) and (6.11).

COROLLARY 2. *If $\mathcal{C} \in \mathbb{Z}^{\Omega}$, then: 1)$\mathbb{N}^0[\mathcal{C}] \neq \emptyset$ is equivalent to the requirement $\forall G \in \mathcal{G} : \ \tilde{N}^0_G[(\mathcal{C} \mid G)] \neq \emptyset$; 2)$\mathbb{N}^0_{\neq\emptyset}[\mathcal{C}] \neq \emptyset$ is equivalent to the requirement $\exists G \in \mathcal{G} : \ N^0_{\neq\emptyset}[G;(\mathcal{C} \mid G)] \neq \emptyset$.*

The proof is obvious by virtue of properties of the mapping (6.11); in particular, see (6.10). Moreover, here the simplest construction of an extension on the basis of (6.3) should be used.

PROPOSITION 6.4. *If $\mathbf{G} \in \mathcal{G}$ and $\alpha \in \mathbb{Z}^{\mathbf{G}}$, then $\tilde{N}^0_{\mathbf{G}}[\alpha] = N^0_{\neq\emptyset}[\mathbf{G};\alpha]$.*

The proof is an obvious combination of definitions of section 3 (in particular, see (3.5),(3.6)) and the basic property of a filter basis; in this connection see (for example) [22, 23]. In terms of Proposition 6.4 it is possible to reformulate Corollary of Proposition 6.3 and previous relations. For example, if $\mathcal{C} \in \mathbb{Z}^{\Omega}$, then the following conditions are equivalent: 1)$\mathbb{N}^0[\mathcal{C}] \neq \emptyset$ (the statement about the existence of nonanticipating set-valued nonempty-valued selectors of $\mathcal{C}$); 2)$N^0_{\neq\emptyset}[G;(\mathcal{C} \mid G)] \neq \emptyset$ under all $G \in \mathcal{G}$. Of course, in Proposition 6.4 we consider the case $\alpha = (\mathcal{C} \mid G)$, where $G \in \mathcal{G}$. Now, we omit many other useful corollaries of Proposition 6.4. The last statement denotes under $\alpha \in \mathbb{Z}^T$ (where $T \in \mathcal{G}$) that for $\beta \in N_0[T;\alpha]$ (3.6) two following properties are equivalent: 1)$(DOM)[\beta] = T$; 2)$(DOM)[\beta] \neq \emptyset$. This equivalence follows (in fact) from (6.6),(6.7).

7. The parallel procedure of MPI. On the basis of localizations of the previous section it is possible to construct a new representation of the basic variant of MPI (see section 5). First we note several absolutely simple circumstances refusing now Condition 6.1. So, unless otherwise stated, $\mathcal{X}$ corresponds only to general suppositions of section 3. We recall (3.11) and (3.12); moreover, $\forall \mathcal{C} \in \mathbb{Z}^{\Omega}\ \forall H \in \mathbb{H}_0$:

$$(7.1) \qquad (\Gamma(\mathcal{C}) \mid H) = \gamma_H((\mathcal{C} \mid H)).$$

The proof of (7.1) follows immediately from the definitions of the families $\mathbb{H}$ and $\mathbb{H}_0$ (see section 6). This property is supplemented by the obvious corollaries of (3.11). Namely, if $\mathcal{C} \in \mathbb{Z}^{\Omega}$, $H \in \Sigma_0$, $h \in H$ and $k \in \mathcal{N}_0$, then $\Gamma^k(\mathcal{C})(h) \subset \gamma^k_H((\mathcal{C} \mid H))(h)$. Moreover, under $\mathcal{C} \in \mathbb{Z}^{\Omega}$, $H \in \Sigma_0$ and $h \in H$ we have $\Gamma^\infty(\mathcal{C})(h) \subset \gamma_H^\infty((\mathcal{C} \mid H))(h)$. Two last statements are realized with the employment of mathematical induction and the property of monotonocity of the mapping (3.4). On the other hand, from (7.1) the important property follows; namely, $\forall \mathcal{C} \in \mathbb{Z}^\Omega\, \forall H \in \mathbb{H}_0\; \forall k \in \mathcal{N}_0$:

$$(\Gamma^k(\mathcal{C}) \mid H) = \gamma_H^k((\mathcal{C} \mid H)). \tag{7.2}$$

As a corollary, from (7.2) we obtain the limit representation. Namely, $\forall \mathcal{C} \in \mathbb{Z}^\Omega\, \forall H \in \mathbb{H}_0$:

$$(\Gamma^\infty(\mathcal{C}) \mid H) = \gamma_H^\infty((\mathcal{C} \mid H)). \tag{7.3}$$

For the cases characterized by the conditions of Theorem 5.1 we obtain (in (7.3)) a fragment of the greatest nonanticipating set-valued selector of the given set-valued mapping. These properties (7.2),(7.3) are correct in the general case of the family $\mathcal{X}$ of section 3. But, new useful statements are realized under Condition 6.1.

So, we postulate, until the end of the given section, that Condition 6.1 is satisfied: $\mathcal{X}$ is a filter basis of X. Then it is possible to use the construction of the coalescence on the basis of (6.8),(6.9). We take into account the following circumstance. Namely, for $\mathcal{C} \in \mathbb{Z}^\Omega$ and $k \in \mathcal{N}_0$ in the form of $(\gamma_G^k((\mathcal{C} \mid G)))_{G\in\mathcal{G}}$ we have an element of the product of all sets $\mathbb{Z}^G$, $G \in \mathcal{G}$. Moreover, under $\mathcal{C} \in \mathbb{Z}^\Omega$ it is true that $(\gamma_G^\infty((\mathcal{C} \mid G)))_{G\in\mathcal{G}}$ is an element of the above-mentioned product.

PROPOSITION 7.1. *Let $\mathcal{C} \in \mathbb{Z}^\Omega$ and $k \in \mathcal{N}_0$. Then $\Gamma^k(\mathcal{C})$ is the sewing of all the fragments $\gamma_G^k((\mathcal{C} \mid G))$, $G \in \mathcal{G}$, in the sense of (6.8):*

$$\Gamma^k(\mathcal{C}) = \square_{G\in\mathcal{G}}\; \gamma_G^k((\mathcal{C} \mid G)).$$

The proof is obtained by the obvious combination of (6.5),(6.9) and (7.2).

THEOREM 7.1. *If $\mathcal{C} \in \mathbb{Z}^\Omega$, then $\Gamma^\infty(\mathcal{C})$ is the sewing of all the fragments $\gamma_G^\infty((\mathcal{C} \mid G))$, $G \in \mathcal{G}$, in the sense of (6.8):*

$$\Gamma^\infty(\mathcal{C}) = \square_{G\in\mathcal{G}}\; \gamma_G^\infty((\mathcal{C} \mid G)).$$

For the proof it is sufficient to use the combination (6.5),(6.9) and (7.3). In spite of the fact that two last statements are established by very simple methods, it is useful to note their importance. Namely, in the form of the procedure of MPI on the basis of Γ (see section 3) we have, in fact, a distinctive analog of some "perceptron" realized by the system of parallel processes characterized by operations of the mappings (analogous to (3.4)) on squares $G \in \mathcal{G}$. Namely, we realize independent iterated procedures on the basis of operators (3.4),(3.5) under $T = G \in \mathcal{G}$. Formally these procedures are defined by powers of γ_G, $G \in \mathcal{G}$. Really, under the given initial mapping $\mathcal{C} \in \mathbb{Z}^{\Omega}$ we obtain the system of sequences of iterations in $\mathbb{Z}^G$, $G \in \mathcal{G}$. In addition, the restrictions $(\mathcal{C} \mid G)$, $G \in \mathcal{G}$, of the (initial) set-valued mapping $\mathcal{C}$ are used in the capacity of starting points. So, we have the system of parallel iterated processes for which the limit elements $\gamma^{\infty}((\mathcal{C} \mid G))$, $G \in \mathcal{G}$, are coalesced in $\Gamma^{\infty}(\mathcal{C})$ (see Theorem 7.1). Of course, here we have a parallel logic procedure, the realization of which (in the form of concrete iterated sequences) are connected with many difficultes.

8. Addition. In this section we consider some questions objectively connected with a construction similar (in fact) to Proposition 6.4. But, first for the general case of the family $\mathcal{X}$ of section 3, we note the following obvious property; namely, $\mathbb{N}$ (3.8) is the set of all mappings $\mathcal{C} \in \mathbb{Z}^{\Omega}$ for each of which $\forall \omega_1 \in \Omega\ \forall \omega_2 \in \Omega\ \forall A \in \mathcal{X}$:

$$((\omega_1 \mid A) = (\omega_2 \mid A)) \Longrightarrow (\{(f \mid A) : f \in \mathcal{C}(\omega_1)\} = \{(f \mid A) : f \in \mathcal{C}(\omega_2)\}).$$

By this relation the natural connection of concrete definitions similar to (1.2) and general definitions of section 3 (see (3.7)) is established. It is known that in the procedures of MPI used for solving DG, the following situation arises sufficiently often. Namely, in some DG under the determination of the value of DG by MPI, the above-mentioned value as a position function is defined after a finite number of iterations. And, what is more, this characteristic number may be highly small (in this connection also it is advisable to recall the known notion of regular DG [11]–[14]). This known circumstance [1]–[6], [8] generates the natural question about conditions for which the corresponding iterated process is stabilized after a finite number of "steps". Of course, this question is remained valid for the considered "direct" version of MPI. Namely, if $\alpha \in \mathbb{Z}^{\Omega}$, then we are interested in the following possibility: $(na)[\alpha] = \Gamma^k(\alpha)$, when $k \in \mathcal{N}$. Of course, the given question was justified in the case $(na)[\alpha] = \Gamma^{\infty}(\alpha)$. The last case takes place (in particular) under conditions used in Theorem 5.1. We consider the given very difficult question only in a specific case corresponding to the situation $\Gamma^{\infty}(\alpha)(\omega) \equiv \emptyset$. But first we consider some auxiliary statements connected (in idea) with constructions of section 6. Unless otherwise stated, then we suppose that Condition 6.1 is correct.

PROPOSITION 8.1. *If* $\mathbf{G} \in \mathcal{G}$ *and* $\mathcal{C} \in \mathbb{Z}^{\mathbf{G}}$, *then the following implication is true:*

$$((DOM)[\gamma_{\mathbf{G}}(\mathcal{C})] \neq \emptyset) \Longrightarrow ((DOM)[\mathcal{C}] = \mathbf{G}).$$

Proof. Fix $\mathbf{G} \in \mathcal{G}$ and $\mathcal{C} \in \mathbb{Z}^{\mathbf{G}}$ for which $(DOM)[\gamma_{\mathbf{G}}(\mathcal{C})] \neq \emptyset$. Choose $\omega \in (DOM)[\gamma_{\mathbf{G}}(\mathcal{C})]$. Then $\omega \in \mathbf{G}$ and $\gamma_{\mathbf{G}}(\mathcal{C})(\omega) \neq \emptyset$. Choose $f \in \gamma_{\mathbf{G}}(\mathcal{C})(\omega)$. Recall (3.7). As a corollary, $f \in \mathcal{C}(\omega)$ has the property

$$\forall A \in \mathcal{X}\ \forall \tilde{\omega} \in (Ge)[\mathbf{G}; \omega \mid A]\ \exists \tilde{f} \in \mathcal{C}(\tilde{\omega}) :\ (f \mid A) = (\tilde{f} \mid A). \tag{8.1}$$

Let $\rho \in \mathbf{G}$. Consider the set $\mathcal{C}(\rho)$. For this we choose (see (6.4)) $\eta \in \Omega$ such that $\mathbf{G}$ is the union of all sets $\Omega_0(\eta \mid E)$, $E \in \mathcal{X}$. Choose $\Sigma_1 \in \mathcal{X}$ and $\Sigma_2 \in \mathcal{X}$ for which $\omega \in \Omega_0(\eta \mid \Sigma_1)$ and $\rho \in \Omega_0(\eta \mid \Sigma_2)$. We use Condition 6.1. Namely, choose $\Sigma \in \mathcal{X}$ for which $\Sigma \subset \Sigma_1 \cap \Sigma_2$. Then (see (3.2)) $(\omega \mid \Sigma) = (\eta \mid \Sigma) = (\rho \mid \Sigma)$. As a corollary, by (3.1) we have $\rho \in (Ge)[\mathbf{G}; \omega \mid \Sigma]$ under $\Sigma \in \mathcal{X}$. By (8.1) we obtain that $\exists \tilde{f} \in \mathcal{C}(\rho) :\ (f \mid \Sigma) = (\tilde{f} \mid \Sigma)$. Then $\mathcal{C}(\rho) \neq \emptyset$. We have $\rho \in (DOM)[\mathcal{C}]$. So, $\mathbf{G} \subset (DOM)[\mathcal{C}]$. As a corollary, $(DOM)[\mathcal{C}] = \mathbf{G}$.

Note that in Proposition 8.1 it is possible to consider the case $\mathcal{C} = (\alpha \mid \mathbf{G})$, where $\alpha \in \mathbb{Z}^{\Omega}$. In this connection we recall (6.5) and (7.1). So, $\forall \alpha \in \mathbb{Z}^{\Omega}\ \forall G \in \mathcal{G}$:

$$((DOM)[(\Gamma(\alpha) \mid G)] \neq \emptyset) \Longrightarrow ((DOM)[(\alpha \mid G)] = G). \tag{8.2}$$

Below, we use (8.2) for an analysis of the basic iterated process. We use the definition of the sequence $(\Gamma^k)_{k \in \mathcal{N}_0}$. Then from (8.2) we have $\forall \mathcal{C} \in \mathbb{Z}^{\Omega}\ \forall G \in \mathcal{G}\ \forall k \in \mathcal{N}_0$:

$$((DOM)[(\Gamma^{k+1}(\mathcal{C}) \mid G)] \neq \emptyset) \Longrightarrow ((DOM)[(\Gamma^k(\mathcal{C}) \mid G)] = G). \tag{8.3}$$

From (8.3) we obtain the following corollary using the definition of Γ^∞. Namely, by (8.3) we have $\forall \mathcal{C} \in \mathbb{Z}^{\Omega}\ \forall G \in \mathcal{G}$:

$$((DOM)[(\Gamma^\infty(\mathcal{C}) \mid G)] \neq \emptyset) \Longrightarrow$$

$$(\forall k \in \mathcal{N}_0 :\ (DOM)[(\Gamma^k(\mathcal{C}) \mid G)] = G). \tag{8.4}$$

It is advisable to connect the properties of kind (8.4) with Theorem 5.2. As a result, we have the statement:

PROPOSITION 8.2. *If* (Y,τ) *is a Hausdorff space and* $\mathcal{C} \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$, *then* $\forall G \in \mathcal{G}$:

$$(G \setminus (DOM)[\Gamma^{\infty}(\mathcal{C})] \neq \emptyset) \Longrightarrow$$

$$(\exists k \in \mathcal{N}: \ (DOM)[(\Gamma^{k}(\mathcal{C}) \mid G)] = \emptyset). \tag{8.5}$$

Proof. Let (Y,τ) be a Hausdorff space and $\mathcal{C} \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$. Fix $G \in \mathcal{G}$ such that the premise of the implication of (8.5) is true. Choose $\lambda \in G \setminus (DOM)[\Gamma^{\infty}(\mathcal{C})]$. Then $\Gamma^{\infty}(\mathcal{C})(\lambda) = \emptyset$. Using the obvious corollary of Theorem 5.2, we choose $n \in \mathcal{N}$ such that $\Gamma^{n}(\mathcal{C})(\lambda) = \emptyset$. Therefore, we have $(DOM)[(\Gamma^{n}(\mathcal{C}) \mid G)] \neq G$. By (8.3) we obviously obtain the property $(DOM)[(\Gamma^{n+1}(\mathcal{C}) \mid G)] = \emptyset$.

THEOREM 8.1. *If* (Y,τ) *is a Hausdorff space and* $\mathcal{C} \in \mathbb{K}^{\Omega} \cup \mathcal{K}^{\Omega}$, *then* $\forall G \in \mathcal{G}$:

$$((DOM)[(\Gamma^{\infty}(\mathcal{C}) \mid G)] = \emptyset) \Longleftrightarrow (\exists k \in \mathcal{N}: \ (DOM)[(\Gamma^{k}(\mathcal{C}) \mid G)] = \emptyset).$$

The proof is obvious and we omit the corresponding reasoning. Consider only the profound interpretation of the last statement. For the simplicity we consider an "one-block" particular case, i.e. we discuss the case $\Omega = G$. A naturality of such supposition follows from statements of section 7. So, let $\Omega \in \mathcal{G}$. Then by identifying Ω and G, we have $(\Gamma^{\infty}(\mathcal{C}) \mid G) = \Gamma^{\infty}(\mathcal{C})$ and $\forall k \in \mathcal{N}_0: \ (\Gamma^{k}(\mathcal{C}) \mid G) = \Gamma^{k}(\mathcal{C})$. Theorem 8.1 realizes (in particular) the following fact: if $\Gamma^{\infty}(\mathcal{C})(\omega) \equiv \emptyset$ (i.e. $(DOM)[\Gamma^{\infty}(\mathcal{C})] = \emptyset$), then for some $k \in \mathcal{N}: \ \Gamma^{k}(\mathcal{C}) = \Gamma^{\infty}(\mathcal{C}) = (na)[\mathcal{C}]$ (of course, here we postulate that (Y,τ) is a Hausdorff space and the following property takes place: $\mathcal{C} \in \mathbb{K}^{\Omega}$ or $\mathcal{C} \in \mathcal{K}^{\Omega}$).

Example 8.1. Consider one concrete variant of the general statement similar (in idea) to Example 3.1. Namely, consider the simplest "scalar" conflict-control system [11]–[14] on the time interval $[0,1]$:

$$\dot{x} = u + v, \quad x(0) = 0. \tag{8.6}$$

Suppose that $u \in [-1,1]$ and $v \in [-1,1]$ are the given geometric constraints on the choice of "instantaneous" controls. Introduce the set W of all Borel functions operating from $[0,1]$ into $[-1,1]$. Moreover, introduce the sets $\mathcal{U}$ and $\mathcal{V}$ of admissible programmed controls of the players I and II respectively. Namely, let $\mathcal{U} \triangleq W$. Suppose, that $\mathcal{V}$ is arbitrary nonempty subset of W, satisfying the following conditions: 1)$\forall V \in \mathcal{V}\ \exists t \in]0,1]\ \forall \xi \in [0,t[$: $V(\xi) = 0$; 2) the control $V_0^+ \in W$, defined as $V_0^+(t) \triangleq 0$ under $t \in [0,1/2[$ and $V_0^+(t) \triangleq 1$ under $t \in [1/2,1]$, is an element of $\mathcal{V}$; 3) the control $V_0^- \in W$, for which $V_0^-(t) \triangleq 0$ under $t \in [0,1/2[$ and $V_0^-(t) \triangleq -1$ under $t \in [1/2,1]$, is an element of $\mathcal{V}$; 4) the control $\mathbf{O} \in W$, for which $\mathbf{O}(t) = 0$ under all $t \in [0,1]$, is an element of $\mathcal{V}$. Postulate that the plyer I strives to the goal $\mid x(1) \mid \geq 1$ and exploits (for this) controls $U \in \mathcal{U}$. The player II has the inverse goal. Introduce the set $C([0,1])$ of all continuous functions from $[0,1]$ into $\mathbb{R}$. In this example we suppose that: $X \triangleq [0,1]$, $\Upsilon \triangleq [-1,1]$, $\mathcal{X} \triangleq \{[0,t[:\ t \in]0,1]\}$, $Y \triangleq \mathbb{R}$, τ is the ordinary $|\cdot|$-topology of $\mathbb{R}$, $\Omega \triangleq \mathcal{V}$, $Z \triangleq C([0,1])$. In this conditions θ is the natural topology of pointwise convergence in $C([0,1])$ (see [22]). In addition, in the considered example (Y,τ) and (Z,θ) are Hausdorff spaces. Consider the mapping (pseudo-strategy) $\mathcal{C}$ acting from $\Omega = \mathcal{V}$ into the family $\mathbb{Z}$ of all subsets of Z. First, we introduce (under $V = v(\cdot) \in \Omega$) the nonempty set $\mathbb{X}_{\mathcal{U}}(V)$ of all functions $\varphi[U;V] \in Z$ defined each as

$$(8.7) \qquad t \longmapsto \int_{[0,t[} U(\xi)\lambda_0(d\xi) + \int_{[0,t[} V(\xi)\lambda_0(d\xi) :\ [0,1] \longrightarrow \mathbb{R}$$

(here λ_0 is the restriction of the Lebesque measure to σ-algebra of Borel subsets of $[0,1]$) under the enumeration of all $U \in \mathcal{U}$. Then (in the correspondence with the above-mentioned goal of the player I) under $V \in \Omega$ we define $\mathcal{C}(V)$ as the set of all $z \in \mathbb{X}_{\mathcal{U}}(V)$ such that $\mid z(1) \mid \geq 1$. In the form of $\mathcal{C}(V)$ we have (under $V \in \Omega$) the sequentially compact nonempty subset of Z equipped with the topology of uniform convergence on $[0,1]$ (note that $\mathcal{C}(V)$ is a compactum in the given topology). As a corollary, if $V \in \Omega$, then $\mathcal{C}(V)$ is a sequentially compact set in (Z,θ), i.e. $\mathcal{C}(V) \in \mathcal{K}$. We have $\mathcal{C} \in \mathcal{K}^{\Omega}$. Moreover, from (6.6) and (6.7) we obtain the property $\mathcal{G} = \{\Omega\}$. So, here an "one-block" problem is considered. In this problem the conditions defining the basic statement of Theorem 8.1 are correct. Note that by Theorem 5.1 in the given example $\Gamma^{\infty}(\mathcal{C}) = (na)[\mathcal{C}]$. But, $(na)[\mathcal{C}](\mathbf{O}) = \emptyset$. Really, let, in contradiction, $z_0 \in (na)[\mathcal{C}](\mathbf{O})$. Then (see (5.3)) $z_0 \in \mathcal{C}(\mathbf{O})$. Therefore, $\mid z_0(1) \mid \geq 1$ and it is possible to indicate $U_0 \in \mathcal{U}$ for which $z_0 = \varphi[U_0;\mathbf{O}]$. So, $z_0(1)$ is integral of U_0 on $[0,1]$. As a corollary, we have that $U_0(t) = 1$ almost everywhere on $[0,1]$ or $U_0(t) = -1$ almost everywhere on $[0,1]$. In addition, $\forall t \in [0,1]$:

$$(8.8) \qquad z_0(t) = \int_{[0,t[} U_0(\xi)\lambda_0(d\xi).$$

For obtaining (8.8) it should be used (8.6) and (8.7). Let $U_0(t) = 1$ almost everywhere on $[0,1]$ (in the sense of λ_0). Note that by (3.2) $V_0^- \in \Omega_0(\mathbf{O} \mid [0,1/2[)$. Therefore, it is possible to choose $z_0^- \in (na)[\mathcal{C}](V_0^-)$ for which $(z_0 \mid [0,1/2[) = (z_0^- \mid [0,1/2[)$. Then $z_0(1/2) = z_0^-(1/2)$, since z_0 and z_0^- are continuous functions. In addition, $z_0^- \in \mathcal{C}(V_0^-)$. As a corollary, $\mid z_0^-(1) \mid \geq 1$. On the other hand,

$$z_0^-(1) = z_0(1/2) + \int\limits_{[\frac{1}{2},1[} U_0^-(t)\lambda_0(dt) - \frac{1}{2}, \tag{8.9}$$

where $U_0^- \in \mathcal{U}$. We use the property: $V_0^-(t) = -1$ under $t \in [1/2,1]$. Recall that $z_0(1/2) = 1/2$ (see (8.8)). From (8.9) we obtain that $1/2 \geq \mid z_0^-(1) \mid$. We have the obvious contradiction. Let $U_0(t) = -1$ almost everywhere (in the sense of λ_0) on $[0,1]$. Recall (see (3.2)) that $V_0^+ \in \Omega_0(\mathbf{O} \mid [0,1/2[)$. Choose $z_0^+ \in (na)[\mathcal{C}](V_0^+)$ such that $(z_0 \mid [0,1/2[) = (z_0^+ \mid [0,1/2[)$. Then by the continuity of z_0 and z_0^+ the equality $z_0(1/2) = z_0^+(1/2)$ takes place. Of course, $z_0^+ \in \mathcal{C}(V_0^+)$. Therefore, $\mid z_0^+(1) \mid \geq 1$. But, in the considered case (see (8.8)) $z_0(1/2) = -1/2$. Then

$$z_0^+(1) = z_0(1/2) + \int\limits_{[\frac{1}{2},1[} U_0^+(t)\lambda_0(dt) + \frac{1}{2} = \int\limits_{[\frac{1}{2},1[} U_0^+(t)\lambda_0(dt), \tag{8.10}$$

where $U_0^+ \in \mathcal{U}$. From (8.10) we obtain the inequality $1/2 \geq \mid z_0^+(1) \mid$. Now we have the contradiction. So, $(na)[\mathcal{C}](\mathbf{O}) = \emptyset$. From Proposition 6.4 we obtain the property $(DOM)[(na)[\mathcal{C}]] = \emptyset$, since in given case $\Omega \in \mathcal{G}$. From Theorem 5.1 we have the equality $(DOM)[\Gamma^\infty(\mathcal{C})] = \emptyset$. By Theorem 8.1, we obtain that $\exists k \in \mathcal{N}: (DOM)[\Gamma^k(\mathcal{C})] = \emptyset$. As a corollary, $\Gamma^\infty(\mathcal{C}) = (na)[\mathcal{C}] = \Gamma^k(\mathcal{C})$ for some $k \in \mathcal{N}$.

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ON STABILIZATION OF SOLUTIONS OF THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS ON THE NETS *

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Abstract. This note is devoted to study sufficient conditions for stabilization of the difference $\lim\limits_{t\to\infty} |u(x,t) - v(x,t)| = 0$, $x \in S$ — nets, where $u(x,t)$ is solution of the Cauchy problem for parabolic equation which is definite on $S \times [0, +\infty)$, S – nets in $\mathbf{E}^N$ and $v(x,t)$ is solution of the Cauchy problem with averaged constant matrix which is definite in all point $x \in \mathbf{E}^N$, $t \geq 0$.

Key Words. Heat equation, Parabolic equation, Cauchy problem, Stabilization, Nets, Knots, Average theory

1. Definitions and the statement of the problems. In the Euclidean space $\mathbf{E}^N$ $(N \geq 2)$ we consider the nets S_ε, $\varepsilon > 0$, which is a union at all of the lines, parallel to coordinate axes with knots of the nets $(n_1\varepsilon, n_2\varepsilon, \dots, n_N\varepsilon)$, $n_k \in \mathbf{Z}$, $(k = 1, \dots, N)$. From this it follows that S_ε is a union of edges of cubes $\square_i^\varepsilon = \{i\varepsilon \leq x_k \leq (i+1)\varepsilon,\ i \in \mathbf{Z},\ k = 1, \dots, N\}$. We define the linear Lebesque measure μ^ε on the lines of S_ε, with normalizating coefficient ε^{N-1}/N. This is a periodic measure μ^ε with period ε, and measure of cell of periodicity is equal ε^N, so $\mu(\square_i^\varepsilon) = \varepsilon^N$. From this it follows that $\mu^\varepsilon \xrightarrow[\varepsilon\to 0]{} dx$: i.e.

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbf{E}^N} \eta(x)\, d\mu^\varepsilon(x) = \int\limits_{\mathbf{E}^N} \eta(x)\, dx, \quad \forall \eta(x) \in C_0^\infty(\mathbf{R}^N).$$

In the half space $\{t \geq 0\} \equiv \{x \in S, t \geq 0\}$ we consider the Cauchy

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problem for parabolic equations of divergence form

$$\text{(1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(a(x)\nabla u), (x,t) \in \{t > 0\}, \\ u\Big|_{t=0} = \varphi(x), x \in S \end{cases}$$

where we assume that the real function $a(x)$ is defined on S for all $t > 0$ and is periodic on each variables with period 1 and satisfies the condition

$$\text{(2)} \qquad \frac{1}{\lambda} \le a(x) \le \lambda, \quad \lambda > 0, \quad x \in \mathbf{E}^N.$$

The symbol ∇ desinate here the differential operator on the net S, which is coincident with $\partial/\partial x_i$ on lines x_i $(i = 1, \ldots, N)$ parallel to axis x_i.

Also we assume that the initial function $\varphi(x)$ is definite on the net S and is bounded function on S.

The Cauchy problem (1) we understand in usual weak sense, that is in sense of integral identity:

$$\text{(3)} \qquad \int_0^{+\infty}\int_S u\frac{\partial \eta}{\partial t}\,d\mu\,dt + \int_S \varphi(x)\eta(x,0)\,d\mu = \int_0^{+\infty}\int_S (a\nabla u, \nabla\eta)\,d\mu\,dt,$$

for all functions $\eta(x,t) \in C_0^\infty(\{t > 0\})$ where function $u(x,t)$ is definite on $\{t > 0\}$ and belonging to $L^2\{S \times [0,T], d\mu \cdot dt\}$ $\forall T > 0$, and $\nabla u(x,t) \in L^2\{S \times [0,T], d\mu \cdot dt\}$ $\forall T > 0$.

The solutions of the problem (1) we takes from class of uniqueness, that is solutions is bounded in each strip $\{0 < t \le T\} \equiv \{S \times (0,T]\}$.

2. Example. If $a(x) = 1$, $N = 2$, then the problem (1) we can interpret in the following equivalence sense

$$\text{(1}'\text{)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \quad (x,t) \in \{t > 0\}, \\ u\Big|_{t=0} = \varphi(x), \quad x \in S, \end{cases}$$

where we assume that

1) S is the usual square net on the plane $\mathbf{E}^2$ with natural linear measure $\mu[i \le x_1 \le i+1, j \le x_2 \le j+1]$ on edges of square with coefficient $1/2$;

2) $u(x,t)$ is continuous function on net S together with knots (n_1, n_2), $n_i \in \mathbf{Z}$;

3) on horizontal and vertical units function $u(x,t)$ have first and second derivative, with is square integrable.

4) derivative du/dx_1 (on horizontal units), du/dx_2 (on vertical units) they can have discontinuity on nodes of network, but a jump of derivative du/dx_1 + jump of derivative $du/dx_2 = 0$ in each nodes of network.

Under this conditions we have, that

$$(1')\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \equiv \begin{cases} \dfrac{d^2u}{dx_1^2} \text{ on horizontal unit, } t>0, \\ \dfrac{d^2u}{dx_2^2} \text{ on vertical unit, } t>0, \end{cases} \\ u\Big|_t = \varphi(x), \quad x \in S, \end{cases}$$

$\varphi(x)$ is bounded initial function on net S.

For this definition of Laplace operator Δ on net S see [1].

Together with problem (1) we consider usual Cauchy problem

$$(4)\qquad \begin{cases} \dfrac{\partial u^0}{\partial t} = L^0 u^0, \quad (x,t) : x \in \mathbf{E}^N,\ t>0 \\ u^0\Big|_{t=0} = \widetilde{\varphi}(x), \quad x \in \mathbf{E}^N, \end{cases}$$

where $L^0 = \sum_{i,j=1}^{N} a_{ij}^0 \partial^2/(\partial x_i \partial x_j)$, $\|a_{ij}^0\|_{N\times N}$ — so called averaged matrix with constant coefficient [2], $\widetilde{\varphi}(x)$ — is bounded initial function on $\mathbf{E}^N$.

The averaged matrix $a^0 = \|a_{ij}^0\|_{N\times N}$ is also simmetric and satisfies the elliptic conditions

$$\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}^0 \xi_i \xi_j \le \lambda |\xi|^2, \quad \lambda > 0.$$

The initial function $\widetilde{\varphi}(x)$ in (4) is fulfillment of initial function $\varphi(x)$ in (1) on S.

We assume that fulfilment function $\widetilde{\varphi}(x)$ $x \in \mathbf{E}^N$, is bounded and satisfies conditions

$$(5)\qquad \int_{\square^i} \varphi(x)\,d\mu(x) = \int_{\square^i} \widetilde{\varphi}(x)\,dx$$

on each cell $\square_i = \{i \le x_k \le i+1;\ k = 1,\dots,N,\ i \in \mathbf{Z}\}$.

We have the following assertions

THEOREM 1. *The solutions of the Cauchy problems (1), (4) satisfies the following property: exist the limit of difference*

$$\lim_{t\to\infty}(u(x,t)-u^0(x,t))=0, \tag{6}$$

on each $x \in S$.

From this closeness theorem we can to obtain the criterium for stabilizations of the solutions of the Cauchy problem (1).

$$\lim_{t\to\infty} u(x,t)=A, \quad x\in S$$

from well known pointwize criterium of stabilization of the solutions of the Cauchy problem (4).

$$\lim_{t\to\infty} u^0(x,t)=A, \quad x\in S\subset \mathbf{E}^N$$

(see [3]–[6]).

THEOREM 2. *If the fulfilment function* $\tilde{\varphi}(x)$ *in (4) is connected with initial function* $\varphi(x)$ *in (1) by conditions (5), then the solutions* $u(x,t)$ *of the Cauchy problem (1) stabilizes on* S

$$\lim_{t\to\infty} u(x,t)=A, \quad x\in S$$

if and only if the following limit of ellipcoidal averaged value of initial function $\varphi(x)$ *exist*

$$\lim_{R\to\infty}\frac{1}{\gamma_N \mathbf{R}^N}\int\limits_{(By,y)\le \mathbf{R}^2}\varphi(y)\,d(\mu)=A,$$

where B — *in inverse matrix for averaged matrix* a^0, γ_N — *is area of the unit ellipsoid in* $\mathbf{E}^N$.

3. Outline of proofs. For fixed $\varepsilon>0$ we consider the compressed net S_ε with variables x/ε, and definite the Cauchy problem for parabolic equation (1)

$$\frac{\partial u^\varepsilon}{\partial t}=\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon), \tag{1_ε}$$

with initial function

$$u^\varepsilon(x,0)=f^\varepsilon(x), \quad f(x)\in C_0^\infty(\mathbf{E}^N)$$

Applying the real Laplace transform to solution of the problem (1_ε) on variable $t > 0$, we obtain following problem in $W^{1,2}(\mathbf{E}^N, d\mu^\varepsilon)$

$$-\operatorname{div}(a^\varepsilon(x)w^\varepsilon) + pw^\varepsilon = f^\varepsilon, \quad p > 0, \tag{7}$$

where $W^{1,2}(\mathbf{E}^N, d\mu^\varepsilon)$ is a closure of functions $w \in C_0^\infty(\mathbf{E}^N)$ in the norm

$$||w||_{W^{1,2}(\mathbf{R}^N, d\mu^\varepsilon)} = \left[\int\limits_{\mathbf{E}^N} (|w^\varepsilon|^2 + |\nabla w^\varepsilon|^2)\, d\mu^\varepsilon\right]^{1/2}$$

and $w^\varepsilon(x, \rho) = \int\limits_0^\infty e^{-pt}u^\varepsilon(x,t)\,dt$, $p > 0$ the Laplace transform of function $u^\varepsilon(x,t)$. Applying the well-known average theorem 6.3 from [1], we obtain, that the solution w^ε of the problem (7) satisfies the following limit relating: for any $\eta \in C_0^\infty(\mathbf{E}^N)$ the limits exists

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbf{E}^N} \eta(x)w^\varepsilon(x,p)\, d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} \eta(x)w^0(x,p)\, dx \tag{8}$$

$$\lim_{\varepsilon\to 0+} \int\limits_{\mathbf{E}^N} [w^\varepsilon(x,p)]^2\, d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} [w^0(x,p)]^2\, dx \tag{9}$$

where w^0 is the solution of the average problem in $W^{1,2}(\mathbf{E}^N, dx)$

$$-\operatorname{div}(a^0\nabla w^0) + pw^0 = f^0, \tag{10}$$

where a^0 is average constant matrix, $f^0 \in C_0^\infty(\mathbf{E}^N)$.

After that we can to apply the well-known Trotter-Kato theorem [7], which imply that the following limits exist

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbf{E}^N} \eta(x)u^\varepsilon(x,t)\, d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} \eta(x)v^0(x,t)\, dx, \quad \forall\eta \in C_0^\infty(\mathbf{E}^N) \tag{11}$$

$$\lim_{\varepsilon\to 0+} \int\limits_{\mathbf{E}^N} [u^\varepsilon(x,t)]^2\, d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} [v^0(x,t)]^2\, dx, \quad \forall\eta \in C_0^\infty(\mathbf{E}^N) \tag{12}$$

for any fixed $t > 0$, where v^0 is the solution of the Cauchy problem (4) with average matrix a^0 and initial function $f^0 \in C_0^\infty(\mathbf{E}^N)$.

This is main property whish we can obtain from standard average theory. But in order to prove theorem 1 we must to bring some refinements in average theory. We have following result

THEOREM 3. *If initial function f in the Cauchy problem (1_ε) satisfies limit condition $f^\varepsilon \in L^\infty(\mathbf{E}^N, d\mu^\varepsilon)$ and the following limit exist*

$$\lim_{\varepsilon\to 0}\int_{\mathbf{E}^N} f^\varepsilon(x)\eta(x)\,d\mu^\varepsilon = \int_{\mathbf{E}^N} f^0(x)\eta(x)\,dx \tag{13}$$

for any $\eta(x) \in C_0^\infty(\mathbf{E}^N)$, then the following limit exist

$$\lim_{\varepsilon\to 0}\int_{\mathbf{E}^N} u^\varepsilon(x,t)\eta(x)\,d\mu^\varepsilon = \int_{\mathbf{E}^N} u^0(x,t)\eta(x)\,dx, \quad t>0 \tag{14}$$

for any $\eta(x) \in C_0^\infty(\mathbf{E}^N)$, where $u^0(x,t)$ is the solutions of the Cauchy problem with initial function $u^0(x,0) = f^0(x)$: i.e.

$$\frac{\partial u^0}{\partial t} = L^0 u^0, \quad u^0\Big|_{t=0} = f^0(x). \tag{15}$$

P r o o f of the theorem 3. Let us assume that limit conditions (13) holds for any function $\eta(x) \in C_0^\infty(\mathbf{E}^N)$. From hypothesis $|f^\varepsilon(x)| < M$ it follows that exists subsequence $\{f^\varepsilon\}$, which is weakly convergence to f^0 ($f^\varepsilon \xrightarrow{\varepsilon\to 0} f^0$ weakly in $L^2(\mathbf{E}^N, d\mu^\varepsilon)$).

Now by applaying Green formula for solutions of the Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t} = L u^\varepsilon, \quad u^\varepsilon\Big|_{t=0} = f^\varepsilon(x) \tag{16}$$

and

$$\frac{\partial v^\varepsilon}{\partial t} = L^0 v^\varepsilon, \quad v^\varepsilon\Big|_{t=0} = \eta(x), \tag{17}$$

where $\eta(x) \in C_0^\infty(\mathbf{E}^N)$, we have, that following equality

$$\int_{\mathbf{E}^N} u^\varepsilon(x,t_0)\eta(x)\,d\mu^\varepsilon = \int_{\mathbf{E}^N} v^\varepsilon(x,t_0) f^\varepsilon(x)\,d\mu^\varepsilon, \quad t_0>0 \tag{18}$$

holds. Passing $\varepsilon \to 0$ in the left of (18) we have

$$\lim_{\varepsilon\to 0}\int_{\mathbf{E}^N} u^\varepsilon(x,t_0)\eta(x)\,d\mu^\varepsilon = \int_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx, \tag{19}$$

where $u^*(x,t_0)$ — some limit point (in weak seance) of sequence $u^\varepsilon(x,t_0)$. Applying the Trotter–Kato theorem [7] in right of (18) we have that following limit exist

$$\lim_{\varepsilon\to 0+}\int\limits_{\mathbf{E}^N} v^\varepsilon(x,t_0)f^\varepsilon(x)\,d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} v^0(x,t_0)f^0(x)\,dx, \tag{20}$$

where $v^0(x,t)$ is the solution of the average Cauchy problem (17) with $\varepsilon = 0$. From (19), (20) it follows that for any $\eta(x) \in C_0^\infty(\mathbf{E}^N)$

$$\int\limits_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx = \int\limits_{\mathbf{E}^N} v^0(x,t_0)f^0(x)\,dx. \tag{21}$$

Applying Green formula in the right side of (21), we have

$$\int\limits_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx = \int\limits_{\mathbf{E}^N} u^0(x,t_0)\eta(x)\,dx, \tag{22}$$

for any $\eta(x) \in C_0^\infty(\mathbf{E}^N)$, $t_0 > 0$. From last equality it is easy to see that

$$u^*(x,t_0) = u^0(x,t_0), \quad t_0 > 0, \quad \text{a.e.} \quad x \in \mathbf{E}^N$$

where $u^0(x,t_0)$ is the solutions of the Cauchy problem (15).

Theorem 3 is proved.

The following statement play very important rôle in the proof of theorem 1.

LEMMA 1. *If initial fulfillment function $\tilde\varphi(x)$ in the Cauchy problem (4) and initial function $\varphi(x)$ in the Cauchy problem (1) satisfies property (5), then limit exist*

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N} \eta(x)\varphi^\varepsilon(x)\,d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} \eta(x)\varphi^0(x)\,dx \tag{23}$$

if and only if the following limit exist

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N} \eta(x)\tilde\varphi^\varepsilon(x)\,dx = \int\limits_{\mathbf{E}^N} \eta(x)\varphi^0(x)\,dx \tag{24}$$

for any $\eta(x) \in C_0^\infty(\mathbf{R}^N)$.

For proof lemma 1 we introduce notation $\Omega \equiv \operatorname{supp}\eta(x)$ — support of functions $\eta(x) \in C_0^\infty(\mathbf{E}^N)$. $N(\varepsilon) = |\Omega|/\varepsilon^N$, where $|\Omega|$ — measure Ω.

Considering $\mathbf{E}^N$ as union of N-dimensional cubes $\square_\varepsilon^i$, we have

$$\Delta = \int\limits_{\mathbf{E}^N} \eta(x)\varphi^\varepsilon(x)\,d\mu^\varepsilon(x) - \int\limits_{\mathbf{E}^N} \eta(x)\widetilde{\varphi}(x)\,dx =$$

$$= \sum_{k=1}^{N} \Bigg[\int\limits_{\square_\varepsilon^k} \eta(x)\varphi^\varepsilon(x)\,d\mu^\varepsilon(x) - \int\limits_{\square_\varepsilon^k} \eta(x)\widetilde{\varphi}(x)\,dx \Bigg].$$

Let x_k — is center of cubes $\square_k^\varepsilon$, then we have identity

$$\Delta = \sum_{k=1}^{N} \Bigg\{ \int\limits_{\square_\varepsilon^k} [\eta(x) - \eta(x_k)]\varphi^\varepsilon(x)\,d\mu^\varepsilon(x) + \eta(x_k) \int\limits_{\square_\varepsilon^k} \varphi^\varepsilon(x)\,dx -$$

$$- \int\limits_{\square_\varepsilon^k} [\eta(x) - \eta(x_k)]\widetilde{\varphi}^\varepsilon(x)\,d\mu^\varepsilon(x) - \eta(x_k) \int\limits_{\square_\varepsilon^k} \widetilde{\varphi}^\varepsilon(x)\,dx \Bigg\}.$$

From condition (5) it follows that

$$\Delta = \sum_{k=1}^{N} \int\limits_{\square_\varepsilon^k} [\eta(x) - \eta(x_k)]\varphi^\varepsilon(x)\,d\mu^\varepsilon(x) - \sum_{k=1}^{N} \int\limits_{\square_\varepsilon^k} [\eta(x) - \eta(x_k)]\widetilde{\varphi}^\varepsilon(x)\,dx.$$

By applying triangle inequality we have

$$|\Delta| \le \sum_{k=1}^{N} \max_{\square_\varepsilon^k} |\eta(x) - \eta(x_k)| \int\limits_{\square_\varepsilon^k} |\varphi^\varepsilon(x)|\,d\mu^\varepsilon(x) +$$

$$+ \sum_{k=1}^{N} \max_{\square_\varepsilon^k} |\eta(x) - \eta(x_k)| \int\limits_{\square_\varepsilon^k} |\widetilde{\varphi}^\varepsilon(x)|\,dx.$$

Taking into account that function $\eta(x)$ is uniformly continuous on support Ω, i.e.

$$\max_{x, x_k \in \square_\varepsilon^k} |\eta(x) - \eta(x_k)| < \frac{\varepsilon}{2M\,|\Omega|}$$

and following evident inequalities

$$\max_{1\le k\le N}\int\limits_{\Box_\varepsilon^k}|\varphi^\varepsilon(x)|\,d\mu^\varepsilon(x)\le M\varepsilon^N,$$

$$\max_{1\le k\le N}\int\limits_{\Box_\varepsilon^k}|\tilde\varphi^\varepsilon(x)|\,dx\le M\varepsilon^N$$

we have:

$$|\Delta|\le\sum_{k=1}^N\frac{\varepsilon}{M|\Omega|}\cdot\varepsilon^N\cdot M.$$

From definition $N(\varepsilon)=|\Omega|/\varepsilon^N$ it follows the proof of Lemma 1.

For proof theorem 1 we consider two Cauchy problem

(25)
$$\frac{\partial u^\varepsilon}{\partial t}=Lu^\varepsilon,\quad u^\varepsilon\Big|_{t=0}=\varphi^\varepsilon(x),$$

(26)
$$\frac{\partial v^\varepsilon}{\partial t}=L^0v^\varepsilon,\quad v^\varepsilon\Big|_{t=0}=\tilde\varphi^\varepsilon(x),$$

where $\tilde\varphi^\varepsilon(x)$ is some fulfilment of initial function $\varphi(x)$, and conditions (5) are holds. Now we put $\varepsilon=\frac{1}{\sqrt t}$, $t>0$. From condition (5) and lemma 1 it follows that the sequances $\{\varphi^\varepsilon(x)\}$ and $\{\tilde\varphi^\varepsilon(x)\}$ have the same weak limit:

$\varphi^\varepsilon\xrightarrow{\varepsilon\to0}\varphi^0$ weakly in $L^2(\mathbf{E}^N,d\mu^\varepsilon)$,

$\tilde\varphi^\varepsilon\xrightarrow{\varepsilon\to0}\varphi^0$ weakly in $L^2(\mathbf{E}^N,d\mu^\varepsilon)$.

It is known [8] that solution $\{u^\varepsilon(x,t)\}$ of the Cauchy problem satisfies uniform Holder conditions, with constant which does not depend on ε. From this condition and theorem 3 it follows that following limit

(27)
$$\lim_{t\to\infty}u^{\frac{1}{\sqrt t}}(0,1)=u^0(0,1)$$

exist. Now we must to apply Poisson formula for solution of the Cauchy problem (26) with constant coefficient, i. e.

$$v^\varepsilon(0,1)=\int\limits_{\mathbf{E}^N}K_0(x,0,1)\tilde\varphi(\varepsilon^{-1}x)\,dx$$

where $K_0(x,y,t)$ is fundamental solutions of (28). Passing $\varepsilon \to 0$ we have

$$\lim_{t\to\infty} v^{\frac{1}{\sqrt{t}}}(0,1) = u^0(0,1). \tag{28}$$

From (27), (28) it follows that theorem 1 is proved.

Proof of the theorem 2 is omitted, and it follows straightforward from theorem 1 and well known criterium of stabilization of the solution of the Cauchy problem for heat equation [3].

After this proof the theorem 2 may be made very easy as in the book [2].

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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH A RETARDED ARGUMENT

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Abstract. Some asymptotic properties are studied for the solutions of a class of impulsive differential equations of second order with retarded argument and fixed moments of impulse effect. Sufficient conditions are found for oscillation of all solutions.

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Key Words. oscillation, impulsive differential equations

1. Introduction. The impulsive differential equations are an object of intensive investigations in regard to the possibilities for their applications in science and technology. Let us mention the monographs Bainov–Simeonov [1], Bainov–Simeonov [2] and Lakshmikantham–Bainov–Simeonov [6] where various properties of the solutions of this type of differential equations were investigated.

However, the oscillation theory of the impulsive functional-differential equations is not yet elaborated in contrast to the oscillation theory of the ordinary differential equations with deviating argument (see the monographs Erbe–Kong–Zhang [3], Györi–Ladas [4], Ladde–Lakshmikantham–Zhang [5] and the bibliography therein).

In the present work we study some asymptotic properties of the solutions of a class of impulsive differential equations of second order with retarded argument and fixed moments of impulse effect. Sufficient conditions are found for oscillation of all solutions of the equation under consideration.

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2. Preliminary notes. Consider the impulsive differential equation of second order with a deviating argument

$$(1) \qquad (r(t)y'(t))' - \sum_{i=1}^{n} p_i(t)y(g_i(t)) = 0, \qquad t \neq \tau_k, k \in N,$$

$$\Delta(r(\tau_k)y'(\tau_k)) - \sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) = 0, \qquad k \in N,$$

with initial condition

$$(2) \qquad y(t) = \varphi(t), \quad t \in E_{t_0} y'(t_0+0) = y_0', \quad y(t_0+0) = y_0,$$

where

$$E_{t_0} = \{t_0\} \cup \{g_i(t); g_i(t) < t_0, t \geq t_0, i \in N_n = \{1, 2, \ldots, n\}\},$$

and $\varphi \in C(E_{t_0}, R)$, t_0 is fixed number,$t_0 \in R_+ = (0, +\infty)$.

Here $\Delta(r(\tau_k)y'(\tau_k)) = r(\tau_k+0)y'(\tau_k+0) - r(\tau_k-0)y'(\tau_k-0)$; $y(\tau_k+0) = y(\tau_k-0) = y(\tau_k)$; $y'(\tau_k-0) = y'(\tau_k)$; $r(\tau_k-0) = r(\tau_k)$; $\tau_1, \tau_2, \ldots$ are the moments of impulse effect.

We denote by $PC(R_+, R)$ the set of all functions $u: R_+ \to R$, which are continuous for $t \in R_+$, $t \neq \tau_k$, $k \in N$, continuous from the left for $t \in R_+$ and have discontinuity of the first kind at the points $\tau_k \in R_+$, $k \in N$.

Introduce the following conditions:

H1. $0 < t_0 < \tau_1 < \tau_2 < \ldots$, $\lim_{k\to+\infty} \tau_k = +\infty$.

H2. $g_i \in C(R_+, R)$, $g_i'(t) \geq 0$, $g_i(t) \leq t$, $\lim_{t\to+\infty} g_i(t) = +\infty$ for $i \in N_n$.

H3. $r \in PC(R_+, R_+)$, $r(\tau_k + 0) > 0$ for $\tau_k \in R_+, k \in N$.

H4. The function $p_i \in PC(R_+, R_+)$, $p_{ki} \geq 0$, $k \in N$, $i \in N_n$.

H5.$\lim_{t\to+\infty} R(t) = +\infty$, where

$$R(t) = \int_0^t \frac{ds}{r(s)}.$$

Definition 1. *The function $y(t)$ is said to be a solution of equation (1) in R_+ if:*

1. $y(t)$ is continuous on R_+

2. $y(t)$ is twice differentiable for $t \in R_+$, $t \neq \tau_k$ and satisfies (1) in R_+.

Definition 2. *The nonzero solution $y(t)$ of the problem (1), (2) is said to be* nonoscillating *if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise the solution $y(t)$ is said to* oscillate.

DEFINITION 3. *The solution $y(t)$ of the equation (1) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and*

$$\sup\{|y(t)|:\ t \geq T\} > 0 \quad \textit{for each } T \geq T_y.$$

DEFINITION 4. *The regular solution $y(t)$ of the equation (1) is said to be eventually positive(eventually negative), if there exists $T > 0$ such that $y(t) > 0$ $(y(t) < 0)$ for $t \geq T$.*

Let S denote the set of all solutions of equation (1) . We introduce the following sets:

$$S^{+\infty} = \{y \in S:\ \lim_{t\to+\infty} y(t) = +\infty, \quad \lim_{t\to+\infty} r(t)y'(t) = +\infty\},$$
$$S^{-\infty} = \{y \in S:\ \lim_{t\to+\infty} y(t) = -\infty, \quad \lim_{t\to+\infty} r(t)y'(t) = -\infty\},$$
$$S^{\sim} = \{y \in S:\ y(t) \text{ is oscillatory solution}\}.$$

3. Main results. THEOREM 1.

Let conditions H1 — H5 *be met.*

Then:

1. $\varphi(t) \geq 0$ on E_{t_0}, $y_0 \geq 0$, $y_0' \geq 0$ imply that the solution $y(t;\varphi, y_0, y_0')$ of equation (1) is a nonnegative and nondecreasing function on $[t_0, +\infty)$.

2. $\varphi(t) \leq 0$ on E_{t_0}, $y_0 \leq 0$, $y_0' \leq 0$ imply that the solution $y(t;\varphi, y_0, y_0')$ of equation (1) is a nonpositive and nonincreasing function on $[t_0, +\infty)$.

Proof. Integrating (1) from t_0 to t $(t \geq t_0)$, we obtain

$$(3)\quad r(t)y'(t) = r(t_0)y'(t_0) + \int_{t_0}^{t} \sum_{i=1}^{n} p_i(s)y(g_i(s))ds + \sum_{t_0 \leq \tau_k < t} \sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)).$$

We divide (3) to $r(t) > 0$, integrate the equality from t_0 to t and obtain

$$(4)\qquad y(t) = y(t_0) + \int_{t_0}^{t} \frac{r(t_0)y_0'}{r(s)} ds +$$

$$+\int_{t_0}^{t} \frac{1}{r(u)} [\int_{t_0}^{u} \sum_{i=1}^{n} p_i(s)y(g_i(s))ds + \sum_{t_0 \leq \tau_k < u} \sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))]du$$

or

$$(5)\qquad y(t) = y(t_0) + r(t_0)y_0'[R(t) - R(t_0)] +$$

$$+\int_{t_0}^{t}[R(t)-R(s)]\sum_{i=1}^{n}p_i(s)y(g_i(s))ds+\sum_{t_0\le\tau_k<t}\sum_{i=1}^{n}[R(t)-R(\tau_k)]p_{ki}y(g_i(\tau_k)).$$

Then (3) and (5) imply the assertion of Theorem 1.

THEOREM 2. *Let conditions* H1 — H5 *be met.*

Then:

1. $\varphi(t)\ge 0$ *on* E_{t_0}, $y_0>0$, $y_0'>0$ *imply that the solution of the equation (1)* $y(t;\varphi,y_0,y_0')\in S^{+\infty}$ *and* $y(t;\varphi,y_0,y_0')>0$ *on* $(t_0,+\infty)$.

2. $\varphi(t)\le 0$ *on* E_{t_0}, $y_0<0$, $y_0'<0$ *imply that the solution of the equation (1)* $y(t;\varphi,y_0,y_0')\in S^{-\infty}$ *and* $y(t;\varphi,y_0,y_0')<0$ *on* $(t_0,+\infty)$.

Proof. From Theorem 1 $y(t)=y(t;\varphi,y_0')\ge 0$ for $t\ge t_0$, and from (5), $y(t)\ge r(t_0)y_0'[R(t)-R(t_0)]$.

From the above inequality as $t\to+\infty$, and from condition H5 it follows that $y\in S^{+\infty}$ and $y(t)>0$ for $t\ge t_0$.

THEOREM 3. *Let conditions* H1 — H5 *be met.Further assume that* $y_1(t)$ *and* $y_2(t)$ *have the same initial function with* $y_{10}'>y_{20}'$. *Then* $y_1(t)>y_2(t), y_1'(t)>y_2'(t)$ *on* $t\ge t_0$ *and* $\lim_{t\to+\infty}(y_1(t)-y_2(t))=+\infty$.

Proof. Consider $y(t)=y_1(t)-y_2(t)$ and note that $y(t)$ is a solution of equation (1) with initial function $\varphi\equiv 0$ and $y_0'=y_{10}'-y_{20}'>0$. From Theorem 2 $y(t)\in S^{+\infty}$ and $y(t)>0$ on $[t_0,+\infty)$ and from (3) we have $y'(t)>0$. The proof is complete.

THEOREM 4. *Let conditions* H1 — H5 *be met.Then for every initial function* φ, *equation (1) has no more than one bounded solution on* $[t_0,+\infty)$.

Proof. For the sake of contradition suppose the opposite. Let $y_1(t)$ and $y_2(t)$ be bounded solutions with $y_{10}'>y_{20}'$. This implies that $\mid y_1(t)-y_2(t)\mid$ is bounded. On the other hand, by Theorem 3 $\mid y_1-y_2\mid\in S^{+\infty}$. Th

e contradiction obtained shows the validity of the theorem.

THEOREM 5. *Let the following conditions hold:*

1. Conditions H1 — H5 *are met.*

2.

$$\int_0^{\infty}R(s)\sum_{i=1}^{\infty}p_i(s)ds+\sum_{k=1}^{\infty}\sum_{i=1}^{n}R(\tau_k)p_{ki}=+\infty.$$

Then all bounded solutions of equation (1) either tend to zero as $t\to+\infty$ *or oscillate.*

Proof. Let $y(t)$ be a positive and bounded solution of equation (1) for $t\ge t_1\ge 0$. It is clear that $y(g_i(t))>0$ for $t\ge t_{1i}\ge t_1, i\in N_n$. Then $(r(t)y'(t))'>0$ and $\Delta(r(\tau_k)y'(\tau_k))>0$ for $t,\tau_k\ge t_2=t_1+\bar{t}$, $\bar{t}=\max\{t_{1i}, i\in N_n\}$.

Therefore $r(t)y'(t)$ is an increasing function for $t \geq t_2$.
The following two cases are possible:
Case 1. There exists $t_3 \geq t_2$ such that $r(t_3)y'(t_3) > 0$;
Case 2. $r(t)y'(t) \leq 0$, for $t \geq t_2$.
We shall consider first case 1.
Since $r(t)y'(t)$ is an increasing function for $t \geq t_3$, it follows that

$$y'(t) \geq \frac{r(t_3)y'(t_3)}{r(t)}, \qquad t \geq t_3.$$

We integrate the last inequality from t_3 to t $(t \geq t_3)$ and conclude

$$y(t) \geq y(t_3) + r(t_3)y'(t_3) \int_{t_3}^{t} \frac{ds}{r(s)}.$$

Now the above inequality as $t \to +\infty$, and condition H5 imply that $\lim_{t\to+\infty} y(t) = +\infty$ which contradicts the assumption that y is a bounded solution of the equation (1).

Next we shall consider case 2. Since $r(t)y'(t)$ is an increasing function for $t \geq t_2$, it follows that there exists the finite limit $\lim_{t\to+\infty} r(t)y'(t) = c \leq 0$.

Let us suppose that $\lim_{t\to+\infty} r(t)y'(t) = c < 0$. Then $r(t)y'(t) < c$ for $t \geq t_2$. Hence

$$y(t) \leq y(t_2) + c \int_{t_2}^{t} \frac{du}{r(u)}.$$

After passing to limit as $t \to +\infty$ in the abobe inequality it follows that $\lim_{t\to+\infty} y(t) = -\infty$, which contradicts the assumption that y is a positive solution. Thus,

$$\lim_{t\to+\infty} r(t)y'(t) = 0. \tag{6}$$

We shall prove that $\lim_{t\to+\infty} y(t) = 0$.

Since $y(t) > 0$, $y'(t) \leq 0$, for $t \geq t_2$, then there exists the finite limit $\lim_{t\to+\infty} y(t) \geq 0$.

Suppose that $\lim_{t\to+\infty} y(t) = A > 0$.

Now, we integrate (1) from t_2 to t and obtain the equality

$$r(t)y'(t) = r(t_2)y'(t_2) + \sum_{t_2\leq\tau_k<t} \sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) + \int_{t_2}^{t} \sum_{i=1}^{n} p_i(s)y(g_i(s))ds. \tag{7}$$

Passing to the limit in (7) as $t \to +\infty$ and having in mind (6), we deduce

$$r(t_2)y'(t_2) = -\sum_{t_2 \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) - \int_{t_2}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))ds.$$

Since t_2 could be made arbitrarily large we obtain

$$r(u)y'(u) = -\sum_{u \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) - \int_{u}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))ds. \tag{8}$$

We divide (8) to $r(t) > 0$, integrate the equality obtained from $\bar{t}$ to t, ($\bar{t} \ge t_2$) and obtain

$$y(t) = y(\bar{t}) - \int_{\bar{t}}^{t}\frac{1}{r(u)}\sum_{u \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))du - \tag{9}$$

$$-\int_{\bar{t}}^{t}\frac{1}{r(u)}\int_{u}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))dsdu$$

Set

$$A(t) = -\int_{\bar{t}}^{t}\frac{1}{r(u)}\sum_{u \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))du$$

and

$$B(t) = -\int_{\bar{t}}^{t}\frac{1}{r(u)}\int_{u}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))dsdu.$$

We shall estimate $A(t)$ and $B(t)$ separately.

$$A(t) = -\int_{\bar{t}}^{t}\frac{1}{r(u)}\Big(\sum_{\bar{t} \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) - \sum_{\bar{t} \le \tau_k < u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))\Big)du =$$

$$= -\int_{\bar{t}}^{t}\frac{1}{r(u)}\sum_{\bar{t} \le \tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))du + \int_{\bar{t}}^{t}\frac{1}{r(u)}\sum_{\bar{t} \le \tau_k < u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))du =$$

$$= -\sum_{\bar t\le\tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))\int_{\bar t}^{t}\frac{du}{r(u)} + \int_{\bar t}^{t}\sum_{\bar t\le\tau_k<u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))dR(u) =$$

$$= -[R(t)-R(\bar t)]\sum_{\bar t\le\tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) + R(u)\sum_{\bar t\le\tau_k<u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))\Big|_{\bar t}^{t} -$$

$$-\int_{\bar t}^{t} R(u)d\Big(\sum_{\bar t\le\tau_k<u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))\Big) =$$

$$= -[R(t)-R(\bar t)]\sum_{\bar t\le\tau_k}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) + R(t)\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) -$$

$$-\int_{\bar t}^{t} R(u)d\Big(\sum_{\bar t\le\tau_k<u}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k))\Big) \le$$

$$\le -[R(t)-R(\bar t)]\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) + R(t)\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n} p_{ki}y(g_i(\tau_k)) -$$

$$-\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n} R(\tau_k)p_{ki}y(g_i(\tau_k)) = \sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}(R(\bar t)-R(\tau_k))p_{ki}y(g_i(\tau_k))$$

or

$$A(t) \le \sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}(R(\bar t)-R(\tau_k))p_{ki}y(g_i(\tau_k)) \tag{10}$$

For $B(t)$ we have

$$B(t) = -\int_{\bar t}^{t}\frac{1}{r(u)}\int_{u}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))dsdu =$$

$$= -\int_{\bar t}^{t}\frac{1}{r(u)}\Big(\int_{\bar t}^{\infty}\sum_{i=1}^{n} p_i(s)y(g_i(s))ds - \int_{\bar t}^{u}\sum_{i=1}^{n} p_i(s)y(g_i(s))ds\Big)du =$$

$$= -\int_{\bar t}^{\infty}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\int_{\bar t}^{t}\frac{du}{r(u)} + \int_{\bar t}^{t}\Big(\int_{\bar t}^{u}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big)dR(u) =$$

$$= -(R(t)-R(\bar t))\int_{\bar t}^{\infty}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds + R(u)\int_{\bar t}^{u}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big|_{\bar t}^{t} -$$

$$-\int_{\bar t}^{t}R(u)d\Big(\int_{\bar t}^{u}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big) =$$

$$= -(R(t)-R(\bar t))\int_{\bar t}^{\infty}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds + R(t)\int_{\bar t}^{t}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds -$$

$$-\int_{\bar t}^{t}R(u)\sum_{i=1}^{n}p_i(u)y(g_i(u))du \le$$

$$\le -(R(t)-R(\bar t))\int_{\bar t}^{t}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds + R(t)\int_{\bar t}^{t}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds -$$

$$-\int_{\bar t}^{t}R(s)\sum_{i=1}^{n}p_i(s)y(g_i(s))ds = \int_{\bar t}^{t}(R(\bar t)-R(s))\sum_{i=1}^{n}p_i(s)y(g_i(s))ds,$$

i.e.

$$B(t) \le \int_{\bar t}^{t}(R(\bar t)-R(s))\sum_{i=1}^{n}p_i(s)y(g_i(s))ds \tag{11}$$

Now (9),(10) and (11) imply

$$y(t) \le y(\bar t) + \sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}(R(\bar t)-R(\tau_k))p_{ki}y(g_i(\tau_k)) +$$

$$+\int_{\bar t}^{t}(R(\bar t)-R(s))\sum_{i=1}^{n}p_i(s)y(g_i(s))ds=$$

$$=y(\bar t)+R(\bar t)\Big[\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}p_{ki}y(g_i(\tau_k))+\int_{\bar t}^{t}\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big]-$$

$$-\Big[\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}R(\tau_k)p_{ki}y(g_i(\tau_k))+\int_{\bar t}^{t}R(s)\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big].$$

From (7) it follows

$$y(t)\le y(\bar t)+R(\bar t)[r(t)y'(t)-r(\bar t)y'(\bar t)]-$$

$$-\Big[\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}R(\tau_k)p_{ki}y(g_i(\tau_k))+\int_{\bar t}^{t}R(s)\sum_{i=1}^{n}p_i(s)y(g_i(s))ds\Big].$$

Since $\lim_{t\to+\infty}y(t)=A$ it follows that

$$y(t)\le y(\bar t)+R(\bar t)[r(t)y'(t)-r(\bar t)y'(\bar t)]-$$

$$-\frac{A}{2}\Big[\sum_{\bar t\le\tau_k<t}\sum_{i=1}^{n}R(\tau_k)p_{ki}+\int_{\bar t}^{t}R(s)\sum_{i=1}^{n}p_i(s)ds\Big].$$

The last inequality,(6) and the condition 2 of the theorem imply $\lim_{t\to+\infty}y(t)=-\infty$ which contradicts the assumption that y is a positive bounded solution of the equation (1).

Therefore $\lim_{t\to+\infty}y(t)=0$.

THEOREM 6. *Let the following conditions hold:*

1. Conditions H1 — H5 *are met.*

2. $r'(t)\ge 0,\quad t\in R_+$.

3.

$$\limsup_{t\to+\infty}\frac{1}{r(t)}\{\int_{g(t)}^{t}(s-g(t))\sum_{i=1}^{n}p_i(s)ds+\sum_{g(t)\le\tau_k<t}[(\tau_k-g(t))\sum_{i=1}^{n}p_{ki}]\}>1$$

where $g(t)=\max\{g_i(t), i\in N_n\}$.

Then all bounded solutions of equation (1) are oscillatory.

Proof. Let $y(t)$ be a bounded and nonoscillatory solution of equation (1). Without loss of generality we assume $y(t)>0$ for $t\ge t_1(t_1\ge t_0)$. Then $y(g_i(t))>0$ for $t\ge t_{1i}\ge t_1, i\in N_n$. Analogously to the proof of Theorem 5 we obtain the inequality $r(t)y'(t)\le 0$, $t\ge t_2=t_1+\bar{t}, \bar{t}=\max\{t_{1i}, i\in N_n\}$.

Integrating (1) from s to t $(t>s\ge t_2)$, we have

$$(12)\, r(t)y'(t)=r(s)y'(s)+\int_{s}^{t}\sum_{i=1}^{n}p_i(\sigma)y(g_i(\sigma))d\sigma+\sum_{s\le\tau_k<t}\sum_{i=1}^{n}p_{ki}y(g_i(\tau_k)).$$

Now we integrate (12) from $g(t)$ to t, and obtain

$$r(t)y'(t)[t-g(t)]=\int_{g(t)}^{t}r(s)y'(s)ds+\int_{g(t)}^{t}[\int_{s}^{t}\sum_{i=1}^{n}p_i(\sigma)y(g_i(\sigma))d\sigma+$$

$$+\sum_{s\le\tau_k<t}\sum_{i=1}^{n}p_{ki}y(g_i(\tau_k))]ds$$

or

$$0\ge r(t)y(t)-r(g(t))y(g(t))-\int_{g(t)}^{t}y(s)dr(s)+$$

$$+\sum_{g(t)\le\tau_k<t}(\tau_k-g(t))\sum_{i=1}^{n}p_{ki}y(g_i(\tau_k))+\int_{g(t)}^{t}(\sigma-g(t))\sum_{i=1}^{n}p_i(\sigma)y(g_i(\sigma))d\sigma\ge$$

$$\ge r(t)y(t)-r(g(t))y(g(t))-y(g(t))[r(t)-r(g(t))]+$$

$$+y(g(t))\int_{g(t)}^{t}[\sigma-g(t)]\sum_{i=1}^{n}p_i(\sigma)d\sigma+y(g(t))\sum_{g(t)\le\tau_k<t}(\tau_k-g(t))\sum_{i=1}^{n}p_{ki}=$$

$$= r(t)y(t) - r(t)y(g(t)) + y(g(t))[\int_{g(t)}^{t} [\sigma - g(t)] \sum_{i=1}^{n} p_i(\sigma) d\sigma +$$

$$+ \sum_{g(t) \le \tau_k < t} (\tau_k - g(t)) \sum_{i=1}^{n} p_{ki}].$$

Dividing the last inequality to $r(t)y(g(t))$, we obtain

$$\frac{y(t)}{y(g(t))} + \left[\frac{1}{r(t)} [\int_{g(t)}^{t} (\sigma - g(t)) \sum_{i=1}^{n} p_i(\sigma) d\sigma + \sum_{g(t) \le \tau_k < t} (\tau_k - g(t)) \sum_{i=1}^{n} p_{ki}] - 1 \right] \le 0.$$

The last inequality contradicts condition 3 of Theorem 6.

COROLLARY 1. *Let the conditions of Theorem 6 be satisfied. Then: 1. The inequality*

$$(r(t)y'(t))' - \sum_{i=1}^{n} p_i(t) y(g_i(t)) \ge 0, \quad t \ne \tau_k, k \in N, \tag{13}$$

$$\Delta(r(\tau_k)y'(\tau_k)) - \sum_{i=1}^{n} p_{ki} y(g_i(\tau_k)) \ge 0, \quad \Delta y(\tau_k) = 0, \quad k \in N$$

has no bounded, eventually positive solution.

2. The inequality

$$(r(t)y'(t))' - \sum_{i=1}^{n} p_i(t) y(g_i(t)) \le 0, \quad t \ne \tau_k, k \in N, \tag{14}$$

$$\Delta(r(\tau_k)y'(\tau_k)) - \sum_{i=1}^{n} p_{ki} y(g_i(\tau_k)) \le 0, \quad \Delta y(\tau_k) = 0, \quad k \in N$$

has no bounded, eventually negative solution.

COROLLARY 2. *Let the following conditions be satisfied:*

1. Conditions H1 — H5 *are met.*

2.

$$\limsup_{t \to +\infty} \frac{1}{r(t)} [\int_{g(t)}^{t} (\sigma - g_i(t)) p_i(\sigma) d\sigma + \sum_{g(t) \le \tau_k < t} (\tau_k - g_i(t)) p_{ki}] > 1$$

for some $i \in N_n$, $g(t) = \max\{g_i(t), i \in N_n\}$.

Then:

1. The inequality (13) has no bounded, eventually positive solution.

2. The inequality (14) has no bounded, eventually negative solution.

3. All bounded solutions of the equation (1) are oscillatory.

Introduce the following conditions:

H6. $g_i \in C(R_+, R)$; $g_i(t) \leq t$ for $t \in R_+$; $\lim_{t\to+\infty} g_i(t) = +\infty$ for $i \in N_n$.

H7. There exists a set $K = \{k_1, k_2, \ldots, k_l\} \subset N_n$ such that $1 \leq k_1 < k_2 < \ldots < k_l \leq n$ and $g'_k(t) \geq 0, k \in K, t \in R_+$.

THEOREM 7. *Let the following conditions hold:*

1. Conditions H1, H4 — H7 *are met.*

2. $r(t) \equiv 1, \quad t \in R_+$.

3.

$$\limsup_{t\to+\infty} \sum_{k\in K} \int_{g^*(t)}^{t} [g_k(t) - g_k(s)]p_k(s)ds > 1,$$

where $g^*(t) = \max\{g_k(t), k \in K\}$.

Then all bounded solutions of equation (1) are oscillatory.

Proof. Let $y(t)$ be a bounded and nonoscillatory solution of equation (1). Without loss of generality we assume $y(t) > 0$ for $t \geq t_1$ ($t_1 \geq t_0$). Then $y(g_i(t)) > 0$ for $t \geq t_2 \geq t_1, i \in N_n$. Analogously to the proof of Theorem 5 we obtain the inequality $y''(t) > 0$ and $y'(t) < 0$ for $t \geq t_3 \geq t_2$. From these observations, we conclude that $y(t)$ is concave up and decreasing for $t \geq t_3$. Therefore, it lies above its tangent. That is, for $\bar{t}, \bar{s} \geq t_3$,

$$y(\bar{t}) + y'(\bar{t})(\bar{s} - \bar{t}) \leq y(\bar{s}).$$

We note that $\lim_{t\to+\infty} g_k(t) = +\infty$,so the above inequality implies that

$$y(g_k(t)) + y'(g_k(t))(g_k(s) - g_k(t)) \leq y(g_k(s))$$

for s, t sufficiently large, say $s, t \geq t_3$ and for all $k \in K$.Multiplying the abobe inequality by $p_k(s)$ and summing up for all $k \in K$, we get

$$\sum_{k\in K} p_k(s)y(g_k(t)) + \sum_{k\in K} y'(g_k(t))(g_k(s) - g_k(t))p_k(s) \leq$$

$$\leq \sum_{k\in K} p_k(s)y(g_k(s)) \leq \sum_{k=1}^{n} p_k(s)y(g_k(s)) = y''(s).$$

Integrating the above inequality, with respect to s, from $g^*(t)$ to t, for t sufficiently large, we obtain

$$\sum_{k\in K} y(g_k(t)) \int_{g^*(t)}^{t} p_k(s)ds + \sum_{k\in K} y'(g_k(t)) \int_{g^*(t)}^{t} (g_k(s) - g_k(t))p_k(s)ds \le$$

$$\le y'(t) - y'(g^*(t)) - \sum_{g^*(t)\le\tau_l<t} \sum_{i=1}^{n} p_{li} y(g_i(\tau_l)).$$

Since $y'(t)$ is increasing and $g_k'(t) \ge 0$ the above inequality, implies

$$y(g^*(t)) \sum_{k\in K} \int_{g^*(t)}^{t} p_k(s)ds + y(g^*(t)) \sum_{g^*(t)\le\tau_l<t} \sum_{k\in K} p_{lk} -$$

$$-y'(g^*(t))\{\sum_{k\in K} \int_{g^*(t)}^{t} (g_k(t) - g_k(s))p_k(s)ds - 1\} \le 0.$$

The last inequality contradicts condition 3 of Theorem 7.

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WHAT SHOULD BE A DISCRETE VERSION OF THE CHANTURIA-KOPLATADZE LEMMA?

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Abstract. Is it possible to formulate a discrete version of the Chanturia-Koplatadze Lemma? Some unsuccessful attempts on this way led to the publication of at least two erroneous papers. We explain the cause of the mistakes and state suitable counterexamples.

Key Words. Delay difference equations, oscillation properties

AMS(MOS) subject classification. 39A10, 39A12

It is known that many phenomena occuring in *continuous* dynamical systems are observable in *discrete* dynamical systems as well. As a result, discrete version of various facts from the theory of *differential* equations is the main source for constructing the theory of *difference* equations. However, there are deep and interesting facts from the theory of difference equations, which do not follow from discretization of "continuous" statements. Similarly, there are many "continuous" facts which do not have discrete analogues.

The following statement is well known in the theory of delay differential equations:

CHANTURIA-KOPLATADZE'S LEMMA 1 (INSIDE OF TH.2 IN [1]). *Let $x(t) > 0$ be an eventually positive solution of the first order differential equation with retarded argument*

$$x'(t) + a(t)x(t-\tau) = 0, \quad \tau > 0, \ t \geq 0 \tag{1}$$

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in which

$$(2) \qquad a(t) \geq 0, \quad \int_{t-\tau}^{t} a(s)ds \geq M > 0 \quad \forall t.$$

Then

$$(3) \qquad x(t) \geq \frac{M^2}{4} x(t-\tau) \quad \forall t \geq t_0 \quad \blacksquare$$

What should be a discrete version of this statement? It turned out that this question is not that simple. Consider the Delay-Difference Equation ($D\Delta E$)

$$(4) \qquad y_{n+1} - y_n + a_n y_{n-k} = 0, \quad k \in \mathrm{N}, \quad n \geq 1$$

The following very important open problem was formulated by G.Ladas in [2] (see Problem 7.11.4 in [8] as well):

It is known that the Ladas-Philos-Sficas condition (see [3])

$$(5) \qquad a_n \geq 0 \ \forall n, \quad \liminf_{n\to\infty} \left(\frac{1}{k}\sum a_{j+n}\right) > \frac{k^k}{(k+1)^{k+1}}$$

implies oscillation of all solutions of Eq.(4). Extend Cond.(5) to Eq.(4) with oscillating *coefficient* $\{a_n\}$ $\blacksquare$

In [4] it was declared that this problem has been solved. Unfortunately, the main results of the paper [4] (including the solution of the Ladas Problem) are wrong. This is because all results are based on an erroneous discrete version of the Chanturia-Koplatadze Lemma.

YU-B.G.ZHANG-QIANS LEMMA 1 (LEMMA 1 IN [4]). *Assume that* $\{y_n\}$ *is an eventually positive solution of Eq.(4) and*

$$(6) \qquad a_n \geq 0 \ \sum_{j=1}^{k+1} a_{n+j} \geq M > 0 \quad \forall n.$$

Then

$$(7) \qquad y_n > \frac{M^2}{4} \cdot y_{n-k} \quad \forall n \quad \blacksquare$$

Its proof is erroneous. Moreover, the statement itself is wrong. Indeed, consider the following

COUNTEREXAMPLE 1. Let in Eq.(4) be $k = 2$, $a_{3m} := 1 - \beta_m$; $a_{3m+i} = 0$, $i = 1,2$; $\lim_{n\to\infty} \beta_m = 0$, $0 < \beta_m < \frac{1}{2}$. Then $\sum_0^2 a_{n+j} > M = \frac{1}{2}$, and Cond.(6) holds. On the other hand, define $\{y_n\}$ by $y_{-2} = 0$, $y_{-1} = y_0 = 1$, $y_1 = y_2 = y_3 = 1$, $y_{3m+1} = y_{3m+2} = y_{3m+3} = \prod_{i=1}^m \beta_i$, $m \ge 1$.

Obviously, the sequence $\{y_n\}$ is an eventually positive solution of Eq.(4) with the initial conditions $y_{-2} = 0$, $y_{-1} = y_0 = 1$. But $\frac{y_n}{y_{n-2}} = \beta_m$ for $n = 3m + i$, $i = 1,2$, that is $\lim_{n\to\infty} \frac{y_n}{y_{n-2}} = 0$ and so, (7) is not valid.

Note that there exists a weeker version of YZQ-Lemma in which the statement is replaced by "then there exists a sequence $n_s \to \infty$ such that $y_{n_s} > \frac{M^2}{4} y_{n_s-k}$" (see [8], page 182 and [9]). But, unfortunately, this fact does not save the main results of [4].

Indeed, the authors assert that the equation

$$\left.\begin{array}{l} y_{n+1} - y_n + a_n y_{n-4} = 0, \ \ n \ge 1, \\ a_{9m+j} = 0.00009, \ \ j = \overline{0,3}; \quad a_{9m+j} = 0.006, \ \ j = \overline{4,8} \end{array}\right\} \tag{8}$$

has oscillatory solutions only (Example 1, [4]).

In view of the well known Erbe-Zhang Theorem (Th.2.3 in [6]), the condition $\sup a_n < \frac{k^k}{(k+1)^{k+1}}$ implies the existence at least one eventually positive solution of Eq.(4). In the case of Eq.(8) $\sup a_n = 0.006 < \frac{4^4}{5^4} \approx 0.082$, and so, Eq.(8) has at least one non-oscillatory solution.

A similar misfortune took place in the work [5] as well. Lemma 2.3 in [5] is a generalization of the YZQ-Lemma and therefore is wrong. In any case, one of the two main results of [5] (Theorem 3.2), is wrong as well. We formulate below this wrong statement for a simplest particular case.

Consider the second order difference equation

$$y_{n+1} - y_n + p_n y_{n-1} = 0, \quad n \ge 1 \tag{9}$$

in which

$$p_n \ge 0, \quad p_{n-1} + p_n \ge d > 0\,. \tag{10}$$

Suppose that

$$\limsup_{n\to\infty} (p_{n-1} + p_n) > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1}, \tag{11}$$

where d is defined by (10). Then every solution of Eq.(9) is oscillatory.

Here is a counterexample to this statement :

COUNTEREXAMPLE 2. Put in (9)

$$p_{2m-1} = \lambda = 0.0001, \quad p_{2m} = \mu = 0.9799 \quad \forall m \,. \tag{12}$$

Then $\inf(p_{n-1}+p_n) = \sup(p_{n-1}+p_n) = d = 0.98$ and
$d > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1-\frac{d^3}{2}}\right)^{-1} = 0.923.$
Then, according to Th.3.2, all solutions of Eq.(9) must be *oscillatory.*

On the other hand, the second order difference equation with 2-periodic coefficient can be solved directly :

$$\text{Eq.(9)} \Leftrightarrow \begin{cases} y_{2m+1} - y_{2m} + \mu y_{2m-1} &= 0 \\ y_{2m+2} - y_{2m+1} + \lambda y_{2m} &= 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y_{2m+2} - (1-\lambda-\mu) y_{2m} + \lambda\mu y_{2m-2} &= 0 \\ y_{2m+1} = y_{2m+2} + \lambda y_{2m} \end{cases}$$

Therefore, the general solution of Eq.(9) is $y_n = C_1 y_n^{(1)} + C_2 y_n^{(2)}$, in which
$y_{2m}^{(j)} = s_j^m$, $y_{2m+1}^{(j)} = s_j^m(s_j+\lambda)$, $j = 1,2$, and $s_{1,2}$ are the roots of the characteristic equation

$$s^2 - (1-\lambda-\mu)s + \lambda\mu = 0. \tag{13}$$

Eq.(13) has two positive roots $s_1 = 0.01142$ and $s_2 = 0.00858$ and so, *all* solutions of Eq.(9) are *non-oscillatory.*

Thus, the Ladas Problem *was not solved* in [4]. However, this Problem has been solved in our work [7] (submitted May 1992). In Remark 2 in [7] this fact was specially declared. One of the possible applications of Th.4 and Cor.4.1 [7] is the following solution of the Ladas's Problem:

THEOREM 1. *Let be* $C > \frac{k^k}{(k+1)^{k+1}}$, $\nu^2 < \frac{2(k+1)^{k+1}}{k^{2k+1}}\left[C - \frac{k^k}{(k+1)^{k+1}}\right]$ *and* $G := \overset{\infty}{\cup}\, [p_n, q_n]$, $q_n - p_n > \frac{\pi}{\nu} + k + 1$, *in which* $\{p_n\} \to \infty$ *and* $\{q_n\}$ *are arbitrary sequences.*

If $a_n \geq C \;\; \forall n \in G$ *then all solutions of Eq.(4) are oscillatory.*

No limitations are imposed on the coefficient $\{a_n\}$ outside G!!

However, the following statement looks right for the discrete version of CKL:

THEOREM 2. *Assume that* $\{y_n\}$ *is an eventually positive solution of Eq.(4) and*

$$a_n \geq 0, \ \sum_{j=1}^{k} a_{n+j} \geq M > 0 \quad \forall n. \tag{14}$$

Then $y_n > \frac{M^2}{4} \cdot y_{n-k} \quad \forall n.$

Proof. For all n consider the following two cases: $a_n \geq \frac{M}{2}$, and $a_n < \frac{M}{2}$. In the first case it is clear that

$$y_n - y_{n+1} = a_n y_{n-k} \geq \frac{M}{2} y_{n-k} \,,$$

$$y_{n-k} - y_{n+1} = \sum_{n-k}^{n} a_i y_{i-k} \geq \frac{M}{2} y_{n-k}$$

and therefore

$$y_n \geq \frac{M}{2} y_{n-k} \geq \frac{M}{2}(y_{n+1} + \frac{M}{2} y_{n-k}) \geq \frac{M^2}{4} y_{n-k}$$

In the second case, there exists n^*, $n+1 \leq n^* \leq n+k$, such that $\sum_{n}^{n^*-1} a_i < \frac{M}{2}$ and $\sum_{n}^{n^*} a_i \geq \frac{M}{2}$.

Hence $\sum_{n^*-k}^{n} a_i = \sum_{n^*-k}^{n^*-1} a_i - \sum_{n+1}^{n^*-1} a_i \geq \frac{M}{2}$.

Then $y_n - y_{n^*+1} = \sum_{n}^{n^*} a_i y_{i-k} \geq y_{n^*-k} \cdot \frac{M}{2}$, $\quad y_{n^*-k} - y_{n+1} = \sum_{n^*-k}^{n} a_i y_{i-k} \geq y_{n-k} \cdot \frac{M}{2}$.

Combining the above two inequalities, we obtain $y_n > y_{n^*-k} \cdot \frac{M}{2} \geq y_{n-k} \cdot \frac{M^2}{4}$.
□

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GENERALIZED SOLUTIONS OF THE CAUCHY PROBLEM FOR INFINITE SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider the infinite system of differential functional equations

$$D_t z_i(t,x) = f_i(t,x,z_{(t,x)}, D_x z_i(t,x)),\ i \in \mathbf{N},\ (t,x) \in [0,a_0] \times R^n$$

where $z = \{z_i\}$ and $z_{(t,x)}$ is the restriction of the function z to the set $[t-b_0,t]\times[x-b,x+b]$ and this restriction is shifted to the set $[-b_0,0] \times [-b,b]$. For this system we prove the existence of weak solution satisfying the initial condition on $[-b_0,0]\times R^n$. We consider the system of integral functional equations equivalent to differential one. Under the additional assumptions on the functions f_i, $i \in \mathbf{N}$, we obtain the existence of classical solution of the Cauchy problem for differential functional system.

Key Words. Cauchy problem, weak and classical solution, method of bicharacteristic, sequence of succesive approximations.

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1. Introduction. For any metric spaces X, Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. We will use vectorial inequalities, with the understanding that the same inequalities hold between their corresponding components.

We will denote by l^∞ the set of all infinite sequences $p = \{p_i\}$, $p_i \in R$, such that

$$|p|_\infty = \sup\{\ |p_i| : i \in \mathbf{N}\ \} < +\infty.$$

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Here N denotes the set of natural numbers.

Let $B = [-b_0, 0] \times [-b, b] \subset R^{1+n}$ where $b_0 \in R_+, b = (b_1, ..., b_n) \in R^n_+$ and $R_+ = [0, +\infty)$. For a function $z : [-b_0, a_0] \times R^n \to l^\infty$, $z = \{z_i\}, a_0 > 0$, and for a point $(t, x) \in [0, a_0] \times R^n$ we define the function $z_{(t,x)} : B \to l^\infty$ by the formula

$$z_{(t,x)}(\tau, s) = z(t + \tau, x + s), \ (\tau, s) \in B.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t - b_0, t] \times [x - b, x + b]$ and this restriction is shifted to the set B.

Let $E_0 = [-b_0, 0] \times R^n$. Suppose that the sequences of functions $f = \{f_i\}, \varphi = \{\varphi_i\}$, where

$$f_i : [0, a_0] \times R^n \times C(B, l^\infty) \times R^n \to R,$$

$$\varphi_i : E_0 \to R$$

are given.

The paper deals with the Cauchy problem

$$D_t z_i(t, x) = f_i(t, x, z_{(t,x)}, D_x z_i(t, x)), \ i \in \mathrm{N}, \tag{1}$$

$$z(t, x) = \varphi(t, x) \text{ on } E_0 \tag{2}$$

where $D_x z_i = (D_{x_1} z_i, ..., D_{x_n} z_i)$. Infinite systems with a deviated argument and differential - integral systems can be derived from (1), (2) by specializing the operator f (see section 7).

We will consider weak solutions of problem (1), (2) according to the following definition ([3]).

DEFINITION 1. *A function $u : [-b_0, a] \times R^n \to l^\infty$ where $0 < a \leq a_0, u = \{u_i\}$, is a solution of the above problem if*

(i) u is continuous on $[-b_0, a] \times R^n$ and there exist the derivatives $D_x u_i$, $i \in \mathrm{N}$, on $[0, a] \times R^n$,

(ii) the function $u(\cdot, x) : [0, a] \to l^\infty$ is absolutely continuous on $[0, a]$ for each $x \in R^n$,

(iii) for each $x \in R^n$ the function u satisfies system (1) for almost all $t \in [0, a]$ and condition (2) holds.

This class of weak solutions is placed between classical solutions and solutions in the Caratheodory sense. What is more both inclusions are strict.

In this time numerous papers were published concerning various problems for hyperbolic functional differential equations. It is not our aim to show a full review of results concerning these problems. We shall mention only those which concern existence results for the Cauchy problem with respect to nonlinear hyperbolic equations. They are [1] - [13]. The following methods are more frequently adopted for proving existence of classical or weak solutions: the method of succesive approximations, the fixed point method based on the Banach theorem, difference methods, the method of bicharacteristics.

The papers [8], [10] - [12] initiated the theory of infinite systems of first order partial functional differential equations. Initial problems on unbounded domains for almost linear systems are considered in [8]. The Chaplygin method of approximation of classical solutions is presented. The paper [12] deals with comparison theorems generated by a generalized Cauchy problem for nonlinear infinite systems. The following questions are discussed: error bounds for an approximate solution, uniqueness of the solution and its continuous dependence on the right hand sides of the system and on the initial functions. Infinite systems of functional differential inequalities on the Haar pyramid were considered in [11]. The paper [10] concerns the existence of classical solutions of the initial value problem for nonlinear functional differential systems. The result is proved by using differential inequalities and by the Banach fixed point theorem. Note that the results of the paper [10] do not cover the existence theory for systems with a deviated argument and integral differential systems of the Volterra type.

We will discuss the question of the existence of solutions to problem (1), (2). Our results are based on the method of bicharacteristics. It consists on linearization of the right hand side of equations with respect to unknown function. In the second step a quasilinear system is constructed for unknown functions and for their spatial derivatives. The system thus obtained is equivalent to a system of functional integral equations of the Volterra type. The existence and uniqueness of solutions of this system will be proved by using integral inequalities and by the Banach fixed point theorem.

Our results are generalizations of the existence theorems from [1], [5], [10].

2. Function spaces. For $x = (x_1, ..., x_n) \in R^n$ we put $|x| = |x_1| + ... + |x_n|$. We will use the symbol l_n^∞ to denote the space of sequences $r = \{r_i\}$, $r_i = (r_{i1}, ..., r_{in}) \in R^n$ for $i \in \mathrm{N}$, such that

$$|r|_{n.\infty} = \sup \{ \, |r_i| : i \in \mathrm{N} \, \} < +\infty.$$

Let $||\cdot||_0$ denote the supremum norm in the space $C(B, l^\infty)$. We will use the symbol $C_L(B, l^\infty)$ to denote the class of all functions $w \in C(B, l^\infty)$ such that

$$||w||_L = \sup\left\{\frac{|w(\tau,s)-w(\tau,\bar{s})|_\infty}{|s-\bar{s}|} : (\tau,s),(\tau,\bar{s}) \in B\right\} < +\infty.$$

Write $||w||_{0.L} = ||w||_0 + ||w||_L$ where $w \in C_L(B, l^\infty)$.

Let $C^{0.1}(B, l^\infty)$ be the set of all continuous functions $w : B \to l^\infty, w = \{w_i\}$, such that the derivatives

$D_s w_i = (D_{s_1} w_i, ..., D_{s_n} w_i)$, $i \in \mathrm{N}$, exist and $D_s w = \{D_s w_i\} \in C(B, l^\infty_n)$. For $w \in C^{0.1}(B, l^\infty)$ we put

$$||w||_1 = ||w||_0 + \max\{\ |D_s w(\tau,s)|_{n.\infty} : (\tau,s) \in B\ \}.$$

We will use the symbol $C^{0.1}_L(B, l^\infty)$ to denote the set of $w \in C^{0.1}(B, l^\infty)$ such that $||w||_{1.L} < +\infty$ where

$$||w||_{1.L} = ||w||_1 + \sup\left\{\frac{|D_s w(\tau,s)-D_s w(\tau,\bar{s})|_{n.\infty}}{|s-\bar{s}|} : (\tau,s),(\tau,\bar{s}) \in B\right\}.$$

The next function spaces are the following. Given $c = (c_0, c_1, c_2) \in R^3_+$, we denote by $C^{1.L}_0[c]$ the set of all functions $\varphi : E_0 \to l^\infty, \varphi = \{\varphi_i\}$, such that

(i) $\varphi \in C(E_0, l^\infty)$, there exist the derivatives $D_x\varphi_i$ on E_0 for $i \in \mathrm{N}$ and $D_x\varphi = \{D_x\varphi_i\} \in C(E_0, l^\infty_n)$,

(ii) $|\varphi(t,x)|_\infty \le c_0$, $|D_x\varphi(t,x)|_{n.\infty} \le c_1$ and

$$|D_x\varphi(t,x) - D_x\varphi(t,\bar{x})|_{n.\infty} \le c_2|x-\bar{x}| \text{ on } E_0.$$

Let $L([0,t], X)$, where $t > 0$, X is the linear normed space, denote the class of all the functions $\zeta : [0,t] \to X$ such that

$$\int_0^t ||\zeta(\tau)||d\tau < +\infty.$$

Let $\varphi \in C^{1.L}_0[c]$ be given and let $a \in (0, a_0], d = (d_0, d_1, d_2) \in R^3_+, \lambda = (\lambda_0, \lambda_1) \in L([0,a], R^2_+)$. We will denote by $C^{1.L}_{a.\varphi}[d,\lambda]$ the set of all functions $z : [-b_0, a] \times R^n \to l^\infty$ such that $z \in C([-b_0,a] \times R^n, l^\infty)$, $z(t,x) = \varphi(t,x)$ on E_0 and

(i) there exists $D_x z$ on $[0,a] \times R^n$,

(ii) $|z(t,x)|_\infty \le d_0$ and $|D_x z(t,x)|_{n.\infty} \le d_1$ on $[0,a] \times R^n$,

(iii) for $t, \bar{t} \in [0,a]$, $x, \bar{x} \in R^n$ we have

$$|z(t,x) - z(\bar{t},x)|_\infty \le \left|\int_t^{\bar{t}} \lambda_0(\tau)d\tau\right|$$

and

$$|D_xz(t,x) - D_xz(\bar{t},\bar{x})|_{n.\infty} \le |\int_t^{\bar{t}} \lambda_1(\tau)d\tau| + d_2|x-\bar{x}|.$$

Let $p = (p_1, p_2) \in R_+^2$, $\mu \in L([0,a], R_+)$. Let us denote by $C_a^{0.L}[p,\mu]$ the class of all functions $v : [0,a] \times R^n \to l_n^\infty$ such that $v \in C([0,a] \times R^n, l_n^\infty)$ and

(i) $|v(t,x)|_{n.\infty} \le p_1$ on $[0,a] \times R^n$,

(ii) for $(t,x), (\bar{t},\bar{x}) \in [0,a] \times R^n$ we have

$$|v(t,x) - v(\bar{t},\bar{x})|_{n.\infty} \le |\int_t^{\bar{t}} \mu(\tau)d\tau| + p_2|x-\bar{x}|.$$

We will prove that under suitable assumptions on f and φ, and for sufficiently small $a \in (0, a_0]$, there exists a solution $\bar{z}$ of problem (1), (2) such that $\bar{z} \in C_{a.\varphi}^{1.L}[d,\lambda]$ and for $\bar{v} = (D_x\bar{z})|_{[0,a]\times R^n}$ we have $\bar{v} \in C_a^{0.L}[p,\mu]$.

3. Bicharacteristics of nonlinear systems. Let Θ be the class of all the functions $\delta : [0,a_0] \times R_+ \to R_+$ such that $\delta(\cdot,\tau) \in L([0,a_0], R_+)$ for every $\tau \in R_+$ and $\delta(\cdot,\tau) : R_+ \to R_+$ is nondecreasing for almost all $\tau \in [0,a_0]$. Write

$$\Omega^{(1)} = [0,a_0] \times R^n \times C^{0.1}(B, l^\infty) \times R^n.$$

We start with assumptions on D_qf.

Assumption H_1. Suppose that D_qf exists on $\Omega^{(1)}$ and

1) $D_qf(\cdot,x,w,q) \in L([0,a_0], l_n^\infty)$ and there is $\beta \in \Theta$ such that

$$|D_qf(t,x,w,q)|_{n.\infty} \le \beta(t, ||w||_1) \text{ on } \Omega^{(1)},$$

2) there is $\gamma \in \Theta$ such that for $(t,x,h,q) \in \Omega^{(1)}$, $\bar{x},\bar{q} \in R^n$, $w \in C_L^{0.1}(B, l^\infty)$ we have

$$|D_qf(t,x,w,q) - D_qf(t,\bar{x},w+h,\bar{q})|_{n.\infty} \le$$

$$\le \gamma(t, ||w||_{1.L})\Big(|x-\bar{x}| + ||h||_1 + |q-\bar{q}|\Big).$$

Suppose that $\varphi \in C_0^{1.L}[c]$ and $z \in C_{a.\varphi}^{1.L}[d,\lambda], u \in C_a^{0.L}[p,\mu]$, $u = \{u_i\}$. Let

$$P_i[z,u](t,x) = \Big(t, x, z_{(t,x)}, u_i(t,x)\Big), \quad i \in \mathbf{N}. \tag{3}$$

For each $i \in \mathrm{N}$ consider the Cauchy problem

$$\eta'(\tau) = -D_q f_i(P_i[z,u](\tau,\eta(\tau)),\ \eta(t) = x \tag{4}$$

where $(t,x) \in [0,a] \times R^n$. Let us denote by $g_i[z,u](\cdot,t,x)$ the Caratheodory solution of (4). The function $g_i[z,u](\cdot,t,x)$ is the i-th bicharacteristic of system (1) corresponding to (z,u). Let

$$g[z,u](\cdot,t,x) = \{g_i[z,u](\cdot,t,x)\}$$

be the set of bicharacteristics of (1) corresponding to (z,u).

Let $||\cdot||_{(t)}$, $||\cdot||_{(n.t)}$ and $||\cdot||_{[n.t]}$ denote the supremum norms in the spaces $C([-b_0,t] \times R^n, l^\infty)$, $C([-b_0,t] \times R^n, l^\infty_n)$ and $C([0,t] \times R^n, l^\infty_n)$ respectively, $t \in [0,a_0]$.

Main properties of bicharacteristics are given in the following lemma.

LEMMA 1. *Suppose that Assumption H_1 is satisfied and the functions*

$$\varphi, \bar\varphi \in C_0^{1.L}[c],\ z \in C^{1.L}_{a.\varphi}[d,\lambda],\ \bar z \in C^{1.L}_{a.\bar\varphi}[d,\lambda],\ u, \bar u \in C_a^{0.L}[p,\mu]$$

are given.

Then the sets of bicharacteristics $g[z,u](\cdot,t,x)$ and $g[\bar z,\bar u](\cdot,t,x)$ exist on $[0,a]$ for every $(t,x) \in [0,a] \times R^n$, they are unique and we have the estimates

$$|g[z,u](\tau,t,x) - g[z,u](\tau,\bar t,\bar x)|_{n.\infty} \le$$
$$\le \left(|\int_t^{\bar t} \beta(\xi,\tilde d)d\xi| + |x-\bar x|\right)\exp\left(\bar d\,|\int_t^\tau \gamma(\xi,|d|)d\xi|\right) \tag{5}$$

and

$$|g[z,u](\tau,t,x) - g[\bar z,\bar u](\tau,t,x)|_{n.\infty} \le \exp\left(\bar d\,|\int_t^\tau \gamma(\xi,|d|)d\xi|\right)\cdot$$
$$\cdot|\int_t^\tau \gamma(\xi,|d|)\Big(||z-\bar z||_{(\xi)} + ||D_x z - D_x \bar z||_{(n.\xi)} + ||u-\bar u||_{[n.\xi]}\Big)d\xi| \tag{6}$$

where

$$\tilde d = d_0 + d_1,\ |d| = d_0 + d_1 + d_2,\ \bar d = 1 + d_1 + d_2 + p_2.$$

Proof. The existence and uniqueness of the solutions of (4) follows from classical theorems. Note that the right hand side of the differential system (4) satisfies the Caratheodory assumptions and the following Lipschitz condition holds

$$|D_q f_i(P_i[z,u](\tau,\eta)) - D_q f_i(P_i[z,u](\tau,\bar\eta))|_{n.\infty} \le$$

$$\leq \gamma(\tau, |d|)\bar{d}\, |\eta - \bar{\eta}| \text{ on } [0, a] \times R^n.$$

The function $g_i[z, u](\cdot, t, x)$ satisfies the integral equation

$$g_i[z, u](\tau, t, x) = x + \int_\tau^t D_q f_i(\ P_i[z, u](\xi, g_i[z, u](\xi, t, x))\ )d\xi, \ i \in \mathbf{N}.$$

It follows from Assumption H_1 that we have the integral inequality

$$|g[z, u](\tau, t, x) - g[z, u](\tau, \bar{t}, \bar{x})|_{n.\infty} \leq |x - \bar{x}| + |\int_t^{\bar{t}} \beta(\xi, \tilde{d})d\xi| +$$

$$+\bar{d}\, |\int_t^\tau \gamma(\xi, |d|)\ |g[z, u](\xi, t, x) - g[z, u](\xi, \bar{t}, \bar{x})|_{n.\infty} d\xi|,$$

$(t, x), (\bar{t}, \bar{x}) \in [0, a] \times R^n$ and we obtain (5) from the Gronwall inequality.

For $z \in C^{1.L}_{a.\varphi}[d, \lambda]$, $\bar{z} \in C^{1.L}_{a,\bar{\varphi}}[d, \lambda]$, $u, \bar{u} \in C^{0.L}_a[p, \mu]$ we have

$$|g[z, u](\tau, t, x) - g[\bar{z}, \bar{u}](\tau, t, x)|_{n.\infty} \leq$$

$$\leq |\int_t^\tau \gamma(\xi, |d|)\Big(||z - \bar{z}||_{(\xi)} + ||D_x z - D_x \bar{z}||_{(n.\xi)} + ||u - \bar{u}||_{[n.\xi]}\Big)d\xi| +$$

$$+\bar{d}\, |\int_t^\tau \gamma(\xi, |d|)\ |g[z, u](\xi, t, x) - g[\bar{z}, \bar{u}](\xi, t, x)|_{n.\infty} d\xi|.$$

From Gronwall inequality we deduce (6). This proves Lemma 1.

4. Integral functional equations. Now we formulate a system of integral functional equations corresponding to problem (1), (2).

We will denote by $CL(X, l^\infty)$ the set of all linear and continuous mappings from normed vector space X into l^∞. Let $|| \cdot ||_X$ be the norm in $CL(X, l^\infty)$. We will consider the space $CL(X, l^\infty)$ with $X = C(B, l^\infty)$, $X = C_L(B, l^\infty)$ or $X = l^\infty$. We formulate now further assumptions on f.

Assumption H_2. Suppose that

1) there is $\alpha \in \Theta$ such that

$$|f(t, x, w, q)|_{n.\infty} \leq \alpha(t, ||w||_0) \text{ on } [0, a_0] \times R^n \times C(B, l^\infty) \times R^n,$$

2) for every $(t, x, w, q) \in \Omega^{(1)}$ there exist the derivative $D_x f(t, x, w, q)$ and the Frechet derivative $D_w f(t, x, w, q) \in CL(C(B, l^\infty), l^\infty)$, assume that

$$|D_x f(t, x, w, q)|_{n.\infty} \leq \beta(t, ||w||_1), \ ||D_w f(t, x, w, q)||_{C(B, l^\infty)} \leq \beta(t, ||w||_1),$$

3) the terms

$$|D_x f(t,x,w,q) - D_x f(t,\bar{x},w+h,\bar{q})|_{n.\infty},$$

$$||D_w f(t,x,w,q) - D_w f(t,\bar{x},w+h,\bar{q})||_{C_L(B,l^\infty)},$$

where $(t,x,h,q) \in \Omega^{(1)}$, $\bar{x},\bar{q} \in R^n$, $w \in C_L^{0.1}(B,l^\infty)$, are bounded from above by

$$\gamma(t,||w||_{1.L})\Big(|x-\bar{x}| + ||h||_1 + |q-\bar{q}|\Big).$$

For simplicity of notation, we have assumed the same estimation for derivatives $D_x f$, $D_w f$, $D_q f$. We have assumed also the Lipschitz condition for these deriatives with the same coefficient.

Suppose that $\varphi \in C_0^{1.L}[c]$. Let $C_{a.\varphi}^{0.L}[p,\mu]$ be the class of all functions $u : [-b_0,a] \times R^n \to l_n^\infty$ such that $u = D_x\varphi$ on E_0 and $u|_{[0,a]\times R^n} \in C_a^{0.L}[p,\mu]$.

Let $\varphi \in C_0^{1.L}[c]$, $z \in C_{a.\varphi}^{1.L}[d,\lambda]$, $u,v \in C_{a.\varphi}^{0.L}[p,\mu]$ be given. Put

$$Q_i(\tau,t,x) = \Big(\tau, g_i(\tau,t,x), z_{(\tau,g_i(\tau,t,x))}, u_i(\tau,g_i(\tau,t,x))\Big),\ i \in \mathbf{N},$$

where $g_i(\cdot,t,x)$ is the i-th bicharacteristic corresponding to (z,u) (for simplicity of notation, we ignore the dependence of g_i on (z,u)).

Write for $(t,x) \in [0,a] \times R^n$

$$F_i[z,u](t,x) = \varphi_i(0,g_i(0,t,x)) +$$

$$+\int_0^t \Big(f_i(Q_i(\tau,t,x)) - \sum_{k=1}^n D_{q_k} f_i(Q_i(\tau,t,x)) u_{ik}(\tau,g_i(\tau,t,x))\Big)d\tau,\ i \in \mathbf{N}$$

and

$$G_{ij}[z,u,v](t,x) = D_{y_j}\varphi_i(0,g_i(0,t,x)) + \int_0^t \Big(D_{y_j} f_i(Q_i(\tau,t,x)) +$$

$$+ D_w f_i(Q_i(\tau,t,x))(v_{.j})_{(\tau,g_i(\tau,t,x))}\Big)d\tau,\ i \in \mathbf{N},\ 1 \le j \le n$$

where

$$v_{.j} = \{v_{ij}\},\ 1 \le j \le n.$$

Moreover we put

$$F[z,u]=\{F_i[z,u]\},$$

$$G_i[z,u,v]=(G_{i1}[z,u,v],...,G_{in}[z,u,v]),\ i\in\mathbf{N},$$

and

$$G[z,u,v]=\{G_i[z,u,v]\}.$$

We will consider the following system of functional integral equations

$$\begin{aligned} z&=F[z,u], & u&=G[z,u,u],\\ z&=\varphi \text{ on } E_0, & u&=D_x\varphi \text{ on } E_0,\\ g_i(\tau,t,x)&=x &+&\int_\tau^t D_qf_i(Q_i(\xi,t,x))d\xi,\ i\in\mathbf{N}. \end{aligned} \tag{7}$$

The above system is obtained in the following way. We introduce an additional unknown function $u=D_xz$ in (1) and we consider the linearization of (1) with respect to u:

$$\begin{aligned} D_tz_i(t,x) &= f_i(P_i[z,u](t,x))+\\ &+\sum_{k=1}^{n}D_{q_k}f_i(P_i[z,u](t,x))(D_{x_k}z_i(t,x)-u_{ik}(t,x)). \end{aligned} \tag{8}$$

For the unknown function u we get the differential system

$$\begin{aligned} D_tu_{ij}(t,x) &= D_{x_j}f_i(P_i[z,u](t,x))+D_wf_i(P_i[z,u](t,x))(D_{x_j}z)_{(t,x)}+\\ &+\sum_{k=1}^{n}D_{q_k}f_i(P_i[z,u](t,x))D_{x_k}u_{ij}(t,x). \end{aligned} \tag{9}$$

Now we put $D_xz=v$ in (9). The differential equations of bicharacteristics for (8) and for (9) are the same and they have form (4). If we consider (8) and (9) along the bicharacteristics $g[z,u](\cdot,t,x)$ we obtain for $i\in\mathbf{N}$, $1\le j\le n$

$$\frac{d}{d\tau}z_i(\tau,g_i(\tau,t,x))=f_i(Q_i(\tau,t,x))-\sum_{k=1}^{n}D_{q_k}f_i(Q_i(\tau,t,x))u_{ik}(\tau,g_i(\tau,t,x)),$$

$$\frac{d}{d\tau}u_{ij}(\tau,g_i(\tau,t,x))=D_{x_j}f_i(Q_i(\tau,t,x))+D_wf_i(Q_i(\tau,t,x))(v_{\cdot j})_{(\tau,g_i(\tau,t,x))}.$$

By integrating on $[0,t]$ with respect to τ and taking $v=u$ we get (7).

5. The sequence of succesive approximations. The proof of the existence of the solution of (7) will be based on the following method of succesive approximations.

Suppose that $\varphi \in C_0^{1.L}[c]$. We put

$$z^{(0)}(t,x) = \varphi(t,x) \quad \text{and} \quad u^{(0)}(t,x) = D_x\varphi(t,x) \text{ on } E_0,$$
$$z^{(0)}(t,x) = \varphi(0,x) \quad \text{and} \quad u^{(0)}(t,x) = D_x\varphi(0,x) \text{ on } [0,a] \times R^n. \tag{10}$$

If $(z^{(m)}, u^{(m)}) \in C_{a.\varphi}^{1.L}[d,\lambda] \times C_{a.\varphi}^{0.L}[p,\mu]$ are known functions, then $u^{(m+1)}$ is a solution of the problem

$$u = G[z^{(m)}, u, u^{(m)}], \qquad u = D_x\varphi \text{ on } E_0 \tag{11}$$

and $z^{(m+1)}$ is defined by

$$z^{(m+1)} = F[z^{(m)}, u^{(m+1)}], \qquad z^{(m+1)} = \varphi \text{ on } E_0. \tag{12}$$

We wish to emphasize that the main difficulty in carrying out of this construction is the problem of the existence of the sequence $\{z^{(m)}, u^{(m)}\}$.

Now we prove the main properties of the sequence $\{z^{(m)}, u^{(m)}\}$.

THEOREM 1. *If Assumptions H_1, H_2 are satisfied, then there exist $a \in (0, a_0]$ and $d = (d_0, d_1, d_2) \in R_+^3$, $p = (p_1, p_2) \in R_+^2$, $\lambda = (\lambda_1, \lambda_2) \in L([0,a], R_+^2)$, $\mu \in L([0,a], R_+)$ such that for any $m \geq 0$ we have*

(i) $z^{(m)}, u^{(m)}$ *are defined on* $[-b_0, a] \times R^n$ *and*

(ii) $z^{(m)} \in C_{a.\varphi}^{1.L}[d,\lambda]$, $u^{(m)} \in C_{a.\varphi}^{0.L}[p,\mu]$,

(iii) $D_x z^{(m)} = u^{(m)}$ on $[0,a] \times R^n$.

Proof. Write

$$\Gamma(t) = \exp\left(\bar{d} \int_0^t \gamma(\tau, |d|)d\tau\right), \ t \in [0,a],$$

$$\Gamma_0(t) = \Gamma(t)\left(c_2 + (1+p_1)\bar{d} \int_0^t \gamma(\tau, |d|)d\tau + p_2 \int_0^t \beta(\tau, \tilde{d})d\tau\right), \ t \in [0,a].$$

Suppose that

$$d_i > c_i, \ i = 0,1,2 \quad \text{and} \quad p_i = d_i, \ i = 1,2. \tag{13}$$

Moreover, suppose that $a \in (0, a_0]$ is small enough to satisfy

$$c_1 + (1+p_1)\int_0^a \beta(\tau, \tilde{d})d\tau \leq p_1, \ \ \Gamma_0(a) \leq p_2,$$
$$\int_0^a \gamma(\tau, |d|)d\tau\left[1 + p_1 + \Gamma(a)\left(c_2 + (1+p_1)\bar{d} \int_0^a \gamma(\tau, |d|)d\tau\right)\right] < 1, \tag{14}$$
$$c_0 + \int_0^a \left(\alpha(\tau, d_0) + p_1\beta(\tau, \tilde{d})\right)d\tau \leq d_0.$$

We define

$$
(15)\qquad
\begin{aligned}
\lambda_0(\tau) &= \alpha(\tau, d_0) + p_1\beta(\tau, \tilde{d}) + \beta(\tau, \tilde{d})\Gamma_1(a)\,\cdot \\
&\quad \cdot\Big(c_1 + (\bar{d} + p_2)\int_0^a \beta(\xi, \tilde{d})d\xi + p_1\bar{d}\int_0^a \gamma(\xi, |d|)d\xi\Big), \\
\lambda_1(\tau) &= \mu(\tau) = \beta(\tau, \tilde{d})(1 + p_1 + \Gamma(a)).
\end{aligned}
$$

We use the method of induction with respect to m. It follows from (10) that the theorem is true for $m = 0$. Suppose that conditions (i), (ii) and (iii) hold for a given $m \geq 0$.

We will prove that there exists $u^{(m+1)} : [-b_0, a] \times R^n \to l_n^\infty$ and $u^{(m+1)} \in C_{a.\varphi}^{0.L}[p, \mu]$. We claim that

$$
(16)\qquad G[z^{(m)}, \cdot, u^{(m)}] \;:\; C_{a.\varphi}^{0.L}[p, \mu] \to C_{a.\varphi}^{0.L}[p, \mu].
$$

Indeed, for $u \in C_{a.\varphi}^{0.L}[p, \mu]$ we have

$$
|G[z^{(m)}, u, u^{(m)}](t, x)|_{n.\infty} \leq c_1 + (1 + p_1)\int_0^t \beta(\tau, \tilde{d})d\tau
$$

on $[0, a] \times R^n$ and

$$
|G[z^{(m)}, u, u^{(m)}](t, x) - G[z^{(m)}, u, u^{(m)}](\bar{t}, \bar{x})|_{n.\infty} \leq
$$

$$
\leq \Big|\int_t^{\bar{t}} \beta(\tau, \tilde{d})(1 + p_1 + \Gamma_0(a))d\tau\Big| + \Gamma_0(a)|x - \bar{x}|
$$

on $[0, a] \times R^n$. It follows from (14) and from the definition of μ that (16) holds.

We conclude from Assumption H_2 and from Lemma 1 that for $u, \bar{u} \in C_{a.\varphi}^{0.L}[p, \mu]$ and $(t, x) \in [0, a] \times R^n$ we have

$$
|G[z^{(m)}, u, u^{(m)}](t, x) - G[z^{(m)}, \bar{u}, u^{(m)}](t, x)|_{n.\infty} \leq C_a||u - \bar{u}||_{(n.a)}
$$

with

$$
C_a = \int_0^a \gamma(\tau, |d|)d\tau\Big[1 + p_1 + \Gamma(a)\Big(c_2 + (1 + d_1 + p_2)(1 + p_1)\int_0^a \gamma(\tau, |d|)d\tau\Big)\Big].
$$

Thus by (14) the operator $G[z^{(m)}, \cdot, u^{(m)}]$ is a contraction and from Banach fixed point theorem it follows that there exists $u^{(m+1)}$ on $[-b_0, a] \times R^n$ and $u^{(m+1)} \in C_{a.\varphi}^{0.L}[p, \mu]$.

Now we prove that for the function $z^{(m+1)}$ given by (12) we have

$$D_x z^{(m+1)}(t,x) = u^{(m+1)}(t,x) \text{ on } [0,a]\times R^n.$$

Write $\Delta(t,x,\bar{x}) = \{\Delta_i(t,x,\bar{x})\}$ and

$$\Delta_i(t,x,\bar{x}) = z_i^{(m+1)}(t,\bar{x}) - z_i^{(m+1)}(t,x) - \sum_{k=1}^n u_{ik}^{(m+1)}(t,x)(\bar{x}_k - x_k)$$

where $t\in[0,a]$, $x,\bar{x}\in R^n$. We prove that there exists $\tilde{c}\in R_+$ such that

$$|\Delta(t,x,\bar{x})|_{n.\infty} \le \tilde{c}\,|x-\bar{x}|^2 \text{ for } t\in[0,a],\ x,\bar{x}\in R^n.$$

Write

$$P_i(\tau) = \Big(\tau, g_i[z^{(m)},u^{(m+1)}](\tau,t,x), z^{(m)}_{(\tau,g_i[z^{(m)},u^{(m+1)}](\tau,t,x))},$$

$$u_i^{(m+1)}(\tau, g_i[z^{(m)},u^{(m+1)}](\tau,t,x))\Big)$$

and denote by $\bar{P}_i(\tau)$ the point given by the above formula with $\bar{x}$ instead of x. According to (12), we have

$$\Delta_i(t,x,\bar{x}) = \varphi_i(0,\bar{g}_i(0)) - \varphi_i(0,g_i(0)) - \sum_{k=1}^n D_{x_k}\varphi_i(0,g_i(0))(\bar{x}_k - x_k) +$$

$$+\int_0^t \Big(f_i(\bar{P}_i(\tau)) - f_i(P_i(\tau))\Big)d\tau +$$

$$+\int_0^t \sum_{k=1}^n \Big(D_{q_k} f_i(P_i(\tau))u_{ik}(\tau,g_i(\tau)) - D_{q_k} f_i(\bar{P}_i(\tau))u_{ik}(\tau,\bar{g}_i(\tau))\Big)d\tau +$$

$$-\int_0^t \sum_{k=1}^n \Big(D_{x_k} f_i(P_i(\tau)) + D_w f_i(P_i(\tau))(u_{\cdot k}^{(m)})_{(\tau,g_i(\tau))}\Big)(\bar{x}_k - x_k)d\tau.$$

Write

$$g_i(\tau) = g_i[z^{(m)},u^{(m+1)}](\tau,t,x),\ \bar{g}_i(\tau) = g_i[z^{(m)},u^{(m+1)}](\tau,t,\bar{x})$$

and

$$T_i(\tau,\sigma) = \Big(\tau, g_i(\tau) + \sigma(\bar{g}_i(\tau) - g_i(\tau)), z^{(m)}_{(\tau,g_i(\tau))} + \sigma(z^{(m)}_{(\tau,\bar{g}_i(\tau))} - z^{(m)}_{(\tau,g_i(\tau))}),$$

$$u_i^{(m+1)}(\tau, g_i(\tau)) + \sigma(u_i^{(m+1)}(\tau, \bar{g}_i(\tau)) - u_i^{(m+1)}(\tau, g_i(\tau)))\Big).$$

Applying the Hadamard mean value theorem to the difference $f_i(\bar{P}(\tau)) - f_i(P_i(\tau))$ we deduce

$$\int_0^t \Big[f_i(\bar{P}_i(\tau)) - f_i(P_i(\tau))\Big] d\tau =$$

$$= \int_0^t \sum_{k=1}^n \int_0^1 D_{x_k} f_i(T_i(\tau,\sigma))d\sigma\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau)\Big)d\tau +$$

$$+ \int_0^t \int_0^1 D_w f_i(T_i(\tau,\sigma))d\sigma\Big(z^{(m)}_{(\tau,\bar{g}_i(\tau))} - z^{(m)}_{(\tau,g_i(\tau))}\Big)d\tau +$$

$$+ \int_0^t \sum_{k=1}^n \int_0^1 D_{q_k} f_i(T_i(\tau,\sigma))d\sigma\Big(u_{ik}^{(m+1)}(\tau, \bar{g}_i(\tau)) - u_{ik}^{(m+1)}(\tau, g_i(\tau))\Big)d\tau.$$

For each $i \in \mathbf{N}$ we will write $\Delta_i(t, x, \bar{x})$ in the form

$$\Delta_i(t,x,\bar{x}) = \Delta_i^{(1)}(t,x,\bar{x}) + \Delta_i^{(2)}(t,x,\bar{x})$$

where

$$\Delta_i^{(1)}(t,x,\bar{x}) =$$

$$= \varphi_i(0, \bar{g}_i(0)) - \varphi_i(0, g_i(0)) - \sum_{k=1}^n D_{x_k}\varphi_i(0, g_i(0))\Big(\bar{g}_i(0) - g_i(0)\Big) +$$

$$+ \int_0^t \sum_{k=1}^n \int_0^1 \Big(D_{x_k} f_i(T_i(\tau,\sigma)) - D_{x_k} f_i(P_i(\tau))\Big)d\sigma\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau)\Big)d\tau +$$

$$+ \int_0^t \sum_{k=1}^n \int_0^1 \Big(D_w f_i(T_i(\tau,\sigma)) - D_w f_i(P_i(\tau))\Big)d\sigma\Big(z^{(m)}_{(\tau,\bar{g}_i(\tau))} - z^{(m)}_{(\tau,g_i(\tau))}\Big)d\tau +$$

$$+ \int_0^t D_w f_i(P_i(\tau))\Big[z^{(m)}_{(\tau,\bar{g}_i(\tau))} - z^{(m)}_{(\tau,g_i(\tau))} - \sum_{k=1}^n (u^{(m)}_{\cdot k})_{(\tau,g_i(\tau))}\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau)\Big)\Big]d\tau +$$

$$+\int_0^t \sum_{k=1}^n \int_0^1 \Big(D_{q_k} f_i(T_i(\tau,\sigma)) - D_{q_k} f_i(P_i(\tau))\Big) d\sigma$$

$$\cdot\Big(u_{ik}^{(m+1)}(\tau,\bar{g}_i(\tau)) - u_{ik}^{(m+1)}(\tau,g_i(\tau))\Big) d\tau$$

and

$$\Delta_i^{(2)}(t,x,\bar{x}) = \sum_{k=1}^n D_{x_k}\varphi_i(0,g_i(0))\Big(\bar{g}_{ik}(0) - g_{ik}(0) - (\bar{x}_k - x_k)\Big) +$$

$$+\int_0^t \sum_{k=1}^n D_{x_k} f_i(P_i(\tau))\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau) - (\bar{x}_k - x_k)\Big) d\tau +$$

$$+\int_0^t D_w f_i(P_i(\tau)) \sum_{k=1}^n (u_{\cdot k}^{(m)})_{(\tau,g_i(\tau))}\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau) - (\bar{x}_k - x_k)\Big) d\tau +$$

$$-\int_0^t \Big(D_{q_k} f_i(\bar{P}_i(\tau)) - D_{q_k} f_i(P_i(\tau))\Big) u_{ik}^{(m+1)}(\tau,g_i(\tau)) d\tau.$$

Our next claim is that $\Delta_i^{(2)}(t,x,\bar{x}) = 0$ for $i \in \mathbf{N}$. It follows from (4) that

$$\bar{g}_i(\tau) - g_i(\tau) - (\bar{x} - x) = \int_\tau^t \Big(D_q f_i(\bar{P}_i(\tau)) - D_q f_i(P_i(\tau))\Big) d\tau$$

and thus we have

$$\Delta_i^{(2)}(t,x,\bar{x}) = \sum_{k=1}^n \int_0^t \Big(D_{q_k} f_i(\bar{P}_i(\tau)) - D_{q_k} f_i(P_i(\tau))\Big)\Big[D_{x_k}\varphi_i(0,g_i(0)) +$$

$$+\int_0^\tau D_{x_k} f_i(P_i(\xi)) d\xi + \int_0^\tau f_i(P_i(\xi))(u_{\cdot k}^{(m)})_{(\xi,g_i(\xi))} d\xi - u_{ik}^{(m+1)}(\tau,g_i(\tau))\Big] d\tau.$$

We have $g_i[z^{(m)},u^{(m+1)}](\xi,\tau,g_i[z^{(m)},u^{(m+1)}](\tau,t,x)) = g_i[z^{(m)},u^{(m+1)}](\xi,t,x)$ for $\xi,\tau \in [0,t]$, therefore we get

$$u_{ik}^{(m+1)}(\tau,g_i(\tau)) = D_{x_k}\varphi_i(0,g_i(0)) +$$

$$+\int_0^\tau \Big(D_{x_k} f_i(P_i(\xi)) + D_w f_i(P_i(\xi))(u_{\cdot k}^{(m)})_{(\xi,g_i(\xi))}\Big) d\xi$$

and consequently $\Delta_i^{(2)}(t,x,\bar{x}) = 0$ for $t \in [0,a]$, $x, \bar{x} \in R^n$, $i \in \mathbf{N}$. Then we have proved that

$$\Delta_i(t,x,\bar{x}) = \Delta_i^{(1)}(t,x,\bar{x}).$$

It follows from the induction hypothesis that $D_x z^{(m)} = u^{(m)}$. An easy computation shows that

$$||z^{(m)}_{(\tau,\bar{g}_i(\tau))} - z^{(m)}_{(\tau,g_i(\tau))} - \sum_{k=1}^{n} (u^{(m)}_{.k})_{(\tau,g_i(\tau))}\Big(\bar{g}_{ik}(\tau) - g_{ik}(\tau)\Big)||_0 \le d_2|\bar{g}_i(\tau) - g_i(\tau)|^2.$$

We conclude from Assumptions H_1 and H_2 that

$$|\Delta_i(t,x,\bar{x})| = |\Delta_i^{(1)}(t,x,\bar{x})| \le c_2\ |\bar{g}_i(0) - g_i(0)|^2 +$$

$$+ \int_0^t \Big(\gamma(\tau,|d|)\bar{d}\ (1 + d_1 + p_2) + \beta(\tau,\tilde{d})d_2\Big)|\bar{g}_i(\tau) - g_i(\tau)|^2 d\tau.$$

It follows from Lemma 1 that $|\bar{g}_i(\tau) - g_i(\tau)| \le \Gamma(a)|x - \bar{x}|$. Then there exists a constant $\tilde{c} > 0$ such that

$$|\Delta(t,x,\bar{x})|_{n.\infty} \le \tilde{c}\ |x - \bar{x}|^2$$

and consequently

$$D_x z^{(m+1)} = u^{(m+1)}. \tag{17}$$

Now we prove that $z^{(m+1)} \in C^{1.L}_{a.\varphi}[d,\lambda]$.
Of course $z^{(m+1)}$ is continuous on $[-b_0,a] \times R^n$ and $z^{(m+1)} = \varphi$ on E_0. It follows from (13), (17) that

$$|D_x z^{(m+1)}(t,x)|_{n.\infty} \le d_1,$$

$$|D_x z^{(m+1)}(t,x) - D_x z^{(m)}(\bar{t},\bar{x})|_{n.\infty} \le |\int_t^{\bar{t}} \lambda_1(\tau)d\tau| + d_2|x - \bar{x}|.$$

Assumptions H_1, H_2 and (14), (15) imply the estimates

$$|z^{(m+1)}(t,x)|_{n.\infty} \le d_0,\ |z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},x)|_{n.\infty} \le |\int_t^{\bar{t}} \lambda_0(\tau)d\tau|,$$

which complete the proof of the theorem.

We can now formulate the result on the convergence of the sequence of succesive approximations.

THEOREM 2. *If Assumptions H_1, H_2 are satisfied, then there exist $a \in (0, a_0]$ and $d = (d_0, d_1, d_2) \in R^3_+$, $p = (p_1, p_2) \in R^2_+$, $\lambda = (\lambda_1, \lambda_2) \in L([0,a], R^2_+)$, $\mu \in L([0,a], R_+)$ such that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on $[-b_0, a] \times R^n$ to $\bar{z} \in C^{1.L}_{a.\varphi}[d, \lambda]$ and $\bar{u} \in C^{0.L}_{a.\varphi}[p, \mu]$ respectively.*

Proof. Suppose that the constants d, p, a and the functions λ, μ satisfy (13) - (15). For $m \geq 1$ and for $t \in [0, a]$ we put

$$Z^{(m)}(t) = \sup\Big\{|z^{(m)}(\bar{t}, x) - z^{(m-1)}(\bar{t}, x)|_{n.\infty} :\ (\bar{t}, x) \in [0, t] \times R^n\Big\},$$

$$U^{(m)}(t) = \sup\Big\{|u^{(m)}(\bar{t}, x) - u^{(m-1)}(\bar{t}, x)|_{n.\infty} :\ (\bar{t}, x) \in [0, t] \times R^n\Big\}.$$

At first we prove that

$$U^{(m+1)}(t) \leq \exp\Big(\int_0^t \delta_1(\tau) d\tau\Big) \int_0^t \Big(\delta_1(\tau) Z^{(m)}(\tau) + \delta_2(\tau) U^{(m)}(\tau)\Big) d\tau \tag{18}$$

where

$$\delta_1(\tau) = \gamma(\tau, |d|)\Gamma(a)\Big[c_2 + \int_0^a \Big(\tilde{d}(1 + p_1)\gamma(\xi, |d|) + p_2 \beta(\xi, \tilde{d})\Big) d\xi\Big],$$

$$\delta_2(\tau) = \delta_1(\tau) + \beta(\tau, \tilde{d}).$$

Write $g_i^{(k)}(\tau) = g_i[z^{(k-1)}, u^{(k)}](\tau, t, x)$, $k = m, m+1$. According to (11), we have

$$|u_{ik}^{(m+1)}(t, x) - u_{ik}^{(m)}(t, x)| \leq c_2 |g_i^{(m+1)}(0) - g_i^{(m)}(0)| +$$

$$+ \int_0^t \gamma(\tau, |d|)(1 + p_1)\Big(|g_i^{(m+1)}(\tau) - g_i^{(m)}(\tau)| + ||z^{(m)}_{(\tau, g_i^{(m+1)}(\tau))} - z^{(m-1)}_{(\tau, g_i^{(m)}(\tau))}||_1 +$$

$$+ |u_i^{(m+1)}(\tau, g_i^{(m+1)}(\tau)) - u_i^{(m)}(\tau, g_i^{(m)}(\tau))|\Big) d\tau +$$

$$+ \int_0^t \beta(\tau, \tilde{d}) ||(u_{.k}^{(m)})_{(\tau, g_i^{(m+1)}(\tau))} - (u_{.k}^{(m-1)})_{(\tau, g_i^{(m)}(\tau))}||_0 d\tau.$$

Using the estimates

$$||z^{(m)}_{(\tau,g_i^{(m+1)}(\tau))}-z^{(m-1)}_{(\tau,g_i^{(m)}(\tau))}||_1 \le Z^{(m)}(\tau)+U^{(m)}(\tau)+(d_1+d_2)|g_i^{(m+1)}(\tau)-g_i^{(m)}(\tau)|,$$

$$|u_i^{(m+1)}(\tau,g_i^{(m+1)}(\tau)-u_i^{(m)}(\tau,g_i^{(m)}(\tau))| \le U^{(m+1)}(\tau)+p_2|g_i^{(m+1)}(\tau)-g_i^{(m)}(\tau)|$$

and

$$|g_i^{(m+1)}(\tau)-g_i^{(m)}(\tau)| \le \Gamma(a)\int_0^t \gamma(\xi,|d|)\Big(Z^{(m)}(\xi)+U^{(m)}(\xi)+U^{(m+1)}(\xi)\Big)d\xi,$$

we obtain the integral inequality

$$U^{(m+1)}(t) \le \int_0^t U^{(m+1)}(\tau)\delta_1(\tau)d\tau + \int_0^t \Big(\delta_1(\tau)Z^{(m)}(\tau)+\delta_2(\tau)U^{(m)}(\tau)\Big)d\tau.$$

The above estimate and Gronwall inequality imply (18).

Now we estimate the function $Z^{(m+1)}$. We conclude from (12) that

$$|z_i^{(m+1)}(t,x)-z_i^{(m)}(t,x)| \le c_1|g_i^{(m+1)}(0)-g_i^{(m)}(0)|+$$

$$+\int_0^t \Big(\beta(\tau,\tilde d)+\gamma(\tau,|d|)p_1\Big)\Big(|g_i^{(m+1)}(\tau)-g_i^{(m)}(\tau)|+$$

$$+||z^{(m)}_{(\tau,g_i^{(m+1)}(\tau))}-z^{(m-1)}_{(\tau,g_i^{(m)}(\tau))}||_1+$$

$$+|u_i^{(m+1)}(\tau,g_i^{(m+1)}(\tau))-u_i^{(m)}(\tau,g_i^{(m)}(\tau))|\Big)d\tau+$$

$$+\int_0^t \beta(\tau,\tilde d)|u_i^{(m+1)}(\tau,g_i^{(m+1)}(\tau))-u_i^{(m)}(\tau,g_i^{(m)}(\tau))|d\tau,$$

and finally that

$$(19)\; Z^{(m+1)}(t) \le \int_0^t \Big[\tilde\delta_1(\tau)\Big(Z^{(m)}(\tau)+U^{(m)}(\tau)\Big)+\tilde\delta_2(\tau)U^{(m+1)}(\tau)\Big]d\tau$$

where

$$\tilde\delta_1(\tau)=\beta(\tau,\tilde d)+\gamma(\tau,|d|)p_1+$$

$$+\gamma(\tau,|d|)\Gamma(a)\Big[c_1+\int_0^a\Big(\beta(\xi,\tilde{d})(\bar{d}+p_2)+\gamma(\xi,|d|)p_1\bar{d}\Big)d\xi\Big],$$

$$\tilde{\delta}_2(\tau)=\tilde{\delta}_1(\tau)+\beta(\tau,\tilde{d}).$$

Writing

$$\tilde{\Gamma}(t)=\exp\Big(\int_0^t\delta_1(\xi)d\xi\Big)\int_0^t\tilde{\delta}_2(\xi)d\xi$$

and using (18) and (19) we obtain

$$\begin{aligned} Z^{(m+1)}(t) \;&\le\; \int_0^t\Big[\Big(\delta_1(\tau)\tilde{\Gamma}(a)+\tilde{\delta}_1(\tau)\Big)Z^{(m)}(\tau)+ \\ &+\Big(\delta_2(\tau)\tilde{\Gamma}(a)+\tilde{\delta}_1(\tau)\Big)U^{(m)}(\tau)\Big]d\tau. \end{aligned} \tag{20}$$

Assume that $a\in(0,a_0]$ is such a small constant that

$$\int_0^a\delta_2(\tau)d\tau\Big[\exp\Big(\int_0^a\delta_1(\tau)d\tau\Big)+\tilde{\Gamma}(a)\Big]+\int_0^a\tilde{\delta}_1(\tau)d\tau \;<\; 1. \tag{21}$$

By (18) and (20) we conclude that there exits $\delta\in(0,1)$ such that

$$U^{(m+1)}(t)+Z^{(m+1)}(t)\le\delta\Big(U^{(m)}(t)+Z^{(m)}(t)\Big),\; t\in[0,a], m\ge 1. \tag{22}$$

From (10) and from Assumptions H_1, H_2 it follows that there is $\bar{c}\in R_+$ such that

$$Z^{(1)}(t)\le\bar{c}+\int_0^t\Big(\alpha(\tau,d_0)+\beta(\tau,\tilde{d})p_1\Big)d\tau,$$

$$U^{(1)}(t)\le\bar{c}+(1+p_1)\int_0^t\beta(\tau,\tilde{d})d\tau,$$

where $t\in[0,a]$. Finally the convergence of the sequence $\{z^{(m)},u^{(m)}\}$ follows from (22).

This completes the proof of Theorem 2.

6. The main theorem. We are able now to state the main result of the paper.

THEOREM 3. *If Assumptions H_1, H_2 are satisfied, then there exists a solution $\bar{z}$ of the problem (1),(2) and $\bar{z}\in C^{1.L}_{a.\varphi}[d,\lambda]$, $D_x\bar{z}\in C^{0.L}_{a.\varphi}[p,\mu]$ with*

$\varphi \in C_0^{1.L}[c]$ *and* $a \in (0, a_0]$, $d \in R_+^3$, $p \in R_+^2$, $\lambda \in L([0,a], R_+^2)$, $\mu \in L([0,a], R_+)$ *satisfying (13) - (15), (21).*

Proof. From Theorem 2 we deduce that the sequence $\{z^{(m)}\}$ converges to $\bar{z}$ and $\{u^{(m)}\}$ converges to $\bar{u}$ uniformly on $[-b_0, a] \times R^n$. Furthemore, we have that $D_x\bar{z}$ exists on $[-b_0, a] \times R^n$ and $D_x\bar{z} = \bar{u}$. Thus we get that

$$\bar{z}_i(t,x) = \varphi_i(0, \tilde{g}_i(0,t,x)) + \int_0^t \Big(f_i(P_i[\bar{z}, D_x\bar{z}](\tau, \tilde{g}_i(\tau,t,x))) +$$

$$- \sum_{k=1}^n D_{q_k} f_i(P_i[\bar{z}, D_x\bar{z}](\tau, \tilde{g}_i(\tau,t,x))) D_{x_k}\bar{z}_i(\tau, \tilde{g}_i(\tau,t,x)) \Big) d\tau$$

where $\tilde{g}_i(\tau,t,x) = g_i[\bar{z}, D_x\bar{z}](\tau,t,x)$ and $P_i[\bar{z}, D_x\bar{z}]$ is given by (3) with $z = \bar{z}$ and $u = D_x\bar{z}$.

For given $x \in R^n$ and $\xi = \tilde{g}_i(0,t,x)$ we have $\tilde{g}_i(\tau,t,x) = \tilde{g}_i(\tau,0,\xi)$ and $x = \tilde{g}_i(t,0,\xi)$. Therefore

$$\text{(23)}\quad \bar{z}_i(t, \tilde{g}_i(t,0,\xi)) = \varphi_i(0,\xi) + \int_0^t \Big(f_i(P_i[\bar{z}, D_x\bar{z}](\tau, \tilde{g}_i(\tau,0,\xi))) +$$

$$- \sum_{k=1}^n D_{q_k} f_i(P_i[\bar{z}, D_x\bar{z}](\tau, \tilde{g}_i(\tau,0,\xi))) D_{x_k}\bar{z}_i(\tau, \tilde{g}_i(\tau,0,\xi)) \Big) d\tau$$

By differentiating (23) with respect to t and puting $\tilde{g}_i(t,0,\xi) = x$, we obtain that $\bar{z}$ satisfies (1) for almost all $t \in [0,a]$ with fixed $x \in R^n$.

Since $\bar{z}$ satisfies initial condition (2) the proof is complete.

If in Theorem 3 we assume that f is continuous, then we get classical solutions for Cauchy problem (1), (2).

7. Some noteworthy particular cases. We wish to emphasize that our hereditary setting contains in particular some well known delay structures.

Given the functions

$$\varphi : E_0 \to l^\infty,\ \tilde{f} : [0,a_0] \times R^n \times l^\infty \times R^n \to l^\infty,\ \tilde{f} = \{\tilde{f}_i\}$$

and

$$\psi_0 : [0,a_0] \to R,\ \psi : [0,a_0] \times R^n \to R^n,$$

we consider the function $f = \{f_i\} : [0,a_0] \times R^n \times C(B, l^\infty) \times R^n \to l^\infty$ defined by

$$\text{(24)}\qquad f(t,x,w,q) \ = \ \tilde{f}(t,x,w(\psi_0(t) - t, \psi(t,x) - x), q).$$

In this case (1), (2) is equivalent to the differential system with deviated argument

$$(25) \qquad D_t z_i(t,x) = \tilde{f}_i(t,x,z(\psi_0(t),\psi(t,x)),D_x z_i(t,x)),\ i\in \mathbf{N}$$

with the initial condition (2).

Now we formulate our existence result for problem (25), (2).

Assumption H_3. Suppose that

1) the function $\tilde{f}$ satisfies the conditions:
there is $\tilde{A}\in R_+$ such that $|\tilde{f}(t,x,\varrho,q)|_\infty \le \tilde{A}$ on $[0,a_0]\times R^n\times l^\infty\times R^n$; the derivatives $D_x\tilde{f}$, $D_q\tilde{f}$ and $D_\varrho\tilde{f}\in CL(l^\infty,l^\infty)$ exist on $[0,a_0]\times R^n\times l^\infty\times R^n$ and there is $\tilde{B}\in R_+$ such that

$$|D_x\tilde{f}(t,x,\varrho,q)|_{n.\infty}\le \tilde{B},\ |D_q\tilde{f}(t,x,\varrho,q)|_{n.\infty}\le \tilde{B},$$

$$||D_\varrho\tilde{f}(t,x,\varrho,q)||_{l^\infty}\le \tilde{B};$$

there is $\tilde{L}\in R_+$ such that for $(t,x,\varrho,q),(t,\bar{x},\bar{\varrho},\bar{q})\in[0,a_0]\times R^n\times l^\infty\times R^n$ the terms

$$|D_x\tilde{f}(t,x,\varrho,q)-D_x\tilde{f}(t,\bar{x},\bar{\varrho},\bar{q})|_{n.\infty},\ |D_q\tilde{f}(t,x,\varrho,q)-D_q\tilde{f}(t,\bar{x},\bar{\varrho},\bar{q})|_{n.\infty},$$

$$||D_\varrho\tilde{f}(t,x,\varrho,q)-D_\varrho\tilde{f}(t,\bar{x},\bar{\varrho},\bar{q})||_{l^\infty}$$

are bounded from above by

$$\tilde{L}\Big(|x-\bar{x}|+|\varrho-\bar{\varrho}|_\infty+|q-\bar{q}|\Big),$$

2) the functions ψ_0 and $\psi=(\psi_1,...,\psi_n)$ satisfy
the conditions:
they are continuous and $(\psi_0(t)-t,\psi(t,x)-x)\in B$ for $(t,x)\in[0,a_0]\times R^n$; the derivative $D_x\psi$ exists on $[0,a_0]\times R^n$ and

$$|D_x\psi(t,x)|\le r_0,\ |D_x\psi(t,x)-D_x\psi(t,\bar{x})|\le r_1|x-\bar{x}|.$$

The theorem reduces to the following one.

THEOREM 4. *Suppose that Assumption H_3 is satisfied and let $\varphi\in C_0^{1,L}[c]$. Then there exist $a\in(0,a_0]$, $d\in R_+^3$, $\lambda\in L([0,a],R_+^2)$ and a function $\bar{z}\in C_{a.\varphi}^{1.L}[d,\lambda]$ such that $\bar{z}$ is a solution of problem (25), (2) on $[-b_0,a]\times R^n$.*

Proof. In order to apply Theorem 3 to the function f given by (24), let us show that the Assumptions H_1 and H_2 are fulfilled.

Observe that in $\Omega^{(1)}$ we have for $i \in \mathbf{N}$, $1 \leq j \leq n$

$$D_{x_j} f_i(t,x,w,q) = D_{x_j} \tilde{f}_i(t,x,w(\psi_0(t)-t,\psi(t,x)-x),q)+$$

$$+D_\varrho \tilde{f}_i(t,x,w(\psi_0(t)-t,\psi(t,x)-x),q)\cdot$$

$$\cdot \sum_{k=1}^{n} D_{s_k} w(\psi_0(t)-t,\psi(t,x)-x)(D_{x_j}\psi_k(t,x)-\delta_{kj}),$$

where δ_{kj} is the Kronecker symbol,

$$D_w f(t,x,w,q)h = D_\varrho \tilde{f}(t,x,w(\psi_0(t)-t,\psi(t,x)-x),q)h(\psi_0(t)-t,\psi(t,x)-x)$$

where $h \in C(B,l^\infty)$ and

$$D_q f(t,x,w,q) = D_q \tilde{f}(t,x,w(\psi_0(t)-t,\psi(t,x)-x),q).$$

Thus it is easy to see that Assumptions H_1 and H_2 are satisfied and the assertion follows as an immediate application of Theorem 3.

If we consider the function defined by

$$f(t,x,w,q) = \tilde{f}(t,x,\int_B w(\tau,s)d\tau ds,\ q)$$

then system (1) reduces to the differential integral system

$$D_t z_i(t,x) = \tilde{f}\Big(t,x,\int_{-b_0}^{0}\int_{-b}^{b} z(t+\tau,x+s)dsd\tau,\ D_x z_i(t,x)\Big).$$

It is easy to formulate sufficient conditions for the existence of the solutions of the above system with the initial condition (2) again as an application of Theorem 3.

Note that we have assumed the Lipschitz condition for the functions $D_x f, D_w f, D_q f$ on some special functions spaces (see Assumptions H_1, H_2). The Lipschitz condition is local with respect to the functional variable. It is important in our considerations.

Let us consider the simplest condition: suppose that there is $L \in R_+$ such that on $\Omega^{(1)}$

$$|D_q f(t,x,w,q) - D_q f(t,\bar{x},w+h,\bar{q})|_{n.\infty} \leq$$

$$\leq L\Big(|x-\bar{x}| + ||h||_1 + |q-\bar{q}|\Big) \tag{26}$$

and analogous inequalities for the derivatives $D_x f, D_w f$.

It is easy to see that equations with a deviated argument (25) do not satisfy conditions of type (26), while they fulfill Assumption H_3.

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PHASE SPACES FOR HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract.
The phase space for quasilinear equations with unbounded delay is constructed. A theorem on the existence, uniqueness and continuous dependence upon initial data is given. Classical solutions of initial problems for equations in a Banach space are investigated. The method of bicharacteristics and integral inequalities are used.

Key Words. unbounded delay, initial problems, local existence.

AMS(MOS) subject classification. 35L50, 35D05.

1. Introduction. For any metric spaces U and V we denote by $C(U,V)$ the class of all continuous functions defined on U and taking values in V. If $z : U \to V$ and $A \subset U$ then $z_{|A}$ denotes the restriction of z to the set A. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Suppose that U and V are domains in R^k and R^m respectively and the function $v : U \to V$ of the variable $y = (y_1, \ldots, y_k)$ is of class C^1. Then $\partial_y v$ denotes the Jacobi matrix of v.

Let $E = [-r_0, 0] \times [-r, r] \subset R^{1+n}$ where $r_0 \in R_+$, $R_+ = [0, +\infty)$, and $r = (r_1, \ldots, r_n) \in R^n_+$. Suppose that $a > 0$, $(t,x) \in [0,a] \times R^n$, $x = (x_1, \ldots, x_n)$, and $z : [-r_0, a] \times R^n \to R^n$. We define a function $z_{(t,x)} : E \to R$ as follows: $z_{(t,x)}(\tau, s) = z(t+\tau, x+s)$, $(\tau, s) \in E$. Assume that $F : [0,a] \times R^n \times C(E,R) \times$

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$R^n \to R$ and $\varphi : [-r_0, 0] \times R^n \to R$ are given functions. In this time numerous papers were published concerning the initial problem

$$\partial_t z(t,x) = F(t,x,z_{(t,x)},\partial_x z(t,x)), \quad z(t,x) = \varphi(t,x) \ \text{ on } \ [-r_0,0] \times R^n,$$

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$ and for adequate initial-boundary value problems. The following questions were considered: functional differential inequalities generated by initial or mixed problems, existence theory of classical or generalized solutions, numerical method of lines for hyperbolic problems, difference methods for initial or initial-boundary value problems. All these problem are such that the set E is bounded. The theory of hyperbolic functional differential problems has been developed in the monograph [6], see also [1], [2].

The papers [3], [5] initiated the investigation of hyperbolic functional differential equations with unbounded delay. Initial and initial - boundary value problems for quasi - linear equations were considered. The set of axioms for phase spaces given there seems to be in final form for Carathéodory solutions.

The theory of ordinary functional differential equations with unbounded delay has been described extensively in the monographs [4], [7].

In the paper we start the investigations of classical solutions of first order partial differential functional equations with unbounded delay. It is the purpose to examine initial data from a general Banach space. We develop a theory of existence, uniqueness and continuous dependence on initial functions for quasilinear equations. The method of bicharacteristics and functional integral inequalities are used.

We formulate the problem. Let B be a Banach space with the norm $\|\cdot\|$. The norm in R^n will also be denoted by $\|\cdot\|$. Namely, for $x = (x_1, \dots, x_n) \in R^n$ we put $\|x\| = |x_1| + \dots + |x_n|$. If $y = (y_1, \dots, y_n) \in B^n$ then we write also $\|y\| = \|y_1\| + \dots + \|y_n\|$. Let $D = (-\infty, 0] \times [-r, r]$, $D \subset R^{1+n}$, $r \in R^n_+$. For a function $z : (-\infty, b] \times R^n \to B$, $b > 0$, and for a point $(t,x) \in [0,b] \times R^n$ we define a function $z_{(t,x)} : D \to B$ by $z_{(t,x)}(\tau, s) = z(t+\tau, x+s)$, $(\tau, s) \in D$. The function $z_{(t,x)}$ is the restriction of z to the set $(-\infty, t] \times [x - r, x + r]$ and this restriction is shifted to the set D. The phase space X for partial differential equations with unbounded delay is a linear space with a norm $\|\cdot\|_X$, consisting of functions mapping the set D into B. Let $a > 0$ be fixed and suppose that

$$\varrho = (\varrho_1, \dots, \varrho) : [0,a] \times R^n \times X \to R^n,$$

$$f : [0,a] \times R^n \times X \to B, \quad \varphi : (-\infty, 0] \times R^n \to B,$$

are given functions. We consider the quasilinear equation

$$\partial_t z(t,x) + \sum_{i=1}^{n} \varrho_i(t,x,z_{(t,x)})\, \partial_{x_i} z(t,x) = f(t,x,z_{(t,x)}) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad (-\infty,0] \times R^n. \tag{2}$$

We will deal with classical solutions of (1), (2). In other words, a function $\bar{u} : (-\infty,b] \times R^n \to B$, where $0 < b \le a$, is a solution of the above problem provides

(i) $\bar{u}_{|[0,b]\times R^n}$ is continuous and the derivatives

$$\partial_t \bar{u}, \quad \partial_x \bar{u} = (\partial_{x_1}\bar{u}, \ldots, \partial_{x_n}\bar{u})$$

exist on $[0,b] \times R^n$,

(ii) the function $\bar{u}$ satisfies equation (1) on $[0,b] \times R^n$ and condition (2) holds.

Differential equations with a deviated argument and differential-integral equations can be derived from a general model by specializing given operators.

2. Definitions and fundamental axioms. Assume that $b > 0$, $t \in [0,b]$ and $w : (-\infty,b] \times [-r,r] \to B$. We define a function $w_{(t)} : D \to B$ by $w_{(t)}(\tau,s) = w(t+\tau,s)$, $(\tau,s) \in D$. For each $t \in [0,b]$ the function $w_{(t)}$ is the restriction of w to the set $(-\infty,t] \times [-r,r]$ and this restriction is shifted to the set D.

If $w : (-\infty,b] \times [-r,r] \to B$, $b > 0$, and $w_{|[0,b]\times[-r,r]}$ is continuous then we put

$$\| w \|_{[0,t]} = \max\{ \|w(\tau,s)\| : (\tau,s) \in [0,t] \times [-r,r] \},\ t \in [0,b].$$

Suppose additionally that the derivatives $\partial_t w$, $\partial_x w = (\partial_{x_1} w, \ldots, \partial_{x_n} w)$ exist and are continuous on $[0,b] \times [-r,r]$. Then we write

$$\| w \|^{(I)}_{[0,t]} = \| w \|_{[0,t]} + \max\{ \| \partial_t w(\tau,s) \| : (\tau,s) \in [0,t] \times [-r,r] \}$$

$$+ \max\{ \| \partial_x w(\tau,s) \| : (\tau,s) \in [0,t] \times [-r,r] \},$$

where $t \in [0,b]$. If the derivatives $\partial_t w$, $\partial_x w$ satisfy the Lipschitz condition with respect to (t,x) on $[0,b] \times [-r,r]$ the we put

$$\mathrm{Lip}\,[\partial_t w, \partial_x w]_{|[0,t]}$$

$$= \sup \left\{ \frac{\|\partial_t w(\tau,s) - \partial_t w(\bar{t},\bar{s})\|}{|t-\bar{t}| + \|s-\bar{s}\|} : \ (\tau,s),(\bar{\tau},\bar{s}) \in [0,t] \times [-r,r] \right\}$$

$$+ \sup \left\{ \frac{\|\partial_x w(\tau,s) - \partial_x w(\bar{t},\bar{s})\|}{|t-\bar{t}| + \|s-\bar{s}\|} : \ (\tau,s),(\bar{\tau},\bar{s}) \in [0,t] \times [-r,r] \right\}$$

and

$$\| w \|^{(I.L)}_{[0,t]} = \| w \|^{(I)}_{[0,t]} + \mathrm{Lip}\,[\partial_t w, \partial_x w]_{|[0,t]}, \quad t \in [0,b].$$

The fundamental axioms on the phase space X are the following.

ASSUMPTION H $[X]$. Suppose that

1) $(X, \|\cdot\|_X)$ is a Banach space and

(i) if $w : (-\infty, b] \times [-r,r] \to B$, $0 < b \leq a$, is a function such that $w_{(0)} \in X$ and $w_{|[0,b]\times[-r,r]}$ is continuous then $w_{(t)} \in X$ for $t \in [0,b]$,

(ii) there are constants K, K_0 independent of w such that

$$\| w_{(t)} \|_X \leq K \| w \|_{[0,t]} + K_0 \| w_{(0)} \|_X, \quad t \in [0,b], \tag{3}$$

2) if $z : (-\infty, b] \times R^n \to B$ is a function such that $z_{(0,x)} \in X$ for $x \in R^n$ and $z_{|[0,b]\times R^n}$ is continuous then the function $(t,x) \to z_{(t,x)}$ is continuous on $[0,b] \times R^n$,

3) the linear subspace $X_I \subset X$ is such that

(i) X_I endowed with the norm $\|\cdot\|_{X_I}$ is a Banach space,

(ii) if $w : (-\infty, b] \times [-r,r] \to B$, $0 < b \leq a$, is a function such that $w_{(0)} \in X_I$ and $w_{|[0,b]\times[-r,r]}$ is of class C^1 then $w_{(t)} \in X_I$ for $t \in [0,b]$, and there are constants $L, L_0 \in R_+$ independent of w such that

$$\| w_{(t)} \|_{X_I} \leq L \| w \|^{(I)}_{[0,t]} + L_0 \| w_{(0)} \|_{X_I}, \quad t \in [0,b], \tag{4}$$

4) the linear subspace $X_{I.L} \subset X_I$ is such that $X_{I.L}$ endowed with the norm $\|\cdot\|_{X_{I.L}}$ is a Banach space and

(i) if $w : (-\infty, b] \times [-r,r] \to B$, $0 < b \leq a$, is a function such that $w_{(0)} \in X_{I.L}$ and $w_{|[0,b]\times[-r,r]}$ is of class C^1 and the derivatives $\partial_t w$, $\partial_x w$ satisfy the Lipschitz condition with respect to (t,x) on $[0,b]\times[-r,r]$ then $w_{(t)} \in X_{I.L}$ for $t \in [0,b]$,

(ii) there are constants $M, M_0 \in R_+$ independent of w such that

$$\| w_{(t)} \|_{X_{I.L}} \leq M \| w \|^{(I.L)}_{[0,t]} + M_0 \| w_{(0)} \|_{X_{I.L}}, \; t \in [0, b]. \tag{5}$$

We will consider the spaces $\Omega = [0, a] \times R^n \times X$, $\Omega_I = [0, a] \times R^n \times X_I$ and $\Omega_{I.L} = [0, a] \times R^n \times X_{I.L}$. We adopt the following notations. If $z : (-\infty, b] \times R^n \to B$, $0 < b \leq a$, is a function such that $z_{|[0,b] \times R^n}$ is continuous then we put

$$\| z \|_{[0,t;x]} = \max \{ \|z(\tau, s)\| : \; (\tau, s) \in [0, t] \times [x - r, x + r] \},$$

and

$$\| z \|_{[0,t;R^n]} = \sup \{\|z(\tau, s)\| : \; (\tau, s) \in [0, t] \times R^n \},$$

where $(t, x) \in [0, b] \times R^n$. Suppose additionally that the derivatives

$$\partial_t z, \;\; \partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$$

exist and are continuous on $[0, b] \times R^n$. Then we write

$$\| z \|^{(I)}_{[0,t;x]} = \| z \|_{[0,t;x]} + \max \{\|\partial_t z(\tau, s)\| : \;\; (\tau, s) \in [0, t] \times [x - r, x + r] \}$$

$$+ \max \{\|\partial_x z(\tau, s)\| : \;\; (\tau, s) \in [0, t] \times [x - r, x + r] \}.$$

For the above z, suppose that the derivatives $\partial_t z$, $\partial_x z$ satisfy the Lipschitz condition with respect to (t, x) on $[0, b] \times R^n$. Then we put

$$\text{Lip}\, [\partial_t z, \partial_x z]_{|[0,t;x]}$$

$$= \sup \left\{ \frac{\|\partial_t z(\tau, s) - \partial_t z(\bar{\tau}, \bar{s})\|}{|\tau - \bar{\tau}| + \|s - \bar{s}\|} : \; (\tau, s), (\bar{\tau}, \bar{s}) \in [0, t] \times [x - r, x + r] \right\}$$

$$+ \sup \left\{ \frac{\|\partial_x z(\tau, s) - \partial_x z(\bar{\tau}, \bar{s})\|}{|\tau - \bar{\tau}| + \|s - \bar{s}\|} : \; (\tau, s), (\bar{\tau}, \bar{s}) \in [0, t] \times [x - r, x + r] \right\}$$

and

$$\| z \|^{(I.L)}_{[0,t;x]} = \| z \|^{(I)}_{[0,t;x]} + \text{Lip}\, [\partial_t z, \partial_x z]_{|[0,t;x]} .$$

If $\varphi : (-\infty, 0] \times R^n \to B$ and $\varphi_{(0,x)} \in X$ for $x \in R^n$ then we write

$$\| \varphi \|_{(X,\infty)} = \sup \{ \| \varphi_{(0,x)} \|_X : x \in R^n \}.$$

LEMMA 1. *Suppose that Assumption H [X] is satisfied.*

I. *If $z : (-\infty, b] \times R^n \to B$, $0 < b \le a$, is a function such that $z_{(0,x)} \in X$ for $x \in R^n$ and $z_{|[0,b]\times R^n}$ is continuous then $z_{(t,x)} \in X$ for $(t,x) \in (0,b] \times R^n$ and*

$$(6) \qquad \| z_{(t,x)} \|_X \le K \| z \|_{[0,t;x]} + K_0 \| z_{(0,x)} \|_X, \quad (t,x) \in [0,b] \times R^n.$$

II. *If $z : (-\infty, b] \times R^n \to B$, $0 < b \le a$, is a function such that $z_{(0,x)} \in X_I$ for $x \in R^n$ and $z_{|[0,b]\times R^n}$ is of class C^1 then $z_{(t,x)} \in X_I$ for $(t,x) \in (0,b] \times R^n$ and*

$$(7) \qquad \| z_{(t,x)} \|_{X_I} \le L \| z \|^{(I)}_{[0,t;x]} + L_0 \| z_{(0,x)} \|_{X_I}, \quad (t,x) \in [0,b] \times R^n.$$

III. *Suppose that $z : (-\infty, b] \times R^n \to B$, $0 < b \le a$, satisfies the conditions*

(i) $z_{(0,x)} \in X_{I.L}$ for $x \in R^n$ and $z_{|[0,b]\times R^n}$ is of class C^1,

(ii) the derivatives $\partial_t z$, $\partial_x z$ satisfy the Lipschitz condition with respect to (t,x) on $[0,b] \times R^n$.

Then $z_{(t,x)} \in X_{I.L}$ for $(t,x) \in (0,b] \times R^n$ and

$$(8) \quad \| z_{(t,x)} \|_{X_{I.L}} \le M \| z \|^{(I.L)}_{[0,t;x]} + M_0 \| z_{(0,x)} \|_{X_{I.L}}, \quad (t,x) \in [0,b] \times R^n.$$

PROOF. Estimates (6) - (8) follow from (3) - (5) for w given by $w(\tau, s) = z(\tau, x + s)$, $(\tau, s) \in (-\infty, b] \times [-r, r]$, with fixed $x \in R^n$.

ASSUMPTION H [φ]. Suppose that $\varphi : (-\infty, 0] \times R^n \to B$, there exist the derivatives $(\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi) = \partial_x \varphi$ and

1) $\varphi_{(0,x)} \in X$ and $(\partial_{x_i}\varphi)_{(0,x)} \in X$, $1 \le i \le n$, for $x \in R^n$,

2) there is $(b_0, b_1, b_2) \in R^3_+$ such that $\| \varphi_{(0,x)} \|_X \le b_0$ for $x \in R^n$ and

$$\| (\partial_{x_i}\varphi)_{(0,x)} \|_X \le b_1, \quad \| (\partial_{x_i}\varphi)_{(0,x)} - (\partial_{x_i}\varphi)_{(0,\bar{x})} \|_X \le b_2 \, \|x - \bar{x}\|,$$

where $1 \le i \le n$, $x, \bar{x} \in R^n$,

3) $\varphi_{(0,x)} \in X_{I.L}$ for $x \in R^n$ and there is $(c_0, c_1, c_3) \in R^3_+$ such that

$$\| \varphi_{(0,x)} \|_{X_I} \le c_0, \quad \| \varphi_{(0,x)} \|_{X_{I.L}} \le c_1, \quad \| \varphi_{(0,x)} - \varphi_{(0,\bar{x})} \|_{X_I} \le c_2 \, \|x - \bar{x}\|$$

where $x, \bar{x} \in R^n$,

4) there is $(s_0, s_1, s_2) \in R^3_+$ such that for $x, \bar{x} \in R^n$ we have

$$\|\varphi(0,x)\| \le s_0, \quad \|\partial_x\varphi(0,x)\| \le s_1, \quad \|\partial_x\varphi(0,x) - \partial_x\varphi(0,\bar{x})\| \le s_2\|x-\bar{x}\|.$$

REMARK 1. *Suppose that the phase space X satisfies the condition: there is a constant $\tilde{C} \in R_+$ such that for each function $w \in X$ we have $\|w(0,x)\| \le \tilde{C}\|w\|_X$, $x \in [-r,r]$. Then conditions 1), 2) of Assumption $H[\varphi]$ imply condition 4).*

We define some function spaces. Let $\varphi : (-\infty, 0] \times R^n \to B$ be given and let $0 < c \le a$, $d = (d_0, d_1, d_2) \in R^3_+$. We denote by $C^{I.L}_{\varphi.c}[d]$ the set of all functions $z : (-\infty, c] \times R^n \to B$ such that

(i) $z(t,x) = \varphi(t,x)$ on $(-\infty, 0] \times R^n$,

(ii) the derivatives $\partial_t z$, $\partial_x z$ exist on $[0,c] \times R^n$ and

$$\|z(t,x)\| \le d_0, \quad \|\partial_t z(t,x)\| + \|\partial_x z(t,x)\| \le d_1 \ \text{ on } \ [0,c] \times R^n,$$

(iii) for $(t,x), (\bar{t},\bar{x}) \in [0,c] \times R^n$ we have

$$\|\partial_t z(t,x) - \partial_t z(\bar{t},\bar{x})\| + \|\partial_x z(t,x) - \partial_x z(\bar{t},\bar{x})\| \le d_2\left[\,|t-\bar{t}| + \|x-\bar{x}\|\,\right].$$

We will prove that under suitable assumptions on ϱ, f and φ and for sufficiently small c with $0 < c \le a$, there exists a solution $\tilde{z} \in C^{I.L}_{\varphi.c}[d]$ of problem (1), (2).

LEMMA 2. *Suppose that Assumptions $H[X]$, $H[\varphi]$ are satisfied and $z \in C^{I.L}_{\varphi.c}[d]$, $0 < c \le a$. Then*

$$\|z_{(t,x)}\|_X \le \kappa_0, \quad \kappa_0 = Kd_0 + K_0 b_0, \tag{9}$$

$$\|z_{(t,x)}\|_{X_I} \le \kappa, \quad \kappa = L(d_0 + d_1) + L_0 c_0, \tag{10}$$

$$\|(\partial_{x_i} z)_{(t,x)}\|_X \le \tilde{\kappa}, \ i = 1, \ldots, n, \quad \tilde{\kappa} = Kd_1 + K_0 b_1 \tag{11}$$

$$\|z_{(t,x)}\|_{I.L} \le \tilde{c}, \quad \tilde{c} = M(d_0 + d_1 + d_2) + M_0 c_1, \tag{12}$$

where $(t,x) \in [0,c] \times R^n$ and

$$\|z_{(t,x)} - z_{(t,\bar{x})}\|_{X_I} \le \left[\,L(d_1 + d_2) + L_0 c_2\,\right] \|x - \bar{x}\|, \tag{13}$$

$$\|(\partial_{x_i} z)_{(t,x)} - (\partial_{x_i} z)_{(t,\bar{x})}\|_X \le (\,Kd_2 + K_0 b_2\,)\,\|x - \bar{x}\|, \tag{14}$$

$$\| z_{(t,x)} - z_{(\bar{t},\bar{x})} \|_X \leq K d_1 (|t - \bar{t}| + \|x - \bar{x}\|) + K_0 b_1 \|x - \bar{x}\|, \tag{15}$$

where $t, \bar{t} \in [0, c]$, $x, \bar{x} \in R^n$.

PROOF. We omit the simple proof of (9) - (12). We prove (13). Suppose that (t, x), $(t, \bar{x}) \in [0, c] \times R^n$ and a function $\tilde{z} : (-\infty, c] \times R^n \to B$ is defined by $\tilde{z}(\tau, s) = z(\tau, s + \bar{x} - x)$, $(\tau, s) \in (-\infty, c] \times R^n$. Then $\tilde{z}_{(t,x)} = z_{(t,\bar{x})}$ and

$$\| z_{(t,x)} - z_{(t,\bar{x})} \|_{X_I} = \| (z - \tilde{z})_{(t,x)} \|_{X_I}$$

$$\leq L \| z - \tilde{z} \|^{(I)}_{[0,t;x]} + L_0 \| \varphi_{(0,x)} - \varphi_{(0,\bar{x})} \|_{X_I} \leq [L(d_1 + d_2) + L_0 c_2] \|x - \bar{x}\|,$$

which proves (13). In a similar way we prove (14) and (15).

Now we give examples of phase spaces.

EXAMPLE 1. *Let X be the class of all functions $w : D \to B$ which are uniformly continuous and bounded on D. For $w \in X$ we write*

$$\| w \|_X = \sup \{ \|w(t, x)\| : \ (t, x) \in D \}. \tag{16}$$

Let $X_I \subset X$ denote the set of all $w \in X$ such that the derivatives $\partial_t w$, $\partial_x w$ exist and are uniformly continuous and bounded on D. For $w \in X_I$ we write

$$\| w \|_{X_I} = \| w \|_X + \sup \{ \|\partial_t w(t, x)\| : \ (t, x) \in D \} \tag{17}$$

$$+ \sup \{ \|\partial_x w(t, x)\| : \ (t, x) \in D \}.$$

Let $X_{I.L} \subset X_I$ be the class of all $w \in X_I$ such that

$$\text{Lip} \, [\partial_t w, \partial_x w]_D < +\infty$$

where

$$\text{Lip} \, [\partial_t w, \partial_x w]_D \tag{18}$$

$$= \sup \left\{ \frac{\|\partial_t w(t, x) - \partial_t w(\bar{t}, \bar{x})\|}{|t - \bar{t}| + \|x - \bar{x}\|} : \ (t, x), \ (\bar{t}, \bar{x}) \in D \right\}$$

$$+ \sup \left\{ \frac{\|\partial_x w(t, x) - \partial_x w(\bar{t}, \bar{x})\|}{|t - \bar{t}| + \|x - \bar{x}\|} : \ (t, x), \ (\bar{t}, \bar{x}) \in D \right\}.$$

Write

$$\| w \|_{X_{I.L}} = \| w \|_{X_I} + \text{Lip} \, [\partial_t w, \partial_x w]_D . \tag{19}$$

Then Assumption $H[X]$ is satisfied with all the constant equal 1.

EXAMPLE 2. *Let X be the class of all functions $w : D \to B$ such that*

(i) w is continuous and bounded on D,

(ii) there exists the limit

$$\lim_{t\to-\infty} w(t,x) = w_0(x) \ \ uniformly\ with\ respect\ to\ \ x \in [-r,r].$$

Let $\| w \|_X$ be defined by (16). Denote by $X_I \subset X$ the set of all $w \in X$ such that

(i) the derivatives $\partial_t w$, $\partial_x w$ exist, they are continuous and bounded on D,

(ii) there exist the limits

$$\lim_{t\to-\infty} \partial_t w(t,x) = w_1(x), \quad \lim_{t\to-\infty} \partial_x w(t,x) = w_2(x)$$

uniformly with respect to $x \in [-r,r]$. For $w \in X_I$ we define the norm $\| w \|_{X_I}$ by (17).

Let $X_{I.L} \subset X_I$ be the class of all $w \in X_I$ which satisfy

$$\mathrm{Lip}\,[\partial_t w, \partial_x w]_D < +\infty.$$

The norm in the space $X_{I.L}$ is defined by (19).

Then Assumption $H[X]$ is satisfied with all the constant equal 1.

EXAMPLE 3. *Let $\gamma : (-\infty, 0] \to (0,\infty)$ be a continuous function. Assume also that γ is nonincreasing on $(-\infty,0]$. Let X be the space of continuous functions $w : D \to B$ such that*

$$\lim_{t\to-\infty} \frac{\|w(t,x)\|}{\gamma(t)} = 0, \quad x \in [-r,r].$$

Put

$$\| w \|_X = \sup\left\{ \frac{\|w(t,x)\|}{\gamma(t)} : \ (t,x) \in D \right\}.$$

Denote by X_I the class of all $w \in X$ such that the derivatives $\partial_t w$, $\partial_x w$ exist, $\partial_t w \in C(D,B)$, $\partial_x w \in C(D,B^n)$ and

$$\lim_{t\to-\infty} \frac{\|\partial_t w(t,x)\|}{\gamma(t)} = 0, \quad \lim_{t\to-\infty} \frac{\|\partial_x w(t,x)\|}{\gamma(t)} = 0,$$

where $x \in [-r,r]$. For $w \in X_I$ we define the norm $\| w \|_{X_I}$ by

$$\| w \|_{X_I} = \| w \|_X + \sup\left\{ \frac{\|\partial_t w(t,x)\|}{\gamma(t)} : \ (t,x) \in D \right\}$$

$$+\sup\left\{\frac{\|\partial_x w(t,x)\|}{\gamma(t)} : (t,x)\in D\right\}.$$

Let $X_{I.L}\subset X_I$ be the space of all $w\in X_I$ which satisfy

$$\operatorname{Lip}\,[\,\partial_t w,\partial_x w\,]_\gamma < +\infty,$$

where

$$\operatorname{Lip}\,[\,\partial_t w,\partial_x w\,]_\gamma = \sup\left\{\frac{\|\partial_t w(t,x)-\partial_t w(\bar t,\bar x)\|}{\gamma(t)\,(|t-\bar t|+\|x-\bar x\|)} : (t,x),(\bar t,\bar x)\in D\right\}$$

$$+\sup\left\{\frac{\|\partial_x w(t,x)-\partial_x w(\bar t,\bar x)\|}{\gamma(t)\,(|t-\bar t|+\|x-\bar x\|)} : (t,x),(\bar t,\bar x)\in D\right\}.$$

We define the norm in the space $X_{I.L}$ by

$$\|\,w\,\|_{X_{I.L}} = \|\,w\,\|_{X_I} + \operatorname{Lip}\,[\,\partial_t w,\partial_x w\,]_\gamma\,.$$

Then Assumption $H\,[X\,]$ is satisfied with

$$K_0=L_0=M_0=1,\quad K=L=M=\frac{1}{\gamma(0)}.$$

Indeed, if $w:(-\infty,b]\times[-r,r]\to B$, $0<b\le a$, is a function such that $w_{(0)}\in X$ and $w\,|_{[0,b]\times[-r,]}$ is continuous then

$$\|w_{(t)}\|_X = \sup\left\{\frac{\|w(t+\tau,x)\|}{\gamma(\tau)} : (\tau,x)\in D\right\}$$

$$\le \|w\|_{[0,t]} + \frac{1}{\gamma(0)}\sup\left\{\frac{\|w_0(\tau,x)\|}{\gamma(\tau)} : (\tau,x)\in D\right\},$$

which proves (3). In a similar way we prove (4) and (5).

EXAMPLE 4. *Let $p\ge 1$ be fixed. Denote by X the class of all functions $w:D\to B$ such that*

(i) w is continuous on $\{\,0\,\}\times[-r,r]$, and for $x\in[-r,r]$ we have

$$\int_{-\infty}^0 \|\,w(\tau,x)\,\|^p\,d\tau < +\infty,$$

(ii) $w(t,\,\cdot\,):[-r,r]\to B$ is continuous for every $t\in(-\infty,0]$.

Write

$$\|\,w\,\|_X = \sup\,\{\,\|w(t,x)\| : (t,x)\in\{\,0\,\}\times[-r,r]\,\}$$

$$+\sup\left\{\left(\int_{-\infty}^{0}\|\,w(\tau,x)\,\|^{p}\,d\tau\right)^{\frac{1}{p}}:\ x\in[-r,r]\right\}.$$

Let $X_I \subset X$ be the set of all $w \in X$ such that the derivatives $\partial_t w$, $\partial_x w$ exist on D and

(i) the functions $\partial_t w$, $\partial_x w$ are continuous on $\{0\} \times [-r,r]$ and for $x \in [-r,r]$ we have

$$\int_{-\infty}^{0}\|\,\partial_t w(\tau,x)\,\|^{p}\,d\tau<+\infty,\quad \int_{-\infty}^{0}\|\,\partial_x w(\tau,x)\,\|^{p}\,d\tau<+\infty,$$

(ii) $\partial_t w(t,\cdot) \in C([-r,r],B)$ and $\partial_x w(t,\cdot) \in C([-r,r],B^n)$ for every $t \in (-\infty,0]$.

Write

$$\|w\|_{X_I}=\|w\|_X+\sup\{\,\|\partial_t w(t,x)\|:\ (t,x)\in\{0\}\times[-r,r]\,\}$$

$$+\sup\{\,\|\partial_x w(t,x)\|:\ (t,x)\in\{0\}\times[-r,r]\,\}$$

$$+\sup\left\{\left(\int_{-\infty}^{0}\|\partial_t w(\tau,x)\|^{p}d\tau\right)^{\frac{1}{p}}:\ x\in[-r,r]\right\}$$

$$+\sup\left\{\left(\int_{-\infty}^{0}\|\partial_x w(\tau,x)\|^{p}\,d\tau\right)^{\frac{1}{p}}:\ x\in[-r,r]\right\}.$$

Let $X_{I.L} \subset X_I$ be the set of all $w \in X_I$ such that Lip $[\partial_t w, \partial_x w]_{[p]} < +\infty$, *where*

$$\text{Lip}\,[\partial_t w,\partial_x w]_{[p]}=\sup\{\,\Gamma(h,x,\bar{x}):\ x,\bar{x}\in[-r,r],x\neq\bar{x},h\neq0\,\}$$

$$+\sup\left\{\tilde{\Gamma}(h,x,\bar{x}):\ x,\bar{x}\in[-r,r],x\neq\bar{x},h\neq0\right\}$$

and

$$\Gamma(h,x,\bar{x})=\left(\int_{-\infty}^{0}\|\partial_t w(\tau+h,x)-\partial_t w(\tau,\bar{x})\|^{p}d\tau\right)^{\frac{1}{p}}(|h|+\|x-\bar{x}\|)^{-1},$$

$$\tilde{\Gamma}(h,x,\bar{x})=\left(\int_{-\infty}^{0}\|\partial_x w(\tau+h,x)-\partial_x w(\tau,\bar{x})\|^{p}d\tau\right)^{\frac{1}{p}}(|h|+\|x-\bar{x}\|)^{-1}.$$

We define the norm in the space $X_{I.L}$ by

$$\| w \|_{X_{I.L}} = \| w \|_{X_I} + \text{Lip}\, [\partial_t w, \partial_x w]_{[p]}.$$

Then Assumption H [X] is satisfied with $K = L = M = 1 + a^{\frac{1}{p}}$, $K_0 = L_0 = M_0 = 1$.

We prove inequality (3). If $w : (-\infty, b] \times [-r, r] \to B$, $0 < b \leq a$, is a function such that $w_{(0)} \in X$ and $w\,|_{[0,b]\times[-r,r]}$ is continuous then

$$\|w_{(t)}\|_X = \sup \left\{ \|w_{(t)}(\tau, x)\| : \ (\tau, x) \in \{0\} \times [-r, r] \right\}$$

$$+ \sup \left\{ \left(\int_{-\infty}^{0} \| w_{(t)}(\tau, x) \|^p \, d\tau \right)^{\frac{1}{p}} : \ x \in [-r, r] \right\}$$

$$\leq \|w\|_{[0,t]} + \sup \left\{ \left(\int_{-\infty}^{0} \| w(\tau, x) \|^p \, d\tau \right)^{\frac{1}{p}} : \ x \in [-r, r] \right\}$$

$$+ \sup \left\{ \left(\int_{0}^{t} \| w(\tau, x) \|^p \, d\tau \right)^{\frac{1}{p}} : \ x \in [-r, r] \right\}$$

$$\leq \|w_{(0)}\|_X + (1 + a^{\frac{1}{p}}) \|w\|_{[0,t]}, \ t \in [0, b],$$

which proves (3). In a similar way we prove (4) and (5).

EXAMPLE 5. *Let X be the class of all functions $w : D \to B$ such that*

(i) w is continuous on $\{0\} \times [-r, r]$ and for every $x \in [-r, r]$ we have

$$V(x) = \sup \left\{ \int_{-(n+1)}^{-n} \| w(t, x) \| \, dt : \ n \in \mathbf{N} \right\} < +\infty,$$

where $\mathbf{N}$ is the set of natural numbers,

(ii) the function $w(t, \cdot) : [-r, r] \to B$ is continuous for every $t \in (-\infty, 0]$.

We define the norm in the space X by

$$\| w \|_X = \sup \{ \|w(t, x)\| : \ (t, x) \in \{0\} \times [-r, r] \} + \sup \{ V(x) : \ x \in [-r, r] \}.$$

Let $X_I \subset X$ be the set of all $w \in X$ such that *(i) the derivatives* $\partial_t w$, $\partial_x w$ *exist on* D *and they are continuous on* $\{0\} \times [-r, r]$,

(ii) $\partial_t w(t, \cdot) \in C([-r, r], B)$, $\partial_x w(t, \cdot) \in C([-r, r], B^n)$ *for every* $t \in (-\infty, 0]$ *and*

$$V_1(x) = \sup \left\{ \int_{-(n+1)}^{-n} \| \partial_t w(t, x) \| \, dt : \quad n \in \mathbf{N} \right\} < \infty,$$

$$V_1(x) = \sup \left\{ \int_{-(n+1)}^{-n} \| \partial_x w(t, x) \| \, dt : \quad n \in \mathbf{N} \right\} < \infty.$$

For $w \in X_I$ *we define the norm*

$$\|w\|_{X_I} = \|w\|_X + \sup \{ \|\partial_t w(t, x)\| : \ (t, x) \in \{0\} \times [-r, r] \}$$

$$+ \sup \{ \|\partial_x w(t, x)\| : \ (t, x) \in \{0\} \times [-r, r] \}$$

$$+ \sup\{ V_1(x) : \ x \in [-r, r] \} + \sup\{ V_2(x) : \ x \in [-r, r] \}.$$

Let $X_{I.L} \subset X_I$ *be the set of all* $w \in X_I$ *such that* $\mathrm{Lip}\, [\partial_t w, \partial_x w]_V < +\infty$ *where*

$$\mathrm{Lip}\, [\partial_t w, \partial_x w]_V$$

$$= \sup \left\{ \int_{-(n+1)}^{-n} \Theta(t, x, \bar{x}, h) \, dt : \ x, \bar{x}, \in [-r, r], x \neq \bar{x}, h \neq 0, n \in \mathbf{N} \right\}$$

$$+ \sup \left\{ \int_{-(n+1)}^{-n} \tilde{\Theta}(t, x, \bar{x}, h) \, dt : x, \bar{x} \in [-r, r], x \neq \bar{x}, h \neq 0, n \in \mathbf{N} \right\}$$

and

$$\Theta(t, x, \bar{x}, h) = \frac{\|\partial_t w(t + h, x) - \partial_t w(t, \bar{x})\|}{|h| + \|x - \bar{x}\|},$$

$$\tilde{\Theta}(t, x, \bar{x}, h) = \frac{\|\partial_x w(t + h, x) - \partial_x w(t, \bar{x})\|}{|h| + \|x - \bar{x}\|}.$$

We define the norm in the space $X_{I.L}$ *by*

$$\|w\|_{X_{I.L}} = \|w\|_{X_I} + \mathrm{Lip}\, [\partial_t w, \partial_x w]_V.$$

Then Assumption $H\,[X\,]$ *is satisfied with* $K = L = M = 1 + a$, $\ K_0 = L_0 = M_0 = 2$.

3. Functional integral equations. Let θ be the class of all functions $\gamma \in C(R_+, R_+)$ which are nondecreasing and $\gamma(0) = 0$. For the matrix $Y = [\,y_{i,j}\,]_{i,j=1,\dots n}$ we denote the norm

$$\|Y\| = \max\left\{\sum_{i=1}^{n} |y_{ij}| : \; 1 \le j \le n\right\}.$$

We will assume that there are the Fréchet derivatives of ϱ_i , $1 \le i \le n$, and f with respect to the functional variable. They are denoted by $\partial_w \varrho_i(t,x,w)$, $1 \le i \le n$, $\partial_w f(t,x,w)$ and

$$\partial_w \varrho(t,x,w) = (\,\partial_w \varrho_1(t,x,w), \dots, \partial_w \varrho_n(t,x,w)\,).$$

The norms of the linear operators $\partial_w \varrho_i(t,x,w)$, $1 \le i \le n$, and $\partial_w f(t,x,w)$ are denoted by $\|\partial_w \varrho_i(t,x,w)\|$ and $\|\partial_w f(t,x,w)$. Write

$$\|\partial_w \varrho(t,x,w)\| = \sum_{i=1}^{n} \|\partial_w \varrho_i(t,x,w)\|.$$

Assumption H $[\,\varrho\,]$. Suppose the following:

1) $\varrho = (\varrho_1, \dots, \varrho_n) \in C(\Omega, R^n)$ is a function of the variables (t,x,w) and there is $\alpha_0 \in \theta$ such that $\|\,\varrho(t,x,w)\,\| \le \alpha_0(q)$ for $(t,x,w) \in \Omega$, $\|w\|_X \le q$,

2) the derivatives

$$\partial_x \varrho(t,x,w) = \left[\,\partial_{x_j} \varrho_i(t,x,w)\,\right]_{i,j=1,\dots,n},$$

and the Fréchet derivatives

$$\partial_w \varrho(t,x,w) = (\,\partial_w \varrho_1(t,x,w), \dots, \partial_w \varrho_n(t,x,w)\,)$$

exist for $(t,x,w) \in \Omega_I$ and there is $\alpha \in \theta$ such that

$$\|\partial_x \varrho(t,x,w)\|, \; \|\partial_w \varrho(t,x,w)\| \le \alpha(q), \; \|\varrho(t,x,w) \; - \varrho(\tilde{t},x,w)\| \le \alpha(q)\,|t - \tilde{t}|,$$

where $(t,x,w) \in \Omega_I$, $\tilde{t} \in [0,a]$, $\|w\|_{X_I} \le q$,

3) there is $\beta \in \theta$ such that the terms

$$\|\,\partial_x \varrho(t,x,w) - \partial_x \varrho(\bar{t},\bar{x},\bar{w})\,\|, \;\; \|\,\partial_w \varrho(t,x,w) - \partial_w \varrho(\bar{t},\bar{x},\bar{w})\,\|$$

where (t,x,w), $(\bar{t},\bar{x},\bar{w}) \in \Omega_{I.L}$, $\|w\|_{X_{I.L}}$, $\|\bar{w}\|_{X_{I.L}} \le q$, are bounded by

$$\beta(q)\,[\,|t - \bar{t}| + \|x - \bar{x}\| + \|w - \bar{w}\|_{X_I}\,].$$

Suppose that φ satisfies Assumption H $[\varphi]$ and $z \in C^{I.L}_{\varphi.c}[d]$. We consider the Cauchy problem

$$\eta'(\tau) = \varrho(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}), \quad \eta(t) = x, \tag{20}$$

where $(t,x) \in [0,a] \times R^n$ and denote by $g[z](\cdot, t, x)$ its solution. The function $g[z]$ is the bicharacteristic of equation (1) corresponding to z. Write

$$P[z](\tau,t,x) = (\tau, g[z](\tau,t,x), z_{(\tau,g[z](\tau,t,x))}).$$

We consider the system of integral functional equations which are generated by (1), (2):

$$z(t,x) = \varphi(0, g[z](0,t,x)) + \int_0^t f(\xi, g[z](\xi,t,x), z_{(\xi,g[z](\xi,t,x))})\, d\xi, \tag{21}$$

$$z(t,x) = \varphi(t,x) \ \text{ on } \ (-\infty, 0] \times R^n, \tag{22}$$

$$g[z](\tau,t,x) = x + \int_t^\tau \varrho(\xi, g[z](\xi,t,x), z_{(\xi,g[z](\xi,t,x))})\, d\xi. \tag{23}$$

REMARK 2. *Note that equation (23) is equivalent to (20) and system (21), (22) is obtained by integration of equation (1) along bicharacteristics.*

LEMMA 3. *Suppose that Assumptions H $[X]$ and H $[\varrho]$ are satisfied and*

1) the functions φ, $\tilde{\varphi} : (-\infty,0] \times R^n \to B$ satisfy Assumption H $[\varphi]$,

2) $c \in (0,a]$ and $z \in C^{I.L}_{\varphi.c}[d]$, $\tilde{z} \in C^{I.L}_{\tilde{\varphi}.c}[d]$.

Then the solutions $g[z](\cdot,t,x)$ and $g[\tilde{z}](\cdot,t,x)$ exist on $[0,c]$, they are unique and

(i) the derivatives $\partial_t g[z]$, $\partial_x g[z]$ exist and are continuous on $[0,c] \times [0,c] \times R^n$ and

$$\| \partial_t g[z](\tau,t,x) \|, \ \| \partial_x g[z](\tau,t,x) \| \le \Gamma_0 \tag{24}$$

where $\Gamma_0 = \max\{1, \alpha_0(\kappa_0)\} \exp[c(1+\tilde{\kappa})\alpha(\kappa)]$ and $\kappa_0, \tilde{\kappa}, \kappa$ are defined in Lemma 2,

(ii) there are p_0, p_1, $\tilde{p}_0$, $\tilde{p}_1 \in R_+$ such that for $(\tau,t,x), (\tau,\bar{t},\bar{x}) \in [0,c] \times [0,c] \times R^n$ we have

$$\| \partial_t g[z](\tau,t,x) - \partial_t g[z](\tau,\bar{t},\bar{x}) \| \le p_0|t-\bar{t}| + p_1\|x-\bar{x}\| \tag{25}$$

and

$$\| \partial_x g[z](\tau,t,x) - \partial_x g[z](\tau,\bar{t},\bar{x}) \| \le \tilde{p}_0|t-\bar{t}| + \tilde{p}_1\|x-\bar{x}\|, \tag{26}$$

$$p_1 = \left[\tilde{\Gamma} + \alpha(\kappa)K_0 b_1\right] \exp\left[c\,\alpha(\kappa)(1+\tilde{\kappa})\right].$$

In the same way, using (29) we get

$$\| \partial_x g[z](\tau,t,x) - \partial_x g[z](\tau,\bar{t},\bar{x}) \| \le \bar{\Gamma}(|t-\bar{t}| + \|x-\bar{x}\|) + \Gamma_0\alpha(\kappa)(1+\tilde{\kappa})|t-\bar{t}|$$

$$+\alpha(\kappa)(1+\tilde{\kappa})\left| \int_t^\tau \| \partial_x g[z](\xi,t,x) - \partial_x g[z](\xi,\bar{t},\bar{x}) \| \, d\xi \right|,$$

where $\bar{\Gamma} = c\Gamma_0\left[(1+\tilde{\kappa})\Gamma + \alpha(\kappa)(Kd_2 + K_0 b_2)\Gamma_0\right]$. This shows that condition (26) is satisfied with

$$\tilde{p}_0 = \left[\bar{\Gamma} + \Gamma_0\alpha(\kappa)(1+\tilde{\kappa})\right] \exp[c\,\alpha(\kappa)(1+\tilde{\kappa})],$$

$$\tilde{p}_1 = \bar{\Gamma}\exp[c\,\alpha(\kappa)(1+\tilde{\kappa})].$$

Now we prove (27). It follows that

$$\| g[z](\tau,t,x) - g[\tilde{z}](\tau,t,x) \|$$

$$\le \alpha(\kappa)\left| \int_t^\tau \left[\|g[z](\xi,t,x) - g[\tilde{z}](\xi,t,x)\| + \|z_{(\xi,g[z](\xi,t,x))} - \tilde{z}_{(\xi,g[\tilde{z}](\xi,t,x))}\|_X \right] d\xi \right|.$$

Since

$$\| z_{(\xi,g[z](\xi,t,x))} - \tilde{z}_{(\xi,g[\tilde{z}](\xi,t,x))} \|_X$$

$$\le \| z_{(\xi,g[z](\xi,t,x))} - z_{(\xi,g[\tilde{z}](\xi,t,x))} \|_X + \| z_{(\xi,g[\tilde{z}](\xi,t,x))} - \tilde{z}_{(\xi,g[\tilde{z}](\xi,t,x))} \|_X$$

$$\le (Kd_1 + K_0 b_1)\|g[z](\xi,t,x) - g[\tilde{z}](\xi,t,x)\|$$

$$+K\| z - \tilde{z} \|_{[0,t;R^n]} + K_0\| \varphi - \tilde{\varphi} \|_{(X,\infty)},$$

then we get the integral inequality

$$\| g[z](\tau,t,x) - g[\tilde{z}](\tau,t,x) \|$$

$$\le \alpha(\kappa)(1 + Kd_1 + K_0 b_1)\left| \int_t^\tau \| g[z](\xi,t,x) - g[\tilde{z}](\xi,t,x) \| \, d\xi \right|$$

$$+\alpha(\kappa)\,K\left|\int_t^T \|\,z-\tilde{z}\,\|_{[0,\xi;R^n]}\,d\xi\right| + c\alpha(\kappa)K_0\|\,\varphi-\tilde{\varphi}\,\|_{(X,\infty)}.$$

Let $\tilde{p} = \alpha(\kappa)\exp[\,c\,\alpha(\kappa)(1+Kd_1+K_0b_1)\,]$. Now we obtain (27) from the Gronwall inequality.

ASSUMPTION H $[\,f\,]$. Suppose the following:

1) $f \in C(\Omega, B)$ is a function of the variables (t,x,w) and

$$\|\,f(t,x,w)\,\| \le \alpha_0(q) \ \text{ for } \ (t,x,w)\in\Omega,\ \|w\|_X \le q,$$

2) the derivatives

$$\partial_x f(t,x,w) = (\,\partial_{x_1} f(t,x,w),\ldots,\partial_{x_n} f(t,x,w)\,)$$

and the Fréchet derivative $\partial_w f(t,x,w)$ exist on Ω_I and

$$\|\,\partial_x f(t,x,w)\,\|,\ \ \|\,\partial_w f(t,x,w)\,\| \le \alpha(q),$$

where $(t,x,w)\in\Omega_I$, $\|w\|_{X_I}\le q$, and

$$\|\,f(t,x,w)-f(\tilde{t},x,w)\,\| \le \alpha(q)\,|t-\tilde{t}|$$

where $(t,x,w)\in\Omega_I$, $\tilde{t}\in[0,a]$, $\|w\|_{X_I}\le q$,

3) the terms

$$\|\,\partial_x f(t,x,w)-\partial_x f(\tilde{t},\tilde{x},\tilde{w})\,\|,\ \ \|\,\partial_w f(t,x,w)-\partial_w f(\tilde{t},\tilde{x},\tilde{w})\,\|,$$

where (t,x,w), $(\tilde{t},\tilde{x},\tilde{w})\in\Omega_{I.L}$, $\|w\|_{X_{I.L}}$, $\|\tilde{w}\|_{X_{I.L}}\le q$, are bounded by

$$\beta(q)\left[\,|t-\tilde{t}|+\|x-\tilde{x}\|+\|w-\tilde{w}\|_{X_I}\right].$$

REMARK 3. *For simplicity of notation, we have assumed the same estimates for ϱ and f and for their corresponding derivatives. We have assumed also the Lipschitz condition for the derivatives of ϱ and f with the same coefficient.*

For a function $z\in C^{I.L}_{\varphi.c}[\,d\,]$, $0<c\le a$, we define a function $F_\varphi[\,z\,]:(-\infty,c]\times R^n\to B$ by

$$F_\varphi[z](t,x) = \varphi(0,g[z](0,t,x)) + \int_0^t f(\xi, g[z](\xi,t,x), z_{(\xi,g[z](\xi,t,x))})\,d\xi$$

on $[0,c]\times R^n$, and $F_\varphi[z](t,x)=\varphi(t,x)$ on $(-\infty,0]\times R^n$.

ASSUMPTION H_0 $[\,\varphi\,]$. Suppose that the consistency condition

$$\partial_t\varphi(0,x)+\sum_{i=1}^n \varrho_i(0,x,\varphi_{(0,x)})\,\partial_{x_i}\varphi(0,x) = f(0,x,\varphi_{(0,x)})$$

is satisfied for $x \in R^n$.

LEMMA 4. *Suppose that Assumptions $H\,[X\,]$, $H\,[\varphi]$, $H_0\,[\varphi]$, $H\,[\varrho]$ and $H\,[f\,]$ are satisfied. Then there is $c \in (0,a]$ and $d = (d_0, d_1, d_2) \in R^3_+$ such that*

$$F_\varphi : C^{I.L}_{\varphi.c}[\,d\,] \to C^{I.L}_{\varphi.c}[\,d\,] \tag{30}$$

and for $z, \tilde{z} \in C^{I.L}_{\varphi.c}[\,d\,]$ we have

$$\|\, F_\varphi[z] - F_\varphi[\tilde{z}]\,\|_{[0,t;R^n]} \le q \,\|\, z - \tilde{z}\,\|_{[0,t;R^n]} \tag{31}$$

with $0 < q < 1$.

PROOF. It follows that

$$\partial_t F_\varphi[z](t,x) = \partial_x \varphi(0, g[z](0,t,x))\partial_t g[z](0,t,x)$$

$$+ f(t,x,z_{(t,x)}) + \int_0^t B(\xi,t,x)\,\partial_t g[z](\xi,t,x)d\xi,$$

and

$$\partial_x F_\varphi[z](t,x) = \partial_x \varphi(0, g[z](0,t,x))\partial_x g[z](0,t,x)$$

$$+ \int_0^t B(\xi,t,x)\,\partial_x g[z](\xi,t,x)\,d\xi,$$

where $(t,x) \in [0,c] \times R^n$. The function B is defined by

$$B(\xi,t,x) = \partial_x f(P[z](\xi,t,x)) + \partial_w f(P[z](\xi,t,x)) \star (\partial_x z)_{(\xi, g[z](\xi,t,x))}$$

and

$$\partial_w f(P[z](\xi,t,x)) \star (\partial_x z)_{(\xi, g[z](\xi,t,x))}$$

$$= (\partial_w f(P[z](\xi,t,x))(\partial_{x_1} z)_{(\xi, g[z](\xi,t,x))}, \ldots, \partial_w f(P[z](\xi,t,x))(\partial_{x_n} z)_{(\xi, g[z](\xi,t,x))}).$$

Let $c \in (0,a]$, $d = (d_0, d_1, d_2) \in R^3_+$ be such constants that

$$\alpha_0(\kappa_0)\,c + s_0 \le d_0,$$

$$2\Gamma_0\,[\,s_1 + c(1+\tilde{\kappa})\alpha(\kappa)\,] + \alpha_0(\kappa_0) \le d_1, \tag{32}$$

$$\tilde{\Gamma} + \bar{\Gamma} + 2s_2\Gamma_0^2 + \tilde{q} + 2\bar{p}\,[s_1 + c\,\alpha(\kappa)(1+\tilde{\kappa})\,] \le d_2,$$

where

$$\bar{p} = (\,\tilde{\Gamma} + \tilde{q}\,)\,\exp[\,c\,\alpha(\kappa)(1+\tilde{\kappa})\,],\ \tilde{q} = \alpha(\kappa)\,\max\{\,K_0 b_1,\ 2\Gamma_0(1+\tilde{\kappa})\,\}.$$

Then we have

$$\|\,F_\varphi[z](t,x)\,\| \le s_0 + c\,\alpha_0(\kappa_0) \le d_0 \tag{33}$$

and

$$\|\,\partial_t F_\varphi[z](t,x)\,\| + \|\,\partial_x F_\varphi[z](t,x)\,\| \le d_1 \tag{34}$$

on $[0,c] \times R^n$. It follows from Assumptions H $[\,\varrho\,]$, H $[\,f\,]$, H $[\,\varphi\,]$ and from Lemma 3 that

$$\|\partial_x\varphi(0, g[z](0,t,x))\,\partial_t g[z](0,t,x) - \partial_x\varphi(0, g[z](0,\bar{t},\bar{x}))\,\partial_t g[z](0,\bar{t},\bar{x})\,\|$$

$$\le s_2\Gamma_0^2(|t-\bar{t}| + \|x-\bar{x}\|) + s_1(p_0|t-\bar{t}| + p_1\|x-\bar{x}\|),$$

and

$$\|\,f(t,x,z_{(t,x)}) - f(\bar{t},\bar{x},z_{(\bar{t},\bar{x})})\,\|$$

$$\le \alpha(\kappa)(1+Kd_1)(|t-\bar{t}| + \|x-\bar{x}\|) + \alpha(\kappa)K_0 b_1\|x-\bar{x}\|$$

and the terms

$$\|\,\partial_x f(P[z](\xi,t,x)) - \partial_x f(P[z](\xi,\bar{t},\bar{x}))\,\|,$$

$$\|\,\partial_w f(P[z](\xi,t,x)) - \partial_w f(P[z](\xi,\bar{t},\bar{x}))\,\|$$

can be estimated by $\Gamma(|t-\bar{t}| + \|x-\bar{x}\|)$. Thus we have

$$\|\,\partial_t F_\varphi[z](t,x) - \partial_t F_\varphi[z](\bar{t},\bar{x})\,\| \le (\,\tilde{\Gamma} + s_2\Gamma_0^2\,)(|t-\bar{t}| + \|x-\bar{x}\|)$$

$$+\Gamma_0\alpha(\kappa)(1+\tilde{\kappa})|t-\bar{t}| + \alpha(\kappa)K_0 b_1\|x-\bar{x}\|$$

$$+[\,s_1 + c\alpha(\kappa)(1+\tilde{\kappa})\,](p_0|t-\bar{t}| + p_1\|x-\bar{x}\|)$$

and

$$\| \partial_x F_\varphi[z](t,x) - \partial_x F_\varphi[z](\bar{t},\bar{x}) \| \le (\bar{\Gamma} + s_2 \Gamma_0^2)(|t-\bar{t}| + \|x-\bar{x}\|)$$

$$+[s_1 + c\alpha(\kappa)(1+\tilde{\kappa})](\tilde{p}_0|t-\bar{t}| + \tilde{p}_1\|x-\bar{x}\|) + \Gamma_0\alpha(\kappa)(1+\tilde{\kappa})|t-\bar{t}|,$$

where (t,x), $(\bar{t},\bar{x}) \in [0,c] \times R^n$. This together with (32) yields

$$(35) \quad \| \partial_t F_\varphi[z](t,x) - \partial_t F_\varphi[z](\bar{t},\bar{x}) \| + \| \partial_x F_\varphi[z](t,x) - \partial_x F_\varphi[z](\bar{t},\bar{x}) \|$$

$$\le d_2(|t-\bar{t}| + \|x-\bar{x}\| | \text{ on } [0,c] \times R^n.$$

Estimates (33)-(35) imply (30). Now we prove (31). Suppose that z, $\tilde{z} \in c^{I.L}_{\varphi.c}[d]$. It follows from Assumption H $[f]$ and from Lemma 3 that

$$\| F_\varphi[z](t,x) - F_\varphi[\tilde{z}](t,x) \| \le s_1 \| g[z](0,t,x) - g[\tilde{z}(0,t,x) \|$$

$$+\alpha(\kappa) \int_0^t \left[\| g[z](\xi,t,x) - g[\tilde{z}](\xi,t,x) \| + \| z_{(\xi,g[z](\xi,t,x))} - \tilde{z}_{(\xi,g[\tilde{z}](\xi,t,x))} \|_X \right] d\xi.$$

It follows from Lemma 2 that

$$\| z_{(\xi,g[z](\xi,t,x))} - \tilde{z}_{(\xi,g[\tilde{z}](\xi,t,x))} \|_X \le K \| z - \tilde{z} \|_{[0,\xi;R^n]}$$

$$+(Kd_1 + K_0 b_1) \| g[z](\xi,t,x) - g[\tilde{z}(\xi,t,x) \|.$$

Then using (27) we have estimate (31) with

$$q = cK\, [s_1\tilde{p} + c\tilde{p}\alpha(\kappa)(1 + Kd_1 + K_0 b_1) + \alpha(\kappa)].$$

For sufficiently small $c \in (0,a]$ we have $q < 1$ and the Lemma is proved.

REMARK 4. *If we assume that $s_0 < d_0$, $2s_1 < d_1$, $2s_2 < d_2$ then there is $c \in (0,a]$ sufficiently small and such that (d_0,d_1,d_2) satisfy (32). Then, under suitable assumptions on X, f, ϱ assertion (30) holds true.*

4. Existence and uniqueness of solutions. Now we formulate main results of the paper.

THEOREM 1. *Suppose that Assumptions $H[X]$, $H[\varphi]$, $H_0[\varphi]$, $H[\varrho]$, $H[f]$ are satisfied. Then there exist $c \in (0,a]$ and $d = (d_0,d_1,d_2) \in R^3_+$ such that problem (1), (2) has exactly one classical solution $u \in C^{I.L}_{\varphi.c}[d]$.*

If $\tilde{\varphi} : (-\infty, 0] \times R^n \to B$ satisfies Assumptions $H\,[\varphi]$ and $H_0\,[\varphi]$ and $\tilde{u} \in C^{I.L}_{\tilde{\varphi}.c}[\,d\,]$ is a solution of equation (1) with the initial condition $z(t,x) = \tilde{\varphi}(t,s)$ on $(-\infty,0] \times R^n$ then there is $\Lambda_c \in R_+$ such that for $t \in [0,c]$ we have

$$\| u - \tilde{u} \|_{[0,t;R^n]} \le \Lambda_c \Big[\| \varphi - \tilde{\varphi} \|_{(X,\infty)} \tag{36}$$

$$+ \sup \{ \|\varphi(0,y) - \tilde{\varphi}(0,y) \| : \; y \in R^n \} \Big].$$

PROOF. It follows from Lemma 4 that there are $c \in (0,a]$ and $d \in R^3_+$ such that the operator F_φ satisfies the conditions: $F_\varphi : C^{I.L}_{\varphi.c}[\,d\,] \to C^{I.L}_{\varphi.c}[\,d\,]$ and there is $q \in (0,1)$ such that condition (31) holds. Then the operator F_φ is a contraction on $C^{I.L}_{\varphi.c}[\,d\,]$ and there is exactly one function $u \in C^{I.L}_{\varphi.c}[\,d\,]$ satisfying equation (21) with initial condition (22). Then we have the identity

$$u(t,x) = \varphi(0, g[u](0,t,x)) + \int_0^t f(\,\xi, g[u](\xi,t,x), u_{(\xi, g[u](\xi,t,x))}\,)\, d\xi$$

for $(t,x) \in [0,c] \times R^n$, which is equivalent to

$$u(t, g[u](t,0,y)) = \varphi(0,y) + \int_0^t f(\,\xi, g[u](\xi,0,y), u_{(\xi, g[u](\xi,0,y))}\,)\, d\xi$$

for $(t,y) \in [0,c] \times R^n$. Differentiating the above identity with respect to t we get

$$\partial_t u(t, g[u](t,0,y)) - f(t, g[u](t,0,y), u_{(t,g[u](t,0,y))}) \tag{37}$$

$$= - \sum_{i=1}^{n} \partial_{x_i} u(t, g[u](t,0,y))\, \varrho_i(\,t, g[u](t,0,y), u_{(t,g[u](t,0,y))}\,),$$

where $(t,y) \in [0,c] \times R^n$. For each $(t,x) \in [0,c] \times R^n$ there is exactly one $y \in R^n$ such that $g[u](t,0,y) = x$. It follows from (37) that u satisfies equation (1) at $(t,x) \in [0,c] \times R^n$. It is clear that u satisfies initial condition (2).

Now we prove relation (36). If $u = F_\varphi[\,u\,]$ and $\tilde{u} = F_{\tilde{\varphi}}[\,\tilde{u}\,]$ then

$$\| u(t,x) - \tilde{u}(t,x) \| \le \| \varphi(0, g[u](0,t,x)) - \varphi(0, g[\tilde{u}](0,t,x)) \|$$

$$+ \| \varphi(0, g[\tilde{u}](0,t,x)) - \tilde{\varphi}(0, g[\tilde{u}](0,t,x)) \|$$

$$+ \alpha(\kappa) \int_0^t \Big[\|g[u](\xi,t,x) - g[\tilde{u}](\xi,t,x)\| + \|u_{(\xi,g[u](\xi,t,x))} - \tilde{u}_{(\xi,g[\tilde{u}](\xi,t,x))}\|_X \Big]\, d\xi.$$

It follows from Lemma 2 that

$$\| u_{(\xi,g[u](\xi,t,x))} - \tilde{u}_{(\xi,g[\tilde{u}](\xi,t,x))} \|_X \le (Kd_1 + K_0 b_1) \| g[u](\xi,t,x) - g[\tilde{u}](\xi,t,x) \|$$

$$+K \| u - \tilde{u} \|_{[0,\xi,g[\tilde{u}](\xi,t,x)]} + K_0 \| \varphi - \tilde{\varphi} \|_{(X,\infty)}$$

and

$$\| g[u](\tau,t,x) - g[\tilde{u}](\tau,t,x) \|$$

$$\le \tilde{s} \left[K \int_0^t \| u - \tilde{u} \|_{[0,\xi;R^n]} \, d\xi + cK_0 \| \varphi - \tilde{\varphi} \|_{(X,\infty)} \right],$$

where $0 \le \tau \le t \le c$, $x \in R^n$ and $\tilde{s} = \alpha(\kappa) \exp[\, c\,\alpha(\kappa)(1 + Kd_1 + K_0 b_1)\,]$. Then we obtain the integral inequality

$$\| u - \tilde{u} \|_{[0,t;R^n]} \le B \int_0^t \| u - \tilde{u} \|_{[0,\xi;R^n]} \, d\xi$$

$$+A \left[\sup \{ \| \varphi(0,y) - \tilde{\varphi}(0,y) \| : \; y \in R^n \} + \| \varphi - \tilde{\varphi} \|_{(X,\infty)} \right], \quad t \in [0,c],$$

where

$$A = \max \{1, \, A_0 \},$$

$$A_0 = cK_0 \, [\, s_1 \tilde{s} + c\,\alpha(\kappa)(1 + Kd_1 + K_0 b_1) + \alpha(\kappa)\,],$$

$$B = K \, [\, s_1 \tilde{s} + c\tilde{s}\alpha(\kappa)(1 + Kd_1 + K_0 b_1) + \alpha(\kappa)\,].$$

It follows from the Gronwall inequality that we have estimate (36) for $\Lambda_c = A \exp[\, cB\,]$. This completes the proof of Theorem 1.

We wish to emphasize that our hereditary setting contains as particular cases some well known delay structures. Given the functions

$$\varphi : (-\infty, 0] \times R^n \to B, \quad F : [0,a] \times R^n \times B \times B \to B,$$

$$G = (G_1, \ldots, G_n) : [0,a] \times R^n \times B \times B \to R^n$$

and

$$\psi_0 : [0,a] \times R^n \to R, \quad \psi = (\psi_1, \ldots, \psi_n) : [0,a] \times R^n \to R^n,$$

we consider the operators ϱ and f given by

$$\varrho(t,x,w) = G(\,t,x,w(0,0),w(\psi_0(t,x)-t,\psi(t,x)-x)\,) \tag{38}$$

$$f(t,x,w) = F(\,t,x,w(0,0),w(\psi_0(t,x)-t,\psi(t,x)-x)\,). \tag{39}$$

In this case (1) is equivalent to the differential equation with a deviated argument

$$\partial_t z(t,x) + \sum_{i=1}^{n} G_i(\,t,x,z(t,x),z(\psi_0(t,x),\psi(t,x))\,)\partial_{x_i} z(t,x) \tag{40}$$

$$= F(\,t,x,z(t,x),z(\psi_0(t,x),\psi(t,x))\,).$$

We consider problem consisting of (40) and (2). Let $\tilde{\Omega} = [0,a] \times R^n \times B \times B$.

ASSUMPTION H $[G,F]$. Suppose that the functions G and F of the variables (t,x,p,q) satisfy the conditions:

1) G and F are continuous and there is $\alpha \in \theta$ such that

$$\|F(t,x,p,q)\|,\ \|G(t,x,p,q)\| \le \alpha(\tau)$$

for $(t,x,p,q) \in \tilde{\Omega}$, $\|p\|,\|q\| \le \tau$,

2) the derivatives $\partial_x G$, $\partial_p G$, $\partial_q G$, $\partial_x F$, $\partial_p F$, $\partial_q F$ exist on $\tilde{\Omega}$ and there is $A \in R_+$ such that

$$\|\partial_x G(\,Q\,)\|,\ \|\partial_p G(\,Q\,)\|,\ \|\partial_q G(\,Q\,)\| \le A,$$

$$\|\partial_x F(\,Q\,)\|,\ \|\partial_p F(\,Q\,)\|,\ \|\partial_q F(\,Q\,)\| \le A,$$

where $Q = (t,x,p,q) \in \tilde{\Omega}$ and G, F satisfy the Lipschitz condition with respect to t with the constant A on $\tilde{\Omega}$,

3) there is $L \in R_+$ such that the derivatives

$$\partial_x G,\ \partial_p G,\ \partial_q G,\ \partial_x F,\ \partial_p F,\ \partial_q F$$

satisfy the Lipschitz condition with respect to (t,x,p,q) on $\tilde{\Omega}$ with the constant L.

ASSUMPTION H $[\,\psi_0,\psi\,]$. Suppose that the functions ψ_0 and ψ satisfy the conditions

1) $\psi_0 \in C([0,a] \times R^n, R)$, $\psi \in C([0,a] \times R^n, R^n)$ and

$$\psi_0(t,x) \le t,\ \ -r \le \psi(t,x) - x \le r \ \text{ on } \ [0,a] \times R^n,$$

2) the derivatives

$$\partial_x\psi_0,\quad \partial_x\psi = \Big[\partial_{x_j}\psi_i\Big]_{i,j=1,\dots,n}$$

exist on $[0,a]\times R^n$ and there is $C_0\in R_+$ such that

$$\|\partial_x\psi_0(t,x)\|\le C_0,\quad \|\partial_x\psi(t,x)\|\le C_0 \ \text{ on } \ [0,a]\times R^n,$$

and ψ_0, ψ satisfy the Lipschitz condition with respect to t with the constant C_0,

3) the derivatives $\partial_x\psi_0$, $\partial_x\psi$ satisfy the Lipschitz condition with respect to (t,x) with a constant L_0.

The main theorem reduces to the following one.

THEOREM 2. *Suppose that Assumptions H [X], H [φ], H_0 [φ], H [G, F], H [ψ_0,ψ] are satisfied. Then there are $c\in(0,a]$, $d=(d_0,d_1,d_1)\in R^3_+$ and a function $u\in C^{I.L}_{\varphi.c}[d]$ such that u is a solution of (40), (2).*

PROOF. It follows from Assumptions H $[G,F]$ and H $[\psi_0,\psi]$ that the operators ϱ and f given by (38), (39) satisfy Assumptions H $[\varrho]$ and H $[f]$. Hence, the assertion follows as an immediate consequence of Theorem 1.

REMARK 5. *If we consider the functions ϱ and f given by*

$$\varrho(t,x,w) = G(t,x,w(0,0),\int_D w(\tau,s)\,d\tau\,ds), \tag{41}$$

$$f(t,x,w) = F(t,x,w(0,0),\int_D w(\tau,s)\,d\tau\,ds), \tag{42}$$

then equation (1) reduces to the differential integral equation

$$\partial_t z(t,x) + \sum_{i=1}^{n} G_i(t,x,z(t,x),\int_D w(t+\tau,x+s)\,d\tau\,ds)\,\partial_{x_i}z(t,x) \tag{43}$$

$$= F(t,x,z(t,x),\int_D w(t+\tau,x+s)\,d\tau\,ds).$$

Suppose that Assumptions H [X], H [φ], H_0 [φ] and H [G, F] are satisfied. Then there are $c\in(0,a]$, $d=(d_0,d_1,d_2)\in R^3_+$ and a function $u\in C^{I.L}_{\varphi.c}[d]$ such that u is a solution of (43), (2).
Indeed, the operators ϱ and f given by (41), (42) satisfy Assumptions H [ϱ] and H [f]. The assertion follows as a consequence of Theorem 1.

REMARK 6. *It is important in Assumptions H [ϱ], H [f] that we have assumed that the derivatives $\partial_x\varrho$, $\partial_w\varrho$, $\partial_x f$, $\partial_w f$ exit on the space Ω_I and that* *these derivatives satisfy the local Lipschitz condition on some special function spaces.*

Let us consider simplest assumptions on ϱ *and* f. *Suppose that*

(A_f) *the derivatives* $\partial_x f$, $\partial_w f$ *exist and are bounded on* Ω *and* f *satisfies the Lipschitz condition with respect to* t *on* Ω,

(B_f) *there is* $L \in R_+$ *such that the terms*

$$\|\partial_x f(t,x,w) - \partial_x f(\bar{t},\bar{x},\bar{w})\|, \ \|\partial_w f(t,x,w) - \partial_w f(\bar{t},\bar{x},\bar{w})\|,$$

are bounded by $L(|t-\bar{t}| + \|x-\bar{x}\| + \|w-\bar{w}\|_X)$ *on* Ω *and that suitable assumptions* (A_ϱ), (B_ϱ) *are satisfied.*

Of course, our results are true under the above stronger assumptions. Now we show that our formulation of Assumptions $H\,[\varrho]$, $H\,[f\,]$ *is important. We show that there is a class of equations (1) satisfying the original assumptions but not satisfying* (A_f), (B_f), (A_ϱ), (B_ϱ).

Let X, X_I, $X_{I.L}$ *be the spaces given in Example 1. Consider equation (40) and the operators* ϱ, f *given by (38), (39). From now we consider the function* f *only. It follows that*

$$\partial_{x_i} f(t,x,w) = \partial_{x_i} F(\,Q\,) + \partial_q F(\,Q\,)\,\partial_t w(\,\tilde{Q}\,)\,\partial_{x_i}\psi_0(t,x)$$

$$+\partial_q F(\,Q\,)\sum_{j=1}^{n} \partial_{x_j} w(\,\tilde{Q}\,)\,(\,\partial_{x_i}\psi_j(t,x) - \delta_{ij}\,),$$

where δ_{ij} *is the Kronecker symbol and .*

$$Q = (\,t, x, w(0,0), w(\psi_0(t,x)-t, \psi(t,x)-x)\,), \quad \tilde{Q} = (\psi_0(t,x)-t, \psi(t,x)-x).$$

Similar considerations apply to ϱ.

Suppose that Assumptions $H\,[G,F\,]$ *and* $H\,[\psi_0,\psi\,]$ *are satisfied. We see at once that Assumptions* $H\,[\varrho]$, $H\,[f\,]$ *are satisfied and that* ϱ, f *given by (38), (39) does not satisfy conditions* (A_f), (B_f), (A_ϱ), (B_ϱ).

Let $a > 0$, $h = (h_1,\ldots,h_n) \in R^n$ and $r = (r_1,\ldots,r_n) \in R^n_+$ be given where $h_i > 0$ for $1 \le i \le n$. Suppose that $\kappa \in \mathrm{N}$, $0 \le \kappa \le n$, is fixed. For each $y = (y_1,\ldots,y_n) \in R^n$ we write $y = (y', y'')$ where $y' = (y_1,\ldots,y_\kappa)$, $y'' = (y_{\kappa+1},\ldots,y_n)$. We have $y' = y$ if $\kappa = n$ and $y'' = y$ if $\kappa = 0$. We define the sets

$$E = [0,a] \times [-h', h') \times (-h'', h''], \ \ B = (-\infty, 0] \times [0, r'] \times [-r'', 0].$$

Let $c = (c_1,\ldots,c_n) = h + r$ and

$$E_0 = (-\infty, 0] \times [-h', c'] \times [-c'', h''],$$

$$\partial_0 E = \big((0,a] \times [-h', c'] \times [-c'', h''] \big) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

Suppose that $z : E^* \to R$ and $(t,x) \in [0,a] \times [-h, h]$ are fixed. We define the function $z_{(t,x)} : B \to R$ as follows $z_{(t,x)}(\tau, s) = z(t+\tau, x+s)$, $(\tau, s) \in B$. Let X be the linear space consisting of functions mapping the set D into B. Suppose that

$$\varrho = (\varrho_1, \ldots, \varrho_n) : E \times X \to R^n, \quad f : E \times X \to B, \quad \varphi : E_0 \cup \partial_0 E \to B,$$

are given functions. We consider the quasilinear equation (1) with the initial boundary condition

$$z(t,x) = \varphi(t,x) \;\text{ on }\; E_0 \cup \partial_0 E. \tag{44}$$

The results of the paper can be extended on the mixed problem (1), (44). The axioms on the phase space X are the fillowing.

ASSUMPTION $\tilde{H}$ $[X]$. Suppose that

(i) conditions 1), 3), 4) of Assumption H $[X]$ is satisfied,

(ii) if $z : E^* \to B$ is a function such that $z_{(0,x)} \in X$ for $x \in [-h, h]$ and z is continuous on $[0,a] \times [-h', c'] \times [-c'', h'']$ then the function $(t,x) \to z_{(t,x)}$ is continuous on $[0,a] \times [-h, h]$.

The above assumption (ii) is an adaptation of condition 2) of Assumption H $[X]$ for mixed problems.

Put $\tilde{E} = (E_0 \cup \partial_0 E) \cap \bar{E}$ where $\bar{E}$ is the closure of E and

$$\Omega = E \times X, \quad \Omega_I = E \times X_I, \quad \Omega_{I.L} = E \times X_{I.L}.$$

ASSUMPTION $\tilde{H}$ $[\varrho, f]$ Suppose that Assumptions H $[\varrho]$ and H $[f]$ are satisfied with the above given Ω, Ω_I, $\Omega_{I.L}$ and for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\varrho_i(t,x,w) \geq \delta(\varepsilon) \text{ for } 1 \leq i \leq \kappa \;\text{ and }\; \varrho_i(t,x,w) \leq -\delta(\varepsilon) \text{ for } \kappa + 1 \leq i \leq n$$

where $(t,x,w) \in E \times X$ and $\|w\|_X \leq \varepsilon$.

We formulate assumptions on φ.

ASSUMPTION $\tilde{H}$ $[\varphi]$. Suppose that $\varphi : E_0 \cup \partial_0 E \to B$, there exist the derivatives $(\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi) = \partial_x \varphi$ and

1) $\varphi_{(t,x)} \in X$ and $(\partial_{x_i}\varphi)_{(t,x)} \in X$, $1 \leq i \leq n$, for $(t,x) \in \tilde{E}$,

2) there is $(b_0, b_1, b_2) \in R^3_+$ such that $\|\varphi_{(t,x)}\|_X \leq b_0$ for $(t,x) \in \tilde{E}$ and

$$\|(\partial_{x_i}\varphi)_{(t,x)}\|_X \leq b_1,$$

$$\| (\partial_{x_i}\varphi)_{(t,x)} - (\partial_{x_i}\varphi)_{(t,\bar{x})} \|_X \leq b_2 \, \|x - \bar{x}\|,$$

where $1 \leq i \leq n$, $(t,x), (t,\bar{x}) \in \tilde{E}$,

3) $\varphi_{(t,x)} \in X_{I.L}$ for $(t,x) \in \tilde{E}$ and there is $(c_0, c_1, c_3) \in R_+^3$ such that

$$\| \varphi_{(t,x)} \|_{X_I} \leq c_0, \quad \| \varphi_{(t,x)} \|_{X_{I.L}} \leq c_1,$$

$$\| \varphi_{(t,x)} - \varphi_{(t,\bar{x})} \|_{X_I} \leq c_2 \, \|x - \bar{x}\|,$$

where $(t,x), (t,\bar{x}) \in \tilde{E}$,

4) there is $(s_0, s_1, s_2) \in R_+^3$ such that

$$\| \varphi(t,x) \| \leq s_0, \quad \|\partial_x \varphi(t,x) \| \leq s_1,$$

$$\| \partial_x \varphi(t,x) - \partial_x \varphi(t,\bar{x}) \| \leq s_2 \|x - \bar{x}\|,$$

where $(t,x), (t,\bar{x}) \in \tilde{E}$,

5) the consistency condition

$$\partial_t \varphi(t,x) + \sum_{i=1}^{n} \varrho_i(t,x,\varphi_{(t,x)}) \partial_{x_i} \varphi(t,x) = f(t,x,\varphi_{(t,x)})$$

is satisfied for $(t,x) \in \tilde{E}$.

THEOREM 3. *If Assumptions $\tilde{H}$ [X], $\tilde{H}$ [ϱ, f] and $\tilde{H}$ [φ] are satisfied then there exists exactly one classical solution of (1), (44). The solution is local with respect to t and it depends continuously of φ.*

The proof of the above Theorem is similar to the proof of Theorem 1. We omit details.

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HOPF BIFURCATIONS AT INFINITY, GENERATED BY BOUNDED NONLINEAR TERMS *

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Abstract. In the paper we present new sufficient conditions for existence of large-amplitude periodic solutions for autonomous equations with a parameter. In contrast to the usual situations, the linear degenerate part of the equation does not depend on the parameter. Therefore the existence of periodic solutions is determined by the asymptotic behavior of bounded nonlinear terms at infinity. We present a new simple method to reduce the original degenerate problem to topologically nondegenerate one. This infinite-dimensional problem is studied by degree theory methods.

AMS(MOS) subject classification. 34A47, 34C23, 58F14

Key Words. Hopf bifurcation, control theory, ODE with delays, hysteresis, stop, periodic solutions, vector field rotation, degree product formula

1. Introduction.

Consider the differential equation[1]

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)F(x,\lambda). \tag{1}$$

Here $L(p,\lambda)$ and $M(p,\lambda)$ are coprime polynomials of degrees ℓ and m, $\ell > m$, with real coefficients, which depend on the scalar parameter $\lambda \in \Lambda = (a,b)$.

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[1] A definition of solution for equation (1) is given in most books on control theory, see, e.g., [2] and [8]. If $M(p,\lambda) \equiv 1$, then (1) is a usual quasilinear ODE.

The continuous nonlinearity $F(x, \lambda)$ is uniformly bounded. We shall use bounded nonlinearities of various types: functional nonlinearities $f(x(t), \lambda)$, where $f(x, \lambda) : \mathbb{R} \times \Lambda \to \mathbb{R}$ is a continuous function, nonlinearities $f(x(t - h), \lambda)$ with the delay $h > 0$, hysteresis nonlinearities. We study sufficient conditions for existence of large-amplitude periodic solutions $x(t)$ of equation (1).

DEFINITION 1. *The number* λ_0 *is called*[2] a Hopf bifurcation point at infinity *(shortly,* a Hopf bifurcation point*) for equation* (1) *with the frequency* w_0 *if for every* $\varepsilon > 0$ *there is a parameter value* $\lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ *such that equation* (1) *with* $\lambda = \lambda_\varepsilon$ *has a periodic solution* $x_\varepsilon(t)$ *of a period* T_ε *satisfying* $|T_\varepsilon - 2\pi/w_0| < \varepsilon$ *with the amplitude* $\max |x_\varepsilon(t)| > \varepsilon^{-1}$.

In other words, λ_0 is a Hopf bifurcation point with the frequency w_0 if for arbitrarily close to λ_0 values of the parameter λ equation (1) has periodic solutions of arbitrarily large amplitudes with periods arbitrarily close to $2\pi/w_0$.

The following result is formulated in [3]. Suppose the polynomial $L(p, \lambda)$ has a pair of simple conjugate roots $\sigma(\lambda) \pm w(\lambda)i$ depending continuously on λ, where $\sigma(\lambda_0) = 0$ and the function $\sigma(\lambda)$ takes values of both sign in every neighborhood of the point λ_0. Suppose $L(kw(\lambda_0)i, \lambda_0) \neq 0$ for $k = 0, 2, 3, \ldots$ Then λ_0 is a Hopf bifurcation point for equation (1) with the frequency $w(\lambda_0)$.

These sufficient conditions for bifurcation point existence use only the information on the linear part of (1). The result holds for equation (1) with any continuous bounded nonlinearity, moreover, it holds for unbounded sublinear[3] nonlinearities. So the nonlinear part of the equation is of no importance under the assumptions above.

In this paper we study equations of the form (1), where the linear part is independent of a parameter and is degenerate, i.e., $L(p, \lambda) = L(p)$, $M(p, \lambda) = M(p)$ and the polynomial $L(p)$ has a pair of pure imaginary roots $\pm w_0 i$. For such equations, the asymptotic behavior of nonlinear terms at infinity is a criterion for λ_0 to be a Hopf bifurcation point.

The paper is organized as follows. Theorem 1 of Section 2 gives sufficient conditions for existence of a Hopf bifurcation point for equations with the delayed term. These conditions are formulated in a simpler form for equations without delays in Theorem 2.

In Section 3 equations with the stop hysteresis nonlinearity ([5]) are studied. The stop is used for simplicity, similar results are valid for equations

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[2] See [3].

[3] The nonlinearity $F(x, \lambda) : E \times \Lambda \to E_1$ is *sublinear* if $\|F(x, \lambda)\|_{E_1} = o(\|x\|_E)$ as $\|x\|_E \to \infty$.

with scalar hysteresis nonlinearities of various types: the Prandtl-Ishlinskii nonlinearities, the Preisach nonlinearities, etc.

In Section 4 we give some remarks on the results obtained. The proofs (Section 5) are based on a new method to reduce the original degenerate problem to a topologically nondegenerate one. The method can also be used to study some classical bifurcation problems with nondegenerate linear part depending on a parameter, in particular to prove the result from [3] above.

2. Equations with delay. Consider the differential equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)[f(x(t),\lambda) + g(x(t-h),\lambda)] \tag{2}$$

with the real coprime polynomials $L(p)$, $M(p)$ of degrees $\ell > m$. The functions $f(x,\lambda) : \mathbb{R}\times\Lambda \to \mathbb{R}$ and $g(x,\lambda) : \mathbb{R}\times\Lambda \to \mathbb{R}$ are continuous with respect to the set of their arguments and uniformly bounded.

Let $L(\pm w_0 i) = 0$. Define

$$\beta \overset{\text{def}}{=} \lim_{w\to w_0} \frac{\Im m[L(wi)M(-wi)]}{\Re e[L(wi)M(-wi)]}, \qquad \alpha \overset{\text{def}}{=} \operatorname{arctg}\beta. \tag{3}$$

The limit in (3) (finite or infinite) always exists. To be definite, we put $\alpha = \pi/2$ if it is infinite (for example, $\alpha = \pi/2$ if the denominator in (3) is the identical zero).

Denote by f_{odd} and g_{odd} the odd components

$$f_{odd}(x,\lambda) = (f(x,\lambda) - f(-x,\lambda))/2, \qquad g_{odd}(x,\lambda) = (g(x,\lambda) - g(-x,\lambda))/2$$

of the functions f and g. Set

$$\begin{aligned} \Psi(\xi,\lambda) &= \int_0^{2\pi} \sin t\, f(\xi\sin t,\lambda)\,dt = 4\int_0^{\pi/2} \sin t\, f_{odd}(\xi\sin t,\lambda)\,dt, \\ \Gamma(\xi,\lambda) &= \int_0^{2\pi} \sin t\, g(\xi\sin t,\lambda)\,dt = 4\int_0^{\pi/2} \sin t\, g_{odd}(\xi\sin t,\lambda)\,dt \end{aligned} \tag{4}$$

and

$$\Phi(\xi,\alpha,\lambda) = \sin\alpha\Psi(\xi,\lambda) + \sin(\alpha + w_0 h)\Gamma(\xi,\lambda).$$

Functions (4) are rather usual for the control theory, they are called *describing functions* (see, e.g., [7]).

THEOREM 1. *Let the following assumptions hold:*

1. *The number w_0 is a root of odd multiplicity K for the polynomial $L(wi)$.*
2. *The relation $L(kw_0 i) \neq 0$ holds for every $k = 0, 2, 3, 4, \dots$*
3. *In every neighborhood of the point λ_0 there are points λ_1 and λ_2 such that*

$$\limsup_{\xi \to +\infty} \Phi(\xi, \alpha, \lambda_1) < 0, \qquad \liminf_{\xi \to +\infty} \Phi(\xi, \alpha, \lambda_2) > 0, \tag{5}$$

where α is given by (3).

Then the value λ_0 is a Hopf bifurcation point for equation (2) *with the frequency w_0.*

Consider the application of Theorem 1 to the problem without the delayed term. Let equation (2) have the form

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(x, \lambda). \tag{6}$$

THEOREM 2. *Let the following conditions hold:*

1. *The number w_0 is a root for both polynomials $\Im m[L(wi)M(-wi)]$ and $L(wi)$ of the same odd multiplicity K.*
2. *The relation $L(kw_0 i) \neq 0$ holds for every $k = 0, 2, 3, 4, \dots$*
3. *For every λ there exists the limit*

$$\psi(\lambda) \stackrel{\text{def}}{=} \lim_{\xi \to +\infty} \Psi(\xi, \lambda). \tag{7}$$

4. *Equation $\psi(\lambda) = 0$ has a solution λ_0 such that the function $\psi(\lambda)$ takes the values of both sign in every neighborhood of the point λ_0.*

Then the value λ_0 is a Hopf bifurcation point for equation (6) *with the frequency w_0.*

Under assumption 1 of Theorem 2 the polynomial $\Im m[L(wi)M(-wi)]$ is nonzero, so at least one of the polynomials $L(p)$ and $M(p)$ is not even. Under this assumption the limit in (3) is distinct from zero, hence $\alpha \neq 0$. Since $\Phi(\xi, \alpha, \lambda) = \sin \alpha \, \Psi(\xi, \lambda)$, conditions 3 and 4 of Theorem 2 imply condition 3 of Theorem 1 and Theorem 2 follows from Theorem 1.

If $\Im m[L(wi)M(-wi)] \equiv 0$, then $\alpha = 0$ and hence $\Phi(\xi, \alpha, \lambda) \equiv 0$. Therefore condition 3 of Theorem 1 is not satisfied for equation (6). In fact, it is an exceptional case. One can show that the identity $\Im m[L(wi)M(-wi)] \equiv 0$ holds iff both polynomials $L(p)$ and $M(p)$ are even. Furthermore, if $L(p) = L(-p)$, $M(p) = M(-p)$, where the polynomial $L(p)$ satisfies conditions 1 and 2 of Theorem 1, then equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)f(x)$$

with any bounded continuous function $f(x)$ has the continuum of periodic cycles $x(t;\xi)$, $\xi \geq \xi_0$ such that $\|x(t;\xi)\|_C \to \infty$ and $T(\xi) \to 2\pi/w_0$ as $\xi \to \infty$, where $T(\xi)$ is the period of the cycle $x(t;\xi)$ and ξ is a parameter. Therefore the identity $\Im m[L(wi)M(-wi)] \equiv 0$ implies that all the values $\lambda \in \Lambda$ are Hopf bifurcation points for equation (6) with the frequency w_0.

A simple example of equation (6) that can be studied with Theorem 2 is $x''' + x'' + x' + x = f(x,\lambda)$.

Sufficient conditions for existence of limit (7) are discussed in Section 4.

3. System with hysteresis. Here we consider the equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)\left(a(\lambda)U(\mu_0)x + f(x,\lambda)\right) \tag{8}$$

with a continuous bounded function $f(x,\lambda) : \mathbb{R} \times \Lambda \to \mathbb{R}$ and a continuous function $a(\lambda)$. By $U(\mu_0)$ we denote the stop nonlinearity with the initial state $\mu_0 \in [-1,1]$. The definition of the stop is given shortly below; for more details and for the general mathematical theory of hysteresis operators, see [5].

For a given initial state μ_0 and for every continuous input $x(t)$, $t \geq t_0$ the operator $\mu(t) = U(\mu_0)x(t)$ determines the state of the stop at each moment $t \geq t_0$. The continuous function $\mu(t)$, $t \geq t_0$ with the values in $[-1,1]$ is at the same time the output of the stop. For monotone continuous inputs,

$$U(\mu_0)x(t) = \begin{cases} \min\{\ \ 1,\ \mu_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ increases,} \\ \max\{-1,\ \mu_0 + x(t) - x(t_0)\} & \text{if } x(t) \text{ decreases.} \end{cases}$$

For each piecewise monotone continuous input the output is calculated with the help of the semigroup identity $U(U(\mu(t_0))x(t_1))x(t) = U(\mu(t_0))x(t)$, $t \geq t_1 \geq t_0$. To define the outputs for any continuous inputs, the operator $U(\mu_0)$ is extended by continuity in the space $C[t_0,t_1]$ of continuous functions from the dense set of piecewise monotone inputs $x(t)$ to the whole space. The correctness of this procedure is proved in [5].

Figure 1 shows the trajectories of the point $\{x(t), U(\mu)x(t)\}$ in the plane $\{x, Ux\}$. The point is always in the closed band $|Ux| \leq 1$, which is the join of the two boundary horizontal lines $Ux = \pm 1$ and continual number of slanting lines $Ux = x - \theta$ with $x \in (\theta - 1, \theta + 1)$ (where $\theta \in \mathbb{R}$ is a parameter). If the initial state μ is not ± 1 the point $\{x(t), U(\mu)x(t)\}$ goes along a slanting

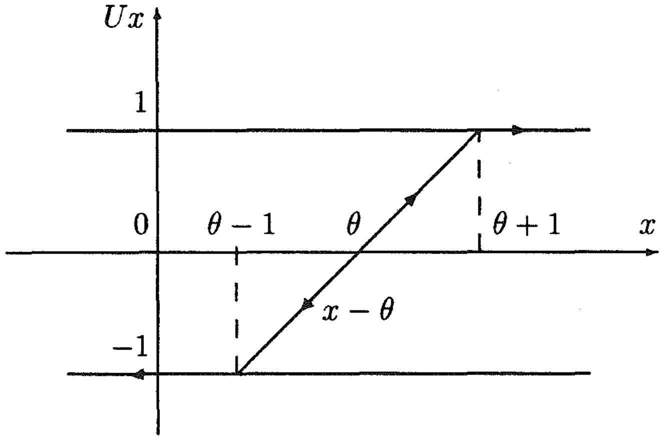

Fig. 1. *Stop* nonlinearity

line: upwards right if $x(t)$ increases and downwards left if $x(t)$ decreases. As the point reaches the horizontal line, it switches to it and goes to the right along the line $Ux = 1$ if $x(t)$ increases and to the left along the line $Ux = -1$ if $x(t)$ decreases. The point switches again to a slanting line as soon as the input $x(t)$ switches from increasing to decreasing or conversely.

The stop $U(\mu_0)x$ is a continuous operator from $[-1, 1] \times C[t_0, t_1]$ to $C[t_0, t_1]$. Moreover, this operator is Lipschitz continuous in both arguments μ_0, $x(t)$ and monotone in the natural sense.

In the following, the initial state μ_0 of the stop is not fixed. A solution $x(t)$ of equation (8) is periodic if both the function $x(t)$ and the variable state $\mu(t) = U(\mu_0)x(t)$ of the stop are periodic with the same period.

Define the number α and the function $\Psi(\xi, \lambda)$ by formulas (3) and (4).

THEOREM 3. *Let assumptions* 1 *and* 2 *of Theorem* 1 *hold and limit* (7) *exist for each* λ. *Suppose the equation*

$$0 = \psi(\lambda) \sin \alpha - 4a(\lambda) \cos \alpha \stackrel{\text{def}}{=} \phi(\lambda)$$

has a solution λ_0 *such that the function* $\phi(\lambda)$ *takes the values of both sign in every neighborhood of the point* λ_0. *Then* λ_0 *is a Hopf bifurcation point for equation* (8) *with the frequency* w_0.

4. Remarks.

4.1. Equations with variable linear part. Consider the equation

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)[f(x(t),\lambda) + g(x(t-h),\lambda)],$$

where the polynomial $L(p,\lambda)$ has the same imaginary roots $\pm w_0 i$ of the same odd multiplicity K for all parameter values and $L(kw_0 i,\lambda) \neq 0$ for every $k = 0,2,3\ldots$ and every λ. If the nonlinearity satisfies the conditions of Theorem 1, then the conclusion of the theorem holds for this equation. The same is true for Theorem 2 applied to the equation

$$L\left(\frac{d}{dt},\lambda\right)x = M\left(\frac{d}{dt},\lambda\right)f(x,\lambda).$$

These facts follow from the proof of Theorem 1 (see Section 5) without any additional argument.

Values (3) may depend on λ in this case.

4.2. Computation of limit (7). For applications of Theorems 2 and 3 it is important to know if limit (7) exists for a given function $f(x,\lambda)$. Consider some sufficient conditions for the limit existence for nonlinearities without parameters (for nonlinearities depending on the parameter we suppose these conditions for every parameter value).

Let $f(x) = f_1(x) + f_2(x) + f_3(x)$, where the function $f_1(x)$ satisfies the Landesman-Lazer conditions, i.e., the finite limits

$$\lim_{x\to+\infty} f_1(x) = f^+, \qquad \lim_{x\to-\infty} f_1(x) = f^-$$

exist; the function $f_2(x)$ is even; the primitive of the function $f_3(x)$ is sublinear:

$$\lim_{x\to\infty} x^{-1}\int_0^x f_3(u)\,du = 0. \tag{9}$$

Then limit (7) exists and

$$\lim_{\xi\to\infty}\int_0^{2\pi} \sin t\, f(\xi\sin t)\,dt = 2(f^+ - f^-).$$

For example, equality (9) holds for all periodic and almost periodic functions $f_3(x)$ with zero average value, for the functions $\sin x^3$, $\sin\sqrt{|x|}$, for every function $f_3(x)$ vanishing at infinity, etc. The sum of the functions satisfying (9) also satisfies (9). Equality (9) is not valid for the function $\sin\ln(1+|x|)$.

4.3. More general result. Natural analogs of Theorems 1 – 3 hold for equations with several delays, for example

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right)\sum_{k=0}^{N} f_k(x(t-h_k),\lambda),$$

and for equations containing both delayed and hysteresis terms.

Consider the more complicated than (2) equation

$$L\left(\frac{d}{dt}\right)x = M\left(\frac{d}{dt}\right) f(x(t), x(t-h), \lambda) \tag{10}$$

with a bounded continuous function $f(x,y,\lambda): \mathbb{R}\times\mathbb{R}\times\Lambda \to \mathbb{R}$. Set

$$\Phi_0(\xi,\alpha,w,\lambda) \stackrel{\text{def}}{=} -\int_0^{2\pi} \cos(t+\alpha)\, f(\xi\sin t, \xi\sin(t-wh),\lambda)\,dt, \tag{11}$$

where α is given by (3).

We say that the function $f(x,y,\lambda)$ satisfies the proper Lipschitz condition in x if

$$|f(x_1,y,\lambda) - f(x_2,y,\lambda)| \le \zeta(|x_1|+|x_2|+|y|)|x_1-x_2|, \qquad \lambda\in\Lambda$$

with $\zeta(r)\to 0$ as $r\to\infty$. Similarly, $f(x,y,\lambda)$ satisfies the proper Lipschitz condition in y if

$$|f(x,y_1,\lambda) - f(x,y_2,\lambda)| \le \zeta(|x|+|y_1|+|y_2|)|y_1-y_2|, \qquad \lambda\in\Lambda$$

with $\zeta(r)$ vanishing at infinity. The following result can be proved by the slightly modified method of the proof of Theorem 1.

THEOREM 4. *Let assumptions* 1 *and* 2 *of Theorem* 1 *hold. Let*

$$f(x,y,\lambda) = f_1(x,y,\lambda) + f_2(x,y,\lambda),$$

where the function f_1 satisfies the proper Lipschitz condition in x and the function f_2 satisfies the proper Lipschitz condition in y. Suppose in every neighborhood of the point λ_0 there are points λ_1 and λ_2 such that for each $R>0$ the relations

$$\limsup_{\xi\to\infty}\ \sup_{|w-w_0|\le R\xi^{-1/K}} \Phi_0(\xi,\alpha,w,\lambda_1) < 0,$$

$$\liminf_{\xi\to\infty}\ \inf_{|w-w_0|\le R\xi^{-1/K}} \Phi_0(\xi,\alpha,w,\lambda_2) > 0$$

hold, where Φ_0 is function (11) *and K is the multiplicity of the root w_0 of the polynomial $L(wi)$. Then λ_0 is a Hopf bifurcation point for equation* (10) *with the frequency w_0.*

5. Proofs.

5.1. Change of variables. We look for periodic solutions of equations (2) and (8) of the periods $2\pi/w$ with unknown w close to w_0. Let us change the time scaling and replace equations (2) and (8) with the equations

$$L\left(w\frac{d}{dt}\right)x = M\left(w\frac{d}{dt}\right)[f(x(t),\lambda) + g(x(t-wh),\lambda)] \tag{12}$$

and

$$L\left(w\frac{d}{dt}\right)x = M\left(w\frac{d}{dt}\right)\big(a(\lambda)U(\mu_0)x(t) + f(x(t),\lambda)\big). \tag{13}$$

Evidently, $x(wt)$ is a $2\pi/w$-periodic solution of equation (2) (respectively, (8)) iff $x(t)$ is a 2π-periodic solution of equation (12) (respectively, (13)).

By assumption, the polynomial $L(wp)$ of the variable p has the roots $\pm i$ for $w = w_0$; furthermore, the relations $M(\pm wi) \neq 0$ and $L(\pm kwi) \neq 0$, $k = 0, 2, 3, \dots$ hold for $w = w_0$, so they hold also for every w from a small neighborhood Ω of the point w_0.

We show that each of equations (12) and (13) has 2π-periodic solutions of the form

$$x(t) = \xi \sin t + z(t), \tag{14}$$

where $z(t)$ is orthogonal in $L^2 = L^2(0, 2\pi)$ to the functions $\sin t$ and $\cos t$. More precisely, for every sufficiently large positive ξ there are the numbers w and λ and the function $z(t)$ such that (14) is a solution of the equation considered. Since ξ is arbitrarily large, so is the amplitude of (14).

Let us stress the following.

First, the shift of time generates the continuum $\{x(t+\varphi),\ \varphi \in \mathbb{R}\}$ of periodic solutions for every given nonconstant periodic solution $x(t)$ of any autonomous equation. That is, together with solution (14) equations (12) and (13) have solutions of the form $\xi \sin(t+\varphi) + z(t+\varphi)$ with any φ. By fixing the phase φ, we choose a unique solution from the continuum $x(t+\varphi)$ (namely, it is the solution orthogonal to $\cos t$ with the positive Fourier coefficient ξ by $\sin t$).

Secondly, the original problem depends on the parameter λ. Unknown solutions of the problem are functions $x(t) = \xi \sin wt + \eta \cos wt + z(wt)$ of unknown period $2\pi/w$ with unknown Fourier coefficients ξ, η and unknown component $z(\cdot)$. Thus, originally we have a problem with the parameter λ and the four unknowns ξ, η, w, $z(\cdot)$; each solution of the problem is included in the continuum of solutions with shifted time. Now the Fourier coefficient ξ is considered as a parameter, the phase φ is fixed (that is, we put $\eta = 0$), and the unknowns are w, λ, $z(t)$. This choice of a parameter and unknowns leads to a problem that can be studied without much difficulty by standard topological methods.

5.2. Topological lemma. For the sequel, we need the following lemma on solution existence for a system of two scalar equations and an equation in the Banach space E.

LEMMA 1. *Consider the system*

$$B_1(w,\lambda,z) = 0, \quad B_2(w,\lambda,z) = 0, \quad z = B_3(w,\lambda,z) \tag{15}$$

with $z \in E$ and scalar $w \in \Omega$, $\lambda \in \Lambda$, where the operators $B_1, B_2 : \Omega\times\Lambda\times E \to \mathbb{R}$ are continuous and the operator $B_3 : \Omega \times \Lambda \times E \to E$ is completely continuous with respect to the set of their arguments. Suppose the operator B_3 maps its domain into a bounded set $Z \subset E$. Suppose there are segments $[w_1, w_2] \subset \Omega$, $[\lambda_1, \lambda_2] \subset \Lambda$ such that $B_1(w_1, \lambda, z) \cdot B_1(w_2, \lambda, z) < 0$ for every $\lambda \in [\lambda_1, \lambda_2]$, $z \in Z$ and $B_2(w, \lambda_1, z) \cdot B_2(w, \lambda_2, z) < 0$ for every $w \in [w_1, w_2]$, $z \in Z$. Then system (15) *has a solution $w \in [w_1, w_2]$, $\lambda \in [\lambda_1, \lambda_2]$, $z \in Z$.*

The proof of Lemma 1 is based on the product formula for vector field rotations (see [4], [6]). Under the assumptions of Lemma 1 the rotation γ_1 of the infinite-dimensional vector field $z - B_3(w,\lambda,z)$ with fixed w, λ on every sphere $\{\|z\|_E = \rho\}$ of a sufficiently large radius ρ equals 1. The rotation γ_2 of the two-dimensional vector field $\{B_1(w,\lambda,z),\ B_2(w,\lambda,z)\}$ with fixed z on the boundary of the rectangular $T = \{w \in (w_1,w_2), \lambda \in (\lambda_1,\lambda_2)\}$ is either 1 or -1. The rotation γ_0 of the vector field

$$\{B_1(w,\lambda,z),\ B_2(w,\lambda,z),\ z - B_3(w,\lambda,z)\}$$

on the boundary of the domain $T \times \{\|z\|_E < \rho\}$ in the space $\mathbb{R} \times \mathbb{R} \times \mathbb{E}$ equals $\gamma_1\gamma_2$, i.e., $|\gamma_0| = 1$. Hence there exists a solution of system (15) in the this domain.

Now we replace equations (12) and (13) with 2π-periodic boundary conditions with systems of form (15). For both equations (12) and (13) system (15) can be constructed in a common way, consider equation (12). We multiply the equation by $\sin t$ and integrate over the segment $[0, 2\pi]$ to obtain the first of equalities (15). Multiplying the equation by $\cos t$ and integrating over the segment $[0, 2\pi]$, we obtain the second scalar equality of (15) (the details are in the next subsections). The equation in a Banach space is constructed as follows.

Denote by $E \subset C$ the space of functions $z(t) : [0, 2\pi] \to \mathbb{R}$, satisfying

$$z(0) = z(2\pi), \qquad \int_0^{2\pi} \sin t \, z(t) \, dt = \int_0^{2\pi} \cos t \, z(t) \, dt = 0$$

and set $\|z\|_E = \|z\|_C$. Consider the linear operator $A(w)$ that maps each function $u(t) \in E$ to a unique 2π-periodic solution $x(t) = A(w)u(t) \in E$ of the equation

$$L\left(w\frac{d}{dt}\right) x(t) = M\left(w\frac{d}{dt}\right) u(t). \tag{16}$$

The operator $A(w)$ existence follows from assumption 2 of Theorem 1. The operator $A(w)$ maps E to $E \cap C^1$. It is a completely continuous operator in the space E and a bounded operator from E to C^1. Moreover, the norms of the operators $A(w)$ are uniformly bounded for all $w \in \Omega$ and the operator $A(w)u : \Omega \times E \to E$ is completely continuous with respect to the set of its arguments w, u.

Denote by $C_0 \subset C$ the space of functions $u(t) : [0, 2\pi] \to \mathbb{R}$, satisfying $u(0) = u(2\pi)$ with the norm $\|u\|_{C_0} = \|u\|_C$. Set

$$Pu(t) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_0^{2\pi} \cos(t - s) \, u(s) \, ds.$$

The operators P and $I - P$ project the space C_0 on the plane $\Pi = \{\xi \sin t + \eta \cos t\}$ and on the subspace E of the space C_0 respectively. Let us extend the operator $A(w)$ to the whole space C_0 by the formula $A(w)u = A(w)(I - P)u$. Each of the projectors P and $I - P$ commutes with the extended operator $A(w)$.

Now the last equation of system (15) can be written as

$$z = A(w)[f(\xi \sin t + z(t), \lambda) + g(\xi \sin t_{wh} + z(t_{wh}), \lambda)];$$

here and henceforth $t_\sigma = t - \sigma$ for $t \geq \sigma$ and $t_\sigma = t - \sigma + 2\pi$ for $t < \sigma$. By construction, $A(w)[f(\xi \sin t + z(t), \lambda) + g(\xi \sin t_{wh} + z(t_{wh}), \lambda)]$ is a completely continuous operator from $\Omega \times \Lambda \times E$ to E for every fixed ξ.

The system constructed can be easily transformed to the system satisfying the conditions of Lemma 1. The main point of the proof of Theorem 1 is to determine the segments $[w_1, w_2]$ and $[\lambda_1, \lambda_2]$ for every large ξ. In the proof of Theorem 3 we also determine the initial stop state μ_0 such that $\mu(t) = U(\mu_0)x(t)$ is a 2π-periodic function for the 2π-periodic solution $x(t)$ of (13).

5.3. Auxiliary lemmas. Let $e(t)$ be a Lipschitz continuous function.

LEMMA 2. *For every $c > 0$ the equality*

$$\lim_{\xi\to\infty} \sup_{z\in C^1,\, \|z\|_{C^1}\le c,\, \lambda\in\Lambda} \left| \int_0^{2\pi} e(t)\big(f(x(t),\lambda) - f(\xi\sin t,\lambda)\big)dt \right| = 0 \tag{17}$$

holds, where $x(t) = \xi\sin t + z(t)$.

LEMMA 3. *For every $c > 0$ the equality*

$$\lim_{\xi\to\infty} \sup_{z\in C^1,\, \|z\|_{C^1}\le c,\, \mu_0\in[-1,1]} \left| \int_0^{2\pi} e(t)\,\big(U(\mu_0)x(t) - \operatorname{sign}(\cos t)\big)dt \right| = 0 \tag{18}$$

holds, where $x(t) = \xi\sin t + z(t)$.

Below we use Lemmas 2 and 3 with $e(t) = \sin t$ and $e(t) = \cos t$.

Lemma 3 is proved in [1]. Let us prove Lemma 2.

Take an arbitrary $\varepsilon > 0$. We need to show that the supremum in (17) is smaller than ε for all sufficiently large ξ, i.e.,

$$\left| \int_0^{2\pi} e(t)\,\big(f(x(t),\lambda) - f(\xi\sin t,\lambda)\big)\,dt \right| < \varepsilon \tag{19}$$

for every $\lambda \in \Lambda$ and $z \in C^1$, $\|z\|_{C^1} \le c$. For this purpose, consider the partition $\cup I_i$ of the segment $[0,2\pi]$, where $I_1 = [0,\pi/2-\delta]$, $I_2 = [\pi/2-\delta,\pi/2+\delta]$, $I_3 = [\pi/2+\delta,3\pi/2-\delta]$, $I_4 = [3\pi/2-\delta,3\pi/2+\delta]$, and $I_5 = [3\pi/2+\delta,2\pi]$ with a small $\delta > 0$. The join $I_2 \cup I_4$ contains the set $\{t \in [0,2\pi] : \cos t = 0\}$. Choose $\delta > 0$ so that

$$2\sup|f(x,\lambda)| \int_{I_2\cup I_4} |e(t)|\,dt < \varepsilon/2 \tag{20}$$

and fix this δ up to the end of the proof. From

$$\inf_{t\in I_1\cup I_3\cup I_5} |\cos t| = \sin\delta > 0$$

it follows that

$$\inf_{t\in I_1\cup I_3\cup I_5} |\xi\cos t + z'(t)| > 1/2\,\xi\sin\delta$$

whenever ξ is sufficiently large, hence the functions $\xi\sin t$ and $\xi\sin t + z(t)$ are strictly monotone in some neighborhood $\tilde{I}_i$ of the segment I_i, $i = 1,3,5$. Therefore the formula $\xi\sin\tau = \xi\sin t + z(t)$, $\tau \in \tilde{I}_i$ defines a strictly monotone function $\tau = \tau(\xi, t)$ of the argument $t \in I_i$ for every large ξ. Let $t = t(\xi, \tau)$ be the inverse function. Consider the integrals

$$\mathcal{J}_i = \int_{I_i} e(t) f(\xi \sin t + z(t), \lambda)\, dt, \qquad i = 1, 3, 5.$$

Changing the variable, we obtain

$$\mathcal{J}_i = \int_{\tau(\xi, a_i)}^{\tau(\xi, b_i)} e(t(\xi, \tau))\, f(\xi \sin \tau, \lambda)\, t'_\tau(\xi, \tau)\, d\tau,$$

where a_i, b_i are the ends of the segment I_i. By construction, $t(\xi, \tau) \to \tau$ and $t'_\tau(\xi, \tau) \to 1$ as $\xi \to \infty$ uniformly in τ. So, the Lipschitz continuity of $e(\cdot)$ implies $e(t(\xi, \tau)) \to e(\tau)$. Also,

$$\tau(\xi, a_i) \to a_i, \quad \tau(\xi, b_i) \to b_i.$$

Hence,

$$\mathcal{J}_i - \int_{a_i}^{b_i} e(\tau) f(\xi \sin \tau, \lambda)\, d\tau \to 0, \qquad i = 1, 3, 5.$$

Together with (20) this proves (19). ∎

5.4. Scalar equations. Let us multiply equation (16) by $\sin t$ (resp., $\cos t$) and integrate over $[0, 2\pi]$. The following lemma writes explicitly the resulting scalar equalities, which allows to write explicitly the scalar equations of system (15).

LEMMA 4. *Suppose the functions* $x(t) = \xi \sin t + z(t)$, $z \in E$, *and* $u(t) \in C_0$ *satisfy* (16). *Then*

$$\text{(21)} \quad \pi\, \Re\mathrm{e} \frac{L(wi)}{M(wi)} \xi = \int_0^{2\pi} \sin t\, u(t)\, dt, \quad \pi\, \Im\mathrm{m} \frac{L(wi)}{M(wi)} \xi = \int_0^{2\pi} \cos t\, u(t)\, dt.$$

Proof. It follows from (16) that

$$L\left(w\frac{d}{dt}\right)(\xi \sin t) = M\left(w\frac{d}{dt}\right) Pu(t).$$

Equivalently,

$$\pi\, \Re\mathrm{e}[L(wi)] \xi \sin t + \pi\, \Im\mathrm{m}[L(wi)] \xi \cos t =$$

$$(\Re\mathrm{e}[M(wi)] \cos t - \Im\mathrm{m}[M(wi)] \sin t) \int_0^{2\pi} \cos s\, u(s) ds \; +$$

$$(\Re\mathrm{e}[M(wi)] \sin t + \Im\mathrm{m}[M(wi)] \cos t) \int_0^{2\pi} \sin s\, u(s) ds,$$

that is

$$\pi\,\Re e[L(wi)]\xi = -\,\Im m[M(wi)]\int_0^{2\pi}\cos s\; u(s)\,ds + \Re e[M(wi)]\int_0^{2\pi}\sin s\, u(s)\,ds,$$
$$\pi\,\Im m[L(wi)]\xi = \Re e[M(wi)]\int_0^{2\pi}\cos s\; u(s)\,ds + \Im m[M(wi)]\int_0^{2\pi}\sin s\, u(s)\,ds.$$

Multiplying the first of these equalities by $\Re e[M(wi)]$, the second one by $\Im m[M(wi)]$ and summing, we obtain the first of equations (21). Summing the first of the equalities multiplied by $-\Im m[M(wi)]$ with the second one multiplied by $\Re e[M(wi)]$, we obtain the second of equations (21). ■

5.5. Proof of Theorem 1. Consider the system

$$\begin{aligned}
&\pi\,\Re e\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi}\sin t\,[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)]\,dt,\\
(22)\quad &\pi\,\Im m\,\frac{L(wi)}{M(wi)}\xi = \int_0^{2\pi}\cos t\,[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)]\,dt,\\
&z = A(w)[f(\xi\sin t + z(t),\lambda) + g(\xi\sin t_{wh} + z(t_{wh}),\lambda)].
\end{aligned}$$

It follows from the definition of the operator $A(w)$ and from Lemma 4 that the function $x(t) = \xi\sin wt + z(wt)$ is a $2\pi/w$-periodic solution of equation (2) whenever the triple $\{w,\lambda,z\}\in\Omega\times\Lambda\times E$ is a solution of system (22) for some $\xi > 0$. Therefore to prove Theorem 1 it is sufficient to show that system (22) has a solution $\{w,\lambda,z\}$ with w and λ arbitrarily close to w_0 and λ_0 for every sufficiently large ξ. Let us transform (22) to the equivalent system satisfying the conditions of Lemma 1.

Rewrite the first two equations of (22) as

$$(23)\quad \pi\frac{\Re e[L(wi)M(-wi)]}{|M(wi)|^2}\xi = \int_0^{2\pi}[\sin t f(x(t),\lambda) + \sin(t+wh)g(x(t),\lambda)]\,dt$$

and

$$(24)\quad \pi\frac{\Im m[L(wi)M(-wi)]}{|M(wi)|^2}\xi = \int_0^{2\pi}[\cos t f(x(t),\lambda) + \cos(t+wh)g(x(t),\lambda)]\,dt,$$

where $x(t) = \xi\sin t + z(t)$. Consider separately the following two situations: first, limit (3) is either infinite or zero; second, limit (3) is a finite number $\beta \neq 0$.

Suppose $\beta = 0$ (the case $\beta = \infty$ is similar and we do not consider it here). Then $\alpha = 0$ and

$$\Phi(\xi, 0, \lambda) = \sin(w_0 h) \int_0^{2\pi} \sin t \, g(\xi \sin t, \lambda) \, dt. \tag{25}$$

It follows from

$$\lim_{w \to w_0} \frac{\Im m[L(wi)M(-wi)]}{\Re e[L(wi)M(-wi)]} = 0$$

that

$$\begin{aligned} \Im m[L(wi)M(-wi)] &= (w - w_0)^{K+N} Q_1(w), \\ \Re e[L(wi)M(-wi)] &= (w - w_0)^{K} \quad Q_2(w), \end{aligned} \tag{26}$$

where $Q_2(w_0) \neq 0$ and either $Q_1(w_0) \neq 0$ or $Q_1(w)$ is the identical zero; N is a positive integer; K is the multiplicity of the root $w_0 i$ of the polynomial $L(p)$. Now equations (23), (24) can be written as

$$\pi(w - w_0)^K \frac{\xi Q_2(w)}{|M(wi)|^2} - \int_0^{2\pi} [\sin t f(x(t), \lambda) + \sin(t + wh) g(x(t), \lambda)] dt = 0, \tag{27}$$

$$\pi(w - w_0)^{K+N} \frac{\xi Q_1(w)}{|M(wi)|^2} - \int_0^{2\pi} [\cos t f(x(t), \lambda) + \cos(t + wh) g(x(t), \lambda)] dt = 0. \tag{28}$$

Let us express the term $(w - w_0)^K$ from equation (27) and substitute in (28) to obtain

$$\begin{aligned} (w - w_0)^N \frac{Q_1(w)}{Q_2(w)} \int_0^{2\pi} \Big(\sin t \, f(x(t), \lambda) + \sin(t + wh) g(x(t), \lambda) \Big) \, dt - \\ \int_0^{2\pi} \Big(\cos t \, f(x(t), \lambda) + \cos(t + wh) g(x(t), \lambda) \Big) \, dt = 0. \end{aligned} \tag{29}$$

We use (27) and (29) as the scalar equations of system (15). The equation in a Banach space is the last equation of (22).

Since $\sup |f(x, \lambda)| + \sup |g(x, \lambda)| < \infty$ and the norms of the operators $A(w) : C_0 \to C^1$ are uniformly bounded, it follows that the nonlinear operator

$$A(w)[f(\xi \sin t + z(t), \lambda) + g(\xi \sin t_{wh} + z(t_{wh}), \lambda)]$$

maps its domain $\Omega \times \Lambda \times E$ onto a bounded subset Z of C^1, so Z is also bounded in E. Thus, the last equation of (22) satisfies the conditions of Lemma 1. The conditions concerning the scalar equations should be verified for $z \in Z$. We take z from the fixed ball $\{\|z\|_{C^1} \leq c\}$ containing Z.

Set

$$(30)\quad w_1 = w_1(\xi) \stackrel{\text{def}}{=} w_0 - R^{1/K}\xi^{-1/K}, \qquad w_2 = w_2(\xi) \stackrel{\text{def}}{=} w_0 + R^{1/K}\xi^{-1/K},$$

where

$$R = 4 \sup_{w\in\Omega} \frac{|M(wi)|^2}{|Q_2(w)|}[\sup|f(x,\lambda)| + \sup|g(x,\lambda)|].$$

Since the number K is odd, the sign of the left-hand side of equation (27) is $(-1)^j \operatorname{sign} Q_2(w_0)$ for $w = w_j$ whenever ξ is sufficiently large and $\lambda \in \Lambda$.

Take any pair of parameter values satisfying the condition 3 of Theorem 1 as the numbers λ_1, λ_2 used in Lemma 1. It follows from Lemma 2 that all the terms in (29) except (25) vanish as $\xi \to \infty$ and $w \to w_0$, hence $\operatorname{sign} \Phi(\xi, 0, \lambda)$ is the sign of the left-hand side of equation (29). By condition 3, the left-hand side of (29) is negative for $\lambda = \lambda_1$ and positive for $\lambda = \lambda_2$ whenever $w \in [w_1(\xi), w_2(\xi)]$ with large enough ξ, which completes the proof for $\beta = 0$.

Now suppose $\beta \neq 0$. This case is similar to the case above. The only difference is that $N = 0$ and $Q_j(w_0) \neq 0$ for both $j = 1, 2$ in formulas (26). Therefore equality (29) can be written in the form

$$\cos\alpha \left[\frac{Q_1(w)}{Q_2(w)} - \beta\right] \int_0^{2\pi} [\sin t\, f(x(t),\lambda) + \sin(t+wh)\, g(x(t),\lambda)]\, dt - \int_0^{2\pi} [\cos(t+\alpha)\, f(x(t),\lambda) + \cos(t+wh+\alpha)\, g(x(t),\lambda)]\, dt = 0,$$

and the sign of the left-hand side coincides with $\operatorname{sign} \Phi(\xi, \alpha, \lambda)$. The further arguments are exactly like above. ■

5.6. Proof of Theorem 3. The proof follows the line of the proof of Theorem 1. The analog of system (22) for equation (8) is

$$(31)\quad \begin{aligned} \pi\,\Re e\,\frac{L(wi)}{M(wi)}\xi &= \int_0^{2\pi} \sin t \big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big)\, dt, \\ \pi\,\Im m\,\frac{L(wi)}{M(wi)}\xi &= \int_0^{2\pi} \cos t \big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big)\, dt, \\ z &= A(w)\big(a(\lambda)U(\mu_0)[\xi\sin t + z(t)] + f(\xi\sin t + z(t),\lambda)\big). \end{aligned}$$

The same transformation as used in the proof of Theorem 1 brings system (31) to the form where the term $(w - w_0)^K \xi$ is principal for the first scalar equation, and the principal term of the second scalar equation is

$$-\int_0^{2\pi} \cos(t + \alpha)\left(a(\lambda)U(\mu_0)[\xi \sin t + z(t)] + f(\xi \sin t + z(t), \lambda)\right) dt.$$

By Lemmas 2 and 3, this expression goes to $\phi(\lambda) = \psi(\lambda)\sin\alpha - 4a(\lambda)\cos\alpha$ as $\xi \to \infty$. Thus, by Lemma 1 system (31) has a solution $\{w(\mu_0), \lambda(\mu_0), z(t; \mu_0)\}$ for each initial stop state $\mu_0 \in [-1, 1]$ whenever $\xi > 0$ is sufficiently large.

It remains to determine a value μ_0 such that the function $\mu(t; \mu_0) = U(\mu_0)(\xi \sin t + z(t; \mu_0))$ is 2π-periodic, i.e., $\mu(2\pi; \mu_0) = \mu_0$; then $x(t; \mu_0) = \xi \sin t + z(t; \mu_0)$ is a 2π-periodic solution of equation (13) for $\lambda = \lambda(\mu_0)$, $w = w(\mu_0)$ and hence $x(wt; \mu_0)$ is a $2\pi/w$-periodic solution of equation (8).

It follows from the semigroup property of the operator $U(\mu)$ that

$$\mu(t; \mu_0) = U(\mu(7\pi/4; \mu_0))x(t; \mu_0), \qquad t \geq 7\pi/4.$$

Since $z(t; \mu_0) \in Z$, where the set Z is bounded in C^1, the relations

$$x'(t; \mu_0) = \xi \cos t + z'(t; \mu_0) \geq 0, \qquad 7\pi/4 \leq t \leq 2\pi, \ \mu_0 \in [-1, 1],$$

and

$$x(2\pi; \mu_0) - x(7\pi/4; \mu_0) \geq 2, \qquad \mu_0 \in [-1, 1], \tag{32}$$

hold for any large ξ. That is, the input $x(t; \mu_0)$ increases on the segment $7\pi/2 \leq t \leq 2\pi$, hence

$$\mu(t; \mu_0) = \min\{1, \mu(7\pi/2; \mu_0) + x(t; \mu_0) - x(7\pi/2; \mu_0)\}, \qquad 7\pi/2 \leq t \leq 2\pi$$

and (32) implies that $\mu(2\pi; \mu_0) = 1$ for every $\mu_0 \in [-1, 1]$. Therefore $x(wt; \mu_0)$ is a $2\pi/w$-solution of equation (8) iff $\mu_0 = 1$. ∎

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FUNCTIONAL DIFFERENTIAL EQUATIONS

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STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING *

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Abstract. The main aim of this paper is to investigate the exponential stability of stochastic functional differential equations with Markovian switching. The Razumikhin argument and the generalized Itô formula will play an important role in this paper. Applying our new results to several important types of equations e.g. stochastic differential delay equations and stochastic differential equations, both with Markovian switching, we obtain a number of very useful results. Several examples are also given for illustration.

Key Words. Lyapunov exponent, Razumikhin argument, Brownian motion, Markov chain, Poisson random measure.

AMS(MOS) subject classification. 60H20, 34D08, 60G48.

1. Introduction. Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models (cf. Arnold [1], Friedman [3], Has'minskii [5] and Mao [10]). The stability of stochastic functional differential equations has been studied by many authors and we here mention Kolmanovskii & Myshkis [7], Kolmanovskii & Nosov [8], Ladde & Lakshmikantham [9], Mao [11] and Mohammed [14] among others. However, there is little work on the stability of stochastic functional differential equations with Markovian switching although there are quite a number

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of papers on the stability of stochastic differential equations with Markovian switching e.g. Basak et al. [2], Ghosh et al. [4] and Skorohod [17].

In this paper we consider a stochastic functional differential equation with Markovian switching of the form

$$dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))dw(t), \tag{1}$$

where $r(t)$ is a Markov chain taking values on $S = \{1, 2, \cdots, N\}$. This equation can be regarded as the result of the following N equations

$$dx(t) = f(x_t, t, i)dt + g(x_t, t, i)dw(t), \quad 1 \le i \le N \tag{2}$$

switching from one to the others according to the movement of the Markov chain. In section 2 we shall quickly establish the existence-and-unique theorem for the solution of the equation and cite the generalized Itô formula. In section 3 we shall apply the Razumikhin argument (cf. Razumikhin [15, 16]) to investigate the exponential stability of equation (3). The general results obtained in this section will then be applied to stochastic differential delay equations and stochastic differential equations in sections 4 and 5, respectively. Finally we give three examples for illustration in section 6.

2. Stochastic Functional Differential Equations with Markovian Switching. Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $||\varphi|| = \sup_{\tau\le\theta\le0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $||A|| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $||\varphi||$). Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued random variables. For $p > 0$ and $t \ge 0$, denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; R^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau \le \theta \le 0\}$ such that $\sup_{-\tau\le\theta\le0} E|\phi(\theta)|^p < \infty$.

Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of R_+ $(:= [0, \infty))$. We stress that almost all sample paths of $r(t)$ are right continuous. It is also very useful to recall that the Markov chain $r(t)$ can be represented as a stochastic integral with respect to a Poisson random measure. Indeed, let Δ_{ij}, $i \neq j$, be consecutive (with respect to the lexicographic ordering on $S \times S$), left closed and right open intervals of the real line each having length λ_{ij}. Define a function

$$\eta : S \times R \to R$$

by

$$\eta(i, y) = \begin{cases} j - i & \text{if } y \in \Delta_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$dr(t) = \int_R \eta(r(t-), y)\nu(dt, dy),$$

where $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, $\mu(\cdot)$ being the Lebesgue measure on R. For more information about this representation please see Skorohod [17].

Consider an n-dimensional stochastic functional differential equation with Markovian switching of the form

$$dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))dw(t) \tag{3}$$

on $t \geq 0$ with initial data $x_0 = \xi$. Here $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}$ which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process. Moreover,

$$f : C([-\tau, 0]; R^n) \times R_+ \times S \to R^n, \quad g : C([-\tau, 0]; R^n) \times R_+ \times S \to R^{n \times m}.$$

For the existence and uniqueness of the solution we impose a hypothesis:

(H1) Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \cdots$, there is an $h_k > 0$ such that

$$|f(\varphi_1, t, i) - f(\varphi_2, t, i)| + |g(\varphi_1, t, i) - g(\varphi_2, t, i)| \leq h_k ||\varphi_1 - \varphi_2||$$

for all $t \geq 0$, $i \in S$ and those $\varphi_1, \varphi_2 \in C([-\tau, 0]; R^n)$ with $||\varphi_1|| \vee ||\varphi_2|| \leq k$, and there is moreover an $h > 0$ such that

$$|f(\varphi, t, i)| + |g(\varphi, t, i)| \leq h(1 + ||\varphi||)$$

for all $\varphi \in C([-\tau, 0]; R^n)$, $t \geq 0$ and $i \in S$.

THEOREM 1. *Under hypothesis (H1), equation (3) has a unique continuous solution $x(t)$ on $t \geq -\tau$. Moreover, for every $p > 0$,*

$$E\Big[\sup_{-\tau \leq s \leq t} |x(s)|^p\Big] < \infty \quad \text{on } t \geq 0. \tag{4}$$

Proof. It is known (cf. Skorohod [17]) that there is a sequence $\{\tau_k\}_{k \geq 0}$ of stopping times such that $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ and $r(t)$ is constant on every interval $[\tau_k, \tau_{k+1})$, i.e. for every $k \geq 0$

$$r(t) = r(\tau_k) \quad \text{on } \tau_k \leq t < \tau_{k+1}.$$

We first consider equation (3) on $t \in [0, \tau_1]$ which becomes

$$dx(t) = f(x_t, t, r(0))dt + g(x_t, t, r(0))dw(t) \tag{5}$$

with initial data $x_0 = \xi$. By the existence-and-unique theorem of stochastic functional differential equations (cf. Mao [11, 12] or Mohammed [14]) we know equation (5) has a unique continuous solution on $[-\tau, \tau_1]$. We next consider equation (3) on $t \in [\tau_1, \tau_2]$ which becomes

$$dx(t) = f(x_t, t, r(\tau_1))dt + g(x_t, t, r(\tau_1))dw(t) \tag{6}$$

with initial data x_{τ_1} given by the solution of equation (5). Again we know equation (6) has a unique continuous solution on $[\tau_1 - \tau, \tau_2]$. Repeating this procedure we see that equation ((3) has a unique solution $x(t)$ on $t \geq -\tau$. To show (4) we note from Hölder's inequality that

$$\left(E\Big[\sup_{-\tau \leq s \leq t} |x(s)|^p\Big]\right)^{\frac{1}{p}} \leq \left(E\Big[\sup_{-\tau \leq s \leq t} |x(s)|^2\Big]\right)^{\frac{1}{2}} \quad \text{if } 0 < p < 2.$$

So we only need to prove (4) in the case of $p \geq 2$. For any $t \in [0, T]$, by Hölder's inequality, Theorem 1.7.2 of Mao [12] and the linear growth condition we can show that

$$E\Big[\sup_{0\leq s\leq t} |x(s)|^p\Big] \leq 3^{p-1}E|\xi(0)|^p + CE\int_0^t (1 + ||x_s||^p)ds,$$

where

$$C = C_{p,T} = 6^{p-1}h^p\Big[T^{p-1} + \Big(\frac{p^3}{2(p-1)}\Big)^{\frac{p}{2}}T^{\frac{p-2}{2}}\Big].$$

Consequently

$$E\Big[\sup_{-\tau\leq s\leq t} |x(s)|^p\Big] \leq E||\xi||^p + E\Big[\sup_{0\leq s\leq t} |x(s)|^p\Big]$$
$$\leq (3^{p-1} + 1)E||\xi||^p + CT + C\int_0^t E\Big[\sup_{-\tau\leq s\leq u} |x(s)|^p\Big]du.$$

An application of the Gronwall inequality implies

$$E\Big[\sup_{-\tau\leq s\leq T} |x(s)|^p\Big] \leq e^{CT}\Big[(3^{p-1} + 1)E||\xi||^p + CT\Big]$$

and the required assertion (4) follows. The proof is complete.

Let $C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times [-\tau, \infty) \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$, define an operator $\mathcal{L}V$ from $C([-\tau, 0]; R^n) \times R_+ \times S$ to R by

$$\begin{aligned} \mathcal{L}V(\varphi, t, i) &= V_t(\varphi(0), t, i) + V_x(\varphi(0), t, i)f(\varphi, t, i) \\ &+ \frac{1}{2}\text{trace}[g^T(\varphi, t, i)V_{xx}(\varphi(0), t, i)g(\varphi, t, i)] \\ &+ \sum_{j=1}^N \gamma_{ij}V(\varphi(0), t, j) \end{aligned} \tag{7}$$

where

$$V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t}, \qquad V_x(x, t, i) = \Big(\frac{\partial V(x, t, i)}{\partial x_1}, \cdots, \frac{\partial V(x, t, i)}{\partial x_n}\Big),$$
$$V_{xx}(x, t, i) = \Big(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

For the convenience of the reader we cite the generalized Itô formula (cf. Skorohod [17]): If $V \in C^{2,1}(R^n \times [-\tau,\infty) \times S; R_+)$, then

$$
\begin{aligned}
(8)\qquad & dV(x(t),t,r(t)) \\
= \ & \mathcal{L}V(x_t,t,r(t))dt + V_x(x(t),t,r(t))g(x_t,t,r(t))dw(t) \\
+ \ & \int_R \Big[V(x(t),t,r(t)+\eta(r(t),y)) - V(x(t),t,r(t))\Big]\tilde{\mu}(dt,dy),
\end{aligned}
$$

where $\eta(\cdot)$ has been defined before while $\tilde{\mu}(dt,dy) = \nu(dt,dy) - \mu(dy)$ is the centered Poisson measure. Integrating both sides from t to $t+h$ and then taking expectation we obtain the following useful formula

$$
\begin{aligned}
(9)\qquad & EV(x(t+h),t+h,r(t+h)) \\
= \ & EV(x(t),t,r(t)) + E\int_t^{t+h} \mathcal{L}V(x_s,s,r(s))ds.
\end{aligned}
$$

3. Exponential Stability. From now on we shall discuss the exponential stability of equation (3). We shall always fix the Markov chain $r(t)$ but let the initial data ξ vary in $C^b_{\mathcal{F}_0}([-\tau,0];R^n)$. The solution of equation (3) is denoted by $x(t;\xi)$ in this paper. For the purpose of stability we may assume, without loss of generality, that $f(0,t,i) \equiv 0$ and $g(0,t,i) \equiv 0$. So equation (3) admits a trivial solution $x(t;0) \equiv 0$.

Let us now establish a Razumikhin-type theorem on the pth moment exponential stability for the stochastic functional differential equation.

THEOREM 2. *Let (H1) hold. Let λ, p, c_1, c_2 be all positive numbers and $q > 1$. Assume that there exists a function $V(x,t,i) \in C^{2,1}(R^n \times [-\tau,\infty) \times S; R_+)$ such that*

$$(10)\qquad c_1|x|^p \le V(x,t,i) \le c_2|x|^p \qquad \text{for all } (x,t,i) \in R^n \times [-\tau,\infty) \times S,$$

and also for all $t \ge 0$

$$(11)\qquad E\Big[\max_{1\le i\le N} \mathcal{L}V(\phi,t,i)\Big] \le -\lambda E\Big[\max_{1\le i\le N} V(\phi(0),t,i)\Big]$$

provided $\phi = \{\phi(\theta) : -\tau \le \theta \le 0\} \in L^p_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$(12)\qquad E\Big[\min_{1\le i\le N} V(\phi(\theta),t+\theta,i)\Big] < qE\Big[\max_{1\le i\le N} V(\phi(0),t,i)\Big]$$

for all $-\tau \le \theta \le 0$. Then for all $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$

$$(13)\qquad E|x(t;\xi)|^p \le \frac{c_2}{c_1}E\|\xi\|^p e^{-\gamma t} \quad \text{on } t \ge 0,$$

where $\gamma = \min\{\lambda, \log(q)/\tau\}$. In other words, the trivial solution of equation (3) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\gamma$.

Proof. Fix the initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ arbitrarily and write $x(t;\xi) = x(t)$ simply. Extend $r(t)$ to $[-\tau, 0)$ by setting $r(t) = r(0)$. Recalling the facts that $x(t)$ is continuous, $E(\sup_{-\tau\le s\le t}|x(s)|^p) < \infty$ for all $t \ge 0$ and $r(t)$ is right continuous, we see easily that $EV(x(t), t, r(t))$ is right continuous on $t \ge -\tau$. Let $\varepsilon \in (0, \gamma)$ be arbitrary and set $\bar{\gamma} = \gamma - \varepsilon$. Define

$$U(t) = \sup_{-\tau\le\theta\le 0}\left[e^{\bar{\gamma}(t+\theta)}EV(x(t+\theta), t+\theta, r(t+\theta))\right] \qquad \text{for } t \ge 0.$$

We claim that

$$(14) \qquad D_+U(t) := \limsup_{h\to 0+}\frac{U(t+h) - U(t)}{t} \le 0 \qquad \text{for all } t \ge 0.$$

Note that for each $t \ge 0$ (fixed for the moment), either $U(t) > e^{\bar{\gamma}t}EV(x(t), t, r(t))$ or $U(t) = e^{\bar{\gamma}t}EV(x(t), t, r(t))$. In the case of former, if follows from the right continuity of $EV(x(\cdot), \cdot, r(\cdot))$ that for all $h > 0$ sufficiently small

$$U(t) > e^{\bar{\gamma}(t+h)}EV(x(t+h), t+h, r(t+h)),$$

hence

$$U(t+h) \le U(t) \quad \text{and} \quad D_+U(t) \le 0.$$

In the other case, i.e. $U(t) = e^{\bar{\gamma}t}EV(x(t), t, r(t))$, we have

$$e^{\bar{\gamma}(t+\theta)}EV(x(t+\theta), t+\theta, r(t+\theta)) \le e^{\bar{\gamma}t}EV(x(t)), t, r(t))$$

for all $-\tau \le \theta \le 0$. So

$$(15) \qquad \begin{aligned} EV(x(t+\theta), t+\theta, r(t+\theta)) &\le e^{-\bar{\gamma}\theta}EV(x(t), t, r(t)) \\ &\le e^{\bar{\gamma}\tau}EV(x(t), t, r(t)) \end{aligned}$$

for all $-\tau \le \theta \le 0$. Note that either $EV(x(t), t, r(t)) = 0$ or $EV(x(t), t, r(t)) > 0$. In the former case, (15) and (10) yield that $x(t+\theta) = 0$ a.s. for all $-\tau \le \theta \le 0$. Recalling the fact that $f(0, t, i) \equiv 0$ and $g(0, t, i) \equiv 0$, one sees that $x(t+h) = 0$ a.s. for all $h > 0$ hence $U(t+h) = 0$ and $D_+U(t) = 0$. On the other hand, in the case of $EV(x(t), t, r(t)) > 0$, (15) implies

$$EV(x(t+\theta), t+\theta, r(t+\theta)) < qEV(x(t), t, r(t))$$

for all $-\tau \le \theta \le 0$, since $e^{\bar\gamma\tau} < q$. Consequently

$$E\Big[\min_{1\le i\le N} V(x(t+\theta), t+\theta, i)\Big] < qE\Big[\max_{1\le i\le N} V(x(t), t, i)\Big]$$

for all $-\tau \le \theta \le 0$. In other words, $x_t \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ satisfying (12). Thus, by condition (11), we have

$$E\Big[\max_{1\le i\le N} \mathcal{L}V(x_t, t, i)\Big] \le -\lambda E\Big[\max_{1\le i\le N} V(x(t), t, i)\Big],$$

which implies

$$E\mathcal{L}V(x_t, t, r(t)) \le -\lambda EV(x(t), t, r(t)). \tag{16}$$

This gives

$$\bar\gamma EV(x(t), t, r(t)) + E\mathcal{L}V(x_t, t, r(t)) \le -(\lambda - \bar\gamma)EV(x(t), t, r(t)) < 0.$$

By the right continuity of the processes involved we hence see that for all $h > 0$ sufficiently small

$$\bar\gamma EV(x(s), s, r(s)) + E\mathcal{L}V(x_s, s, r(s)) \le 0 \qquad \text{if } t \le s \le t+h.$$

By Itô's formula, we can then derive that

$$e^{\bar\gamma(t+h)}EV(x(t+h), t+h, r(t+h)) - e^{\bar\gamma t}EV(x(t), t, r(t))$$

$$= \int_t^{t+h} e^{\bar\gamma s}\Big[\bar\gamma EV(x(s), s, r(s)) + E\mathcal{L}V(x_s, s, r(s))\Big]ds \le 0.$$

That is

$$e^{\bar\gamma(t+h)}EV(x(t+h), t+h, r(t+h)) \le e^{\bar\gamma t}EV(x(t), t, r(t)).$$

So it must hold that $U(t+h) = U(t)$ for all $h > 0$ sufficiently small, and hence $D_+U(t) = 0$. Inequality (14) has therefore been proved. It now follows from (14) immediately that

$$U(t) \le U(0) \qquad \text{for all } t \ge 0.$$

By the definition of $U(t)$ and condition (10) one sees

$$E|x(t)|^p \le \frac{c_2}{c_1}E||\xi||^p e^{-\bar\gamma t} = \frac{c_2}{c_1}E||\xi||^p e^{-(\gamma-\varepsilon)t}.$$

Since ε is arbitrary, the required inequality (13) must hold. The proof is complete.

The following theorem gives the sufficient conditions for the almost sure exponential stability.

THEOREM 3. *Let $p \geq 1$ and assume that there is a constant $K > 0$ such that for all $\phi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$, $t \geq 0$ and $i \in S$,*

$$E[|f(\phi,t,i)|^p + |g(\phi,t,i)|^p] \leq K \sup_{-\tau\leq\theta\leq 0} E|\phi(\theta)|^p. \tag{17}$$

Then (13) implies that

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t;\xi)| \leq -\frac{\gamma}{p} \quad a.s. \tag{18}$$

for all $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$. In other words, under condition (17), the pth moment exponential stability implies the almost sure exponential stability.

Proof. Fix any $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$ and again write $x(t;\xi) = x(t)$. For each integer $k \geq 2$,

$$\begin{aligned} E\|x_{k\tau}\|^p &= E\Big(\sup_{0\leq h\leq\tau} |x((k-1)\tau+h)|^p\Big) \\ &\leq 3^{p-1}\Big(E|x((k-1)\tau)|^p + E\Big[\int_{(k-1)\tau}^{k\tau} |f(x_s,s,r(s))|ds\Big]^p \\ &+ E\Big[\sup_{0\leq h\leq\tau} \Big|\int_{(k-1)\tau}^{(k-1)\tau+h} g(x_s,s,r(s))dw(s)\Big|^p\Big]\Big). \end{aligned} \tag{19}$$

By Hölder's inequality,

$$E\Big[\int_{(k-1)\tau}^{k\tau} |f(x_s,s,r(s))|ds\Big]^p \leq \tau^{p-1}\int_t^{t+\tau} E|f(x_s,s,r(s))|^p ds. \tag{20}$$

But by (17) and (13) we can derive that

$$E|f(x_s,s,r(s))|^p \leq \sum_{i\leq i\leq N} E|f(x_s,s,i)|^p$$

$$\leq NK \sup_{-\tau\leq\theta\leq 0} E|x(s+\theta)|^p \leq \frac{NKc_2}{c_1} E\|\xi\|^p e^{-\gamma(s-\tau)}.$$

Substituting this into (20) yields

$$E\Big[\int_{(k-1)\tau}^{k\tau} |f(x_s,s,r(s))|ds\Big]^p \leq \frac{NKc_2\tau^p}{c_1} E\|\xi\|^p e^{-(k-2)\tau\gamma}. \tag{21}$$

On the other hand, by the Burkholder-Davis-Gundy inequality (cf. Karatzas & Shreve [6] or Mao [11]), we have that

$$(22)\qquad \begin{aligned} J := E\Big[\sup_{0\le h\le\tau} \Big|\int_{(k-1)\tau}^{(k-1)\tau+h} g(x_s,s,r(s))dw(s)\Big|^p\Big] \\ \le\ C_pE\Big(\int_{(k-1)\tau}^{k\tau} |g(x_s,s,r(s))|^2ds\Big)^{p/2} \\ \le\ C_pE\Big[\Big(\sup_{(k-1)\tau\le s\le k\tau} |g(x_s,s,r(s))|\Big)\int_{(k-1)\tau}^{k\tau} |g(x_s,s,r(s))|ds\Big]^{p/2}, \end{aligned}$$

where C_p is a positive constant dependent of p only. Let $\varepsilon \in (0, 1/3^{p-1}KN)$ be sufficiently small for

$$(23)\qquad \frac{3^{p-1}KN\varepsilon}{1-3^{p-1}KN\varepsilon} < e^{-\gamma\tau}.$$

Using the elementary inequality $|ab| \le \varepsilon a^2 + b^2/4\varepsilon$ we derive from (3.13) that

$$(24)\qquad \begin{aligned} J\ \le\ & \varepsilon E\Big(\sup_{(k-1)\tau\le s\le k\tau} |g(x_s,s,r(s))|^p\Big) \\ +\ & \frac{C_p^2}{4\varepsilon}E\Big[\int_{(k-1)\tau}^{k\tau} |g(x_s,s,r(s))|ds\Big]^p. \end{aligned}$$

In the same way as (21) we can show that

$$(25)\qquad E\Big[\int_{(k-1)\tau}^{k\tau} |g(x_s,s,r(s))|ds\Big]^p \le \frac{NKc_2\tau^p}{c_1}E||\xi||^pe^{-(k-2)\tau\gamma}.$$

Note also from condition (17) that

$$|g(x_s,s,r(s))|^p \le \sum_{i=1}^N |g(x_s,s,i)|^p \le KN||x_s||^p.$$

Consequently

$$(26)\quad E\Big(\sup_{(k-1)\tau\le s\le k\tau} |g(x_s,s,r(s))|^p\Big) \le KN(E||x_{(k-1)\tau}||^p + E||x_{k\tau}||^p).$$

Substituting (25) and (26) into (24) yields

$$(27)\qquad \begin{aligned} J\ \le\ & \varepsilon KN(E||x_{(k-1)\tau}||^p + E||x_{k\tau}||^p) \\ +\ & \frac{NKc_2C_p^2\tau^p}{4\varepsilon c_1}E||\xi||^pe^{-(k-2)\tau\gamma}. \end{aligned}$$

Making use of (23), we can now substitute (13), (21) and (27) into (19) to obtain that

$$E||x_{k\tau}||^p \leq e^{-\gamma\tau}E||x_{(k-1)\tau}||^p + Ce^{-(k-2)\tau\gamma}, \tag{28}$$

where C is a constant independent of k. By induction we can easily show from (28) that

$$E||x_{k\tau}||^p \leq (kCe^{2\gamma\tau} + E||\xi||^p)e^{-k\tau\gamma}. \tag{29}$$

Finally we can show the required assertion (18) from (29) in the same way as in the proof of Theorem 2.2 in Mao [13]. The proof is therefore complete.

4. Stochastic Differential Delay Equations with Markovian Switching. A special but important case of equation (3) is the stochastic differential delay equation with Markovian switching of the form

$$\begin{aligned} dx(t) &= F(x(t), x(t-\delta(t)), t, r(t))dt \\ &+ G(x(t), x(t-\delta(t)), t, r(t))dw(t) \end{aligned} \tag{30}$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. Here $\delta : R_+ \to [0, \tau]$ is Borel measurable while

$$F : R^n \times R^n \times R_+ \times S \to R^n \quad \text{and} \quad G : R^n \times R^n \times R_+ \times S \to R^{n\times m}.$$

We impose a hypothesis:

(H2) Both F and G satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \cdots$, there is an $h_k > 0$ such that

$$\begin{aligned} &|F(x, y, t, i) - F(\bar{x}, \bar{y}, t, i)| + |G(x, y, t, i) - G(\bar{x}, \bar{y}, t, i)| \\ &\quad \leq h_k(|x - \bar{x}| + |y - \bar{y}|) \end{aligned}$$

for all $t \geq 0$, $i \in S$ and those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}| \leq k$, and there is moreover an $h > 0$ such that

$$|F(x, y, t, i)| + |G(x, y, t, i)| \leq h(1 + |x| + |y|)$$

for all $x, y \in R^n$, $t \geq 0$ and $i \in S$.

If we define, for $(\varphi, t, i) \in C([-\tau, 0]; R^n) \times R_+ \times S$,

$$f(\varphi, t, i) = F(\varphi(0), \varphi(-\delta(t)), t, i), \quad g(\varphi, t, i) = G(\varphi(0), \varphi(-\delta(t), t, i), \tag{31}$$

then equation (30) becomes equation (3) and (H2) implies (H1). So by Theorem 1, equation (30) has a unique global solution which is again denoted by $x(t;\xi)$. Assume furthermore that $F(0,0,t,i) \equiv 0$ and $G(0,0,t,i) \equiv 0$.

If $V \in C^{2,1}(R^n \times [-\tau,\infty) \times S; R_+)$, define an operator LV from $R^n \times R^n \times R_+ \times S$ to R by

$$(32) \qquad \begin{aligned} LV(x,y,t,i) &= V_t(x,t,i) + V_x(x,t,i)F(x,y,t,i) \\ &+ \frac{1}{2}\text{trace}[G^T(x,y,t,i)V_{xx}(x,t,i)G(x,y,t,i)] \\ &+ \sum_{j=1}^{N} \gamma_{ij} V(x,t,j). \end{aligned}$$

Note that the operators $\mathcal{L}V$ and LV have the following relationship

$$(33) \qquad \mathcal{L}V(\varphi,t,i) = LV(\varphi(0),\varphi(-\delta(t)),t,i).$$

To state our new result, let us introduce one more notation. Let $L^p_{\mathcal{F}_t}(\Omega;R^n)$ denote the family of all $\mathcal{F}_t$-measurable R^n-valued random variables X such that $E|X|^p < \infty$.

THEOREM 4. *Let (H2) hold. Let λ, p, c_1, c_2 be all positive numbers and $q > 1$. Assume that there exists a function $V(x,t,i) \in C^{2,1}(R^n \times [-\tau,\infty) \times S; R_+)$ such that*

$$(34) \quad c_1|x|^p \le V(x,t,i) \le c_2|x|^p \quad \textit{for all } (x,t,i) \in R^n \times [-\tau,\infty) \times S,$$

and also for all $t \ge 0$

$$(35) \qquad E\Big[\max_{1\le i\le N} LV(X,Y,t,i)\Big] \le -\lambda E\Big[\max_{1\le i\le N} V(X,t,i)\Big]$$

provided $X, Y \in L^p_{\mathcal{F}_t}(\Omega;R^n)$ satisfying

$$(36) \qquad E\Big[\min_{1\le i\le N} V(Y,t-\delta(t),i)\Big] < qE\Big[\max_{1\le i\le N} V(X,t,i)\Big].$$

Then the trivial solution of equation (30) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than

$$-\gamma = -\min\{\lambda, \log(q)/\tau\}.$$

Proof. We know that by definition (31) equation (30) becomes (3). To apply Theorem 2 we need to show that (12) implies (11). Let $t \ge 0$ and

$\phi = \{\phi(\theta) : -\tau \le \theta \le 0\} \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ satisfying (12). In particular, $\varphi(0), \varphi(-\delta(t)) \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ and

$$E\Big[\min_{1\le i\le N} V(\phi(-\delta(t)), t-\delta(t), i)\Big] < qE\Big[\max_{1\le i\le N} V(\phi(0), t, i)\Big].$$

That is, condition (36) is satisfied with $X = \phi(0)$ and $Y = \phi(-\delta(t))$. By (35) we have

$$E\Big[\max_{1\le i\le N} LV(\phi(0), \phi(-\delta(t)), t, i)\Big] \le -\lambda E\Big[\max_{1\le i\le N} V(\phi(0), t, i)\Big].$$

This, together with (33), yields

$$E\Big[\max_{1\le i\le N} \mathcal{L}V(\phi, t, i)\Big] \le -\lambda E\Big[\max_{1\le i\le N} V(\phi(0), t, i)\Big]$$

which is (11). Hence all the assumptions of Theorem 2 are satisfied and the conclusions follow. The proof is complete.

We now use this theorem to establish a useful result.

THEOREM 5. *Let (H2) hold. Let p, c_1, c_2 be positive numbers and $\lambda_1 > \lambda_2 \ge 0$. Assume that there exists a function $V(x,t,i) \in C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$ such that (34) is satisfied and, moreover, for all $x, y \in R^n$ and $t \ge 0$,*

$$(37)\quad \max_{1\le i\le N} LV(x, y, t, i) \le -\lambda_1 \max_{1\le i\le N} V(x, t, i) + \lambda_2 \min_{1\le i\le N} V(y, t-\delta(t), i).$$

Then the trivial solution of equation (30) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-(\lambda_1 - q\lambda_2)$ with $q > 1$ being the unique root of $\lambda_1 - q\lambda_2 = \log(q)/\tau$.

Proof. For $t \ge 0$ and $X, Y \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ satisfying

$$E\Big[\min_{1\le i\le N} V(Y, t-\delta(t), i)\Big] < qE\Big[\max_{1\le i\le N} V(X, t, i)\Big],$$

we drive from condition (37) that

$$\begin{aligned} & E\Big[\max_{1\le i\le N} LV(X, Y, t, i)\Big] \\ \le\ & -\lambda_1 E\Big[\max_{1\le i\le N} V(X, t, i)\Big] + \lambda_2 E\Big[\min_{1\le i\le N} V(Y, t-\delta(t), i)\Big] \\ \le\ & -(\lambda_1 - q\lambda_2) E\Big[\max_{1\le i\le N} V(X, t, i)\Big]. \end{aligned}$$

In other words, (35) is satisfied with $\lambda = \lambda_1 - q\lambda_2$. So the conclusions follow from Theorem 4 and the proof is complete.

Let us now give a result on the almost sure exponential stability for the delay equation (30).

THEOREM 6. *Let $p \geq 1$. Assume that there is a constant $K > 0$ such that for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$,*

$$|F(x,y,t,i)| \vee |G(x,y,t,i)| \leq K(|x| + |y|). \tag{38}$$

Then the pth moment exponential stability of the trivial solution of equation (30) implies the almost sure exponential stability.

Proof. To apply Theorem 3 we need to verify (17). With definition (31), for $\phi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$, $t \geq 0$ and $i \in S$, we compute that

$$\begin{aligned} & E[|f(\phi,t,i)|^p + |g(\phi,t,i)|^p] \\ = \; & E[|F(\phi(0), \phi(-\delta(t)), t, i)|^p + |G(\phi(0), \phi(-\delta(t)), t, i)|^p] \\ \leq \; & 2K^p E(|\phi(0) + \phi(-\delta(t))|)^p \leq 2^{p+1} K^p \sup_{-\tau \leq \theta \leq 0} E|\phi(\theta)|^p. \end{aligned}$$

Now the conclusion follows from Theorem 3. The proof is complete.

5. Stochastic Differential Equations with Markovian Switching. If the coefficients F and G of equation (30) are independent of the past state $x(t - \delta(t))$, equation (30) reduces to the stochastic differential equation with Markovian switching of the form

$$dx(t) = F(x(t), t, r(t))dt + G(x(t), t, r(t))dw(t) \tag{39}$$

on $t \geq 0$. Of course, F and G are now functions from $R^n \times R_+ \times S$ to R^n and $R^{n \times m}$, respectively; and the initial value $x(0) = \xi$ is a bounded $\mathcal{F}_0$-measurable R^n-valued random variable. We still assume that both F and G satisfy the local Lipschitz condition and the linear growth condition so the equation has a unique global solution which is again denoted by $x(t;\xi)$. We also assume that $F(0,t,i) \equiv 0$ and $G(0,t,i) \equiv 0$. For $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, the operator LV is now from $R^n \times R_+ \times S$ to R with a simpler form

$$\begin{aligned} LV(x,t,i) \; = \; & V_t(x,t,i) + V_x(x,t,i)F(x,t,i) \\ & + \frac{1}{2}\text{trace}[G^T(x,t,i)V_{xx}(x,t,i)G(x,t,i)] \\ & + \sum_{j=1}^N \gamma_{ij} V(x,t,j). \end{aligned} \tag{40}$$

The following result follows directly from Theorem 5.

THEOREM 7. *Let p, λ, c_1, c_2 be positive numbers. Assume that there exists a function $V(x,t,i) \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that*

$$c_1|x|^p \le V(x,t,i) \le c_2|x|^p$$

and

$$\max_{1\le i\le N} LV(x,t,i) \le -\lambda \max_{1\le i\le N} V(x,t,i)$$

for all $(x,t,i) \in R^n \times R_+ \times S$. Then the trivial solution of equation (39) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\lambda$.

We now establish a result on the almost sure exponential stability.

THEOREM 8. *Assume that there is a constant $K > 0$ such that for all $(x,t,i) \in R^n \times R_+ \times S$,*

$$|F(x,t,i)| \vee |G(x,t,i)| \le K|x|. \tag{41}$$

Then the pth ($p > 0$) moment exponential stability of the trivial solution of equation (39) implies the almost sure exponential stability.

Proof. If $p \ge 1$, the conclusion follows from Theorem 6 directly. We now prove the result in the case of $0 < p < 1$. Fix the initial value ξ arbitrarily and write $x(t;\xi) = x(t)$. By the property of pth moment exponential stability, there is a pair of positive constants M and γ such that

$$E|x(t)|^p \le Me^{-\gamma t} \quad \text{on } t \ge 0. \tag{42}$$

Let $\sigma > 0$ be small enough for

$$(3\sigma K)^p(1 + C_p) \le \frac{1}{2}, \tag{43}$$

where $C_p = (32/p)^{p/2}$ is the constant given by the Burkholder–Davis–Gundy inequality (cf. Mao [12]). Note that for any $a, b, c \ge 0$,

$$(a+b+c)^p \le [3(a \vee b \vee c)]^p = 3^p(a^p \vee b^p \vee c^p) \le 3^p(a^p + b^p + c^p).$$

Let $k = 1, 2, \cdots$. Using the Burkholder–Davis–Gundy inequality and (41)–(43) we compute that

$$E\Big[\sup_{(k-1)\sigma\le t\le k\sigma} |x(t)|^p\Big]$$

$$\begin{aligned}
&\le 3^pE|x((k-1)\sigma)|^p + 3^pE\Big(\int_{(k-1)\sigma}^{k\sigma}|F(x(s),s,r(s))|ds\Big)^p \\
&+ 3^pE\Big[\sup_{(k-1)\sigma\le t\le k\sigma}\Big|\int_{(k-1)\sigma}^{t} G(x(s),s,r(s))dw(s)\Big|^p\Big] \\
&\le 3^pMe^{-\gamma(k-1)\sigma} + (3\sigma K)^pE\Big[\sup_{(k-1)\sigma\le t\le k\sigma}|x(t)|^p\Big] \\
&+ 3^pC_pE\Big(\int_{(k-1)\sigma}^{k\sigma}|G(x(s),s,r(s))|^2ds\Big)^{\frac{p}{2}} \\
&\le 3^pMe^{-\gamma(k-1)\sigma} + (3\sigma K)^p(1+C_p)E\Big[\sup_{(k-1)\sigma\le t\le k\sigma}|x(t)|^p\Big] \\
&\le 3^pMe^{-\gamma(k-1)\sigma} + \frac{1}{2}E\Big[\sup_{(k-1)\sigma\le t\le k\sigma}|x(t)|^p\Big].
\end{aligned}$$

Consequently

$$E\Big[\sup_{(k-1)\sigma\le t\le k\sigma}|x(t)|^p\Big] \le 2M3^pe^{-\gamma(k-1)\sigma}. \tag{44}$$

We can then show from (44) that

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\gamma}{p} \quad \text{a.s.}$$

in the same way as in the proof of Theorem 2 of Mao [13]. The proof is therefore complete.

6. Examples. In this section we shall discuss a number of examples to illustrate our theory. In the following examples we shall omit mentioning the initial data.

EXAMPLE 1. Consider a linear stochastic functional differential equation with Markovian switching of the form

$$dx(t) = A(r(t))x(t)dt + g(x_t, r(t))dw(t) \tag{45}$$

on $t \ge 0$. Here $w(t)$ is a scalar Brownian motion, $A : S \to R^{n\times n}$ and we shall write $A(i) = A_i$, while $g : C([-\tau,0];R^n)\times S \to R^n$ is defined by

$$g(\varphi, i) = \int_{-\tau}^{0} B_i\varphi(\theta)d\theta$$

with B_i's being all $n\times n$ matrices. Assume that there are symmetric positive-definite matrices Q_i, $1 \le i \le k$, such that

$$-\lambda_1 := \max_{1\le i\le N}\lambda_{\max}\Big(Q_iA_i + A_i^TQ_i + \sum_{j=1}^{N}\gamma_{ij}Q_j\Big) < 0, \tag{46}$$

where and in the sequel $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and smallest eigenvalue of matrix A, respectively. We now claim that if

$$\tau^2 < \frac{\lambda_1 \lambda_2}{\lambda_3 \lambda_4}, \tag{47}$$

then the trivial solution of equation (45) is mean square exponentially stable, where

$$\lambda_2 = \min_{1\le i\le N} \lambda_{\min}(Q_i), \quad \lambda_3 = \max_{1\le i\le N} \lambda_{\max}(Q_i),$$

and

$$\lambda_4 = \max_{1\le i\le N} ||B_i^T Q_i B_i||.$$

To show this, we let $V(x,t,i) = x^T Q_i x$. The operator $\mathcal{L}V$ has the form

$$\begin{aligned} \mathcal{L}V(\varphi,t,i) &= \varphi^T(0)(Q_iA_i + A_i^TQ_i)\varphi(0) + g^T(\varphi,i)Q_ig(\varphi,i) \\ &+ \sum_{j=1}^N \gamma_{ij}\varphi^T(0)Q_j\varphi(0) \\ &= \varphi^T(0)\Big(Q_iA_i + A_i^TQ_i + \sum_{j=1}^N \gamma_{ij}Q_j\Big)\varphi(0) \\ &+ \Big(\int_{-\tau}^0 \varphi(\theta)d\theta\Big)^T B_i^TQ_iB_i\Big(\int_{-\tau}^0 \varphi(\theta)d\theta\Big) \\ &\le -\lambda_1|\varphi(0)|^2 + \tau\lambda_4\int_{-\tau}^0 |\varphi(\theta)|^2 d\theta. \end{aligned} \tag{48}$$

By (47) we can choose $q > 1$ for

$$\lambda_1 - \frac{\tau^2 q \lambda_3 \lambda_4}{\lambda_2} > 0. \tag{49}$$

For any $t \ge 0$ and $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$E\Big[\min_{1\le i\le N} \phi^T(\theta)Q_i\phi(\theta)\Big] < qE\Big[\max_{1\le i\le N} \phi^T(0)Q_i\phi(0)\Big]$$

on $-\tau \le \theta \le 0$, we have

$$\lambda_2 E|\phi(\theta)|^2 < q\lambda_3 E|\phi(0)|^2 \quad \text{on } -\tau \le \theta \le 0$$

and hence, by (48),

$$E\Big[\max_{1\le i\le N}\mathcal{L}V(\phi,t,i)\Big] \le -\lambda_1 E|\phi(0)|^2+\tau\lambda_4\int_{-\tau}^0 E|\phi(\theta)|^2d\theta$$
$$\le -\Big(\lambda_1-\frac{\tau^2 q\lambda_3\lambda_4}{\lambda_2}\Big)E|\phi(0)|^2.$$

Recalling (49) we see all assumptions of Theorem 2 are satisfied with $p=2$, and therefore the trivial solution of equation (45) is mean square exponentially stable. To show the almost sure exponential stability, we verify condition (17): For $\phi\in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$ and $i\in S$

$$E[|A_i\phi(0)|^2+|g(\phi,i)|^2]$$
$$\le ||A_i||^2E|\phi(0)|^2+\tau||B_i||^2\int_{\tau}^0 E|\phi(\theta)|^2d\theta$$
$$\le \Big[\max_{1\le i\le N}(||A_i||^2+\tau^2||B_i||^2)\Big]\sup_{-\tau\le\theta\le 0}E||\phi(\theta)||^2.$$

Hence, by Theorem 3, the trivial solution of equation (45) is also almost surely exponentially stable.

EXAMPLE 2. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S=\{1,2\}$ with generator

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}.$$

Assume that $w(t)$ and $r(t)$ are independent. Let $\delta: R_+\to[0,\tau]$ be Borel measurable. Consider a one-dimensional stochastic differential delay equation with Markovian switching of the form

$$dx(t)=\alpha(r(t))x(t)dt+\sigma(x(t-\delta(t)),t,r(t))dw(t) \tag{50}$$

on $t\ge 0$, where

$$\alpha(1)=\frac{1}{4},\quad \alpha(2)=-3,$$

while $\sigma: R\times R_+\times S\to R$ satisfying

$$|\sigma(y,t,1)|\le\frac{|y|}{8},\quad |\sigma(y,t,2)|\le\frac{|y|}{4}.$$

It is interesting to point out that equation (50) can be regarded as the result of the following two equations

$$dx(t)=\frac{1}{4}x(t)dt+\sigma(x(t-\delta(t)),t,1)dw(t) \tag{51}$$

and

$$(52)\qquad dx(t) = -3x(t)dt + \sigma(x(t-\delta(t)), t, 2)dw(t)$$

switching to each other according to the movement of the Marknov chain $r(t)$. We know that when $2 \le p \le 2.5$, equation (51) is not pth moment exponentially stable while equation (52) is. We shall now show that by switching the equations from one to the other the overall behaviour, i.e. equation (50) will be stable. To examine the pth moment exponential stability, we construct a function $V : R \times [-\tau, \infty) \times S \to R_+$ by

$$V(x,t,i) = \begin{cases} \beta|x|^p & \text{if } i = 1, \\ |x|^p & \text{if } i = 2, \end{cases}$$

where $\beta > 0$ is a constant to be determined. It is easy to show that the operator LV from $R \times R \times R_+ \times S$ to R has the form

$$\begin{aligned} &LV(x,y,t,i) \\ &= \begin{cases} -\left[\beta\left(1-\frac{p}{4}\right)-1\right]|x|^p + \frac{\beta p(p-1)}{2}|x|^{p-2}|\sigma(y,t,1)|^2 & \text{if } i=1, \\ -(3p+1-\beta)|x|^p + \frac{p(p-1)}{2}|x|^{p-2}|\sigma(y,t,2)|^2 & \text{if } i=2. \end{cases} \end{aligned}$$

Hence, bearing in mind that $2 \le p \le 2.5$,

$$\begin{aligned} LV(x,y,t,1) &\le -\left(\beta\left[1-\frac{p}{4}\right]-1\right)|x|^p + \frac{\beta p(p-1)}{128}|x|^{p-2}|y|^2 \\ &\le -\left(\beta\left[1-\frac{p}{4}-\frac{(p-1)(p-2)}{128}\right]-1\right)|x|^p + \frac{\beta(p-1)}{64}|y|^p \\ &\le -(0.369\beta - 1)|x|^p + 0.024\beta|y|^p, \end{aligned}$$

where we have used the elementary inequality

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b \quad \text{if } a, b \ge 0,\ \alpha \in (0,1).$$

Similarly

$$\begin{aligned} LV(x,y,t,2) &\le -(3p+1-\beta)|x|^p + \frac{p(p-1)}{32}|x|^{p-2}|y|^2 \\ &\le -\left[3p+1-\beta-\frac{(p-1)(p-2)}{32}\right]|x|^p + \frac{(p-1)}{16}|y|^p \\ &\le -(6.976-\beta)|x|^p + 0.094|y|^p. \end{aligned}$$

Choosing β for

$$0.369\beta - 1 = 6.976 - \beta,$$

i.e. $\beta = 5.826$, we then have

$$LV(x,y,t,i) \leq \begin{cases} -1.15x^2 + 0.139y^2 & \text{if } i = 1, \\ -1.15x^2 + 0.094y^2 & \text{if } i = 2. \end{cases}$$

Consequently

$$\begin{aligned} & \max_{i=1,2} LV(x,y,t,i) \leq -1.15x^2 + 0.139y^2 \\ = \; & -0.197\Big[\max_{i=1,2} V(x,t,i)\Big] + 0.139\Big[\min_{i=1,2} V(y,t,i)\Big]. \end{aligned}$$

By Theorem 5 we see that the trivial solution of equation (50) is pth moment exponentially stable, and by Theorem 6, it is also almost surely exponentially stable.

EXAMPLE 3. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1,2\}$ with generator $\Gamma = (\gamma_{ij})_{2\times 2}$, where

$$-\gamma_{11} = \gamma_{12} > 0 \quad \text{and} \quad -\gamma_{22} = \gamma_{21} > 0.$$

Assume that $w(t)$ and $r(t)$ are independent. Consider a one-dimensional linear stochastic differential equation with Markovian switching of the form

$$dx(t) = \alpha(r(t))x(t)dt + \sigma(r(t))x(t)dw(t) \tag{53}$$

on $t \geq 0$. Here $\alpha(i)$ and $\sigma(i)$, $i = 1,2$, are all constants and we shall write $\alpha(i) = \alpha_i$ and $\sigma(i) = \sigma_i$. Assume that there is a number $p > 0$ such that

$$1 + \frac{p[(1-p)\sigma_1^2 - 2\alpha_1]}{2\gamma_{12}} > \left(1 + \frac{p[(1-p)\sigma_2^2 - 2\alpha_2]}{2\gamma_{21}}\right)^{-1} > 0. \tag{54}$$

We shall now show that the trivial solution of equation (6.8) is pth moment exponentially stable. Choose a positive constant β such that

$$1 + \frac{p[(1-p)\sigma_1^2 - 2\alpha_1]}{2\gamma_{12}} > \beta > \left(1 + \frac{p[(1-p)\sigma_2^2 - 2\alpha_2]}{2\gamma_{21}}\right)^{-1} > 0. \tag{55}$$

This implies

$$\lambda_1 := \frac{p}{2}[(1-p)\sigma_1^2 - 2\alpha_1] + \gamma_{12}(1-\beta) > 0 \tag{56}$$

and

$$\lambda_2 := \frac{\beta p}{2}[(1-p)\sigma_2^2 - 2\alpha_2] + \gamma_{21}(\beta - 1) > 0. \tag{57}$$

Define $V : R \times R_+ \times S \to R_+$ by

$$V(x,t,i) = \beta_i |x|^p$$

with $\beta_1 = 1$ and $\beta_2 = \beta$. Then the operator LV has the form

$$LV(x,t,i) = \Big[p\alpha_i\beta_i + \frac{1}{2}p(p-1)\beta_i\sigma_i^2 + \gamma_{i1}\beta_1 + \gamma_{i2}\beta_2\Big]|x|^p$$

It is then easy to show that

$$LV(x,t,1) = -\lambda_1|x|^p \quad \text{and} \quad LV(x,t,2) = -\lambda_2|x|^p.$$

Consequently

$$\begin{aligned} &\max\{LV(x,t,1), LV(x,t,2)\} \\ \le\ & -(\lambda_1 \wedge \lambda_2)|x|^p = -\frac{\lambda_1 \wedge \lambda_2}{1 \vee \beta}\max\{V(x,t,1), V(x,t,2)\}. \end{aligned}$$

By Theorems 7 and 8 we conclude that the trivial solution of equation (53) is pth moment exponentially stable and is also almost surely exponentially stable.

As a special case, let us put

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \sigma_1 = \sigma_2 = 2, \quad \gamma_{12} = 1, \quad \gamma_{21} = 3. \tag{58}$$

We can regard equation (53) as the result of the following two equations

$$dx(t) = x(t)dt + 2x(t)dw(t) \tag{59}$$

and

$$dx(t) = 2x(t)dt + 2x(t)dw(t) \tag{60}$$

switching to each other according to the movement of the Markov chain $r(t)$. It is known that equation (60) is not almost surely exponentially stable although equation (59) is. On the other hand, choosing $p = 0.2$ we have

$$1 + \frac{p[(1-p)\sigma_1^2 - 2\alpha_1]}{2\gamma_{12}} = 1.12 > \left(1 + \frac{p[(1-p)\sigma_2^2 - 2\alpha_2]}{2\gamma_{21}}\right)^{-1} = 1.04.$$

That is, condition (54) is satisfied. Hence equation (53) with coefficients defined by (58) is almost surely exponentially stable. In other words, for equation (53) to be stable, it is not necessary to require each of equations (59) and (60) be stable.

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PERIODICITY AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF ONE DIFFERENCE EQUATION

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Abstract. In this paper we study the periodicity and the asymptotic behavior of positive solutions of the difference equation $x_{n+1} = A + x_{n-k}\sum_{s=0}^{k-1}\frac{1}{x_{n-s}}$, $n \in \{0,1,...\}$, $k = 1,2,...$ and $A \in [0,1] \cup (k,\infty)$.

1. Introduction. In [1] Amlech, Grove, Ladas and Georgiou studied the global stability, the boundedness and the periodicity of the positive soloutions of the difference equation

(1) $$x_{n+1} = A + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, ...$$

where $A \in [0,\infty)$, $x_{-1}, x_0 \in (0,\infty)$.

In this paper we concider the difference equation of the form

(2) $$x_{n+1} = A + x_{n-k}\sum_{s=0}^{k-1}\frac{1}{x_{n-s}}, \quad n \in \{0,1,...\}$$

where $x_{-k},...,x_0 \in (0,\infty)$, $A \in [0,1] \cup (k,\infty)$, and $k = 1,2,...$ It is obvious that for $k = 1$ equation (2) reduces to (1).

First we find necessary and sufficient conditions so that equation (2) has solutions of pariod $k+1$. Also we find conditions so that a positive solution of (2) tend to the unique positive equilibrium $c = A + k$ as $n \to \infty$. Moreover if $0 \le A < 1$ we prove that (2) has unbounded soloutions, if $A = 1$ we find conditions so that a positive solution of (2) tends to a period $k+1$

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solution of (2) and if $A > k$ the positive equilibrium is qlobally asymptotically stable. Finally for $k = 2$ (resp. $k = 3$) we prove that if $A > 1$, the positive equilibrium $A + 2$ (resp. $A + 3$) is locally asymptotically stable.

2. Main Results. We prove now our main results. In the first proposition we study the periodic nature of the positive solutions of (2).

PROPOSITION 1.

(i) Suppose that (2) has a nontrivial positive solution x_n of period $k+1$. Then $A = 1$.

(ii) Let $A = 1$. A solution x_n of (1) is periodic of period $k+1$ if and only if

$$\sum_{s=0}^{k} \frac{1}{x_{-s}} = 1. \tag{3}$$

Proof (i) Suppose that (1) has a nontrivial solution x_n of period $k+1$. Then

$$x_{k+1-s} = x_{-s}, \quad s = 0, 1, ..., k. \tag{4}$$

Then from (2) and (4) it follows that for $i, j \in \{0, 1, ..., k\}$

$$x_{-i} = x_{k+1-i} = A + x_{-i} \sum_{s=0}^{k-1} \frac{1}{x_{k-i-s}}$$

$$x_{-j} = x_{k+1-j} = A + x_{-j} \sum_{s=0}^{k-1} \frac{1}{x_{k-j-s}}$$

from which it is obvious that

$$\begin{aligned} x_{-i}x_{-j} &= Ax_{-j} + x_{-i}x_{-j} \sum_{s=0}^{k-1} \frac{1}{x_{k-i-s}} \\ x_{-j}x_{-i} &= Ax_{-i} + x_{-j}x_{-i} \sum_{s=0}^{k-1} \frac{1}{x_{k-j-s}}. \end{aligned} \tag{5}$$

Without loss of generality we may suppose that $j > i$. Then since x_n is periodic of period $k+1$ we get

$$\sum_{s=0}^{k-1} \frac{1}{x_{k-i-s}} - \sum_{s=0}^{k-1} \frac{1}{x_{k-j-s}} = \frac{1}{x_{-j}} - \frac{1}{x_{-i}}. \tag{6}$$

Therefore relations (5) and (6) imply that

$$(A-1)(x_{-i}-x_{-j})=0, \quad i,j\in\{0,1,..,k\}$$

from which the proof of (i) follows immediately.

(ii) Let $A=1$ and x_n be a positive solution of (2) of period $k+1$. Then we have $x_1=x_{-k}$. So from (2) relation (3) holds.

Suppose now that (3) is satisfied. Then from (2) and (3) we have

$$x_1=x_{-k}. \tag{7}$$

Therefore from (2) and (7) it follows that

$$x_2=1+x_{1-k}\sum_{s=0}^{k-1}\frac{1}{x_{1-s}}=1+x_{1-k}\left(\frac{1}{x_1}-\frac{1}{x_{1-k}}-\frac{1}{x_{-k}}\right)+x_{1-k}=x_{1-k}.$$

Working inductively we can easily prove that $x_{n+k+1}=x_n$, $n=-k,-k+1,\ldots$. This completes the proof of the proposition.

In the following proposition we find conditions so that a positive solution of (2) tends to the positive equilibrium $c=A+k$.

PROPOSITION 2. *Let x_n be a positive solution of (2). Suppose that there exists an $m\in\{-k,-k+1,\ldots\}$ such that*

$$(i)\quad x_n\geq A+k \quad \text{or} \quad (ii)\quad x_n<A+k,\quad n\geq m. \tag{8}$$

Then x_n tends to the positive equilibrium $c=A+k$ of (2).

Proof Let x_n be a positive solution of (2) such that (i) of (8) is satisfied. Then from (2) we take

$$x_{n+1}\leq A+\frac{k}{A+k}x_{n-k},\quad n>m+k-1. \tag{9}$$

We concider the difference equation

$$u_{n+1}=A+\frac{k}{A+k}u_{n-k},\quad n>m+k-1. \tag{10}$$

We can easily prove that the general solution of (10) has the form

$$u_n=\left(\frac{k}{A+k}\right)^{\frac{n}{k+1}}\sum_{s=0}^{v}\left(c_s\cos\frac{2sn\pi}{k+1}+d_s\sin\frac{2sn\pi}{k+1}\right)+A+k \tag{11}$$

where

$$v = \begin{cases} \dfrac{k+1}{2} & \text{if } k \text{ is an odd number} \\ \dfrac{k}{2} & \text{if } k \text{ is an even number,} \end{cases} \tag{12}$$

$c_s, d_s \in (0, \infty)$. Therefore from (11) it is obvious that

$$\lim_{n\to\infty} u_n = A + k. \tag{13}$$

We concider the solution u_n of (10) such that

$$u_{m+i} = x_{m+i}, \qquad i = 0, 1, ..., k. \tag{14}$$

From (9) and (10) it follows that

$$x_{n+1} - u_{n+1} \leq \frac{k}{A+k}(x_{n-k} - u_{n-k}), \quad n > m + k - 1. \tag{15}$$

Then from (14) and (15) we can easily prove by induction that

$$x_n \leq u_n, \qquad n \geq m. \tag{16}$$

So from (i) of (8), (13) and (16) it is obvious that x_n tends to c as $n \to \infty$. Simirarly we can prove that if (ii) of (8) holds then x_n tends to c as $n \to \infty$. This completes the proof of the proposition.

In the following proposition we prove that if $0 \leq A < 1$ then the equation (2) has unbounded solutions.

PROPOSITION 3. *Suppose that* $0 \leq A < 1$. *Let* x_n *be a positive solution of (2) such that*

$$\sum_{s=1}^{k} \frac{1}{x_{-s}} < 1 - A, \qquad x_0 < 1. \tag{17}$$

Then

$$\lim_{n\to\infty} x_{(k+1)n+m} = \infty, \qquad m \in \{1, 2, ..., k\}, \qquad \lim_{n\to\infty} x_{(k+1)n+k+1} = A. \tag{18}$$

Proof We prove by induction that for $n = 0, 1, ...$

$$x_{(k+1)n+m} > A + x_{(k+1)(n-1)+m}, \; m \in \{1, 2, ..., k\}, \; x_{(k+1)n+k+1} < 1. \tag{19}$$

From (2) and (17) we get for $m \in \{1, 2, ..., k\}$

$$(20) \qquad x_m = A + x_{m-k-1}\sum_{s=0}^{k-1}\frac{1}{x_{m-1-s}} > A + \frac{x_{m-k-1}}{x_0} > A + x_{m-k-1}.$$

Moreover relations (2), (17) and (20) imply that

$$(21) \qquad \begin{aligned} x_{k+1} &= A + x_0\sum_{s=0}^{k-1}\frac{1}{x_{k-s}} < A + x_0\sum_{s=0}^{k-1}\frac{1}{A + x_{-s-1}} \\ &< A + x_0\sum_{s=0}^{k-1}\frac{1}{x_{-s-1}} < 1. \end{aligned}$$

Therefore from (20) and (21) relations (19) hold for $n = 0$. Furthermore from (2), (21) for $m \in \{1, 2, ..., k\}$ we get

$$(22) \qquad x_{k+1+m} = A + x_m\sum_{s=0}^{k-1}\frac{1}{x_{k+m-s}} > A + \frac{x_m}{x_{k+1}} > A + x_m.$$

Then from (2), (17), (20), (21) and (22) it follows that

$$(23) \qquad \begin{aligned} x_{2(k+1)} &= A + x_{k+1}\sum_{s=0}^{k-1}\frac{1}{x_{2k+1-s}} < A + \sum_{s=0}^{k-1}\frac{1}{A + x_{k-s}} \\ &< A + \sum_{s=0}^{k-1}\frac{1}{x_{-s-1}} < 1. \end{aligned}$$

Therefore from (22), (23) relations (19) hold for $n = 1$.

Suppose that (19) hold for all $n \leq r$. Then from (2) we get for $m \in \{1, 2, ..., k\}$

$$(24) \qquad \begin{aligned} x_{(k+1)(r+1)+m} &= A + x_{(k+1)r+m}\sum_{s=0}^{k-1}\frac{1}{x_{(k+1)r+k+m-s}} \\ &> A + x_{(k+1)r+m}\frac{1}{x_{k+1)(r+1)}} > A + x_{(k+1)r+m}. \end{aligned}$$

Moreover from (2), (24) and since (19) hold for $n \leq r$ we have

$$(25) \qquad \begin{aligned} x_{(k+1)(r+2)} &= A + x_{(k+1)(r+1)}\sum_{s=0}^{k-1}\frac{1}{x_{(k+1)(r+1)+k-s}} \\ &< A + x_{(k+1)(r+1)}\sum_{s=0}^{k-1}\frac{1}{x_{(k+1)r+k-s}} \\ &< A + \sum_{s=0}^{k-1}\frac{1}{x_{-s-1}} < 1. \end{aligned}$$

Therefore from (20) - (25) relations (19) hold.

Suppose first that $A \neq 0$. Then from (19) it is obvious that

$$\lim_{n\to\infty} x_{(k+1)n+m} = \infty, \qquad m \in \{1,2,...,k\}. \tag{26}$$

Moreover from (2), (19) it follows that

$$x_{(k+1)(n+1)} - A = x_{(k+1)n} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k-s}} < \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k-s}}. \tag{27}$$

From (26) and (27) it follows that

$$\lim_{n\to\infty} x_{(k+1)(n+1)} = A. \tag{28}$$

Using (26) and (28) relations (18) hold when $A \neq 0$.

Suppose now $A = 0$. Using (17), (19) for $A = 0$ there exist

$$\lim_{n\to\infty} x_{(k+1)n+m} = L_m, \qquad m = 1,2,...,k, \qquad L_m \in (1,\infty]. \tag{29}$$

Suppose that there exists an $r \in \{1,2,...,k\}$ such that $L_r \neq \infty$. Without loss of generality we may suppose that

$$L_1 \neq \infty. \tag{30}$$

From (2) we get

$$x_{(k+1)(n+1)+1} = x_{(k+1)n+1} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k+1-s}}. \tag{31}$$

Then from (19), (29), (30), (31) there exists the $\lim_{n\to\infty} x_{(k+1)n} = L_0$ and

$$L_0 = 1, \quad \sum_{s=2}^{k} \frac{1}{L_s} = 0$$

from which it follows that

$$L_0 = 1, \quad L_s = \infty, \quad s = 2,...,k. \tag{32}$$

Furthermore from (2) we get

$$x_{(k+1)(n+1)} = x_{(k+1)n} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k-s}}. \tag{33}$$

Hence relations (32) and (33) imply that $L_1 = 1$ which is a contradiction. Therefore (26) are satisfied. Moreover from (19), (26), (33) we have (28) hold with $A = 0$. This completes the proof of the proposition.

In the following proposition if $A = 1$ we find conditions so that a positive solution of (2) tends to a period $k+1$ solution of (2). We need two lemmas.

LEMMA 1. *Suppose $A = 1$. Let x_n be a positive solution of (2) such that for $n \geq n_0$ where $n_0 \in \{-k, -k+1, ...\}$*

(34) $x_{(k+1)n+\tau_i} \leq M_i,\ i \in \{1, 2, ..., m\},\ \tau_i \in \{1, 2, ..., k+1\},\ 2 \leq m \leq k$

M_i are positive constants which satisfy

$$\sum_{s=1}^{m} \frac{1}{M_s} = 1. \tag{35}$$

Then, it holds,

$$\begin{aligned}
&\lim_{n\to\infty} x_{(k+1)n+\tau_i} = M_i, \quad i \in \{1, 2, ..., m\}, \quad \tau_i \in \{1, 2, ..., k+1\}, \quad 2 \leq m \leq k,\\
&\lim_{n\to\infty} x_{(k+1)n+\mu_j} = \infty, \quad \mu_j \neq \tau_i, \quad j \in \{1, 2, ..., k+1-m\},\\
&\mu_j \in \{1, 2, ..., k+1\}.
\end{aligned} \tag{36}$$

Proof From (2), (34) we have for $i \in \{1, ..., m\},\ \ 2 \leq m \leq k$

$$\begin{aligned}
x_{(k+1)(n+1)+\tau_i} &= 1 + x_{(k+1)n+\tau_i} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k+\tau_i-s}}\\
&= 1 + x_{(k+1)n+\tau_i} \left(\sum_{s=1}^{\tau_i-1} \frac{1}{x_{(k+1)(n+1)+s}} + \sum_{s=\tau_i+1}^{k+1} \frac{1}{x_{(k+1)n+s}} \right)\\
&= 1 + x_{(k+1)n+\tau_i} \left(\sum_{s=1}^{i-1} \frac{1}{x_{(k+1)(n+1)+\tau_s}} + \sum_{s=i+1}^{m} \frac{1}{x_{(k+1)n+\tau_s}} \right.\\
&\left. + \sum_{s=1}^{\tau_i} \frac{1}{x_{(k+1)(n+1)+\mu_s}} + \sum_{s=\tau_i+1}^{k+1-m} \frac{1}{x_{(k+1)n+\mu_s}} \right)\\
&\geq 1 + \sum_{s=1}^{i-1} \frac{1}{x_{(k+1)(n+1)+\tau_s}} + \sum_{s=i+1}^{m} \frac{1}{x_{(k+1)n+\tau_s}}\\
&\geq 1 + \sum_{s=1, s\neq i}^{m} \frac{1}{M_s}.
\end{aligned} \tag{37}$$

where $r_i = \max\{j,\ \mu_j \leq \tau_i - 1\}$, $j \in \{1, 2, ..., k+1-m\}$. From (35) it follows that

$$w_i = \sum_{s=1, s\neq i}^{m} \frac{1}{M_s} < 1. \tag{38}$$

Then from (2), (34), (35) and (37) we get

$$x_{(k+1)(n+1)+\tau_i} \geq 1 + w_i, \quad x_{(k+1)(n+2)+\tau_i} \geq 1 + w_i + w_i^2 \tag{39}$$

and working indectively we can prove that

$$x_{(k+1)(n+r)+\tau_i} \geq \sum_{\nu=0}^{r} w_i^{\nu} = \frac{1 - w_i^{r+1}}{1 - w_i}. \tag{40}$$

Furthermore from (35), (38) we have

$$\lim_{n\to\infty} \frac{1 - w_i^{n+1}}{1 - w_i} = \frac{1}{1 - w_i} = M_i. \tag{41}$$

Therefore relations (34), (40) and (41) imply that

$$\lim_{n\to\infty} x_{(k+1)n+\tau_i} = M_i, \quad i \in \{1, ..., m\}, \quad 2 \leq m \leq k. \tag{42}$$

Moreover from (2) and (42) it follows that

$$M_i = 1 + M_i \sum_{s=1, s\neq i}^{m} \frac{1}{M_s} + M_i \sum_{j=1}^{k+1-m} \frac{1}{\lim_{n\to\infty} x_{(k+1)n+\mu_j}}. \tag{43}$$

Then using (35) and (43) we get

$$\lim_{n\to\infty} x_{(k+1)n+\mu_j} = \infty, \mu_j \neq \tau_i, j \in \{1, ..., k+1-m\}, \mu_j \in \{1, ..., k+1\}. \tag{44}$$

From (42) and (44) relations (36) hold. This completes the proof of the lemma.

LEMMA 2. *Suppose that* $A = 1$. *Then the following relations hold for* $m = 2, 3, ...$ *and* $n = 0, 1, ...$,

$$x_{(k+1)n+m} - x_{(k+1)(n-1)+m} = \frac{a(n,m)}{b(n,m)} \left(x_{(k+1)n+m-1} - x_{(k+1)(n-1)+m-1} \right) \tag{45}$$

where

$$a(n,m) = \sum_{j=0}^{k-1} \prod_{s=0, s\neq j}^{k-1} x_{(k+1)(n-1)+s+m}, \qquad b(n,m) = \prod_{s=1}^{k} x_{(k+1)(n-1)+s+m}.$$

Proof From (2) we have

$$x_{(k+1)n+m} - x_{(k+1)(n-1)+m} = 1 + x_{(k+1)(n-1)+m}\sum_{s=1}^{k}\frac{1}{x_{(k+1)(n-1)+m+s}}$$

$$-x_{(k+1)(n-1)+m} = \frac{1}{b(n,m)}\Bigg(\sum_{j=0}^{k}\prod_{s=0,s\neq j}^{k} x_{(k+1)(n-1)+s+m}$$

$$(46)\quad -\prod_{s=0}^{k-1} x_{(k+1)(n-1)+s+m} - \prod_{s=-1}^{k-1} x_{(k+1)(n-1)+s+m}\sum_{r=0}^{k-1}\frac{1}{x_{(k+1)(n-1)+m+r}}\Bigg)$$

$$= \frac{1}{b(n,m)}\left(\sum_{j=0}^{k-1}\prod_{s=0,s\neq j}^{k} x_{(k+1)(n-1)+s+m} - \sum_{j=0}^{k-1}\prod_{s=-1,s\neq j}^{k-1} x_{(k+1)(n-1)+s+m}\right)$$

$$= \frac{a(n,m)}{b(n,m)}\left(x_{(k+1)n+m-1} - x_{(k+1)(n-1)+m-1}\right).$$

From (46) the proof of the lemma is completed.

PROPOSITION 4. *Let $A = 1$ and x_n be a positive solution of (2). Then the following statements are true:*

(i) if

$$(47)\quad x_{-k}\left(1 - \sum_{s=0}^{k-1}\frac{1}{x_{-s}}\right) \le 1, \quad \left(\textit{resp. } x_{-k}\left(1 - \sum_{s=0}^{k-1}\frac{1}{x_{-s}}\right) \ge 1\right)$$

then $x_{(k+1)n+m}$, $m = 1, ..., k+1$ are increasing (resp. decreasing) sequences.

(ii) Suppose that (36) do not hold. Then there exist

$$(48)\quad \lim_{n\to\infty} x_{(k+1)n+m} = M_m, \quad m \in \{1,2,...,k+1\}, \quad M_m \in (1,\infty),$$

and

$$(49)\qquad \sum_{s=1}^{k+1}\frac{1}{M_s} = 1.$$

(iii) Suppose that (36) are not satisfied. Then the positive solution x_n of (2) tends to a period $k+1$ solution.

Proof (i) Suppose that the first inequality of (47) holds. Then from (2) we can easily find that

$$(50)\qquad x_{-k} \le x_1.$$

Suppose that

$$x_{(k+1)(n-1)+1} \leq x_{(k+1)n+1}. \tag{51}$$

From Lemma 2 we can prove that

$$\begin{aligned} x_{(k+1)(n+1)+1} - x_{(k+1)n+1} &= x_{(k+1)n+k+2} - x_{(k+1)(n-1)+k+2} \\ &= \frac{a(n,k+2)}{b(n,k+2)}\left(x_{(k+1)n+k+1} - x_{(k+1)(n-1)+k+1}\right) = \dots \\ &= \prod_{s=2}^{k+2} \frac{a(n,s)}{b(n,s)}\left(x_{(k+1)n+1} - x_{(k+1)(n-1)+1}\right). \end{aligned} \tag{52}$$

Relations (51) and (52) imply that

$$x_{(k+1)n+1} \leq x_{(k+1)(n+1)+1}. \tag{53}$$

Then since (50), (51) and (53), by induction method we have that $x_{(k+1)n+1}$ is an incresing function. Then using (52) we can easily prove that $x_{(k+1)n+m}$, $m = 1, \dots, k+1$ are increasing functions. Simirarly if the second ralation of (47) holds we can prove that $x_{(k+1)n+m}$, $m = 1, \dots, k+1$ are decreasing functions.

(ii) From (i) we have that $x_{(k+1)n+m}$, $m = 1, \dots, k+1$ are increasing or decreasing functions. First we suppose that $x_{(k+1)n+m}$, $m = 1, \dots, k+1$ are increasing functions. Then (48) hold where $M_i \in (1, \infty]$, $i = 1, 2, \dots, k+1$. Suppose first that

$$M_i = \infty, \quad i = 1, \dots, k+1. \tag{54}$$

From (2) we get for $m = 1, \dots, k+1$

$$x_{(k+1)(n+1)+m} = 1 + x_{(k+1)n+m} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k+m-s}}. \tag{55}$$

Then from (54) and (55) for an $L > k$ there exists a n_0 such that for $m = 1, \dots, k+1$ and $n \geq n_0$

$$h(n,m) = x_{(k+1)n+m} \sum_{s=0}^{k-1} \frac{1}{x_{(k+1)n+k+m-s}} > L. \tag{56}$$

Then from (56) we have for $m = 1, 2, \dots, k+1$

$$k x_{(k+1)n+m} \max\left\{\frac{1}{x_{(k+1)n+k+m-s}}, \quad s = 0, \dots, k-1\right\} \geq h(n,m) > L$$

from which we get for $m = 1, 2, ..., k+1$

$$x_{(k+1)n+m} > T \min\left\{x_{(k+1)n+k+m-s},\ \ 0 \le s \le k-1\right\} \tag{57}$$

where

$$T = Lk^{-1} > 1. \tag{58}$$

We fix a $n \in \{0, 1, ...\}$. We suppose that

$$\min\left\{x_{(k+1)n+k+1-s},\ \ 0 \le s \le k-1\right\} = x_{(k+1)n+k+1-r} \tag{59}$$

where $r \in \{0, ..., k-1\}$. Moreover from (57) it follows that

$$x_{(k+1)n+k+1-r} > T \min\left\{x_{(k+1)n+2k+1-s-r}, 0 \le s \le k-1\right\}. \tag{60}$$

From (59) and since $x_{(k+1)n+m}$, $m = 1, ..., k+1$ are increasing functions we have

$$x_{(k+1)n+k+1-r} \le \begin{cases} x_{(k+1)n+k+1-s}, & 0 \le s \le r-1 \\ x_{(k+1)n+2k+2-s}, & r+1 \le s \le k-1. \end{cases} \tag{61}$$

Using (58), (60) and (61) it follows that

$$\min\left\{x_{(k+1)n+2k+1-s-r},\ \ 0 \le s \le k-1\right\} = x_{(k+1)n+k+2}. \tag{62}$$

Therefore from relations (60) and (62) we get

$$x_{(k+1)n+k+1-r} > Tx_{(k+1)n+k+2}. \tag{63}$$

Then relations (57) for $m = 1$, (59) and (63) imply that

$$x_{(k+1)n+1} > T^2 x_{(k+1)(n+1)+1}$$

which is a contradiction since (58) holds and $x_{(k+1)n+1}$ is an increasing function. Therefore (54) does not hold.

Suppose now that there exists a $m \in \{1, ..., k+1\}$ such that

$$M_m \ne \infty \quad \text{and} \quad M_i = \infty, \quad i \in \{1, ..., k+1\}, \quad i \ne m. \tag{64}$$

Then from (55) and (64) we get $M_m = 1$ which is a contradiction. So since (36) do not hold we have that (48) hold.

Suppose now $x_{(k+1)n+m}$, $m = 1, ..., k+1$ are decreasing functions. Consider that there exists a $m \in \{1, ..., k+1\}$ such that $M_m = 1$. Then from (55) we get

$$1 = 1 + \sum_{s=0}^{k-1} \frac{1}{\lim_{n\to\infty} x_{(k+1)(n+1)+m-1-s}}$$

from which we have that

$$\lim_{n\to\infty} x_{(k+1)n+m+s} = \infty, \quad s = 1, ..., k$$

which is contradiction. Therefore (48) are satisfied.

Using (55) for $m = 1$ and (48) we can easily prove that (49) is satisfied.

(iii) Using Proposition 1 and relations (48) and (49) the proof of (iii) follows immediately. This completes the proof of the proposition.

In the following proposition we prove that if $A > k$ the positive equilibrium c is globally asymptotically stable.

PROPOSITION 5. *Suppose that $A > k$. Then the positive equilibrium of (2) c is globally asymptotically stable.*

Proof First we prove that c is locally asymptotically stable. The linearized equation of (2) about the positive equilibrium $c = A + k$ is the following

$$y_{n+1} + \frac{1}{A+k} \sum_{s=n-k+1}^{n} y_s - \frac{k}{A+k} y_{n-k} = 0 \tag{65}$$

From Remark 1.3.1 [2,p.12] (65) is asymptotically stable if

$$\frac{2k}{A+k} < 1$$

which is true since $A > k$. Therefore $c = A + 1$ is locally asymptotically stable.

It remains to show that every positive solution of (2) tends to c as $n \to \infty$. First we prove that x_n is bounded above and below. From (2) it is obvious that

$$x_n \geq A, \quad n \geq 1. \tag{66}$$

Therefore from (2) and (66) we get

$$x_{n+1} \leq A + \frac{k}{A} x_{n-k}, \quad n \geq k. \tag{67}$$

We can prove that the solution of the equation

$$u_{n+1} = A + \frac{k}{A} u_{n-k}, \quad n \geq k \tag{68}$$

such that $u_s = x_s, \quad s = 0, 1, ..., k$ has the form

$$u_n = \left(\frac{k}{A}\right)^{\frac{n}{k+1}} \sum_{s=0}^{v} \left(c_s \cos \frac{2sn\pi}{k+1} + d_s \sin \frac{2sn\pi}{k+1}\right) + \frac{A^2}{A-k} \tag{69}$$

where v is defined in (12), c_s, d_s depends on $x_s, \quad s = 0, 1, ..., k$. Using (67), (68) and arguing in Proposition 2 we can prove that

$$x_n \leq u_n, \quad n = 0, 1, ... \tag{70}$$

Then from relations $A > k$, (69) and (70) it follows that x_n is a bounded function. Therefore we can set

$$\limsup_{n\to\infty} x_n = L, \quad \liminf_{n\to\infty} x_n = M. \tag{71}$$

From (2) and (71) we have

$$L \leq A + \frac{kL}{M}, \quad M \geq A + \frac{kM}{L} \tag{72}$$

and so

$$(A-k)(L-M) \leq 0. \tag{73}$$

Then since $A > k$, relation (73) imply that $L = M$ and so there exists the $\lim_{n\to\infty} x_n$. From (2) we obtain that $\lim_{n\to\infty} x_n = c$. This completes the proof of the proposition.

In the last proposition we prove that for $k = 2, 3$ if $A > 1$, then the positive equilibrium of (2) is locally asymptotically stable.

PROPOSITION 6. *Let $A > 1$. Then for $k = 2, 3$ the positive equilibrium $c = A + k$ of (2) is locally asymptotically stable.*

Proof Suppose first $k = 2$. Then equation (65) reduces to the equation

$$y_{n+1} + \alpha_1(y_n + y_{n-1}) + \alpha_2 y_{n-2} = 0, \quad \alpha_1 = \frac{1}{A+2}, \quad \alpha_2 = -\frac{2}{A+2}. \tag{74}$$

From Lemma 4 [4] (74) is asymptotically stable if and only if

$$|\alpha_1 + \alpha_2| < \alpha_1 + 1, \quad |\alpha_1 - 3\alpha_2| < 3 - \alpha_1, \quad \alpha_2^2 + a_1 - a_1 a_2 - 1 < 0. \tag{75}$$

We can easily prove that (75) hold if and only if $A > 1$ from which we take that the positive equilibrium c of (2) for $k = 2$ is locally asymptotically stable.

Finally suppose that $k = 3$. Then equation (65) reduces to the equation

$$(76) \quad y_{n+1}\alpha_1(y_n + y_{n-1} + y_{n-2}) + \alpha_2 y_{n-3} = 0, \ \alpha_1 = \frac{1}{A+3}, \ \alpha_2 = -\frac{3}{A+3}.$$

From Lemma 1 [3] (76) is asymptotically stable if and only if the following conditions are satisfied:

$$(77) \quad \begin{array}{l} (i)\ \lambda_1 \neq 0, \quad (ii)\ \dfrac{\lambda_2}{\lambda_1} > 0, \quad (iii)\ \dfrac{\lambda_3}{\lambda_1} > 0, \quad (iv)\ \dfrac{\lambda_5}{\lambda_1} > 0, \\ (v)\ \dfrac{\lambda_4}{\lambda_1}\left(\dfrac{\lambda_2\lambda_3}{\lambda_1^2} - \dfrac{\lambda_4}{\lambda_1}\right) > \dfrac{\lambda_2^2\lambda_5}{\lambda_1^3} \end{array}$$

where

$$\lambda_1 = 1, \quad \lambda_2 = \frac{4(A+6)}{A+3}, \quad \lambda_3 = \frac{6A-2}{A+3}, \quad \lambda_4 = \frac{4(A+6)}{A+3}, \quad \lambda_5 = \frac{A-1}{A+3}.$$

We can easily prove that (77) hold if and only if $A > 1$. Therefore the positive equilibrium c of (2) for $k = 3$ is is locally asymptotically stable. This completes the proof of the proposition.

REMARK 1. *It remains an open problem concerning the stability of the positive equilibrium of equation (2) if* $1 < A \leq k$.

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ROBUSTNESS OF AN ANALOG DYNAMIC MEMORY SYSTEM TO A CLASS OF INFORMATION TRANSMISSION CHANNELS PERTURBATIONS*

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Abstract. ADDAMS System Inc. (1990–1993) suggested a memory system using a simple feedback loop circuit for an analog signal. The circuit includes a time-delay element with no gain. If the perturbations and noises can be neglected, then this circuit is capable of permanently storing a signal, whose time duration is shorter than the time-delay function in this loop. In this paper, we shall discuss the stability of signal and the role of perturbations in the feedback system.

1. Introduction. In the early 1950s, a simple linear feedback loop circuit was suggested as a possible mechanism for the implementation of an analog memory system. Consider the system M shown in Figure 1, where $\mathcal{D}_h$ denotes an ideal linear delay with delay time $h > 0$.

The input $\xi(t)$ and output $x(t)$ for this system obey the relationship $x(t) = \xi(t-h)$, and the transfer function is e^{-ph} [13]. Here we assume that $\xi(t)$ is zero outside of the interval $-h \le t < 0$ and $x(t) \equiv 0$ for $t < 0$, ensuring $x(t) = \xi(t-h) + x(t-h)$. Then the identity $x(t+Nh) \equiv \xi(t)$, $-h \le t < 0$, holds for all positive integers N. Notably, the system M is capable, in principle, of permanently storing a signal $\xi(t)$ $(-h \le t < 0)$ whose time duration is shorter than the time-delay h in the system M.

* This project is related to the operations of *ADDAMS System Inc.*, Japan.

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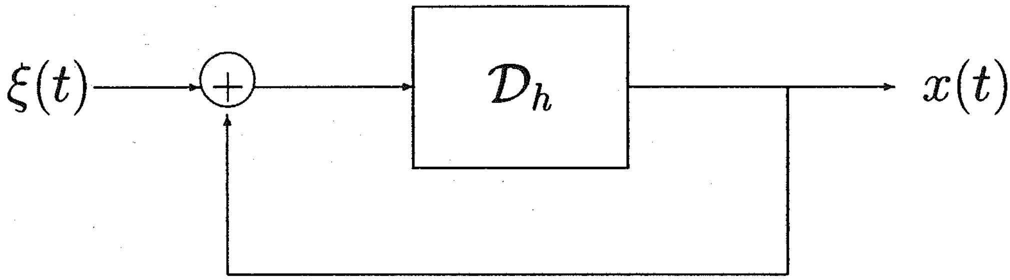

FIG. 1. *Simple feedback loop with time delay* $\mathcal{D}_h$

This idea has attracted industrial interest as a potential alternative memory mechanism to semiconductor materials and other electronic devices. The attraction of this alternative mechanism is that the hardware needed to build such a memory mechanism is well suited to commercial applications.

In 1988, *ADDAMS Inc.* performed an experiment with Analog Dynamic Memory *ADM*[1] following from this idea [3]. Optoelectronic elements such as light fibers and Charge Coupled Device (CCD) elements were chosen in this experiment to ensure low losses in signal transmission. This experiment was successful in demonstrating the implementation of such an *ADM* system, and was involved with several patents [1, 2]. However, gradual losses of memorized signals were observed as the time t increased, and the signal approached a flattened value for increased time. This signal loss has been one of the serious problems in the further development of the *ADM* system. Here some questions arise: although the memory system discussed here is simple in its structure, it is not straightforward to state that such a memory system is stable with respect to variations in device parameters and perturbations.

This motivates the analysis of the robustness of the *ADM* system considered in this work. We focus attention on the role of perturbation in the information transmission channels. This perturbation is intrinsically distributed along the channels, but using the method of lumping [6], p.6 we represent it as a finite dimensional linear system $\mathcal{W}_\varepsilon$ with a transfer function $W_\varepsilon(p)$ in the feedback path of the system $M_{\varepsilon,h}$ in Figure 2.

Concentration and separation of resistance, inductance and capacitance features of the information transmission channel (which is the essence of the lumping method) introduce some errors into the system description. However, it seems that this method is quite adequate in the analysis of

[1] *ADM* stands for "Analog Dynamic Memory", and it is the trademark of the *ADDAMS System Inc.*, Japan.

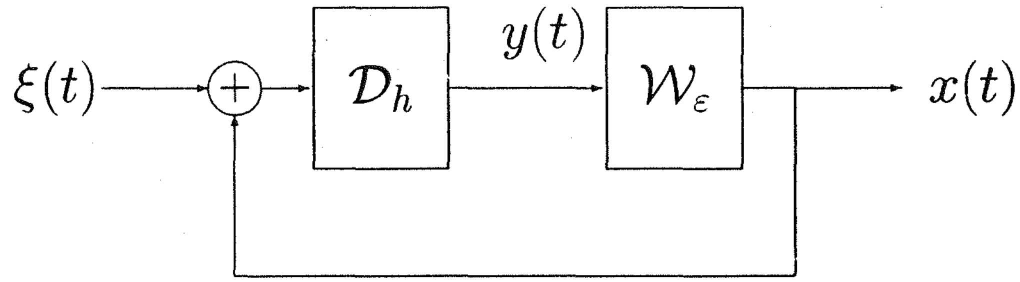

FIG. 2. *A feedback loop system considered in the paper*

the ADM system. Here we consider a case when the element $\mathcal{W}_\varepsilon$ is a general finite dimensional linear system, to be described by the relationships $\varepsilon z' = Az + fy(t)$, $x(t) = \langle g, z(t)\rangle$ where ε is a (small) positive parameter which characterizes the overall quality of the channel, A is a $d \times d$-matrix, $f, g \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. The vector variable z represents here the internal dynamics of the lumped inductance link $\mathcal{W}_\varepsilon$, whereas the scalar variable $x = \langle g, z\rangle$ is the observable output of this link. Below we suppose that A is a *Hurwitz* matrix, that is the eigenvalues of the matrix A lie in the open left hand side of the complex plane, and the conditions

$$\det(-A) = 1, \qquad \left\langle g, A^{-1}f\right\rangle = -1 \tag{1.1}$$

are satisfied. The first condition (1.1) can always be satisfied by an appropriate scaling, since $\det(-M) > 0$ for any Hurwitz matrix M; the second one means that the channel "does not spoil the constant signals" in the sense that for a constant input $y(t) \equiv y_* =$ const the function $x(t) \equiv y_*$ represents a possible output of the linear element $\mathcal{W}_\varepsilon$ (the corresponding internal dynamics $z(t)$ is actually given by $z(t) \equiv -A^{-1}f$). This second condition (1.1) holds in most situations, for instance it holds for the simplest inductance link with the transfer function $W(p) = (\varepsilon p + 1)^{-1}$. It means that the element $\mathcal{W}_\varepsilon$ works as a low pass filter, which is a typical feature of inductance elements.

With the choice of the lumped inductance element described above, the output $x(t)$ of the closed loop system is defined by the relationship

$$x(t) = \langle g, z(t)\rangle \tag{1.2}$$

where the restrictions of the function $z(t)$, $Nh \le t < (N+1)h$ are constructed inductively in N as the solutions of the vector differential equations

$$\varepsilon z' = Az + y(t-h)f, \qquad z(Nh) = z_N, \tag{1.3}$$

where $y(t)$ coincides with $\xi(t)$ for $N = 0$ and $y(t)$ coincide with $x(t-h)$ for $N > 0$; the initial conditions y_N coincides with an initial internal state y_0 of the inductance link for $N = 0$ and is defined by the continuity condition

$$z_N = z(Nh - 0) = \lim_{t \to Nh} z(t) \tag{1.4}$$

for the positive integer N. This description can be also reworded as follows. The function $z(t)$, $t \geq 0$, is the solution of the system of the delay differential equations

$$\varepsilon z' = Az + x(t-h)f, \qquad x(t) = \langle g, z(t) \rangle \tag{1.5}$$

subjected the initial conditions

$$x(t) = \xi(t), \quad -h \leq t < 0, \qquad z(0) = z_0. \tag{1.6}$$

Thus from the mathematical point of view we will be interested in the long term behaviour of the solution $z(t)$ of the system (1.2)–(1.4), or, what is the same, the system (1.5), (1.6), and especially in the behaviour of the function (1.2) which represents the output of our closed loop system. The main attention will be paid to the following two questions. Firstly, we will investigate the asymptotic behaviour of solutions as $t \to \infty$. Under some simple technical restrictions we establish that the solutions are "flattening": they approach some constant functions. We will calculate these limits and also evaluate the convergence rate. Results will be formulated as Theorem 1 and Proposition 2.1, in the next section and will be proved in Sections 3 and 5. Secondly, we will investigate the rate of deformations of the stored signal. There are basically two sources of the deformations: on the one hand, an unfortunate choice of the initial condition z_0 will spoil the stored signal from the very beginning. However it could be expected that these deformations should fade as time increases. On the other hand, for rather large t the output signal should be deformed gradually by, loosely speaking, the same forces which would eventually flatten the output. We will estimate the rate of deformations due to both causes and will clarify the nature of the corresponding deformations. The results will be formulated as Theorem 2, and will be proved in Section 4.

The results will be formulated in terms of the transfer function $W(p) = \left\langle g, (pI - A)^{-1} f \right\rangle$. One can say that even this transfer function is not known exactly in many situations. *However, the whole point here is that the qualitative character of the process does not depend on detailed information concerning the function $W(p)$; moreover, even the constants which govern the* *rates of deformation etc, will be given in terms of only the first few derivatives of the corresponding transfer function $W(p)$ at $p = 0$.*

To conclude the introduction, we note that the system with the block diagram represented by Figure 1 appears as a fragment of many other technical and biological objects (for instance, it is not dissimilar from some phenomenological models of short-time human memory).

2. Main results.

2.1. Notations. It will be convenient from now on to fix the matrix A and the vectors f, g, however, we consider ε and h as the parameters.

Denote by

$$W(p) = \left\langle g, (pI - A)^{-1} f \right\rangle. \tag{2.1}$$

This function is a rational function of the form $W(p) = \frac{M(p)}{L(p)}$ where $L(p) = \det(pI - A) = p^d + a_{d-1}p^{d-1} + \ldots + a_1 p + a_0$ and $M(p) = b_{d-1}p^{d-1} + \ldots + b_1 p + b_0$ are some polynomials. The second condition (1.1) can be rewritten as

$$W(0) = \frac{a_0}{b_0} = 1. \tag{2.2}$$

Thus by the first condition (1.1)

$$a_0 = b_0 = 1. \tag{2.3}$$

Let $\mathcal{L}_h$ be the set of all integrable scalar functions $\eta : [-h, 0] \to \mathbb{R}$.

For each initial function $\xi(t) \in \mathcal{L}_h$ define

$$E_{\varepsilon,h}(\xi, z_0) = \left(1 + \frac{\varepsilon}{h}(a_1 - b_1)\right)^{-1} \left(E_h(\xi) - \varepsilon \left\langle g, A^{-1} z_0 \right\rangle\right), \tag{2.4}$$

where $E_h(\xi)$ is the mean value of $\xi(t) \in \mathcal{L}_h$, $E_h(\xi) = \frac{1}{h}\int_{-h}^{0} \xi(s)\, ds$. The first multiplier in (2.4) is finite and positive for sufficiently small ε.

2.2. Rate of flattening of the output signal. Let us come back to the analysis of the initial problem (1.2)–(1.4) describing the dynamics of the system shown at Figure 2.

We will understand solutions of the equation (1.3) in the classical sense: a solution is an absolutely continuous function which satisfies the equation for almost all $t \geq 0$. Then the system (1.2)–(1.4) has the unique solution $z_{\varepsilon,h}(t; \xi, z_0)$ for any integrable ξ and any $z_0 \in \mathbb{R}^d$; this is continuously differentiable for $t > h$. Note that the restrictions of the function $z(t) = z_{\varepsilon,h}(t; \xi, z_0)$

to the intervals $[Nh, (N+1)h)$ can also be defined inductively in N by the explicit formulas

$$z(t) = e^{A(t-Nh)/\varepsilon} z_N + \frac{1}{\varepsilon} \int_{Nh}^{t} e^{A(t-s)/\varepsilon} f y(s)\, ds, \quad Nh \le t < (N+1)h,$$

where $y(t)$ coincides with $\xi(t)$ for $N = 0$ and $y(t)$ coincides with $\langle g, z(t-h)\rangle$ for $N > 0$, whereas z_N coincides with an initial internal state z_0 for $N = 0$ and is defined by the continuity condition (1.4) for the positive integer N. These formulas, however, will not be of much help below.

We will also use the following notation

$$f_* = -A^{-1} f, \qquad \alpha = (a_1^2 - b_1^2) - 2(a_2 - b_2) \tag{2.5}$$

and

$$\gamma_* = \frac{2\pi^2}{h^4}\alpha. \tag{2.6}$$

Note in passing that by (2.3) the constant α in (2.5) can be compactly expressed in terms of the transfer function (2.1):

$$\alpha = W''(0) - (W'(0))^2. \tag{2.7}$$

Recall, that we always assume that the eigenvalues of the matrix A lie in the open left hand side of the complex plane, and the conditions

$$\det(-A) = 1, \qquad \langle g, f_*\rangle = 1 \tag{2.8}$$

are satisfied.

THEOREM 1. *Let $\alpha > 0$ and for $-\infty < \omega < \infty$ and $\omega \neq 0$ let*

$$|W(i\omega)| < 1. \tag{2.9}$$

Then there exists a positive $\sigma(\gamma)$, $\gamma < \gamma_*$, such that the estimate

$$\lim_{t\to\infty} \left(e^{\gamma\varepsilon^2 t} |z_{\varepsilon,h}(t;\xi,z_0) - E_{\varepsilon,h}(\xi,z_0) f_*| \right) = 0 \tag{2.10}$$

holds for any

$$\gamma < \gamma_*, \qquad \varepsilon \in (0, \sigma(\gamma)h], \qquad \xi(\cdot) \in \mathcal{L}_h, \qquad z_0 \in \mathbb{R}^d. \tag{2.11}$$

Note that the condition (2.9) is a standard assumption in absolute stability theory, see [13] and that the limit relationship (2.10) is not dissimilar to some estimates in [7].

We will be particularly interested in the output signal

$$x_{\varepsilon,h}(t;\xi,z_0) = \langle g, z_{\varepsilon,h}(t;\xi,z_0)\rangle . \tag{2.12}$$

COROLLARY 2.1. *Under the conditions of the theorem above the limit*

$$\lim_{t\to\infty} x_{\varepsilon,h}(t;\xi,z_0) = E_{\varepsilon,h}(\xi,z_0)$$

holds for all sufficiently small positive ε *and*

$$\lim_{t\to\infty}\left(e^{\gamma\varepsilon^2 t}\left|x_{\varepsilon,h}(t;\xi,z_0) - E_{\varepsilon,h}(\xi,z_0)\right|\right) = 0. \tag{2.13}$$

The estimate (2.13) is the best possible in a natural sense:

PROPOSITION 2.1. *For each positive* γ *satisfying the estimate*

$$\gamma > \frac{2\pi^2}{h^4}\alpha \tag{2.14}$$

and each positive h, σ *there exists a positive* $\varepsilon < \sigma h$, *a function* $\xi_\varepsilon(t)$, *a vector* $z_\varepsilon \in \mathbb{R}^d$ *and a sequence* $t_n \to \infty$ *such that*

$$\lim_{n\to\infty}\left(e^{\gamma\varepsilon^2 t_n}\left|x_{\varepsilon,h}(t_n;\xi_\varepsilon,z_\varepsilon) - E_{\varepsilon,h}(\xi_\varepsilon,z_\varepsilon)\right|\right) = \infty. \tag{2.15}$$

These assertions yield the following informal observations:

- The eventual effect of small inductance is to cause the output signal $x_{\varepsilon,h}(t;\xi,z_0)$ to decay to the constant signal $E_{\varepsilon,h}(\xi,z_0)$, which for small ε almost (but not quite) coincides with $E_h(\xi)$.
- The rate of this decay is quite slow, essentially similar to that of the function $e^{-2\pi^2(\varepsilon/h^2)^2 t}$.
- Notably, the exponent $r(t) = -2\pi^2\left(\frac{\varepsilon}{h^2}\right)^2 t$ of e above depends quadratically on the ratio ε/h^2: two-fold decreasing of this ratio, by increasing the delay parameter h or decreasing the inductance parameter ε, causes an approximate four-fold decreasing of $|r(t)|$.

2.3. Rate of deforming of the stored signal. Recall some definitions concerning the linear control system

$$z' = Az + fy(t), \qquad x = \langle g, z\rangle , \tag{2.16}$$

see further details in [13].

A particular state z_0 is defined to be *controllable (on* $[0,1]$) if there exists a continuous control $y(t)$, $t \in [0,1]$ such that the solution $z(t)$ of the initial problem $z' = Az + fy(t)$, $z(0) = z_0$ satisfies $z(1) = 0$. The linear system (2.16) is *completely controllable* if given z_0 there exists a continuous control $u(t)$, $t \in [0,1]$ such that the solution $z(t)$ of the equation (2.16) satisfies $z(1) = 0$. A particular state z_0 is defined to be *unobservable* if $\langle g, z(t)\rangle \equiv 0$, $t \in [0,1]$, where $z(t)$ is the solution of the differential equation $z' = Az$, $z(0) = z_0$. The linear system (2.16) is *completely observable* if no state is unobservable.

The function

$$G(t) = \left\langle g, e^{At} f\right\rangle, \qquad t \geq 0, \tag{2.17}$$

is said to be the *impulse response* of the linear system (2.16). Its Laplace transform is $W(p)$. By the definition, $G(t)$ is a scalar function which under simple additional conditions [10] is known to be non-negative. Loosely speaking, $G(t)$ is the output of the linear system (2.16) corresponding to the input $x(t) = \delta(t)$ and the initial condition $z(0) = z_0$, where $\delta(t)$ is the Dirac impulse function. The importance of the function (2.17) is explained by the fact that the function $y(t) = \langle g, z(t)\rangle$, where $z(t)$ is the solution of the initial problem $z' = Az + fu(t)$, $z(0) = 0$, can be represented as $y(t) = \int_0^t G(t-s)u(s)\,ds$.

Let $q(r)$, $r \geq 0$ be a continuous increasing function satisfying $q(0) = 0$. Consider the set C_q of the functions $\eta(t)$, $-1 \leq t \leq 0$, satisfying

$$|\eta(t_1) - \eta(t_1)| \leq q(t_2 - t_1), \qquad -1 \leq t_1 < t_2 \leq 0.$$

That is, C_q is the class of functions which are equi-continuous with the modulus of continuity q. Denote by $C_{q,h}$ the totality of the functions $\xi(t) = \eta(t/h)$ with $\eta \in C_q$. We denote by $\tilde{\xi}$ the h-periodic extension of a function $\xi(t) \in C_{q,h}$.

THEOREM 2. *Suppose that the impulse response* (2.17) *is non-negative and the linear system* (2.16) *is completely controllable and completely observable. Then there exist positive constants $\sigma, c_1 > 0$ and, for each $\delta > 0$, there exists $c_0(\delta) > 0$ such that for $\varepsilon < \sigma h$, the estimate*

$$\left|z_{\varepsilon,h}(t + Nh; \xi, z_0) - \tilde{\xi}\left(t - \omega_{\varepsilon,h,N}\right) f_*\right| \leq \delta + c_0(\delta)\left(\frac{\varepsilon}{h}\right)^2 N + c_1 \frac{|z_0 - \xi(0) f_*|}{\sqrt{N}}$$

holds for $\xi \in C_{q,h}$, all positive integers N and all $t \in [-h, 0]$, where

$$\omega_{\varepsilon,h,N} = (a_1 - b_1)\frac{\varepsilon}{h}N, \qquad f_* = -A^{-1}f. \tag{2.18}$$

COROLLARY 2.2. *Let the assumptions of Theorem 2 hold and let*

$$x_{\varepsilon,h}(t; \xi, z_0) = \langle g, z_{\varepsilon,h}(t; \xi, z_0)\rangle$$

be an output of the system. Then there exist positive constants $\sigma, c_1 > 0$ *and, for each* $\delta > 0$, *there exists* $c_0(\delta) > 0$ *such that for* $\varepsilon < \sigma h$, *the estimate*

$$|x_{\varepsilon,h}(t + Nh + \omega_{\varepsilon,h,N}; \xi, z_0) - \xi(t)| \leq \delta + c_0(\delta)\left(\frac{\varepsilon}{h}\right)^2 N + c_1 \frac{|z_0 - \xi(0)f_*|}{\sqrt{N}}$$

holds for all positive integers N *and all* $-h \leq t \leq 0$.

This assertion concerns the rate of divergence of the signals

$$x_{\varepsilon,h}(t + Nh; \xi, z_0), \qquad -h \leq t \leq 0, \quad N = 1, 2, \ldots,$$

from the initial function $\xi(t)$, rather than the rate of the flattening as in the previous subsections. The main features of this rate are:

- The divergence $\mathrm{dis}_0(N) = \max_{t \in [-h,0]} |x_{\varepsilon,h}(t + Nh; \xi, \xi_0 f_*) - \xi(t)|$ is of magnitude $N\varepsilon$.
- Much of the divergence $\mathrm{dis}_0(N)$ is due to phase delays: if we allow a correction $\omega_{\varepsilon,h,N}$ of the phase of the signal, then the rate of divergence becomes that of an arithmetic progression with a difference of the order of $(\varepsilon/h)^2$.
- In an experimental context, the phase delay can be most easily examined by introducing a signal $\xi(t)$ with a marked peak at the beginning of the interval $[-h, 0)$. The phase shifts of the signal are then easily observed in the positive time interval $t > 0$.
- The divergence $\max_{t \in [-h,0]} |x_{\varepsilon,h}(t + Nh; \xi, \xi_0 f_*) - \xi(t)|$, which is due to the initial state discrepancy $|z_0 - \xi(0)f_*|$, fades with the rate, which is comparable with that of the function $N^{-1/2}$, and which is uniform for small values of ε/h.

3. Proof of Theorem 1.

3.1. Outline of the proof. First of all we note that the investigation of system $M_{\varepsilon,h}$ with the block diagram shown in Figure 2 can be reduced to the analysis of a similar system with $h = 1$, using the change of time $t = \tau h$. Thus it will be sufficient to consider the case $h = 1$. From now on we suppose that the equality $h = 1$ holds; we will also omit the bottom index h when appropriate: e.g. we use the notations $x_\varepsilon(t; \xi, z_0)$, $z_\varepsilon(t; \xi, z_0)$, $E_\varepsilon(\xi)$, $\mathcal{L}$ instead of $x_{\varepsilon,1}(t; \xi, z_0)$, $z_{\varepsilon,1}(t; \xi, z_0)$, $E_{\varepsilon,1}(\xi)$, $\mathcal{L}_1$.

Denote by $\mathcal{L}(d)$ the Banach space of integrable function $z : [-1, 0] \to \mathbb{R}^d$ endowed with the norm $\|z\|_{\mathcal{L}} = \int_{-1}^0 |z(s)|\, ds$. We will also use the product space $\widehat{\mathcal{L}} = \mathcal{L}(d) \times \mathbb{R}^d$ of the pairs $\widehat{z} = (z(\cdot), z_0)$ with the usual norm $\|\widehat{z}\|_{\widehat{\mathcal{L}}} = \|z(\cdot)\|_{\mathcal{L}} + |z_0|$. Introduce the auxiliary operator $S_\varepsilon : \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$ which is given by

$$S_\varepsilon(z(t), z_0) = (z_\varepsilon(t + 1; \xi, z_0), z_*), \qquad -1 \leq t \leq 0, \tag{3.1}$$

where $\xi(t) = \langle g, z(t) \rangle$, $z_* = z_\varepsilon(1; \xi, z_0)$.

Clearly, $z_\varepsilon(t + N; \xi, z_0) = z_N(t)$, $-1 \le t \le 0$, where $z_N(\cdot)$ is the first component of the pair $\widehat{z}_N = S_\varepsilon^N(z(\cdot), z_0)$, while $z(\cdot)$ is any function, satisfying almost everywhere the identity $\langle g, z(t) \rangle \equiv \xi(t)$, $-1 \le t \le 0$. We will also use the notation $E_\varepsilon(\widehat{z}) = E_{\varepsilon,1}(\langle g, z(\cdot) \rangle, z_0)$ or, what is the same by (2.8),

$$E_\varepsilon(\widehat{z}) = (1 + \varepsilon(a_1 - b_1))^{-1} \left(\int_{-1}^{0} \langle g, z(t) \rangle \, ds - \varepsilon \left\langle g, A^{-1} z_0 \right\rangle \right), \tag{3.2}$$

and

$$\widehat{z}_* = (z_*(\cdot), f_*) \quad \text{and} \quad z_*(t) \equiv f_* \tag{3.3}$$

(here, as always, f_* is defined by the first equation (2.5): $f_* = -A^{-1} f$).

It suffices to prove the following statement instead of Theorem 1:

LEMMA 3.1. *Let $\alpha > 0$ and (2.9) hold. Then there exists a positive $\sigma(\gamma)$, $\gamma < 2\pi^2\alpha$, such that the limit*

$$\lim_{N \to \infty} e^{\gamma \varepsilon^2 N} \left\| S_\varepsilon^N \widehat{z} - E_\varepsilon(\widehat{z}) \widehat{z}_* \right\|_{\widehat{\mathcal{L}}} = 0 \tag{3.4}$$

holds for all

$$\gamma < 2\pi^2\alpha, \qquad \varepsilon \in (0, \sigma(\gamma)], \qquad \widehat{z} \in \widehat{\mathcal{L}}. \tag{3.5}$$

The proof of this last assertion will be given in several steps. Firstly, in the next subsection we give a general description of the eigenvalues of the operator S_ε in Lemma 3.2; then we investigate in more detail the invariant subspaces of the operator S_ε and its adjoint corresponding to the eigenvalue 1. Secondly, in Subsection 3.3 we consider the asymptotic behaviour of the eigenvalues of the operator S_ε, which are different from 1, as $\varepsilon \to 0$. Finally, in Subsection 3.4 we complete the proof of Lemma 3.1.

3.2. Spectral analysis of the operator S_ε. We denote by S_ε^c the complexification of the operator S_ε.

LEMMA 3.2. *The non-zero eigenvalues λ of the operator S_ε^c belong to the set of numbers*

$$\lambda = e^w \tag{3.6}$$

where w is a root of the characteristic equation

$$L(\varepsilon w) = e^{-w} M(\varepsilon w). \tag{3.7}$$

This assertion can be extracted from the general theory [9], but it is convenient to give a straightforward elementary proof.

We commence with a simple auxiliary statement. Denote by B the square $d \times d$-matrix with the elements $b_{ij} = f_i g_j$.

LEMMA 3.3. *The characteristic polynomial of the matrix $A + \lambda B$ is given by $L - \lambda M$.*

Proof: By continuity reasons, it is sufficient to consider the case when the roots of the polynomial $L - \lambda M$ are not roots of the polynomial L:

$$|L(p)| + |L(p) + \lambda M(p)| > 0 \tag{3.8}$$

for all complex p and all roots of the polynomial $L - \lambda M$ are simple.

Let p be a root of $L - \lambda M$:

$$L(p) - \lambda M(p) = 0. \tag{3.9}$$

By (3.8)

$$L(p) \neq 0 \tag{3.10}$$

and (3.9) can be rewritten as

$$\frac{\lambda M(p)}{L(p)} = \lambda W(p) = 1. \tag{3.11}$$

By (3.8) $L(p) - \lambda M(p) = 0$; thus the matrix $A - pI$ is invertible and we can consider the non-zero vector

$$x = \lambda\,(pI - A)^{-1} f. \tag{3.12}$$

Then $\langle g, x\rangle = \lambda \left\langle g, (pI - A)^{-1} f\right\rangle = \lambda W(p)$ by (2.1), which implies

$$\langle g, x\rangle = 1 \tag{3.13}$$

by (3.11). The vector (3.12) satisfies by the definition the equation $px - Ax = \lambda f$ which can be rewritten as $Ax + \lambda f\,\langle g, x\rangle = px$ by (3.13) and, further, as $(A + \lambda B)x = px$. Thus, we have proved that each root of the polynomial $L - \lambda M$ is an eigenvalue of the matrix $A + \lambda B$. Since we have assumed that all roots of the polynomial $L - \lambda M$ are simple, we conclude that $L - \lambda M$ is the characteristic polynomial of $A + \lambda B$ and the lemma is proved. □

Let us proceed with the proof of Lemma 3.2. Let $\hat{z} = (z(\cdot), z_0)$ be an eigenvector of the complexification S_ε^c of the operator S_ε with a non-zero complex eigenvalue λ. Then by definition, the function $z(t)$, $-1 \leq t \leq 0$, satisfies the differential equation $\varepsilon z' = Az + \lambda^{-1}Bz = (A + \lambda^{-1}B)z$ with the initial condition $z(-1) = z_0$, and the equation $z(0) = \lambda z_0$. That is, $z_0 = \lambda^{-1}e^{(A+\lambda^{-1}B)/\varepsilon}z_0$. Thus λ is an eigenvalue of the matrix $e^{(A+\lambda^{-1}B)/\varepsilon}$ which in turn means that $\lambda = e^{v/\varepsilon}$ where v is an eigenvalue of $A + \lambda^{-1}B$. By Lemma 3.3, v satisfies the equation $L(v) = \lambda^{-1}M(v)$. After the substitution $w = v/\varepsilon$ the two last displayed formula come to the relationships (3.6), (3.7). So we have proved that every non-zero eigenvalue of the shift operator can be written in the form (3.6) where w is a root of the characteristic equation (3.7). The lemma is thus proved. □

The characteristic equation (3.7) has a zero root; therefore there is at least one eigenvector of the operator S_ε with the eigenvalue 1.

LEMMA 3.4. *The pair* $\widehat{z}_* = (z_*(\cdot), A^{-1}f)$ *with* $z_*(t) \equiv A^{-1}f$ *is an eigenvector of* S_ε *with the eigenvalue 1. Any other eigenvector of the operator* S_ε *corresponding to the eigenvalue* 1 *is collinear with* $\widehat{z}_*$.

Proof: Indeed, let $\widehat{z} = (z(t), z_0) \in \widehat{\mathcal{L}}$ be the corresponding eigenvector. Then $z(t)$ satisfies the equation $\varepsilon z' = (A+B)z$ for $-1 \le t \le 0$, and $z(0) = z(-1) = z_0$. Thus z_0 is an eigenvector of the matrix $e^{(A+B)/\varepsilon}$ with the eigenvalue 1 which means in turn that the vector z_0 is annihilated by the operator $A+B$: $Az_0 + f\langle g, z_0\rangle = 0$. This implies immediately that z_0 is collinear with $A^{-1}f$. The equality $z(t) \equiv z_0$ follows and we have proved that any eigenvector of the operator S_ε corresponding to the eigenvalue 1 is collinear with the pair $\widehat{z}_* = (z_*(\cdot), A^{-1}f)$ with $z_*(t) \equiv A^{-1}f$. The fact that this pair $\widehat{z}_*$ is an eigenvector is a trivium. □

Introduce the functional

$$\Phi_\varepsilon(\widehat{z}) = \int_{-1}^{0} \langle g, z(s)\rangle \, ds - \varepsilon \left\langle g, A^{-1}z_0\right\rangle \tag{3.14}$$

for $\widehat{z} \in \widehat{\mathcal{L}}$.

LEMMA 3.5. *The identity* $\Phi_\varepsilon(S_\varepsilon\widehat{z}) \equiv \Phi_\varepsilon(\widehat{z})$ *is valid.*

Proof: Let $\widehat{z} = (z(\cdot), z_0) \in \widehat{\mathcal{L}}$ and $S_\varepsilon\widehat{z} = (y(\cdot), y_0)$. By the definition $\varepsilon y' = Ay + \langle g, z(t)\rangle f$. Or, because A is invertible, $\varepsilon A^{-1}y' = y - \langle g, z(t)\rangle f_*$ where, as usual, $f_* = -A^{-1}f$. Integrating we obtain

$$\varepsilon A^{-1}(y(0) - y(-1)) = \int_{-1}^{0} y(s)\, ds - \left\langle g, \int_{-1}^{0} z(s)\, ds\right\rangle f_*.$$

Taking the scalar product with g we arrive at

$$\varepsilon \left\langle g, A^{-1}(y_0 - z_0)\right\rangle = \int_{-1}^{0} \langle g, y(s)\rangle \, ds - \langle g, f_*\rangle \left\langle g, \int_{-1}^{0} z(s)\, ds\right\rangle$$

since $y(-1) = z_0$, $y(0) = y_0$. Because of the equality $\langle g, f_* \rangle = 1$, see the second equality (2.8), this can be rewritten as

$$\varepsilon \left\langle g, A^{-1}(y_0 - z_0) \right\rangle = \int_{-1}^{0} \langle g, y(s) \rangle \, ds - \left\langle g, \int_{-1}^{0} z(s) \, ds \right\rangle$$

and, further,

$$\int_{-1}^{0} \langle g, y(s) \rangle \, ds - \varepsilon \left\langle g, A^{-1} y_0 \right\rangle = \int_{-1}^{0} \langle g, z(s) \rangle \, ds - \varepsilon \left\langle g, A^{-1} z_0 \right\rangle .$$

The last displayed equality can be rewritten as $\Phi_\varepsilon(\widehat{y}) = \Phi_\varepsilon(\widehat{z})$, by (3.14), and the lemma is proved. □

COROLLARY 3.1. *The hyper-plane E^s defined by $\Phi_\varepsilon(\widehat{z}) = 0$ is invariant for the operator S_ε. If ε is sufficiently small, then the restriction $S_\varepsilon|_{E^s}$ has no unit eigenvalues.*

Proof: By the previous lemma the hyperplane $\Phi_\varepsilon(\widehat{z}) = 0$ is invariant for the operator S_ε. On the other hand, any non-zero eigenvector of this operator with the eigenvalue 1 is collinear to $\widehat{z}$ by Lemma 3.4, and does not belong to this hyper-plane for all sufficiently small ε. The corollary is proved. □

3.3. Auxiliary estimates. Now we investigate carefully the roots of the equation (3.7) which are different from 0. This subsection and Subsection 4.2 below are in line with some simple fragments of the Lambert's W-function theory, that plays an important role in delay equations [15] and many other applications.

LEMMA 3.6. *Suppose that the assumptions of Theorem* 1 *are satisfied. Let $\gamma_1 > 0$ satisfy the inequality*

$$\gamma_1 < 2\pi^2 \alpha. \tag{3.15}$$

Then there exists an $\varepsilon(\gamma_1)$ such that the real part $\Re w$ of each non-zero solution w of the characteristic equation (3.7) *satisfies the inequality*

$$\Re w < -\gamma_1 \varepsilon^2 \tag{3.16}$$

for $\varepsilon < \varepsilon(\gamma_1)$.

Proof: We first show that any non-zero solution of the equation (3.7) belongs to the left hand side of the complex plane. Let us rewrite the equation (3.7) as

$$\frac{M(\varepsilon w)}{L(\varepsilon w)} = e^w. \tag{3.17}$$

The function $W(\varepsilon w) = \frac{M(\varepsilon w)}{L(\varepsilon w)}$ is analytic in the right hand side of the complex plane because the roots of the polynomial $L(p)$ lie in the open left hand side of the complex plane. Additionally, $|W(p)| \leq 1$ on the imaginary axes and $\lim_{|p|\to\infty} |W(p)| = 0$. Thus by the Maximum Modulus Principle, [12], Th. 14.4, p. 296, $\left|\frac{M(\varepsilon w)}{L(\varepsilon w)}\right| \leq 1$ in the right hand side of the complex plane. On the other hand, $|e^w| > 1$ in the right hand side. The last two inequalities show that the non-zero solutions of (3.17), and consequently (3.7), belong to the left hand side of the plane.

Now let us prove that for small ε, each non-zero solution w of (3.7) satisfies (3.16). Suppose this assertion is false, in which case there exists $\varepsilon_n \to 0$ with the corresponding solutions w_n satisfying

$$0 > \Re w_n > -\gamma_1 \varepsilon_n^2. \tag{3.18}$$

We now show that the imaginary parts of the solutions w_n of (3.17) are bounded, in particular,

$$|\Im w_n| < \gamma_2 < 2\pi \tag{3.19}$$

for large n, where γ_2 will be constructed below and $\Im w$ denotes the imaginary part of a complex number w.

Denote $u_n = w_n \varepsilon_n$. The estimate (3.18) takes the form

$$0 > \Re u_n > -\gamma_1 \varepsilon_n^3. \tag{3.20}$$

The numbers u_n satisfy the equation

$$\frac{L(u)}{M(u)} = e^{-u/\varepsilon_n}. \tag{3.21}$$

The real parts $\Re u_n$ converge to zero by (3.20) and by the estimate (2.9) we arrive at the relationship

$$\lim_{n\to\infty} u_n = 0. \tag{3.22}$$

Thus to prove (3.19) it suffices to establish the estimate

$$|\Im u_n| < \gamma_2 \varepsilon_n < 2\pi \varepsilon_n \tag{3.23}$$

for large n, *providing that* (3.22) *holds.*

By Taylor's expansion of the function $H(u) = L(u)/M(u)$ at 0

$$H(u) = H(0) + H'(0)u + \frac{1}{2}H''(0)u^2 + O(u^3).$$

By inspection,

$$H(0) = 1, \quad H'(0) = a_1 - b_1, \quad H''(0) = 2\left[(a_2 - b_2) - (a_1 - b_1)b_1\right]$$

and $\alpha = (H'(0))^2 - H''(0) > 0$ by assumption. Therefore, by (3.22), for small u,

$$|\Im H(u)| \geq |H'(0)|\ |\Im u| - O\left(|\Im u|^3 + |\Re u|\right)$$

and

$$|\Re H(u)| \geq 1 - \frac{1}{2} H''(0)\ |\Im u|^2 - O\left(|\Im u|^3 + |\Re u|\right).$$

So

$$|H(u)|^2 \geq 1 + \left(|H'(0)|^2 - H''(0)\right) |\Im u|^2 - O\left(|\Im u|^3 + |\Re u|\right)$$

and, finally,

$$|H(u)|^2 \geq 1 + \alpha\, |\Im u|^2 - O\left(|\Im u|^3 + |\Re u|\right) \tag{3.24}$$

by (2.7).

The relationships (3.24), (3.20) and (3.22) result in

$$\left|\frac{L(u_n)}{M(u_n)}\right| \geq 1 + \frac{1}{2}\alpha\, |\Im u_n|^2 - O\left(|\Im u_n|^3 + \varepsilon_n^3\right). \tag{3.25}$$

On the other hand,

$$\left|e^{-u_n/\varepsilon_n}\right| \leq 1 + \Re\left(\frac{u_n}{\varepsilon_n}\right) + O\left(\frac{\Re u_n}{\varepsilon_n}\right)^2$$

by (3.22). By the inequality (3.20) we can further write down

$$\left|e^{-u_n/\varepsilon_n}\right| \leq 1 + \gamma_1\varepsilon_n^2 + O\left(\varepsilon_n^3\right). \tag{3.26}$$

From (3.25), (3.26) and (3.21) we get $1+\frac{1}{2}\alpha\, |\Im u_n|^2 \leq 1+\gamma_1\varepsilon_n^2+O(|\Im u_n|^3+\varepsilon_n^3)$. Solving this inequality for $|\Im u_n|^2$ we get

$$|\Im u_n|^2 \leq \frac{2\gamma_1}{\alpha}\varepsilon_n^2 + O\left(|\Im u_n|^3 + \varepsilon_n^3\right),$$

which proves (3.23) for each γ_2 satisfying $\sqrt{2\gamma_1/\alpha} < \gamma_2 < 2\pi$ for sufficiently large n. The set of γ_2 satisfying the last inequality is not empty, because $\sqrt{2\gamma_1/\alpha} < 2\pi$ by (3.15). Thus (3.19) is proved.

We complete the proof as follows. Any limit point of the sequence w_n should be a solution of the limit equation $1 = e^{-w}$, that is should be of the form $2\pi ik$ with an integer k. Taking into account the estimate (3.19) for $\Im w_n$ above, we conclude that the sequence w_n converges to 0. Note that the solutions w_n should satisfy the equation

$$F(w, \varepsilon_n) = 0 \tag{3.27}$$

where $F(w, \varepsilon) = \frac{L(\varepsilon w)}{M(\varepsilon w)} - e^{-w}$. The function F is smooth and $F'_w(0, 0) = 1 \neq 0$. Therefore by the implicit function theorem ([4], pp. 482), the equation (3.7) has a unique *small* solution $w(\varepsilon)$ for all small ε and, in particular, a unique small solution for all ε_n with large n. On the other hand, $w = 0$ satisfies (3.27) for any ε, and therefore w_n should be zero for all ε_n with large n. This yields a contradiction with the left side of the inequality (3.18).

This proves that for small ε each non-zero solution w of the equation (3.7) satisfies (3.16). □

Lemmas 3.6 and 3.2 imply

COROLLARY 3.2. *Let $\gamma_1 > 0$ satisfy the inequality* (3.15). *Then there exists $\varepsilon(\gamma)$ such that the eigenvalues of the operator $S_\varepsilon|_{E^s}$ satisfy the estimate $|\lambda| < e^{-\gamma_1 \varepsilon^2}$, $0 < \varepsilon \leq \varepsilon(\gamma_1)$.*

3.4. Completion of the proof. We will fix a positive γ satisfying the first inequality (3.5), that is $\gamma < 2\pi^2\alpha$. Clearly, S_ε is a compact operator in $\widehat{\mathcal{L}}$. Therefore, its spectrum consists of its eigenvalues and of the point 0 (which also could be an eigenvalue). By Lemma 3.2 the set of eigenvalues coincides with the set $\mathcal{W}(\varepsilon)$ of roots of the characteristic equation. Thus, by Corollaries 3.2 and 3.1 the spectrum $\sigma(S_\varepsilon)$ of the operator S_ε consists of the two disjoint sets $\{1\}$ and $\{\lambda \in \sigma(S_\varepsilon) : |\lambda| < 1\}$. Moreover, by Lemma 3.4 and Corollary 3.1 the space $\widehat{\mathcal{L}}$ can be decomposed into a direct sum $\widehat{\mathcal{L}} = E^1 \oplus E^s$, where E^1 is the one-dimensional subspace of elements which are collinear with $\widehat{z}_* = (z_*(\cdot), f_*)$, whereas the hyper-plane E^s is defined by the equation $\Phi_\varepsilon(\widehat{z}) = 0$, and the functional Φ_ε is defined by (3.14). The projector onto E^1 along the hyper-plane E^s is then given by

$$\operatorname{Pr} \widehat{z} = \frac{\Phi_\varepsilon(\widehat{z})}{\Phi_\varepsilon(\widehat{z}_*)} \widehat{z}_*. \tag{3.28}$$

On the other hand, $\Phi_\varepsilon(\widehat{z}_*) = \Phi_\varepsilon(z_*(t), f_*)$ can be rewritten as

$$\Phi_\varepsilon(\widehat{z}_*) = \int_{-1}^{0} \langle g, f_* \rangle \, ds - \varepsilon \left\langle g, A^{-1} f_* \right\rangle = 1 + \varepsilon \left\langle g, A^{-2} f \right\rangle$$

where the first equation holds by (3.14) and the second one by the second equation (2.8). On the other hand, $\langle g, A^{-2}f\rangle = -W'(0)$ whereas $-W'(0) = a_1 - b_1$ by (2.2). By the last two equalities $\Phi_\varepsilon(\widehat{z}_*) = 1+\varepsilon(a_1-b_1)$. Thus, by (3.14)

$$\frac{\Phi_\varepsilon(\widehat{z})}{\Phi_\varepsilon(\widehat{z}_*)} = (1+\varepsilon(a_1-b_1))^{-1}\left(\int_{-1}^{0}\langle g, z(s)\rangle\,ds - \varepsilon\left\langle g, A^{-1}z_0\right\rangle\right).$$

That is, (3.28) can be rewritten as

$$\operatorname{Pr}\widehat{z} = E_\varepsilon(\widehat{z})\widehat{z}_* \tag{3.29}$$

by (3.2).

Clearly, $S_\varepsilon|_{E^s}$ is a compact operator. Therefore, its spectrum consists of its eigenvalues and of the point 0 (which could also be an eigenvalue). Thus by Corollary 3.2 there exists $\varepsilon_1(\gamma_1)$ such that

$$|\lambda| \le e^{-\gamma_1\varepsilon^2}, \qquad 0<\varepsilon\le\varepsilon_1(\gamma_1), \quad \lambda\in\sigma(S_\varepsilon)$$

with $\gamma_1 = (\gamma+\gamma_0)/2$. Therefore, there exists an equivalent norm $\|\cdot\|_\varepsilon$, $0<\varepsilon\le\varepsilon_1(\gamma_1)$, on E^s in which the operator $S_\varepsilon|_{E^s}$ *contracts* with a factor

$$q_\varepsilon < e^{-\gamma\varepsilon^2} < 1 \tag{3.30}$$

(see Proposition 9.6, [5], p. 83). The estimate

$$\left\|S_\varepsilon^N(\widehat{z}-\operatorname{Pr}\widehat{z})\right\|_\varepsilon \le q_\varepsilon^N\|\widehat{z}-\operatorname{Pr}\widehat{z}\|_\varepsilon, \qquad 0<\varepsilon\le\varepsilon_1(\gamma_1),$$

holds for each $\widehat{z} = (z(t), z_0) \in \widehat{\mathcal{L}}$. Taking into account (3.29), (3.30) and the linearity of the operator S_ε we arrive at

$$\lim_{N\to\infty} e^{\gamma\varepsilon^2 N}\left\|S_\varepsilon^N\widehat{z} - E_\varepsilon(\widehat{z})\widehat{z}_*\right\|_\varepsilon = 0, \qquad 0<\varepsilon\le\varepsilon_1(\gamma_1), \quad \widehat{z}\in\widehat{\mathcal{L}}.$$

This estimate implies the estimate (3.4) immediately, since the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\widehat{\mathcal{L}}}$ are equivalent. Lemma 3.1 is proved and so is the theorem. □

4. Proof of Theorem 2.

4.1. Monotonicity of the operator S_ε. We recall that complete controllability is equivalent to the requirement that the system of vectors

$$f, Af, \ldots, A^{d-1}f \tag{4.1}$$

is linearly independent, whereas complete observabillity is equivalent to the requirement that the system of vectors

$$g, A^T g, \ldots, (A^T)^{d-1} g, \tag{4.2}$$

where A^T denotes the adjoint matrix, is linearly independent. This assertion is the famous Kalman Criterion for complete controllability and complete observabillity, see further details in [13].

Recall that a closed convex set $\mathcal{K} \subset \mathbb{R}^d$ is a *cone* if together with any point u it contains the ray $B(u) = \{\lambda u : \lambda \geq 0\}$ and, on the other hand, $v, -v \in \mathcal{K}$ implies $v = 0$.

Denote

$$\mathcal{K} = \operatorname{conv}\left\{y \in \mathbb{R}^d : y = \lambda e^{At} f, 0 \leq \lambda, t < \infty\right\}. \tag{4.3}$$

LEMMA 4.1. *$\mathcal{K}$ is a cone* [10] *in $\mathbb{R}^d$ and $A^{-1}f$ belongs to the interior of $\mathcal{K}$. Additionally,*

$$e^{At}\mathcal{K} \subseteq \mathcal{K}, \qquad t \geq 0, \tag{4.4}$$

and

$$\langle g, z\rangle \geq 0, \qquad z \in \mathcal{K}. \tag{4.5}$$

Proof: Let us show that $\mathcal{K}$ is a cone in $\mathbb{R}^d$: that is we show (a) that $\mathcal{K}$ is a closed convex set $\mathcal{K} \subset \mathbb{R}^d$, (b) that together with any point u it contains the ray $B(u)$ and, (c) that on the other hand, $v, -v \in \mathcal{K}$ implies $v = 0$. The properties (a) and (b) follow immediately from (4.3). It remains to show that $v, -v \in \mathcal{K}$ implies $v = 0$. Let us denote by E the following subset of $\mathcal{K}$: $E = \{z \in \mathcal{K} : -z \in \mathcal{K}\}$. Clearly E is a linear subspace in $\mathbb{R}^d$. Further, by the definition E is invariant with respect to the operator e^{At}, $t \geq 0$. Since $A = \frac{d}{dt} e^{At}|_{t=0}$, the linear space E is also invariant for A. Therefore there exists a non-zero eigenvector $p \in E$ of the complexification A_* of the operator A; in particular , $A_* p = \lambda p, \ldots A_*^{d-1} p = \lambda^{d-1} p$ for some complex number λ. On the other hand, $\langle g, p\rangle = 0$ follows from (4.5) which has been established at the beginning of the proof. Thus $\langle g, A^j p\rangle = 0, \quad j = 0, 1, \ldots, d-1$. This can be rewritten as $\left\langle \left(A^T\right)^j g, p\right\rangle = 0, \quad j = 0, 1, \ldots, d-1$. This contradicts the hypothesis that the linear system is completely observable.

Let us establish that the cone $\mathcal{K}$ has some interior points. Let us suppose the contrary. Then there exists a non-zero vector q which satisfies the equation

$$\left\langle q, e^{At} f\right\rangle = 0 \tag{4.6}$$

for all $t \geq 0$. The equation (4.6) implies $\left\langle q, A^k e^{At} f\right\rangle = 0,\ k = 0,1,2,\ldots$. Therefore $\left\langle q, A^k f\right\rangle = 0$, $k = 0,1,2,\ldots$. That is, the vectors $f, Af, \ldots, A^{d-1}f$ are linearly dependent and, by the Kalman Criterion the system (2.16) is not completely controllable.

Let us show that the vector $f_* = -A^{-1}f$ belongs to the interior of $\mathcal{K}$. To this end it suffices to mention the formula $A^{-1}f = -\int_0^\infty e^{At} f\, dt$ and to make use of the linear independence of the vectors (4.2) which has been mentioned at the beginning of this section. The inclusion (4.4) follows from the definition of $\mathcal{K}$ and the inequality (4.5) follows from the assumption that the impulse response is non-negative. The lemma is proved. □

Denote by $\mathcal{C}$ the Banach space of continuous functions $z : [-1,0] \to \mathbb{R}^d$ with the usual norm and designate by $\widehat{\mathcal{C}}$ the corresponding product space $\widehat{\mathcal{C}} = \mathcal{C} \times \mathbb{R}^d$. Denote by $\widehat{\mathcal{K}}$ the cone in $\mathcal{L}$ defined by

$$\widehat{\mathcal{K}} = \{(z(\cdot), z_0) : z(s) \in \mathcal{K},\ -1 \leq s \leq 0,\ z_0 \in \mathcal{K}\}.$$

LEMMA 4.2. *The operator S_ε is monotone with respect to the cone $\widehat{\mathcal{K}}$ in the sense that the inclusion*

$$S_\varepsilon(\widehat{\mathcal{K}}) \subseteq \widehat{\mathcal{K}} \tag{4.7}$$

holds.

Proof: Let $\widehat{z} = (z(\cdot), z_0) \in \widehat{\mathcal{K}}$. Denote $S_\varepsilon \widehat{z}$ by $\widehat{y} = (y(\cdot), y_0)$. We should prove the inclusion $(y(\cdot), y_0) \in \widehat{\mathcal{K}}$. By the definition $y(0) = y_0$, thus it suffices to establish the inclusion

$$y(s) \in \mathcal{K} \tag{4.8}$$

for any $s \in [-1, 0]$.

By the definition $y(t) = e^{A(t+1)/\varepsilon} z_0 + \frac{1}{\varepsilon}\int_{-1}^t e^{A(t+1-s)/\varepsilon} f\,\langle g, z(s)\rangle\, ds$. The inclusion $e^{A(t+1)/\varepsilon} z_0 \in \mathcal{K},\ -1 \leq t \leq 0$, follows from (4.4), whereas the inclusion $\frac{1}{\varepsilon}\int_{-1}^t e^{A(t+1-s)/\varepsilon} f\,\langle g, z(s)\rangle\, ds \in \mathcal{K}, \quad -1 \leq t \leq 0$, follows from (4.4) and (4.5). The last three displayed formulas imply (4.8) and the lemma is proved. □

Denote by $\widehat{\mathcal{C}}$ the set of pairs $\widehat{z} = (z(\cdot), z_0) \in \mathcal{L}$ where the first coordinate is a continuous function. We endow this space with the norm $\|\widehat{z}\| = |z_0| + \max_{t\in[-1,0]} |z(t)|$. Let $\widehat{B}_\mathcal{C}$ be the corresponding unit ball: $\widehat{B}_\mathcal{C} = \left\{\widehat{z} \in \widehat{\mathcal{C}} : \|\widehat{z}\|_{\widehat{\mathcal{C}}} \leq 1\right\}$ and let $\widehat{z}_*$ be defined as usual by (3.3). The number $\Delta = \min\left\{\lambda \geq 0 : \widehat{B}_\mathcal{C} \subseteq \left(\lambda\widehat{z}_* - \widehat{\mathcal{K}}\right) \cap \left(-\lambda\widehat{z}_* + \widehat{\mathcal{K}}\right)\right\}$ is finite by Lemma 4.1. On the other hand, $S(\widehat{z}_*) = \widehat{z}_*$ by Lemma 3.4.

Thus, Lemma 4.2 implies the following assertion.

COROLLARY 4.1. *The inequality* $\left\|S_\varepsilon^N(\widehat{z})\right\|_{\widehat{C}} \leq \Delta\|\widehat{z}\|_{\widehat{C}}$ *holds for all positive integers* N.

4.2. Estimates. LEMMA 4.3. *Let* k *be a positive integer. Then for small* $\varepsilon > 0$ *there exist eigenvalues* $\lambda_{\varepsilon,k} = e^{w_{\varepsilon,k}}$ *of the operator* S_ε^c *with the approximations*

$$|w_{\varepsilon,k} - 2\pi ik\,(1 - (a_1 - b_1)\varepsilon)| \leq O(\varepsilon^2) \tag{4.9}$$

and with corresponding eigenvectors $\widehat{z}_{\varepsilon,k} = (z_{\varepsilon,k}(t), f_{\varepsilon,k})$ *where*

$$f_{\varepsilon,k} = \left(\varepsilon w_{\varepsilon,k} I - A\right)^{-1} f, \tag{4.10}$$

and $z_{\varepsilon,k}(t) = e^{w_{\varepsilon,k}t} f_{\varepsilon,k}$, $-1 \leq t < 0$.

Proof: To show that there exists a function $w_{\varepsilon,k}$ with the required properties we will investigate the solutions $w_{\varepsilon,k}$ of equation (3.7) which are close to $w_* = 2\pi ik$. By definition these solutions satisfy the equation $F(w,\varepsilon) = 0$ with $F(w,\varepsilon) = \frac{L(\varepsilon w)}{M(\varepsilon w)} - e^{-w}$. Since the relationships

$$F(w_*,0) = 0, \qquad F_w'(w_*,0) = 1 \neq 0 \tag{4.11}$$

hold, the implicit function theorem is applicable. That is, for small ε there exists a unique solution $w_{\varepsilon,k} = w_k(\varepsilon)$ which is close to w_*. The relationships $w_k'(0) = -\frac{F_\varepsilon'(w_*,0)}{F_w'(w_*,0)}$, and

$$F_w'(w_*,0)w_k''(0) = -F_{ww}''(w_*,0)w_k'(0)^2 - 2F_{w\varepsilon}''(w_*,0)w_k'(0) - F_{\varepsilon\varepsilon}''(w_*,0)$$

are valid ([4], pp. 483). Evidently, $F_\varepsilon'(w_*,0) = (a_1 - b_1)w_*$, that is, $w_k'(0) = -(a_1 - b_1)w_* = -2\pi ik(a_1 - b_1)$. Thus we have proved that the characteristic equation (3.7) has roots $w_{\varepsilon,k} = w_k(\varepsilon)$ with the approximate representation $|w_{\varepsilon,k} - 2\pi ik\,(1 - (a_1 - b_1)\varepsilon)| \leq O(\varepsilon^2)$. Formula (4.10) is correct for small ε by (2.9). It is a straightforward calculation to verify that the numbers $\lambda_{\varepsilon,k} = e^{w_{\varepsilon,k}}$ are the eigenvalues of S_ε^c with the corresponding eigenfunctions, and the assertion of the lemma follows. □

Now we give a 'decomplexified' version of the previous lemma. Denote by $f_{\varepsilon,k}^{re}$, $f_{\varepsilon,k}^{im}$ the real and imaginary parts of the vector (4.10): $f_{\varepsilon,k}^{re} = \Re f_{\varepsilon,k}$, $\quad f_{\varepsilon,k}^{im} = \Im f_{\varepsilon,k}$. Note immediately the limits

$$\lim_{\varepsilon\to 0} f_{\varepsilon,k}^{re} = f_*, \qquad \lim_{\varepsilon\to 0} f_{\varepsilon,k}^{im} = 0, \tag{4.12}$$

which follow from (4.10). Denote further

$$z_{\varepsilon,k}^{re}(t) = e^{\Re w_{\varepsilon,k}t}\left(f_{\varepsilon,k}^{re}\cos\left(\Im w_{\varepsilon,k}t\right) - f_{\varepsilon,k}^{im}\sin\left(\Im w_{\varepsilon,k}t\right)\right), \tag{4.13}$$

$$z^{im}_{\varepsilon,k}(t) = e^{\Re w_{\varepsilon,k}t}\left(f^{re}_{\varepsilon,k}\sin(\Im w_{\varepsilon,k}t) + f^{im}_{\varepsilon,k}\cos(\Im w_{\varepsilon,k}t)\right) \tag{4.14}$$

where $w_{\varepsilon,k}$ are the same as in Lemma 4.3. Denote now $\xi^{re}_{\varepsilon,k}(t) = \left\langle g, z^{re}_{\varepsilon,k}(t)\right\rangle$, and $\xi^{im}_{\varepsilon,k}(t) = \left\langle g, z^{im}_{\varepsilon,k}(t)\right\rangle$.

COROLLARY 4.2. *Let k be a positive integer and $w_k(\varepsilon)$ be as in the lemma above. Then the estimates*

$$\left|z^{re}_{\varepsilon}\left(t+N, \xi^{re}_{\varepsilon,k}, f^{re}_{\varepsilon,k}\right) - z^{re}_{\varepsilon,k}\left(t - N(a_1-b_1)\varepsilon\right)\right| < c_k\varepsilon^2 N, \tag{4.15}$$

$$\left|z^{im}_{\varepsilon}\left(t+N, \xi^{im}_{\varepsilon,k}, f^{im}_{\varepsilon,k}\right) - z^{im}_{\varepsilon,k}\left(t - N(a_1-b_1)\varepsilon\right)\right| < c_k\varepsilon^2 N \tag{4.16}$$

hold where the c_k are constants independent from ε and N.

Proof: Let $\widehat{z}^{re}_{\varepsilon,k} = \left(z^{re}_{\varepsilon,k}(t), z^{re}_{\varepsilon,k}(0)\right)$, $\widehat{z}^{im}_{\varepsilon,k} = \left(z^{im}_{\varepsilon,k}(t), z^{im}_{\varepsilon,k}(0)\right)$ and let $z^{re}_{\varepsilon,k,N}, z^{im}_{\varepsilon,k,N}$ be the first components of the pairs $S^N_\varepsilon \widehat{z}^{re}_{\varepsilon,k}, S^N_\varepsilon \widehat{z}^{im}_{\varepsilon,k}$. Then the equalities

$$z^{re}_{\varepsilon,k,N}(t) = e^{\Re w_{\varepsilon,k}(N+t)}\left(f^{re}_{\varepsilon,k}\cos(\Im w_{\varepsilon,k}(N+t)) - f^{im}_{\varepsilon,k}\sin(\Im w_{\varepsilon,k}(N+t))\right),$$

$$z^{im}_{\varepsilon,k,N}(t) = e^{\Re w_{\varepsilon,k}(N+t)}\left(f^{re}_{\varepsilon,k}\sin(\Im w_{\varepsilon,k}(N+t)) + f^{im}_{\varepsilon,k}\cos(\Im w_{\varepsilon,k}(N+t))\right)$$

hold as the projections of the corresponding complex equalities $\left(S^c_\varepsilon\right)^N \widehat{z}_{\varepsilon,k} = e^{w_{\varepsilon,k}N}\widehat{z}_{\varepsilon,k}$ which reflect the fact that the pairs $\widehat{z}_{\varepsilon,k}$ are the eigenvectors of S^c_ε with the eigenvalues $e^{w_{\varepsilon,k}}$, see the lemma above. The last two displayed equalities and the estimates (4.9) imply the inequalities

$$\left|z^{re}_{\varepsilon,k,N}(t) - z^{re}_{\varepsilon,k}\left(t - N(a_1-b_1)\varepsilon\right)\right| < c_k\varepsilon^2 N,$$

$$\left|z^{im}_{\varepsilon}(t) - z^{im}_{\varepsilon,k}\left(t - N(a_1-b_1)\varepsilon\right)\right| < c_k\varepsilon^2 N.$$

On the other hand, by the definition

$$z^{re}_{\varepsilon,k,N}(t) = z^{re}_{\varepsilon}\left(t+N, \xi^{re}_{\varepsilon,k}, f^{re}_{\varepsilon,k}\right), \qquad z^{im}_{\varepsilon,k,N}(t) = z^{im}_{\varepsilon}\left(t+N, \xi^{im}_{\varepsilon,k}, f^{im}_{\varepsilon,k}\right)$$

and the corollary is thus proved. □

4.3. Constant $c_0(\delta)$. LEMMA 4.4. *For each $\delta > 0$ there exists a constant $c_0(\delta)$ such that for sufficiently small ε, the estimate*

$$(4.17)\quad \left|z_\varepsilon\left(t+N;\xi,\xi(0)f_*\right)-\tilde{\xi}\left(t-N(a_1-b_1)\varepsilon\right)f_*\right|\le\frac{\delta}{2}+c_0(\delta)\varepsilon^2N$$

holds for $-1 \le t \le 0$ for all positive integers N.

Proof: Let the function $\xi(t)$ and the number $\delta > 0$ be given.

The periodic extension $\tilde{\xi}(t)$, $-\infty < t < \infty$ of the function ξ can be uniformly approximated (see, for instance, [11], p. 230, Example 2) with an arbitrary precision by a trigonometric polynomial of the form:

$$(4.18)\quad \eta(t)=\alpha_0+\sum_{k=1}^{K(\delta)}\alpha_k\cos\left(2\pi kt\right)+\beta_k\sin\left(2\pi kt\right).$$

Below we will fix a function (4.18) which satisfies the estimates

$$(4.19)\quad |f_*|\cdot|\eta(t)-\tilde{\xi}(t)|<\frac{\delta}{4},\qquad -1\le t\le 0,$$

$$(4.20)\quad |f_*|\cdot|\eta(t)-\tilde{\xi}(t)|<\frac{\delta_1}{4},\qquad -1\le t\le 0,$$

where $\delta_1 = \frac{\delta}{4\Delta}$, and Δ is the same as in Corollary 4.1. Then, for small $\varepsilon > 0$, we have $\|\hat{z}_0 - \hat{z}_\varepsilon\|_{\hat{C}} < \frac{\delta_1}{4}$ where $\hat{z}_0 = (\xi(t)f_*, \xi(0)f_*)$, $\hat{z}_\varepsilon = (z_\varepsilon(t), z_\varepsilon(0))$ with the choice

$$(4.21)\quad z_\varepsilon(t)=\alpha_0+\sum_{k=1}^{K(\delta)}\alpha_k z^{re}_{\varepsilon,k}(t)+\beta_k z^{im}_{\varepsilon,k}(t),$$

see (4.13), (4.14). By (4.12) $\lim_{\varepsilon\to 0}|z_\varepsilon(t)-\eta(t)f_*| = 0$, and the inequalities (4.19), (4.20) imply

$$(4.22)\quad |z_\varepsilon(t)-\xi(t)f_*|<\frac{\delta}{4},\qquad -1\le t\le 0,$$

$$(4.23)\quad |z_\varepsilon(t)-\xi(t)f_*|<\frac{\delta_1}{4},\qquad -1\le t\le 0,$$

for sufficiently small $\varepsilon > 0$. Therefore, the estimate (4.20) implies

$$(4.24)\quad \left\|S^N_\varepsilon\hat{z}_0-S^N_\varepsilon\hat{z}_\varepsilon\right\|_C\le\frac{\delta}{4},$$

for all positive integers N, by the equality $\delta_1 = \frac{\delta}{4\Delta}$, and Corollary 4.1.

By the definition of the operator S_ε, the first component of the pair $S_\varepsilon^N \widehat{z}_0$ coincides with the function $z_\varepsilon(t+N;\xi,\xi(0)f_*)$, $-1 \le t \le 0$. Therefore (4.24) implies $|z_{\varepsilon,N}(t) - z_\varepsilon(t+N;\xi,\xi(0)f_*)| < \frac{\delta}{4}$, $-1 \le t \le 0$, where $z_{\varepsilon,N}(t)$ denotes the first component of the pair $S_\varepsilon^N \widehat{z}_\varepsilon$. Using again the definition of the operator S_ε we can rewrite this as

$$|z_\varepsilon(t+N,\xi_\varepsilon,f_\varepsilon) - z_\varepsilon(t+N;\xi,\xi(0)f_*)| < \frac{\delta}{4}, \qquad -1 \le t \le 0,$$

where $\xi_\varepsilon(t) = \langle g, z_\varepsilon(t)\rangle$, $f_\varepsilon = z_\varepsilon(0)$.

On the other hand, by Corollary 4.2, the estimate

$$|z_\varepsilon(t+N,\xi_\varepsilon,f_\varepsilon) - z_\varepsilon(t - N(a_1-b_1)\varepsilon)| < c_1(\delta)\varepsilon^2 N, \qquad -1 \le t \le 0,$$

with a suitable $c_1(\delta)$ holds as a linear combination of the estimates (4.15) and (4.15), see (4.21). (Actually, the constant $c_1(\delta)$ above depends on K, but the number K was defined in turn by δ). Additionally,

$$\left|\tilde{\xi}(t - N(a_1-b_1)\varepsilon)f_* - z_\varepsilon(t - N(a_1-b_1)\varepsilon)\right| < \frac{\delta}{4} + c_2(\delta)\varepsilon^2 N, \quad -1 \le t \le 0,$$

by (4.22), (4.21) and the estimates (4.9).

The last three inequalities result in the estimate

$$\left|z_\varepsilon(t+N;\xi,\xi(0)f_*) - \tilde{\xi}(t - N(a_1-b_1)\varepsilon)f_*\right| < \frac{\delta}{2} + c_0(\delta)\varepsilon^2 N, \quad -1 \le t \le 0,$$

with $c_0(\delta) = c_1(\delta) + c_2(\delta)$ for all positive integers N. The last estimate coincides with (4.17) and the lemma is proved. □

4.4. Constant c_1. LEMMA 4.5. *Under the conditions of Theorem 2, for each $\delta > 0$ there exists a constant $c_1 > 0$ such that the estimate*

$$|z_\varepsilon(t+N;0,z_0)| \le \frac{\delta}{2} + \frac{c_1|z_0|}{\sqrt{N}}, \qquad -1 \le t \le 0, \tag{4.25}$$

holds for sufficiently small ε for all positive integers N.

Proof: Firstly we consider the functions

$$x_{\varepsilon,N,}(t;z_0) = \langle g, z_\varepsilon(t+N;0,z_0)\rangle, \qquad -1 \le t \le 0, \quad N = 1,2,\ldots.$$

LEMMA 4.6. *For each $\delta_2 > 0$ there exists a constant $c_2 > 0$ such that for sufficiently small ε, the estimate $|x_{\varepsilon,N}(t;z_0)| \le \delta_2 + c_2|z_0|/\sqrt{N}$, $-1 \le t \le 0$, holds for any given positive integer N.*

Proof: Denote $G_0(t) = \left\langle g, e^{At} f \right\rangle$, $t \geq 0$, and

$$G_n(t) = \int_0^t G_{n-1}(s) G_0(t-s)\, ds, \qquad n = 1, 2, \ldots.$$

The third moment $\mu_3 = \int_0^\infty t^3 G_0(t)\, dt$ exists and the corresponding characteristic function $\varphi(\zeta) = \int_0^\infty e^{i\zeta t} G_0(t)\, dt$ belongs to L_2 by the Plancherel identity. Thus Theorem 1, p. 533, [8] is applicable and the functions $\sqrt{n} G_n((t - nM)\sigma\sqrt{n})$ with $M = \int_0^\infty t G_0(t)\, dt$, $\sigma^2 = \int_0^\infty (t-M)^2 G_0(t)\, dt$, approach uniformly the normal density $\mathcal{N}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$.

In particular,

$$\eta_N(t) := \int_0^{1/\varepsilon} \xi_1(s) G_{N-1}(t-s)\, ds \leq \frac{c}{\sqrt{N}} \int_0^{1/\varepsilon} |\xi_1(s)|\, ds \tag{4.26}$$

with a suitable c, where

$$\xi_1(t) = \langle g, z_\varepsilon(t\varepsilon; 0, z_0) \rangle, \qquad t \in [0, 1/\varepsilon]. \tag{4.27}$$

Taking into account the explicit formula $\xi_1(t) = e^{At} z_0$, $t \in [0, 1/\varepsilon]$, we observe $\int_0^{1/\varepsilon} |\xi_1(s)|\, ds \leq |A^{-1}| \cdot |z_0|$ and (4.26) becomes

$$\eta_N(t) \leq \frac{c\,|A^{-1}| \cdot |z_0|}{\sqrt{N}}. \tag{4.28}$$

Below, c_2 is a fixed constant satisfying $c_2 > c\,|A^{-1}|$. Clearly, for each N the function $\xi_{\varepsilon,N}(t, z_0) = \langle g, z_\varepsilon(t\varepsilon + (N-1)/\varepsilon; 0, z_0) \rangle$, $t \in [0, 1/\varepsilon]$, is close to the convolution (4.26):

$$|\xi_{\varepsilon,N}(t, z_0) - \eta_N(t)| < \delta_2 \tag{4.29}$$

for any given δ_2, N for all sufficiently small $\varepsilon > 0$. Thus the function $\xi_{\varepsilon,N}(t, z_0)$ satisfies the estimate $|\xi_{\varepsilon,N}(t, z_0)| \leq \delta_2 + c_2 |z_0| / \sqrt{N}$, $t \in [0, 1/\varepsilon]$, for any given δ_2, N for all sufficiently small $\varepsilon > 0$. This means, in turn, that the function

$$x_{\varepsilon,N}(t, z_0) = \langle g, z_\varepsilon(t + N; 0, z_0) \rangle = \xi_{\varepsilon,N}(-t/\varepsilon), \qquad -1 \leq t \leq 0,$$

satisfies the analogous estimate

$$|x_{\varepsilon,N}(t, z_0)| \leq \delta_2 + c_2 \frac{|z_0|}{\sqrt{N}}, \qquad -1 \leq t \leq 0, \tag{4.30}$$

again for any given δ_2, N for all sufficiently small $\varepsilon > 0$.

Lemma 4.6 is proved. □

Denote $z_{\varepsilon,N}(z_0) = z_\varepsilon(N; 0, z_0)$, $N = 1, 2, \ldots$.

COROLLARY 4.3. *There exists a constant* $c_3 > 0$ *such that for each* $\delta_3 > 0$ *the estimate* $|z_{\varepsilon,N}(z_0)| \le \delta_3 + c_3|z_0|/\sqrt{N}$, $-1 \le t \le 0$, *holds for sufficiently small* ε *for any given positive integer* N.

Proof: Indeed, by the definition

$$z_{\varepsilon,N}(z_0) = e^{At/\varepsilon} z_{N-1} + \frac{1}{\varepsilon}\int_{-1}^{0} G_0(-s/\varepsilon)x_{\varepsilon,N-1}(s, z_0)\, ds.$$

The influence of the first addend is negligible for sufficiently small positive ε and the result follows from Lemma 4.6 above. □

Denote $z_{\varepsilon,N}(t; z_0) = z_\varepsilon(t + N; 0, z_0)$, $-1 \le t \le 0$, $N = 1, 2, \ldots$.

COROLLARY 4.4. *There exists a constant* $c_4 > 0$ *such that for each* $\delta_4 > 0$ *the estimate*

$$(4.31) \qquad |z_{\varepsilon,N}(t; z_0)| \le \frac{\delta_4}{2} + c_4\frac{|z_0|}{\sqrt{N}}, \qquad -1 \le t \le 0,$$

holds for sufficiently small ε *for any given positive integer* N.

Proof: By the definition,

$$z_{\varepsilon,N}(t, z_0) = e^{At/\varepsilon} z_{N-1} + \int_{-1}^{0} G_0((t-s)/\varepsilon)x_{\varepsilon,N-1}(t, z_0)\, ds.$$

Now the assertion follows from Lemma 4.6 and Corollary 4.3. □

Let us complete the proof of Lemma 4.5. We define $c_1 = c_4$, where c_4 is the same as in the previous corollary. Denote by N_* a positive integer satisfying $\delta/(3\Delta) + c_1|z_0|/\sqrt{N_*} < \delta/2\Delta$, where Δ is the same as in the previous subsection. By Corollary 4.4 the estimate

$$(4.32) \qquad |z_{\varepsilon,N}(t; z_0)| \le \frac{\delta}{3\Delta} + c_1\frac{|z_0|}{\sqrt{N}}, \qquad -1 \le t \le 0,$$

holds for small ε for $N \le N_*$. In particular, $|z_{\varepsilon,N_*}(t; z_0)| < \delta/2\Delta$, $-1 \le t \le 0$, holds for small ε. Therefore, by Corollary 4.1

$$(4.33) \qquad |z_{\varepsilon,N}(t; z_0)| < \frac{\delta}{2}, \qquad -1 \le t \le 0, \quad N > N_*.$$

On the other hand, by (4.32)

$$(4.34) \qquad |z_{\varepsilon,N_*}(t; z_0)| < \frac{\delta}{2} + c_1\frac{|z_0|}{\sqrt{N}}, \qquad -1 \le t \le 0, \quad N \le N_*,$$

holds for sufficiently small ε. Combining the last two estimates we arrive at

$$|z_\varepsilon(t + N; 0, z_0)| \le \frac{\delta}{2} + c_1\frac{|z_0|}{\sqrt{N}}, \qquad -1 \le t \le 0, \quad N = 1, 2, \ldots,$$

which coincides with (4.25), and the lemma is proved. □

4.5. Completion of the proof. The function $z_\varepsilon(t;\xi,z_0)$ is by the definition linear with respect to the last two arguments. In particular,

$$z_\varepsilon\left(t+N;\xi,z_0\right)=z_\varepsilon\left(t+N;\xi,\xi(0)f_*\right)+z_\varepsilon\left(t+N;0,z_0-\xi(0)f_*\right).$$

Therefore, the assertion of Theorem 2 follows from lemmas 4.4 and 4.5. The theorem is proved. □

5. Proof of Proposition 2.1. It suffices to consider the case $h=1$. Let $w_{\varepsilon,1}$ and $z_{\varepsilon,1}$ be the same as in Lemma 4.3, that is $e^{w_{\varepsilon,1}}$ is the eigenvalue of the operator S_ε^c with the approximation

$$|w_{\varepsilon,1}-2\pi i(1-(a_1-b_1)\varepsilon)|\le O(\varepsilon^2)$$

and with the corresponding eigenvector $\widehat{z}_{\varepsilon,1}=(z_{\varepsilon,1}(t),f_{\varepsilon,1})$ where $f_{\varepsilon,1}=\left(\varepsilon w_{\varepsilon,1}I-A\right)^{-1}f$, and $z_{\varepsilon,1}(t)=z_{\varepsilon,1}e^{w_{\varepsilon,1}t},\quad -1\le t<0$. We will investigate the function $w_{\varepsilon,1}$ more carefully. Denote $w_*=2\pi i$. By definition the equation $F(w_{\varepsilon,1},\varepsilon)=0$ holds. Since $F'_w(2\pi i,0)=1$, the implicit function theorem is applicable, as was mentioned earlier in Subsection 4.2. In particular for small $\varepsilon>0$ the function $w(\varepsilon)=w_{\varepsilon,1}$ satisfies the relations

$$w'(0)=-\frac{F'_\varepsilon(2\pi i,0)}{F'_w(w_*,0)},$$

and

$$F'_w(w_*,0)w''(0)=-F''_{ww}(w_*,0)w'(0)^2-2F''_{w\varepsilon}(w_*,0)w'(0)-F''_{\varepsilon\varepsilon}(w_*,0)$$

([4], pp. 483). That is, $w'(0)=-w_*(a_1-b_1)$ and

$$\begin{aligned}w''(0) &= -2(a_2-b_2)\varepsilon^2+2(a_1-b_1)b_1\varepsilon^2+1\\ &\quad+\left(2(a_1-b_1)-4(a_1-b_1)b_1w_*^2\right)(a_1-b_1)w_*\\ &\quad+2(a_2-b_2)w_*^2-2(a_1-b_1)b_1w_*^2.\end{aligned}$$

The leading two terms of the Taylor expansion for the function $w_{\varepsilon,1}=w(\varepsilon)$ gives the approximation of $\Re w_{\varepsilon,1}$

$$\begin{aligned}\Re w_{\varepsilon,1} &\approx \left[-(a_2-b_2)\varepsilon^2+(a_1-b_1)b_1\varepsilon^2+1/2\right](a_1-b_1)^2w_*^2\varepsilon^2\\ &\quad-\left(a_2-b_2)w_*^2-(a_1-b_1)b_1w_*^2\right)\varepsilon^2.\end{aligned}$$

and hence,

$$\lim_{\varepsilon\to0}\Re w_{\varepsilon,1}/\varepsilon^2=\left[(a_1-b_1)^2/2-(a_2-b_2)+(a_1-b_1)b_1\right]w_*^2=-2\pi^2\alpha,$$

Now we complete the proof of the proposition. Let $\sigma > 0$ and γ satisfy the estimate $\gamma > 2\pi^2\alpha$. Then, by the last equation, we can choose $\varepsilon \in (0, \sigma)$ satisfying $\gamma + \Re w_{\varepsilon,1} > 0$. It remains to define

$$\xi_\varepsilon(t) = \left\langle g, \Re z_{\varepsilon,1} e^{\Re w_{\varepsilon,1} t} \right\rangle, \qquad -1 \le t < 0,$$

and $z_\varepsilon = \Re f_{\varepsilon,1}$. The proposition is proved. □

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ON PROBLEM OF ROBOTIC HAND CONTROL UNDER CONSTRAINTS OF PROGRAM TYPE *

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Abstract.
The control circuit for single link manipulator limited with constraints of the program type is considered. Program constraints come in a natural way to robot motion analysis as demands — programs that have to be performed during manipulation. Their mathematical representations are differential equations, not always linear and of the first order. Nonintegrable constraints called nonholonomic ones are considered and program motion equations for a single link manipulator have been obtained. Program motion that has been defined is desirable from the practical point of view but it can be unstable or does not ensure desirable speed of robot hand responds. Sufficient conditions for the almost surely asymptotic stability and exponential 2–stability (with a priori given speed) of a control system under various random perturbations have been obtained.

Key Words. Stability of stochastic-functional differential equations, robotic hand control, program constraints

AMS(MOS) subject classification. 34K50, 60H30, 93C85, 93E15.

1. Program constraints formulation. Program constraints come from designing, technological, operating or other kinematical or dynamical requirements put upon a manipulator system motion. For example we need a velocity or acceleration of a certain link point i.e. its gripper to fulfill additional

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conditions like being a given time function. Such conditions, necessary during operating, generate constraints which mathematical representations are differential equations, not always linear and of the first order [3]. Nonholonomic constraints are often considered but they are linear the most often and of the first order [2]. Nonholonomic constraints influence the modification of dynamical analysis of a system because the number of degrees of freedom changes.

When additional constraints are put upon a system we can say about program motion. This program has to be realized during a system motion and when it can be formulated in the form of differential equations that the most often happens we have to consider both program and other constraints, coming from dependent coordinates for instance. To get our considerations clear we assume that the program and program motion are formulated in independent coordinates that means we do not have to deal with position constraints coming from the dependent coordinates selection. Program formulation and calculations of program coordinates, i.e. coordinates that fulfill program equations, do not provide an information about the stability of the program motion.

The paper deals with mathematical modelling and stability conditions of systems limited with nonholonomic constraints of the program type. Solutions of program motion equations give a system dynamical characteristics need to fulfill the prescribed program but they do not give a receipt for control conditions. One has to select them and the question here is what control force or torque u should be applied to a system. In the paper values of control parameters are found. The program constraint equation example shown herein is simple enough to show the influence of program constraints upon a system stability.

Independent coordinates form the smallest set of coordinates necessary to describe a system. They do not generate additional constraint equations so the only constraints we have to consider are program ones. Mathematical formulation of program constraint equations depends on their order; it means on the order of the highest coordinate derivative in it. We can require a system link or a point on it to move according to a certain trajectory that is written as:

$$g_a(t, q_1, q_2, ..., q_k) = 0, \tag{1}$$

where q_i, $i=1, ..., k$ are independent coordinates and the number a of these equations is less than k, and $k - a$ is the number of degrees of freedom. Equations (1) are geometrical program constraints and their mathematical formulation is the same as for other geometrical constraints but the interpretation is different. They indicate an extra requirement that has to be fulfilled during a system motion. The same way we formulate kinematical program constraints that are the first order ones:

$$f_b(t, q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k) = 0,$$

where the number b of constraint equations is less than k. Again, the mathematical formulation of the first order program constraints is the same as kinematical ones known from classical analytical mechanics but their meaning is different. We add additional requirement put upon a system. Generally, program constraints of the higher orders can be formulated in the following way:

$$G_\beta = (t, q_i, \dot{q}_i, ..., q_k^{(p)}) = 0, \qquad (2)$$

where p is the order of the constraints, β is the number of constraint equations and $\beta<k$. Constraint equations can be nonlinear but differentiating them with respect to time we can always get equations linear to the highest order of one of variables. One has to pay for it and to handle equations of higher orders then the given ones. The concept of equations (2) is helpful in the general formulation of kinematics and dynamics of constraint systems and it is used here to be free of nonlinear constraint equations [3].

2. The control problem formulation. We consider one–link manipulator made of an ideally rigid homogeneous rectilinear rod of length l and mass M (see [1]). One end of the rod is connected with the fixed base via an ideal cylindric hinge O, while at the other end a weight (to be moved) of mass m is made fast. The control torque u is applied to the axis of hinge O.

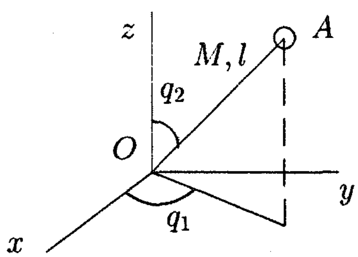

Figure 1.

If we would like to control the angle q ($q = q_1$ or q_2) with the aid of standard proportional differential (PD) regulators the control rule has the form $u(t) = -bq(t) - c\dot{q}(t)$, where $b, c > 0$. But in the process of its realizations several nonlinearities, aftereffects and random noises may appear due to the nonlinearities and hysteresis phenomenon of the executive mechanisms and the inaccurate locating of the system's coordinate. The last one demands to account all the states of the system from the start of the job up to the moment t. Moreover we should take into consideration the random nature of the measurement. Taking into account all the above we consider the following correction of proportional–differential regulators control law (see [10, 11, 12])

$$u = -bq(t) - c\dot{q}(t) + g[t, q_t^0, \dot{q}_t^0] + \sigma[t, q_t^0, \dot{q}_t^0]\dot{w}_t, \tag{3}$$

where g, σ are some nonlinear functionals such that for each $t \geq 0$ $g(t, \bullet, \bullet)$ and $\sigma(t, \bullet, \bullet)$ are mappings from $C([0,t], R) \times C([0,t], R)$ to R, $\dot{w}_t$ is a white noise, $q_t^0 = \{q(s),\ s \in [0,t]\}$ and $\dot{q}_t^0 = \{\dot{q}(s),\ s \in [0,t]\}$.

For some robot hand tasks when high speed of its response is not demanded, it is sufficient to find control parameter values which guarantee asymptotic stability of a system model only. But for some services like high speed pressing or in robot–based punching manufacturing, one needs to construct control u that supplies its high accuracy, response speed and insensitivity to the wide range influence of disturbances.

The paper is organized as follows: in section 3 we find the values of control parameters $b, c > 0$ and the restrictions on the noises g and σ that ensure the almost surely asymptotic stability of correspondent control system. In section 4 we find the values of control parameters $b, c > 0$ and restrictions on the noises g and σ (which are a little bit stronger then in section 3) that ensure desirable speed of exponential 2–stability. More precisely, for given numbers $T, \varepsilon > 0$ the mathematical expectations of $q^2(t)$ and $\dot{q}^2(t)$ are less then ε for all $t \geq T$.

For the investigation of the problems mentioned above we use Lyapunov–Krasovskii functional method (see [1, 5]), integral and differential inequalities, some approaches from the theory of random process, especially criterion of convergence of nonnegative semimartingales (see [6]). The similar approach was used in [10], [11] and [12] for the investigation of the plane motion of robot hand without any constraints.

3. Program motion formulation. The system we consider is constrained with program type constraints. We put task requirements upon the gripper. In the example under consideration, we demand that the velocity components of the end effector – the gripper A have to be connected by the give time function $f(t)$. This requirement is simple enough to illustrate the problem that has been stated but other requirements can be put upon a system in the same way.

Independent coordinates chosen here are shown in Figure 1: q_1 and q_2 are the two angles describing motion of the link. The equation of program constraint formulated for the considered case has the following form:

$$\dot{q}_1 = \dot{q}_2 f(t). \tag{4}$$

We get the nonholonomic constraint equation of the first order.

Program motion equations derivation method is based on so called generalized program motion equations that has been developed for nonholonomic systems [3, 4]. For the one link manipulator and constraint equation (4) program motion equations are:

$$\begin{aligned} &\left(\frac{1}{3}M + m\right) l^2 \left[f(t)\ddot{q}_1 + \ddot{q}_2\right] + \left(m + \frac{M}{2}\right) gl \sin q_2 = u \\ &\dot{q}_1 = f(t)\,\dot{q}_2. \end{aligned} \tag{5}$$

To simplify next considerations we eliminate dependent coordinate q_1 from equations (5) and get:

$$M_1 l^2 \left[1 + f^2(t)\right] \ddot{q}_2 + M_1 l^2 f(t)\dot{f}(t)\dot{q}_2 + M_2 gl \sin q_2 = u,$$

where:

$$M_1 = \left(\frac{1}{3}M + m\right), \quad M_2 = \left(m + \frac{M}{2}\right).$$

Let:

$$F(t) = \frac{1}{M_1 l^2 \left[1 + f^2(t)\right]}, \quad F_1(t) = \frac{f(t)\dot{f}(t)}{1 + f^2(t)}, \quad F_2(t) = \frac{M_2 g}{M_1 l \left[1 + f^2(t)\right]} \tag{6}$$

and then

$$\ddot{q}_2 + F_1(t)\dot{q}_2 + F_2(t)\sin q_2 = F(t)u. \tag{7}$$

Equations (7) can be written for $q_2 = y$ as the following system:

$$\begin{aligned} &\dot{y} = z \\ &\dot{z} + F_1(t)z + F_2(t)\sin y = F(t)u. \end{aligned} \tag{8}$$

We consider u in the form

$$u = -by - cz - g(t, y_t^0, z_t^0) - \sigma(t, y_t^0, z_t^0)\dot{w}_t. \tag{9}$$

Then (8) takes the following shape

$$(10)\qquad \begin{aligned}\dot y &= z\\ \dot z &= -(F_1+cF)z - bFy - F_2\sin y - Fg(t,y_t^0,z_t^0) - F\sigma(t,y_t^0,z_t^0)\dot w_t.\end{aligned}$$

We assume that there exist constants $c_i > 0$, $i=1,...,7$, such that for $t \in [0,\infty)$

I) $c_1 < F(t) < c_2$, $F'(t) < c_3$;

II) $c_4 < F_1(t) < c_5$, $F_1'(t) < c_6$;

III) $F_2(t) \le c_7$.

From conditions I)–III) it is seen that $F(t) < c_2$ and $c_2 = \frac{3}{(M+3m)l^2}$. If $F(t) > c_1$ it means that f^2 is bounded. $F_1(t) > c_4$ indicates that $f(t)f'(t) > 0$. All the above conditions can be satisfied for some $c_i > 0$ if $k_1 < f < k_2$, $k_3 < f' < k_4$, $k_5 < f'' < k_6$ for some $k_i > 0$. As it was stated, we looked for stability conditions for a system limited with constraints (4) and motion according to program do not have to be stable, in general. Program that secures stable motion has to be selected. From the practical point of view it is possible to assume we have chosen $f(t)$ function the way that it fulfills the above conditions.

Now we put restrictions on noises:

IV) $\left|g(t,y_t^0,z_t^0)\right|^2 \le \int\limits_0^\infty \left(|y(t-s)|^2 + |z(t-s)|^2\right) dR(s),$

$\left|\sigma(t,y_t^0,z_t^0)\right|^2 \le \int\limits_0^\infty \left(|y(t-s)|^2 + |z(t-s)|^2\right) dR(s),$

where $R(s)$ is a nondecreasing function, $R(\infty)-R(0) = \mathbf{R}$, we mean $y(s) = z(s) = 0$ for $s < 0$.

As Lyapunov–Krasovskii functional we consider the following expression

$$(11)\qquad \begin{aligned}W(u,v,t) &= 2F(t)bu^2 + v^2 + [v + (F(t)c + F_1(t))u]^2\\ &\quad + H\int\limits_0^\infty dR(s)\int\limits_{t-s}^t \left(|y(\tau)|^2 + |z(\tau)|^2\right) d\tau,\end{aligned}$$

where y and z are the solutions of (10) and constant H will be chosen later.

It is not difficult to see that system (10) has a unique solution (for every initial conditions y_0, z_0), if functions g and σ satisfy Lipschitz or Osgood type conditions with respect to the second and third variable (see, for example, [8, 9]).

4. A.s. asymptotic stability conditions for the control system. First of all we estimate the stochastic differential $dW(y(t),z(t),t)$ along the trajectories of system (10) using Ito formula (see [6]) and conditions I)–IV):

$$\begin{aligned}
dW(y(t),z(t),t) \le & \Big\{ z^2\big[2(Fc+F_1) - 4(F_1+cF) + \mathbf{R}H\big] \\
& + y^2\big[-2bF(Fc+F_1) + 2F'b \\
& + 2(Fc+F_1)(F'c+F_1') + \mathbf{R}H + 2F_2(Fc+F_1)\big] \\
& + yz\big[4Fb + 2(Fc+F_1)^2 - 4bF + 4F_2 - 2(Fc+F_1)^2 \\
& + 2(F'c+F_1')\big] + [4z + 2(Fc+F_1)y][-Fg(...) - F\sigma(...)] \\
& - H\int_0^\infty \big[y^2(t-s)+z^2(t-s)\big]dR(s) + 2F^2\sigma^2 \Big\} dt + dm_t. \qquad (12)
\end{aligned}$$

Here

$$m_t = \int_0^t \big[4z(s) + 2\big(F(s)c + F_1(s)\big)y(s)\big]F(s)\sigma(...)dw_s \qquad (13)$$

is a martingale (see [6]). We estimate

$$\begin{aligned}
&\big[4z + 2\big(Fc+F_1\big)y\big]\big[-Fg(...) - F\sigma(...)\big] \\
&\qquad \le 16F^2z^2 + 4(Fc+F_1)^2F^2y^2 + 2F^2\int_0^\infty [y^2(t-s) + z^2(t-s)]dR(s);
\end{aligned}$$

$$2F^2\sigma^2 \le 2F^2\int_0^\infty [y^2(t-s) + z^2(t-s)]dR(s);$$

$$yz\big(4F_2 + 2(F'c+F_1')\big) \le 2F_2^2y^2 + 2z^2 + y^2(F'c+F_1)^2.$$

Therefore (12) can be rewritten in the form

$$\begin{aligned}
dW_t \le & \Big\{ - z^2[2(Fc+F_1) - \mathbf{R}H - 1] \\
& - y^2\big[2bF^2c + 2bF^2c + 2bFF_1 - 2F'b \\
& - 2(Fc+F_1)(F'c+F_1') - 2F_2(Fc+F_1) - 4(Fc+F_1)^2 \\
& - (F'c+F_1)^2 - 2F_2^2 - \mathbf{R}H\big] \\
& + (4F^2 - H)\int_0^\infty [y^2(t-s) + z^2(t-s)]dR(s) \Big\} dt + dm_t. \qquad (14)
\end{aligned}$$

Due to I) $4F^2 < 4c_2^2$ and we can take

$$H = 4c_2^2 + 1.$$

Due to I)–II) we have

$$2(Fc + F_1) - \mathbf{R}H - 1 > 2(c_1c + c_4) - \mathbf{R}H - 1 = P_1(c). \tag{15}$$

We put

$$b = P_2(c), \tag{16}$$

where $P_2(c)$ is some polynomial of the second order with positive coefficients. Then the coefficient of y^2 in (14) can be estimated from the below by polynomial $P_3(c)$ of the third order with the positive coefficient of c^3. Inequality (14) takes the form

$$dW_t \leq \left\{-P_1(c)z^2 - P_3(c)y^2\right\}dt + dm_t, \tag{17}$$

where $P_1(c)$, $P_3(c)$ are positive for sufficiently large c. Applying the lemma of the convergence of nonnegative semimartingale (see [6, 7, 10, 11, 12]) to (17) and acting in the same way as in [7], [10], [11] or [12] we obtain

$$P\left\{\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0\right\} = 1, \tag{18}$$

where $P\{\bullet\}$ is the probability measure.

Thus we have proved the theorem.

THEOREM 1. *Let conditions I)–IV) be fulfilled and control parameters b and c are large enough, then solution of system (10) is a.s. asymptotically stable.*

COROLLARY 1. *For every constraints satisfying conditions I)–III) corresponding program motion with control realized by proportional–differential regulators is asymptotically stable for every noises in the control system, satisfying conditions IV) and sufficiently large control parameters values b and c.*

5. Exponential 2–stability for the control system. In this section we show that control parameters b and c can be chosen so large that solution of system (10) is exponential 2–stable with sufficiently large speed. In this case we reinforce a little the restrictions IV) on the disturbances. This will give us ability for given numbers T, ε make $E[y^2(t) + z^2(t)]$ less then ε for $t \geq T$. It means that robot hand belongs to ε–neighborhood of the equilibrium in time T. It ensures arbitrary desirable speed of robot hand response if ε and T are small enough.

Below we give definition of exponential 2–stability from [1].

DEFINITION 1. *Trivial solution of system (10) is called to be exponential 2–stable if for some positive constants A_1, A_2 the following equality is true*

$$E\Big(|y(t)|^2 + |z(t)|^2\Big) \le A_1\Big(E|y(0)|^2 + E|z(0)|^2\Big)\exp(-A_2 t), \quad t \ge 0.$$

At first we suppose there are no any noises in control system: $g(...) = \sigma(...) = 0$. Let

$$\Psi(u,v,t) = 2F(t)bu^2 + v^2 + \Big[v + (F(t)c + F_1(t))u\Big]^2.$$

Using I)–III) we can estimate

$$\begin{aligned} \Psi(u,v,t) &\le \Big[2c_2 b + 2[c_2 c + c_5)^2\Big]u^2 + 3v^2 \\ &\le \Big[2c_2 P_2(c) + 2(c_2 c + c_5)^2\Big]u^2 + 3v^2 \\ &= \overline{P_2}(c)u^2 + 3v^2 \le \frac{1}{\overline{P_1}(c)}\Big[P_1(c)v^2 + P_3(c)u^2\Big]. \end{aligned} \tag{19}$$

Here $\overline{P_2}(c)$, $\overline{P_1}(c)$ are some polynomials of the second and first order respectively with positive leading coefficients; $P_1(c)$, $P_2(c)$ are defined in (15) and (16). It is not difficult to see that (19) is true for sufficiently large c.

Carrying the calculations similar to that of the previous section (see (12) and (14)) and taking into account $f = \sigma = 0$, (17) and (19), we get that differential of $\Psi(y(t), z(t), t)$ along the trajectories of system (10) satisfies the following inequalities

$$d\Psi_t \le -\overline{P_1}(c)\Psi_t dt, \qquad d\Big[e^{\overline{P_1}(c)}\Psi_t\Big] \le 0. \tag{20}$$

From (20) we have

$$\begin{gathered} e^{\overline{P_1}(c)t}\Psi_t \le \Psi_0 \le \overline{P_2}(c)y^2(0) + 3z^2(0), \\ y^2(t) + z^2(t) \le \Psi_t \le e^{-\overline{P_1}(c)t}\Psi_0 \le e^{-\overline{P_1}(c)t}(P_2(c) + 3)(y^2(0) + z^2(0)), \end{gathered}$$

that is we get exponential 2–stability of solution of system (10) for $A_1 = \overline{P_2}(c) + 3$, $A_2 = \overline{P_1}(c)$ according to the definition 1.

We suppose that the noise exists and the following conditions are fulfilled

V) $\Big|g(t, y_t^0, z_t^0)\Big|^2 \le \alpha(t)\int\limits_0^\infty \Big(|y(t-s)|^2 + |z(t-s)|^2\Big)\, dR(s),$

$$\left|\sigma(t, y_t^0, z_t^0)\right|^2 \leq \alpha(t) \int_0^\infty \left(y^2(t-s) + z^2(t-s)\right) dR(s),$$

where R is from IV) and $\alpha(t) = e^{-(\overline{P_1}(c)+1)t}$.

Acting in the same way as in (12) and (14) for sufficiently large c we get

$$d\Psi_t \leq -\overline{P_1}(c)\Psi_t dt + 4F^2\alpha(t) \int_0^\infty (y^2(t-s) + z^2(t-s))dR(s) + dm_t,$$

where m_t is defined in (13). From (17) we can get the following estimations

$$(21)\quad E\left[z^2(t) + y^2(t)\right] \leq EW_t \leq EW_0 \leq \left(\overline{P_2}(c) + 3\right)\left(Ey^2(0) + Ez^2(0)\right).$$

It is not difficult to see that

$$d\left[e^{\overline{P_1}(c)t}\Psi_t\right] \leq 4c_2^2 \int_0^\infty \left(y^2(t-s) + z^2(t-s)\right)dR(s)\, e^{\overline{P_1}(c)t}\alpha(t)dt + e^{\overline{P_1}(c)t}dm_t.$$

Therefore, using (21) we have

$$\begin{aligned} &Ee^{\overline{P_1}(c)t}\Psi_t \\ &\quad\leq EW_0 + 4c_2^2 \int_0^t \int_0^\infty \left(Ey^2(\tau-s) + Ez^2(\tau-s)\right)dR(s)e^{\overline{P_1}(c)\tau}\alpha(\tau)d\tau \\ &\quad\leq EW_0 + 4c_2^2 \cdot EW_0\mathbf{R} \int_0^\infty e^{-s}ds \\ &\quad\leq \left(\overline{P_2}(c) + 3\right)\left(Ey^2(0) + Ez^2(0)\right)(1 + 4c_2^2\mathbf{R}), \end{aligned}$$

$$\begin{aligned} (22)\quad &E\left[z^2(t) + y^2(t)\right] \\ &\quad\leq e^{\overline{P_1}(c)t}\left(\overline{P_2}(c) + 3\right)(1 + 4c_2^2\mathbf{R})\left(Ey^2(0) + Ez^2(0)\right). \end{aligned}$$

We have got exponential 2–stability of solution of system (10) for $A_1 = (\overline{P_2}(c)+3)\times\ (1 + 4c_2^2\mathbf{R})$ and $A_2 = \overline{P_1}(c)$.

THEOREM 2. *If conditions I)–III), V) are fulfilled for every ε, $T > 0$ we can find control parameters values b and c such that $Eq_2^2(t) + E\dot{q}_2^2(t) \leq \varepsilon$ for $t \geq T$, where q_2 is the solution of equation (7).*

REMARK 1. *The presence of function $\alpha(t)$ in the conditions V) seems unnatural by the first view. But we need to have it only for producing our mathematical calculations. In real life time t varies on finite interval $[0, T_0]$ and for every actual situation of robot operation we can estimate number T_0*

(for example $T_0 \leq 24$ hours, or $T_0 \leq 1$ hour, or $T_0 \leq 30$ min). Therefore we really need the fulfillment of condition V) only for $t \in [0, T_0]$ with sufficiently small h instead of $\alpha(t)$, $h \leq e^{-(P_1(c)+1)T_0}$. For $t > T_0$ we can demand everything we want, it is only the mathematical abstraction.

6. Conclusions. The problem of control motion of a robotic hand limited with constraints of the program type has been considered. Program constraints have been included to dynamic system considerations and motion stability were investigated. It has been proved that one can choose program conditions satisfying stability conditions that is not obvious because any program can be thought or demanded but not each of them can secure stable motion of a system. It has also been shown that if constraints satisfy certain conditions and control system is constructed on proportional–differential regulators, we can find control parameters b and c for every noises in the control system satisfying also certain limitations. It is important from the practical point of view because noises can play the role of real disturbances here that can occur in real robot working environment.

Moreover for given $\varepsilon, T > 0$ we can find control parameter values b and c that ensure the transition of gripper in ε–neighborhood of equilibrium state in time T.

Program motion equations are nonlinear, disturbances that appear in control, depend on previous states of robot hand and contain random component. To prove stability (and exponential stability) and to find control parameter values b and c for providing it we construct special Lyapunov–Krasovskii functional and estimate its stochastic differential. Using some methods of theory of stochastic process (as lemma on convergence of nonnegative semimartingale, stochastic differential inequalities) we obtain our results from this estimation.

In this paper we consider only disturbances having linear estimation of its growth and continuous type noises, which are expressed by Ito integral with respect to the Wiener process. Some developing of the work is intended, namely to relax this restrictions in particular to consider discontinuous noises. Investigations of another type constraints with motion simulation is also desirable.

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FUNCTIONAL DIFFERENTIAL EQUATIONS

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GLOBAL ATTRACTOR FOR STRONGLY DAMPED NONLINEAR WAVE EQUATIONS

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Abstract. We show the uniformly boundedness of the global attractor for large strong damping and obtain a more precise estimate of the upper bound of the Hausdorff dimension of attractor for strongly damped nonlinear wave equations. The obtained Hausdorff dimension decreases as the strong damping grows for large damping, which conforms to the physical intuition.

Key Words. Wave equation, Global attractor, Hausdorff dimension

AMS(MOS) subject classification. 35B40, 35L70

1. Introduction. We consider the strongly damped nonlinear wave equation

$$u_{tt} - \alpha\Delta u_t - \Delta u + h(u_t) + f(u) = g, \quad x \in \Omega, \quad t > 0 \tag{1}$$

with the homogeneous Dirichlet boundary condition

$$u(x,t)|_{x\in\partial\Omega} = 0, \quad t > 0 \tag{2}$$

and the initial value conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where $u = u(x,t)$ is a real-valued function on $\Omega \times [0,+\infty)$, Ω is an open bounded set of R^n ($n \in N$) with a smooth boundary $\partial\Omega$, $\alpha > 0$, $g \in L^2(\Omega)$, $f(u)$, $h(v) \in C^1(R;R)$, $D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$.

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Let $G(s) = \int_0^s f(r)dr$. We make the following assumptions on functions $G(s)$, $f(s)$ and $h(s)$:

(i)

$$\liminf_{|s|\to+\infty} \frac{G(s)}{s^2} \geq 0, \quad \forall s \in R. \tag{4}$$

(ii) There exist two constants $c_1 > 0$, $c_2 > 0$ such that

$$\liminf_{|s|\to+\infty} \frac{sf(s) - c_1 G(s)}{s^2} \geq 0, \quad \forall s \in R, \tag{5}$$

$$|f'(s)| \leq c_2(1 + |s|^p) \text{ with } \begin{cases} 0 \leq p < \infty, & n = 1, 2, \\ 0 \leq p < \frac{4}{n-2}, & n \geq 3, \end{cases} \quad \forall s \in R. \tag{6}$$

(iii) For every $M > 0$, there exists $c_3 = c_3(M)$ such that

$$||f'(u_1) - f'(u_2)||_{L(H_0^1(\Omega), L^2(\Omega))} \leq c_3 ||u_1 - u_2||^{\delta_1} \tag{7}$$

for any $u_1,\ u_2 \in H_0^1(\Omega)$, $||u_1|| \leq M$, $||u_2|| \leq M$, where $\delta_1 > 0$, $||\cdot||$ and $||\cdot||_{L(H_0^1(\Omega),L^2(\Omega))}$ denote the norms of $H_0^1(\Omega)$ and $L(H_0^1(\Omega), L^2(\Omega))$ (the space of linear continuous operators from $H_0^1(\Omega)$ into $L^2(\Omega)$), respectively.

(iv) There exist two constants β_1, β_2 such that

$$-\alpha\lambda_1 < \beta_1 \leq h'(s) \leq \beta_2 < +\infty, \quad \forall s \in R. \tag{8}$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ on Ω with condition (2).

(v) For every $M' > 0$, there exists $c_4 = c_4(M')$ such that

$$||h'(v_1) - h'(v_2)||_{L(L^2(\Omega),L^2(\Omega))} \leq c_4 |v_1 - v_2|^{\delta_2} \tag{9}$$

for $v_1,\ v_2 \in L^2(\Omega)$, $|v_1| \leq M'$, $|v_2| \leq M'$, where $\delta_2 > 0$, $|\cdot|$ and $||\cdot||_{L(L^2(\Omega),L^2(\Omega))}$ denote the norms of $L^2(\Omega)$ and $L(L^2(\Omega), L^2(\Omega))$ (the space of linear continuous operators from $L^2(\Omega)$ into $L^2(\Omega)$), respectively.

Two examples of equation (1) with conditions (4)-(9) are the perturbed sine-Gordon equation ($f(u) = \sin u$) [3] and the perturbed wave equation occurring in quantum mechanics ($f(u) = |u|^p u$) [5].

The asymptotic behavior of solutions for strongly damped nonlinear wave equations has been studied by many authors [1,2,4,6-9], of those, Ghidaglia & Marzocchi [1] proved the existence and finiteness of the Hausdorff dimension of the global attractor for (1)-(3). However, the obtained results showed that the bound of the global attractor and the upper bound of dimension of attractor in the phase space are both directly proportional to the coefficient α of strong damping for large α and tends to infinity as $\alpha \to +\infty$, which is hard to comprehend intuitively. S. Zhou [9] obtained an upper bound of the Hausdorff dimension of the attractor for the following equation with Dirichlet boundary condition

$$(10) \qquad \frac{\partial^2 u}{\partial t^2} - \alpha\Delta\frac{\partial u}{\partial t} - \Delta u = f(u, u_t) + g, \quad x \in \Omega, \quad t > 0,$$

when the strong damping α is not very small and the function $f(u, v)$ satisfies

$$(11) \qquad \begin{cases} f(u,v) \in C^1(R \times R; R), & |f(u,v)| \le k_0 + k_1|u|^{\delta_0}, \\ |f_1(u,v)| \le k_2, \quad |f_2(u,v)| \le k_3, & \forall (u,v) \in R \times R, \end{cases},$$

where $k_i \ge 0, i = 0,1,2,3,\ 0 \le \delta < 1$, i.e., $f(u, v)$ is sublinear in u, uniformly bounded in v and its partial derivatives in u and v are both uniformly bounded. The system (10)-(11) does not include the equation (1) with conditions (4)-(9) because the function $h(v)$ and $f(u)$ in (1) do not satisfy conditions (11).

For system (1)-(3) with the conditions (4)-(9), by applying a similar technique in [9], i.e., by introducing a new norm in the phase space (which is equivalent to the usual norm), and by carefully estimating and splitting the positivity of the linear operator in the corresponding evolution equation of the first order in time, we will point out that the global attractor is uniformly bounded for large strong damping α in the phase space and show that the upper bound of the Hausdorff dimension of attractor decreases as α grows for large α. In the linear damping, i.e., $h(s) = \beta s$, the Hausdorff dimension of attractor is uniformly bounded for large α and bounded β. The main results are as follows.

Let $E = H_0^1(\Omega) \times L^2(\Omega)$, $E_0 = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ and endowed them with the usual norms $\|\cdot\|_{H_0^1 \times L^2}$ and $\|\cdot\|_{E_0}$, then the solution of system (1)-(3) with conditions (4)-(9) exists globally and uniquely, and defines a continuous semigroup of mappings:

$$S(t) : \{u_0, u_1\} \to \{u(t), u_t(t)\}, \quad E \to E$$

for any $t \ge 0$. Moreover, the semigroup $\{S(t), t \ge 0\}$possesses a global attractor $\mathbf{B}_0$ in E [3-4].

Suppose

$$(12) \qquad \begin{aligned} &\gamma_1 = 4 + (\alpha\lambda_1 + \beta_1)\alpha + \tfrac{\beta_2^2}{\lambda_1}, \quad \gamma_2 = (\alpha\lambda_1 + \beta_1)\alpha + \tfrac{\beta_2^2}{\lambda_1}, \\ &\sigma = \tfrac{\lambda_1\alpha + \beta_1}{\gamma_1 + \sqrt{\gamma_1\gamma_2}} \end{aligned}$$

$$(13) \qquad \begin{array}{l} \alpha \geq \alpha_0 > 0, \ -\alpha_0\lambda_1 < \beta_1 \leq \beta_2 < +\infty, \\ \beta_2 \geq |\beta_1| + \min\{\frac{1}{\alpha}, \frac{\alpha\lambda_1+\beta_1}{2}\}, \end{array}$$

Consider the global attractor $\mathbf{B_0}$ for system (1)-(3) with assumptions (4)-(9), we have

THEOREM 1. *There exists a positive constant* $M_0 = M_0(\alpha_0, \beta_1, \beta_2)$ *such that*

$$||\varphi||_{H_0^1\times L^2} = (|\nabla u|^2 + |v|^2)^{1/2} \leq M_0, \quad \forall \varphi = (u, v)^T \in \mathbf{B_0},$$

and if $\beta_1 \geq 0$, *then the Hausdorff dimension* d_H *of the global attractor* $\mathbf{B_0}$ *in* E *satisfies:*

$$(14) \qquad d_H \leq \min\left\{m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{\rho_0-1} \leq \frac{\alpha\sigma}{k^2}\right\}.$$

where

$$(15) \qquad \rho_0 = \begin{cases} \frac{(n-2)p-2}{2}, & \frac{2}{n-2} \leq p < \frac{4}{n-2}, \quad n \geq 3, \\ 0, & n = 1, 2 \ \text{ or } \ 0 \leq p \leq \frac{2}{n-2}, n \geq 3, \end{cases}$$

and $k = k(\alpha_0, \beta_1, \beta_2)$ *is a positive constant which is independent of* α.

THEOREM 2. *If (13) holds and* p *in (6) satisfies*

$$(16) \qquad \begin{cases} 0 \leq p < \infty, & \text{when} \quad n = 1, 2, \\ 0 \leq p \leq 2, & \text{when} \quad n = 3, \\ p = 0, & \text{when} \quad n \geq 4, \end{cases} \qquad \forall s \in R,$$

then

$$(17) \qquad d_H \leq \min\left\{m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-1} \leq \frac{(\alpha\lambda_1+\beta_1)\sigma}{k^2}\right\},$$

in which $k = k(\alpha_0, \beta_1, \beta_2)$ *is a positive constant which depends on* α_0, β_1 *and* β_2. *If* $p = 0$ $(\forall n \in N)$, *then* $k = c_2$.

THEOREM 3. *If* $\alpha \geq \alpha_0 > 0$, *the function* $h(s)$ *is linear, i.e.,* $h(s) = \beta s$, *and* p *in (6) satisfies (16), then for* $\beta \geq 0$, *there exists a positive constant* $M_0 = M_0(\alpha_0)$ *which is independent of* α *and* β *such that*

$$(18) \qquad ||\varphi||_{H_0^1\times L^2} = (|\nabla u|^2 + |v|^2)^{1/2} \leq M_0, \forall \varphi = (u, v) \in \mathbf{B}.$$

and for $0 \leq \beta \leq \beta_0 < +\infty$, *there exists a positive constant* $k_0 = k_0(\alpha_0, \beta_0)$ *such thatthe Hausdorff dimension* d_H *of the global attractor* $\mathbf{B_1}$ *in* E *satisfies:*

(19) $$d_H \le \min\left\{ m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-1} \le \frac{(\alpha\lambda_1+\beta)\sigma_0}{k_0^2} \right\},$$

in which

(20) $$\sigma_0 = \sigma \text{ when } \beta_1 = \beta_2 = \beta.$$

If $p = 0$ $(\forall n \in N)$, *then for any* $\alpha > 0$, $\beta \ge 0$,

(21) $$d_H \le \min\left\{ m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-1} \le \frac{(\alpha\lambda_1+\beta)\sigma_0}{c_2^2} \right\}.$$

In above theorems, $\{\lambda_j : 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots\}$ are the eigenvalues of operator $-\Delta$ with the homogeneous Dirichlet boundary condition on Ω.

It is easy to see from Theorem 1 that the upper bound of d_H in the right side of (14) is a decreasing function of α and remains small for large strong damping α because the quantity $\frac{\alpha\sigma}{k^2}$ in (14) increases as α grows and

(22) $$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{\rho_0-1} = 0, \qquad \lim_{\alpha\to+\infty}\frac{\alpha\sigma}{k^2} = \frac{1}{2k^2} > 0.$$

Therefore, the Hausdorff dimension d_H in (15) is uniformly bounded for large α. Similarly, it is not difficult to see that the dimension in (19) is uniformly bounded for large α and bounded β.

2. Preliminaries. It is well known that the operator $A = -\Delta$is self-adjoint, positive and linear from $D(A) \to L^2(\Omega)$, the eigenvalues $\{\lambda_i\}_{i\in N}$ of A satisfy:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots, \quad \text{and} \quad \lambda_m \to +\infty \text{ as } m \to +\infty.$$

Let

$$E = H_0^1(\Omega) \times L^2(\Omega), \quad E_0 = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega),$$

$$E_1 = H_0^1(\Omega) \times H_0^1(\Omega),$$

$$(u,v) = \int_\Omega u\imath\, dx, \quad |u| = (u,u)^{\frac{1}{2}}, \quad \forall u,\ v \in L^2(\Omega),$$

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \|u\| = ((u,u))^{\frac{1}{2}}, \quad \forall u, \quad v \in H_0^1(\Omega),$$

$$(y_1, y_2)_{H_0^1 \times L^2} = ((u_1, u_2)) + (v_1, v_2), \quad |y|_{H_0^1 \times L^2} = (y, y)^{\frac{1}{2}}_{H_0^1 \times L^2},$$
$$\forall y_i = (u_i, v_i)^T, \quad y = (u,v)^T \in H_0^1(\Omega) \times L^2(\Omega), \quad i = 1, 2$$

and

$$(y_1, y_2)_{E_0} = (Au_1, Au_2) + (v_1, v_2), \quad \forall y_i = (u_i, v_i)^T \in E_0, \quad i = 1, 2,$$
$$|y|_{E_0} = (y, y)^{\frac{1}{2}}_{E_0}, \quad \forall y = (u, v)^T \in E_0$$

denote the usual inner products and norms in $L^2(\Omega)$, $H_0^1(\Omega)$, E and E_0, respectively.

We define a new weight inner product and norm in $E = H_0^1(\Omega) \times L^2(\Omega)$ as

(23) $$(\varphi, \psi)_E = \mu((u_1, u_2)) + (v_1, v_2), \quad |\varphi|_E = (\varphi, \varphi)^{\frac{1}{2}}_E$$

for any $\varphi = (u_1, v_1)^T, \psi = (u_2, v_2)^T \in E$, where μ is chosen as

(24) $$\mu = \frac{4 + (\alpha\lambda_1 + \beta_1)\alpha + \frac{\beta_2^2}{\lambda_1}}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \frac{\beta_2^2}{\lambda_1}}.$$

Since

(25) $$\mu \in (\frac{1}{2}, 1),$$

the norm $|\cdot|_E$ in (23) is equivalent to the usual norm $|\cdot|_{H_0^1 \times L^2}$ in E.

It is convenient to reduce (1) to an evolution equation of the first order in time. Let

(26) $$\varphi = (u, v)^T, \quad v = u_t + \varepsilon u,$$

where ε is chosen as

(27) $$\varepsilon = \frac{\lambda_1 \alpha + \beta_1}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \frac{\beta_2^2}{\lambda_1}},$$

then the system (1)-(3) can be written as

(28) $$\varphi_t + H(\varphi) = F(\varphi), \quad \varphi(0) = (u_0, u_1 + \varepsilon u_0)^T,$$

where

$$(29)\qquad \begin{aligned} F(\varphi) &= \begin{pmatrix} 0 \\ -f(u)+g \end{pmatrix}, \\ H(\varphi) &= \begin{pmatrix} \varepsilon u - v \\ Au - \varepsilon(\alpha A - \varepsilon)u + (\alpha A - \varepsilon)v + h(v - \varepsilon u) \end{pmatrix}. \end{aligned}$$

LEMMA 1. *Suppose*

$$-\alpha\lambda_1 < \beta_1 \le \beta_2 < +\infty \quad \text{and} \quad \beta_2 \ge |\beta_1| + \min\{\frac{1}{\alpha}, \quad \frac{\alpha\lambda_1 + \beta_1}{2}\}.$$

For any $\varphi = (u,v)^T \in E_1$,

$$(30)\qquad (H(\varphi),\varphi)_E \ge \sigma|\varphi|_E^2 + \frac{\alpha}{2}||v||^2 + \frac{\beta_1}{2}|v|^2 \ge \sigma|\varphi|_E^2 + \frac{\alpha\lambda_1+\beta_1}{2}|v|^2,$$

where σ *is as in (12).*

Proof. Since E_1 is dense in $D(A) \times D(A)$, we only need to prove the Lemma for any $\varphi = (u,v)^T \in D(A) \times D(A)$.

For any $\varphi = (u,v)^T \in D(A)\times D(A)$, by (8), (23), (28), (29), the Poincare ineuality and $\mu = 1 - \varepsilon\alpha$, we have

$$(31)\qquad \begin{aligned} &(H(\varphi),\varphi)_E - \sigma|\varphi|_E^2 - \tfrac{\alpha}{2}||v||^2 - \tfrac{\beta_1}{2}|v|^2 \\ &= (\varepsilon - \sigma)\mu||u||^2 + \alpha(Av,v) - \tfrac{\alpha}{2}||v||^2 + (h(v-\varepsilon u),v) \\ &-(\varepsilon + \sigma + \tfrac{\beta_1}{2})|v|^2 + \varepsilon^2(u,v) \\ &\ge (\varepsilon-\sigma)\mu||u||^2 + (\tfrac{\alpha\lambda_1+\beta_1}{2} - \varepsilon - \sigma)|v|^2 - \tfrac{\varepsilon\beta_2}{\sqrt{\lambda_1\mu}} \cdot \sqrt{\mu}||u|| \cdot |v|. \end{aligned}$$

By (12) and (26), elementary computation shows

$$\begin{aligned} (1 + \tfrac{\alpha\lambda_1+\beta_1}{2}\alpha + \tfrac{\beta_2^2}{4\lambda_1})\varepsilon^2 \quad &-(\tfrac{\alpha\lambda_1+\beta_1}{2} + \tfrac{\alpha\lambda_1+\beta_1}{2}\alpha\sigma - \alpha\sigma^2)\varepsilon \\ &+\tfrac{\alpha\lambda_1+\beta_1}{2}\sigma - \sigma^2 = 0, \end{aligned}$$

thus,

$$4(\varepsilon-\sigma)(\frac{\alpha\lambda_1+\beta_1}{2} - \varepsilon - \sigma) \ge \frac{\varepsilon^2{\beta_2}^2}{\mu\lambda_1}.$$

By (31),

$$(H(\varphi),\varphi)_E \ge \sigma|\varphi|_E^2 + \frac{\alpha}{2}||v||^2 + \frac{\beta_1}{2}|v|^2 \text{ for } \varphi = (u,v)^T \in D(A) \times D(A).$$

The proof is comlpeted. □

3. Boundedness Of The Global Attractor. Under the assumptions (4)-(9) in section 1, we have known that for every $g \in L^2(\Omega)$, for all $T > 0$ and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, there exists an unique function $u(t)$ such that

(32) $$(u, u_t) \in C(R_+; H_0^1(\Omega)) \times [C(R_+; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))]$$

satisfying (1)-(3) which defines a continuous semigroup of mapping:

$$S(t) : \{u_{0,} u_1\} \mapsto \{u, u_t\}, \quad E \to E, \quad \forall t \geq 0$$

and the semigroup $\{S(t), t \geq 0\}$possesses a global attractor $\mathbf{B_0}$, and the upper bound of the Hausdorff dimension of attractor $\mathbf{B_0}$ is finite (see e.g. [3] for detail). However, the bound of the global attractor $\mathbf{B_0}$ and the upper bound of dimension of $\mathbf{B_0}$ in E are both directly proportional to the strong damping α for large α. Moreover, these two bounds tend to infinity as $\alpha \to +\infty$, which does not conforms to the physical intuition.

In this section, we will point out the uniformly boundedeness of the global attractor $\mathbf{B_0}$ in E for suitable large α. It is easy to see that the semigroup

(33) $$S_\varepsilon(t) : (u_0, u_1 + \varepsilon u_0)^T \to (u(t), u_t(t) + \varepsilon u(t))^T, E \to E,$$

defined by (28) has the following relation with $S(t)$:

(34) $$S_\varepsilon(t) = R_\varepsilon S(t) R_{-\varepsilon},$$

in which $R_\varepsilon : \{u, v\} \to \{u, v + \varepsilon u\}$is an isomorphism of E.

By (34), the semigroup $\{S_\varepsilon(t), t \geq 0\}$ possesses a global attractor $\mathbf{B} = R_\varepsilon \mathbf{B}_0$, $\mathbf{B}$ and $\mathbf{B}_0$ have the same dimension. So, we only need consider the equivalent system (28).

LEMMA 2. *If (13) is satisfied, then the global attractor* $\mathbf{B}$ *is included in the bounded ball* B_0 *of* E, $B_0 = B_E(0, \sqrt{2}M_0)$, *centered at* 0 *of radius* $\sqrt{2}M_0$, *where* $M_0 = M_0(\alpha_0, \beta_1, \beta_2)$ *is independent of* α.

Proof. Write $\overline{G}(u) = \int_\Omega G(u)dx$. Let $\varphi(t) = (u(t), v(t))^T$ be a solution of the system (28) in which $v(t) = u_t(t) + \varepsilon u(t)$ with the initial value $\varphi(0) = (u_0, u_1 + \varepsilon u_0)^T \in E$, by (32),

(35) $$\varphi(t) \in C(R_+; H_0^1(\Omega)) \times [C(R_+; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))], \ \forall T > 0.$$

Taking the inner product $(\cdot, \cdot)_E$ of (28) with $\varphi = (u, v)^T$, we find

(36) $$\frac{1}{2}\frac{d}{dt}[|\varphi|_E^2 + 2\overline{G}(u)] + (H(\varphi), \varphi)_E + \varepsilon(f(u), u) = (g, v).$$

By the assumptions (4), (5) and the Poincare inequality, there exist two positive constants k_1, $k_2 \geq 0$ such that

$$(37) \qquad \overline{G}(u) + \frac{1}{8+32c_1}||u||^2 + k_1 \geq 0, \quad \forall u \in H_0^1(\Omega)$$

$$(38) \qquad (u, f(u)) - c_1\overline{G}(u) + \frac{1}{8}||u||^2 + k_2 \geq 0, \quad \forall u \in H_0^1(\Omega).$$

By (12) and (27), $\frac{1}{2}\varepsilon < \sigma < \varepsilon$. By (37),

$$(39)\ \overline{G}(u) + \frac{1}{8}||u||^2 + k_1 \geq 0,\ \overline{G}(u) + \frac{1}{32c_1}||u||^2 + k_1 \geq 0,\ \forall u \in H_0^1(\Omega).$$

Let

$$(40) \qquad y = |\varphi|_E^2 + 2\overline{G}(u) + 2k_1 \geq \text{by}(23)(25)(39) \geq \frac{1}{4}|\varphi|_E^2 \geq 0.$$

Write $\tilde{G}(u) = \overline{G}(u) + \frac{1}{32c_1}||u||^2 + k_1 \geq 0$. By (30), (38), (40), we have

$$(41) \quad \begin{aligned} &(H(\varphi),\varphi)_E + \varepsilon(f(u),u) \\ &\geq \tfrac{\varepsilon}{2}(\tfrac{1}{2}||u||^2 + |v|^2) + \varepsilon c_1\tilde{G}(u) - \tfrac{3\varepsilon}{32}||u||^2 - \varepsilon(k_2 + c_1k_1) + \tfrac{\alpha\lambda_1+\beta_1}{2}|v|^2 \\ &\geq \tfrac{1}{32}\varepsilon\theta[|\varphi|_E^2 + 2\overline{G}(u) + \tfrac{1}{16c_1}||u||^2 + 2k_1] - \varepsilon(k_2 + c_1k_1) + \tfrac{\alpha\lambda_1+\beta_1}{2}|v|^2 \\ &\geq \tfrac{1}{2}\rho y - \varepsilon(k_2 + c_1k_1) + \tfrac{\alpha\lambda_1+\beta_1}{2}|v|^2, \end{aligned}$$

where

$$(42) \quad \rho = \frac{1}{32}\varepsilon\theta, \quad \theta = \min(1, 16c_1) = \begin{cases} 1, & \text{when} \quad c_1 \geq \frac{1}{16}, \\ 16c_1, & \text{when} \quad 0 < c_1 < \frac{1}{16}. \end{cases}$$

By (36) and (41),

$$\frac{d}{dt}y + \rho y \leq \frac{1}{\alpha\lambda_1 + \beta_1}|g|^2 + 2\varepsilon(c_1k_1 + k_2),$$

By the Gronwall inequality,

$$(43) \quad \begin{aligned} |\varphi|_E^2 &\leq 4y(t) \\ &\leq 4y(0)e^{-\rho t} + 4\left(\tfrac{1}{(\alpha\lambda_1+\beta_1)\rho}|g|^2 + \tfrac{2\varepsilon(c_1k_1+k_2)}{\rho}\right)(1 - e^{-\rho t}). \end{aligned}$$

and

$$(44) \quad \lim_{t\to+\infty}\sup|\varphi|_E^2 \leq 4\left(\frac{1}{(\alpha\lambda_1 + \beta_1)\rho}|g|^2 + \frac{2\varepsilon(c_1k_1 + k_2)}{\rho}\right) = M_{\alpha,\beta_1,\beta_2}^2,$$

where

$$
(45)\quad \begin{aligned}
M^2_{\alpha,\beta_1,\beta_2} &= \frac{64}{\theta}\left\{\frac{4+2(\alpha\lambda_1+\beta_1)\alpha+\frac{\beta_2^2}{\lambda_1}}{(\alpha\lambda_1+\beta_1)^2}|g|^2+2(c_1k_1+k_2)\right\}\\
&\le \frac{64}{\theta}\left\{\left(\frac{2}{\lambda_1}+\frac{2\alpha_0}{\alpha_0\lambda_1+\beta_1}+\frac{4+\frac{\beta_2^2}{\lambda_1}}{(\alpha_0\lambda_1+\beta_1)^2}\right)|g|^2+2(c_1k_1+k_2)\right\}\\
&= M_0^2.
\end{aligned}
$$

From (43), (44) and (45), we complete the proof. □

4. Differentiability Of The Semigroup. In this section we assume that the functions $f(u), h(v)$ satisfy the conditions (4)-(9) in section 1. To estimate the Hausdorff dimension of the global attractor of the semigroup $S(t), t \ge 0$ in E, we need to consider the differentiability of $S(t),\ \ t \ge 0$.

LEMMA 3. *Consider the linearized equation of (1) with initial-boundary conditions:*

$$
(46)\quad \begin{cases}
U_{tt}+\alpha AU_t+AU+h(u_t)U_t+f'(u)U=0, & x\in\Omega, \quad t>0,\\
U(x,t)|_{x\in\partial\Omega}=0, \quad t>0, & \\
U(x,0)=\xi, \quad U_t(x,0)=\eta, \quad x\in\Omega, &
\end{cases}
$$

where $u=u(x,t)$ is a solution of (1)-(3). Then (46) is a well-posed problem in E, the mapping $S(t)$ defined by (10) is Fréchet differentiable on E for any $t>0$, its differential at $\varphi=(u_0,u_1)^T$ is the linear operator on $E:(\xi,\eta)^T\mapsto (U(t),V(t))^T$, where $U(t)$ is the solution of (46).

Proof. It is clear from the assumptions in the first section that the problem (46) is a well-posed problem in E. We first consider the Lipschitz property of $S(t)$ on the bounded sets of E. Let

$$\varphi_0=(u_0,u_1)^T\in E, \quad \tilde\varphi_0=\varphi_0+(\xi,\eta)^T=(u_0+\xi,u_1+\eta)^T\in E$$

with

$$(47)\quad |\varphi_0|_E\le r_0, \quad |\tilde\varphi_0|_E\le r_0.$$

and

$$S(t)\varphi_0=\varphi(t)=(u(t),u_t(t))^T, \quad S(t)\tilde\varphi_0=\tilde\varphi(t)=(\tilde u(t),\tilde u_t(t))^T.$$

By the hypothesis (6), the mean value theorem and the Sobolev embedding theorem

$$(48)\quad H_0^\nu(\Omega)\subset D(A^{\frac{\nu}{2}})\subset H^\nu(\Omega)\subset L^q(\Omega)\subset L^2(\Omega)\subset L^{q'}(\Omega)\subset H^{-\nu}(\Omega),$$

where

$$(49)\qquad \frac{1}{q} = \frac{1}{2} - \frac{\nu}{n}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \nu \in [0,1],$$

for every $r > 0$, there exists a positive constant $c_5 = c_5(r)$ such that

$$(50)\qquad |A^{\delta-\frac{1}{2}} f(u)| \le c_5, \quad \forall u \in H_0^1(\Omega), \quad ||u|| \le r,$$

where

$$(51)\qquad \delta = \begin{cases} \frac{4-(n-2)p}{4}, & \frac{2}{n-2} \le p < \frac{4}{n-2},\ n \ge 3, \\ \frac{1}{2}, & n = 1,2 \text{ or } 0 \le p \le \frac{2}{n-2},\ n \ge 3 \end{cases}$$

and p is as in (6). From (38),

$$(52)\qquad \overline{G}(u) \le \frac{1}{c_1}[|A^{\delta-\frac{1}{2}} f(u)| \cdot |A^{\frac{1}{2}-\delta} u| + \frac{1}{8}||u||^2 + k_2], \quad \forall u \in H_0^1(\Omega)$$

By (47), (50) and (52), there exists a constant $c_6 = c_6(r_0) > 0$ such that $\overline{G}(u_0) \le c_6$. By (40),

$$(53)\qquad y(0) = |(u_0, u_1 + \varepsilon u_0)^T|_E^2 + 2\overline{G}(u_0) + 2k_1 \le c_7(r_0).$$

By (43) and (53), there exists a constant $r' = r'(r_0)$ such that

$$(54)\qquad |\varphi(t)|_E \le r', \quad |\tilde{\varphi}(t)|_E \le r', \quad \forall t \ge 0.$$

The difference

$$(55)\qquad \psi = \tilde{u} - u$$

satisfies

$$(56)\qquad \psi_{tt} + \alpha A \psi_t + A\psi + h(\tilde{u}_t) - h(u_t) + f(\tilde{u}) - f(u) = 0.$$

For $n = 1$, $H_0^1(\Omega) \subset L^\infty(\Omega) \subset L^1(\Omega) \subset H^{-1}(\Omega) \subset (H_0^1(\Omega))'$, there exists a constant $c_8 = c_8(r') > 0$ such that

$$(57)\qquad \begin{aligned} |A^{-\frac{1}{2}}[f(\tilde{u}(t)) - f(u(t))]| &\le c' |f'(u + \vartheta_1(\tilde{u} - u))(\tilde{u} - u)|_{L^1} \\ &\le c_2' \textstyle\int_\Omega (1 + |u + \vartheta_1(\tilde{u} - u)|^p)(\tilde{u} - u)dx \\ &\le c_8 |\tilde{u}(t) - u(t)|, \quad \forall t \ge 0. \end{aligned}$$

where $\vartheta_1 \in (0,1)$.

For $n = 2$, $H_0^1(\Omega) \subset L^q(\Omega) \subset H^{-1}(\Omega) \subset (H_0^1(\Omega))'$, $q > 0$, there exists a constant $c_8' = c_8'(r') > 0$ such that

$$\begin{aligned} |A^{-\frac{1}{2}}[f(\tilde{u}(t)) - f(u(t))]| &\leq c'|f'(u + \vartheta_1(\tilde{u} - u))(\tilde{u} - u)|_{L^{\frac{3}{2}}} \\ &\leq c_2' \left(\int_\Omega [(1 + |u + \vartheta_1(\tilde{u} - u)|^p)(\tilde{u} - u)]^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \\ &\leq c_8' |\tilde{u}(t) - u(t)|, \quad \forall t \geq 0. \end{aligned} \tag{58}$$

For $n \geq 3$, by (6), there exists a constant $c_9 = c_9(r')$ such that

$$\begin{aligned} |A^{-\frac{1}{2}}[f(\tilde{u}(t)) - f(u(t))]| &\leq |A^{-\frac{1}{2}}[f'(u + \vartheta_1(\tilde{u} - u))(\tilde{u} - u)]| \\ &\leq |f'(u + \vartheta_1(\tilde{u} - u))(\tilde{u} - u)|_{L^{\frac{2n}{n+2}}} \\ &\leq \left(\int_\Omega [(1 + |u + \vartheta_1(\tilde{u} - u)|^p)(\tilde{u} - u)]^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{2n}} \\ &\leq c_9 ||\tilde{u}(t) - u(t)||, \quad \forall t \geq 0, \end{aligned}$$

(59)
By (8),

$$\begin{aligned} |h(\tilde{u}_t) - h(u_t)| &= |h'(u_t + \vartheta_2(\tilde{u}_t - u_t))(\tilde{u}_t - u_t)| \\ &\leq \beta_2 |\tilde{u}_t - u_t|, \quad \vartheta_2 \in (0, 1). \end{aligned} \tag{60}$$

Taking the inner product of (56) with $\psi_t = \tilde{u}_t - u_t$ in $L^2(\Omega)$, by (57), (58), (59), (60), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\psi_t|^2 + ||\psi||^2) + \alpha||\psi_t||^2 \\ &= (-(h(\tilde{u}_t) - h(u_t) + f(\tilde{u}) - f(u)), \psi_t) \\ &\leq |h(\tilde{u}_t) - h(u_t)||\psi_t| + |A^{-\frac{1}{2}}(f(\tilde{u}) - f(u))|||\psi_t|| \\ &\leq c_{10}(|\psi_t|^2 + ||\psi||^2), \end{aligned}$$

where $c_{10} = c_{10}(r') > 0$, i.e.,

$$\frac{d}{dt}(|\psi_t|^2 + ||\psi||^2) \leq 2c_{10}(|\psi_t|^2 + ||\psi||^2), \quad \forall t \geq 0.$$

So, we have the Lipschitz property

$$\begin{aligned} ||\tilde{\varphi}(t) - \varphi(t)||^2_{H_0^1 \times L^2} &= |\tilde{u}_t(t) - u_t(t)|^2 + ||\tilde{u}(t) - u(t)||^2 \\ &\leq (|\eta|^2 + ||\xi||^2)e^{2c_{10}t}, \quad \forall t \geq 0. \end{aligned} \tag{61}$$

Consider the difference $\theta = \tilde{u} - u - U$, with U the solution of the linearized system (46). Obviously, $\theta(0) = \theta_t(0) = 0$ and

$$\theta_{tt} + \alpha A\theta_t + A\theta = d, \tag{62}$$

where

$$d = f(u) - f(\tilde{u}) - f'(u)(u-\tilde{u}) - f'(u)\theta \\ +h(u_t) - h(\tilde{u}_t) - h'(u_t)(u_t-\tilde{u}_t) - h'(u_t)\theta_t.$$

By the hypothesis (7) and (9), there exist two constants $c_{11}(r')$, $c_{12}(r')$ such that

$$|f'(s\tilde{u}+(1-s)u)) - f'(u)|_{L(H_0^1(\Omega),L^2(\Omega))} \le c_{11}s^{\delta_1}||\tilde{u}-u||^{\delta_1}, \quad \forall s \in [0,1].$$

and

$$|h'(s\tilde{u}_t+(1-s)u_t)) - f'(u_t)|_{L(L^2(\Omega),L^2(\Omega))} \le c_{12}s^{\delta_2}|\tilde{u}_t-u_t|^{\delta_2}, \quad \forall s \in [0,1].$$

We observe that

$$\begin{aligned} &|f(u) - f(\tilde{u}) - f'(u)(u-\tilde{u})| \\ &= \textstyle\int_0^1 |\{f'(s\tilde{u}+(1-s)u)) - f'(u)\}(u-\tilde{u})|ds \\ &\le c_{14}(r')||\tilde{u}-u||^{1+\delta_1}, \end{aligned} \tag{63}$$

$$\begin{aligned} &|h(u_t) - h(\tilde{u}_t) - h'(u_t)(u_t-\tilde{u}_t)| \\ &= \textstyle\int_0^1 |\{h'(s\tilde{u}_t+(1-s)u_t)) - f'(u_t)\}(u_t-\tilde{u}_t)|ds \\ &\le c_{15}(r')|\tilde{u}_t-u_t|^{1+\delta_2}, \end{aligned} \tag{64}$$

and

$$|A^{-\frac{1}{2}}f'(u)\theta| \le c_{16}(r')||\theta||. \tag{65}$$

Taking the scalar product of each side of (61) with θ_t in $L^2(\Omega)$, by (8), (63), (64) and (65),

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(|\theta_t|^2 + ||\theta||^2) + \alpha||\theta_t||^2 \\ &= |\theta_t|(c_{14}||\tilde{u}-u||^{1+\delta_1} \\ &+c_{15}|\tilde{u}_t-u_t|^{1+\delta_2}) + \beta_2|\theta_t|^2 + c_{16}||\theta||\cdot||\theta_t|| \end{aligned}$$

i.e.,

$$\begin{aligned} \tfrac{d}{dt}(|\theta_t|^2 + ||\theta||^2) \le\ & c_{17}(r')(|\theta_t|^2 + ||\theta||^2) \\ & +c_{18}(r')(||\tilde{u}(t)-u(t)||^{2+2\delta_1} + |\tilde{u}_t(t)-u_t(t)|^{2+2\delta_2}), \end{aligned}$$

By the Gronwall's inequality and (61), we obtain

$$\begin{aligned} &|\theta_t(t)|^2 + ||\theta(t)||^2 \\ &\le \tfrac{c_{18}}{c_{17}}e^{c_{17}t}\cdot \textstyle\int_0^t(||\tilde{u}(t)-u(t)||^{2+2\delta_1} + |\tilde{u}_t(t)-u_t(t)|^{2+2\delta_2})d\tau \\ &\le c_{19}e^{c_{20}t}\cdot[(|\eta|^2+||\xi||^2)^{1+\delta_1} + (|\eta|^2+||\xi||^2)^{1+\delta_2}], \forall t \ge 0, \end{aligned}$$

where $c_{19}, c_{20} > 0$, that is,

$$|\tilde{\varphi}(t) - \varphi(t) - U(t)|^2_{H_0^1 \times L^2} \le c_{19} e^{c_{20} t}[|(\xi, \eta)^T|_E^{2+2\delta_1} + |(\xi, \eta)^T|_E^{2+2\delta_2}],$$

therefore,

$$\frac{|\tilde{\varphi}(t) - \varphi(t) - U(t)|^2_{H_0^1 \times L^2}}{|(\xi, \eta)^T|^2_{H_0^1 \times L^2}} \le c_{19} e^{c_{20} t} \cdot [|(\xi, \eta)^T|_E^{2\delta_1} + |(\xi, \eta)^T|_E^{2\delta_2}] \to 0,$$

as $(\xi, \eta)^T \to 0$ in E. The proof is comlpeted. □

5. Proof Of Theorem 1. In this section, we will show that the Hausdorff dimension of the global attractor $\mathbf{B}_0$ of the semigroup $S(t), t \ge 0$ associated with the system (1)-(3) decreases as α grows, and complete the proof of Theorem 1.

Since $\mathbf{B}$ (the global attractor of $S_\varepsilon(t), t \ge 0$) and $\mathbf{B}_0$ (the global attractor of $S(t), t \ge 0$) have the same dimension, we only need to estimate the Hausdorff dimension of the global attractor $\mathbf{B}$ in E, i.e., we only need to consider the system (28). To estimate the Hausdorff dimension of the global attractor $\mathbf{B}$ in E, we consider the first variation equation of (28) with initial condition

$$\Psi' + H'(\varphi)\Psi = F'(\varphi)\Psi, \quad \Psi(0) = (\xi, \eta)^T \in E \tag{66}$$

where $\Psi = (U, V)^T \in E$ and $\varphi = (u, v)^T \in E$ is a solution of (28),

$$\begin{aligned} H'(\varphi) &= \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon\alpha A + \varepsilon^2 I - \varepsilon h'(v - \varepsilon u) I & \alpha A + h'(v - \varepsilon u) I - \varepsilon I \end{pmatrix}, \\ F'(\varphi) &= \begin{pmatrix} 0 & 0 \\ -f'(u) & 0 \end{pmatrix}. \end{aligned} \tag{67}$$

It is easy to show from Lemma 3 that (66) is a well-posed problem in E, the mapping $S_\varepsilon(t)$ is Fréchet differentiable on E for any $t > 0$, its differential at $\varphi = (u_0, u_1 + \varepsilon u_0)^T$ is the linear operator on $E, (\xi, \eta)^T \mapsto (U(t), V(t))^T$, where $(U, V)^T$ is the solution of (66).

LEMMA 4. *Consider the system (28). Let Φ denote a set of m vectors $\{\Phi_1, \Phi_2, \cdots, \Phi_m\}$ which are orthonormal in $(E, |\cdot|_E)$. If*

$$\lim_{t \to +\infty} \sup_{\Phi \subset E} \sup_{\varphi \in \mathbf{B}} \frac{1}{t} \int_0^t \sum_{j=1}^m ((-H'(\varphi(\tau)) + F'(\varphi(\tau)))\Phi_j(\tau), \Phi_j(\tau))_E d\tau < 0, \tag{68}$$

then the Hausdorff dimension of the global attractor $\mathbf{B}$ is less than or equal to m.

Proof. This is a direct consequence of Theorem V. 3.3., equations (V. 3.47)-(V. 3.49) of [5]. □

LEMMA 5. *For any orthonormal family of elements of* $(E, |\cdot|_E)$, $(\xi_j, \eta_j)^T$, $j = 1, 2, \cdots, m$, *we have*

$$\sum_{j=1}^{m} |A^{\frac{1}{2}\nu}\xi_j|^2 \leq \mu^{\nu-1}\sum_{j=1}^{m}\lambda_j^{\nu-1} \leq 2\sum_{j=1}^{m}\lambda_j^{\nu-1}, \quad \nu \in [0, 1). \tag{69}$$

Proof. This is a direct consequenceof Lemma VI. 6.3 of [5] and (25). □

LEMMA 6. *Suppose the functions* $f(u)$, $h(v)$ *satisfy the assumptions (4)-(9). If* $\beta_1 \geq 0$ *and (13) holds, then the Hausdorff dimension* $d_H(\mathbf{B})$ *of the global attractor* $\mathbf{B}$ *for system (28) in* $(E, |\cdot|_E)$ *satisfies*

$$d_H(\mathbf{B}) \leq \min\left\{ m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{\rho_0-1} < \frac{\alpha\sigma}{k^2} \right\}. \tag{70}$$

where $\rho_0 \in [0, 1)$ *is defined by (15),* $k = k(\alpha_0, \beta_1, \beta_2)$ *is a positive constant which is independent of* α, σ *is as in (12).*

Proof. Let $m \in N$ be fixed. Consider m solutions $\Psi_1, \Psi_2, \cdots, \Psi_m$ of (66). At a given time τ, let $Q_m(\tau)$ denote the orthogonal projection in E onto the space spanned by $\Psi_1, \Psi_2, \cdots, \Psi_m$. Let $\Phi_j(\tau) = (\xi_j, \eta_j)^T \in E, j = 1, 2, \cdots, m$, be an orthonormal basis of

$$Q_m(\tau)E = \text{span}\{\Psi_1(\tau), \Psi_2(\tau), \cdots, \Psi_m(\tau)\}.$$

with respect to the inner product $(\cdot,\cdot)_E$ and norm $|\cdot|_E$.

Suppose

$$\varphi(\tau) = (u(\tau), v(\tau))^T \in \mathbf{B}. \tag{71}$$

then $|\varphi(\tau)|_E \leq \sqrt{2}M_0$ (M_0 is defined by Lemma 2). Similar to the proof of Lemma 1 and by $|\Phi_j|_E = 1$, we have

$$-(H'(\varphi)\Phi_j, \Phi_j)_E \leq -\sigma - \frac{\alpha}{2}||\eta_j||^2 - \frac{\beta_1}{2}|\eta_j|^2. \tag{72}$$

where σ is as in (12). By (23) and (67),

$$\begin{aligned}(F'(\varphi(\tau))\Phi_j(\tau), \Phi_j(\tau))_E &= (-f'(u(\tau))\xi_j(\tau), \eta_j(\tau)) \\ &\leq |A^{-\frac{1}{2}}f'(u(\tau))\xi_j(\tau)| \cdot ||\eta_j||\end{aligned} \tag{73}$$

For $n = 1$, 2, similar to (57) and (58), we easily obtain that

$$|A^{-\frac{1}{2}}f'(u(\tau))\xi_j(\tau)| \leq c_{21}(M_0)|\xi(\tau)|, \quad c_{21}(M_0) > 0. \tag{74}$$

For $n \geq 3$,similar to (59),

$$(75)\quad \begin{aligned} |A^{-\frac{1}{2}}f'(u(\tau))\xi_j(\tau)| &\leq c''|f'(u(\tau))\xi_j(\tau)|_{L^{\frac{2n}{n+2}}} \\ &\leq c_2''\left(\int_\Omega[(1+|u(\tau)|^p)\xi_j(\tau)]^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}} \\ &\leq c_{22}(M_0)|A^{\frac{1}{2}\rho_0}\xi_j(\tau)|, \quad c_{22}(M_0)>0, \quad \forall\tau\geq 0. \end{aligned}$$

where ρ_0 is as in (15). Thus, there exists a constant $k = \max\{c_{21}(M_0), c_{22}(M_0)\}> 0$ which depends on α_0, β_1, β_2 such that

$$(76)\quad (F'(\varphi(\tau))\Phi_j(\tau), \Phi_j(\tau))_E \leq k|A^{\frac{1}{2}\rho_0}\xi_j(\tau)|\cdot||\eta_j||, \quad \forall\tau\geq 0.$$

Hence, by (69), (72) and (76), we have

$$(77)\ \sup_{\varphi\in\mathbf{B}}\sum_{j=1}^{m}((-H'(\varphi)+F'(\varphi))\Phi_j,\Phi_j)_E \leq -\frac{mk^2}{\alpha}\left(\frac{\alpha\sigma}{k^2}-\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{\rho_0-1}\right).$$

If

$$(78)\quad \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{\rho_0-1} < \frac{\alpha\sigma}{k^2},$$

then by (77),

$$\sup_{\varphi\in\mathbf{B}}\sum_{j=1}^{m}((-H'(\varphi)+F'(\varphi))\Phi_j,\Phi_j)_E < 0.$$

By Lemma 4, the proof is comlpeted. □

Combining with Lemma 2 and Lemma 6, we complete the proof of Theorem 1.

6. Proof Of Theorem 2. We suppose the conditions in Theorem 2 are satisfied. In this case, Lemma 1 holds.

LEMMA 7. *For any initial value u_0, $u_1 \in \mathbf{B}$, the solution of system (1)-(3) $u(t)$ can be decomposed into $u(t) = z(t) + w((t)$, where $z(t)$ satisfies*

$$(79)\quad \mu||z(t)||^2 + |z_t(t)+\varepsilon z(t)|^2 \leq 2M_0^2\exp(-2\sigma_1 t), \quad \forall t\geq 0.$$

and $w(t)$ satisfies

$$(80)\quad \mu|Aw(t)|^2 + ||w_t(t)+\varepsilon w(t)||^2 \leq c_{21}(\alpha_0,\beta_1,\beta_2), \quad \forall t\geq 0.$$

where $c_{21}(\alpha_0,\beta_1,\beta_2)$ is a constant, M_0 is as in Lemma 2 and

$$(81)\quad \sigma_1 = \frac{\lambda_1\alpha}{4+\alpha^2\lambda_1+\alpha\sqrt{4\lambda_1+\alpha^2\lambda_1^2}}.$$

Proof. Let the initial values $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ such that

$$|\varphi(0)|_E^2 = \mu||u_0||^2 + |u_1 + \varepsilon u_0|^2 \leq 2M_0^2. \tag{82}$$

By (38),

$$\begin{aligned} \overline{G}(u) &\leq \tfrac{1}{c_1}[(f(u), u) + \tfrac{1}{8}||u||^2 + k_2] \\ &\leq \tfrac{1}{c_1}[|f(u)| \cdot |u| + \tfrac{1}{8}||u||^2 + k_2], \quad \forall u \in H_0^1(\Omega), \end{aligned} \tag{83}$$

By (6) and Sobolev embedding theorem, for every $r > 0$, there exists a positive constant $c_{22} = c_{22}(r)$ such that $|f(u)| \leq c_{22}$, $\forall u \in H_0^1(\Omega)$, $||u|| \leq r$. By (83) and (82), there exists a constant $c_{23} = c_{23}(M_0) > 0$ such that $y(0) \leq c_{23}(M_0)$. By the absorbing property (44) of the semigroup $S(t), t \geq 0$, the solution of (28) $\varphi(t) = (u(t), v(t))^T$ in which $v = u_t + \varepsilon u$ satisfies:

$$|\varphi(t)|_E^2 = \mu||u(t)||^2 + |v(t)|^2 \leq c_{24}(M_0), \quad \forall t \geq 0. \tag{84}$$

where $c_{24}(M_0)$ is independent of α.

Let $u = w + z$,where w and z satisfy:

$$\begin{cases} w_{tt} - \alpha\Delta w_t - \Delta w = b(t), \\ w(0) = 0, \quad w_t(0) = 0, \\ b(t) = g - f(u(t)) - h(u_t(t)) \end{cases} \tag{85}$$

and

$$\begin{cases} z_{tt} - \alpha\Delta z_t - \Delta z = 0, \\ z(0) = u_0, \quad z_t(0) = u_1. \end{cases} \tag{86}$$

Let $\phi = (z, z_t + \varepsilon_1 z)^T$, where $\varepsilon_1 = \frac{\lambda_1\alpha}{4+\alpha^2\lambda_1}$, then (86) can be written as

$$\phi_t + \Lambda_1\phi = 0, \quad \phi(0) = (u_0, u_1 + \varepsilon_1 u_0)^T, \tag{87}$$

where

$$\Lambda_1 = \begin{pmatrix} \varepsilon_1 I & -I \\ A - \varepsilon_1\alpha A + \varepsilon_1^2 I & \alpha A - \varepsilon_1 I \end{pmatrix}. \tag{88}$$

Similar to Lemma 1, we have that for any $\phi = (z, z')^T \in E_1$,

$$(\Lambda_1\varphi, \varphi)_E \geq \sigma_1|\varphi|_E^2 + \frac{\alpha}{2}||z'||^2 \geq \sigma_1|\varphi|_E^2 + \frac{\alpha\lambda_1}{2}|z'|^2, \tag{89}$$

where $\sigma_1$is as in (81). By (87) and (89), we obtain (79). It follows that

$$\cup_{t\geq 0,(u_0,u_1)\in \mathbf{B}}\{z(t), z_t(t)\}$$

is contained in a ball which tends strongly to zero.

In the following, we prove that $w(t)$ satisfies (80). Setting $\zeta = A^{\frac{1}{2}}w$, $\varsigma = \zeta_t + \varepsilon_1\zeta$, then (85) can be written as

$$\psi_t + \Lambda_1\psi = B(t), \quad \psi = (\zeta, \varsigma)^T, \quad B(t) = (0, A^{\frac{1}{2}}b(t))^T \tag{90}$$

Taking the inner product $(\cdot,\cdot)_E$ of (90) with $\psi = (\zeta, \varsigma)^T$, we have

$$\frac{d}{dt}|\psi|_E^2 + (\Lambda_1\psi, \psi)_E = (\varsigma, A^{\frac{1}{2}}b) = (A^{\frac{1}{2}}\varsigma, b) \leq ||\varsigma|| \cdot |b|. \tag{91}$$

By emdedding theorem and (6), (16), there exists a constant $c_{25}(M_0) > 0$ such that $|b(t)| \leq c_{25}(M_0)$, $\forall t \geq 0$. By (89) and (91), we find

$$\frac{d}{dt}|\psi(t)|_E^2 + 2\sigma_1|\psi(t)|_E^2 \leq \frac{c_{25}(M_0)}{\alpha}, \quad \forall t \geq 0. \tag{92}$$

By the Gronwall inequality, (85) and (92), we obtain (80), where

$$c_{21}(\alpha_0, \beta_1, \beta_2) = \frac{c_{25}(M_0)}{2}\left[1 + \frac{4}{\alpha_0^2\lambda_1} + \sqrt{1 + \frac{4}{\alpha_0^2\lambda_1}}\right]. \tag{93}$$

The proof is comlpeted. □

Proof of Theorem 2. By the embedding theorem and (80), the continuous norm of $w(t)$ is uniformly bounded with respect to α, i.e., there exists a constant $c_{26} = c_{26}(\alpha_0, \beta_1, \beta_2)$ such that

$$|w(t)|_C \leq c_{26}. \tag{94}$$

Hence, for any $u(t) \in H_0^1(\Omega) \cap \mathbf{B}$, $\xi(t) \in H_0^1(\Omega)$, we have

$$|f'(u(t)))\xi(t)| \leq c_2\left(\int [1 + |z(t) + w(t)|^p]^2\xi^2(t)dx\right)^{\frac{1}{2}}.$$

For $n = 1$, 2, it is easy to see that $|f'(u(t)))\xi(t)| \leq c_{27}(M_0)|\xi(t)|$. For $n = 3$, by (6), (79), (94), the Hölder inequality, (79), (94) and $0 \leq p \leq 2$,

$$\begin{aligned} |f'(u(t)))\xi(t)|^2 &\leq c_{28}\int[1 + z^2(t) + w^2(t)]\xi^2(t)dx \leq c_{29}[||z(t)||^4||\xi(t)||^2 \\ &\quad +(1 + |w(t)|_C^2)|\xi(t)|^2] \\ &\leq c_{30}(M_0)(e^{-4\sigma_1 t}||\xi(t)||^2 + |\xi(t)|^2), \quad c_{30}(M_0) > 0. \end{aligned}$$

So, by (73), we have

$$\begin{aligned}(F'(\varphi(\tau))\Phi_j(\tau),\Phi_j(\tau))_E &= (-f'(u(\tau))\xi_j(\tau),\eta_j(\tau))\\ &\le |f'(u(\tau))\xi_j(\tau)||\eta_j(\tau)|\\ &\le [c_{30}(M_0)(e^{-4\sigma_1\tau}||\xi_j(\tau)||^2+|\xi_j(\tau)|^2)]^{\frac{1}{2}}\cdot|\eta_j(\tau)|\\ &\le \tfrac{c_{30}(M_0)}{2(\alpha\lambda_1+\beta_1)}(e^{-4\sigma_1\tau}||\xi_j(\tau)||^2+|\xi_j(\tau)|^2)\\ &\quad+\tfrac{\alpha\lambda_1+\beta_1}{2}|\eta_j(\tau)|^2.\end{aligned}$$

(95)

Thus, by $||\xi_j(\tau)||^2 \le 2$ and (75), (95)

$$\begin{aligned}q_m &= \lim_{t\to+\infty}\sup_{\Phi\subset E}\sup_{\varphi\in\mathbf{B}}\\ &\quad\tfrac{1}{t}\int_0^t\sum_{j=1}^m((-H'(\varphi(\tau))+F'(\varphi(\tau)))\Phi_j(\tau),\Phi_j(\tau))_E d\tau\\ &\le \lim_{t\to+\infty}\left(-m\sigma+\tfrac{c_{30}(M_0)m}{4\sigma_1(\alpha\lambda_1+\beta_1)t}(1-e^{-4\sigma_1 t})+\tfrac{c_{30}(M_0)}{(\alpha\lambda_1+\beta_1)}\sum_{j=1}^m\lambda_j^{-1}\right)\\ &\le -\tfrac{mc_{30}(M_0)}{(\alpha\lambda_1+\beta_1)}\left(\tfrac{(\alpha\lambda_1+\beta_1)\sigma}{c_{30}(M_0)}-\tfrac{1}{m}\sum_{j=1}^m\lambda_j^{-1}\right).\end{aligned}$$

If

$$\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{-1} < \frac{(\alpha\lambda_1+\beta_1)\sigma}{c_{30}(M_0)}, \tag{96}$$

then $q_m < 0$, that is, (17) holds. The proof of Theorem 2 is comlpeted.

7. Proof Of Theorem 3. In this case, the equation (1) is

$$u_{tt}-\alpha\Delta u_t-\Delta u+\beta u_t+f(u)=g, \quad x\in\Omega, \quad t>0 \tag{97}$$

Similar to the proof of the Theorem 2 where $\beta_1=\beta_2=\beta$, we complete the proof of the Theorem 3.

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