FUNCTIONAL DIFFERENTIAL EQUATIONS

Guest Editors
Ferenc Hartung and Mihály Pituk

VOLUME 11, 2004
No. 1-2

DEDICATED TO ISTVÁN GYÖRI
ON THE OCCASION OF HIS SIXTIETH BIRTHDAY

THE RESEARCH AUTHORITY
THE COLLEGE OF JUDEA & SAMARIA
ARIEL, ISRAEL
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This Special Issue of Functional Differential Equations is dedicated to Professor István Győri on the occasion of his sixtieth birthday. It contains selected papers mainly from those presented at the conference “Delay Differential and Difference Equations with Applications”, Veszprém, Hungary, August 25–29, 2003. In the conference dedicated to the sixtieth birthday of István Győri, there were 61 participants from 23 countries presenting 55 talks.

István Győri graduated from Attila József University in Szeged, Hungary in 1968. Until 1993, he worked as the head of the Computing Centre of the Albert Szent-Györgyi Medical University in Szeged, at the same time he was also a lecturer in math at the Attila József University. In 1976 he received his Candidate in Science degree, and in 1992 he received the scientific degree Doctor in Science in mathematics. Since 1993 he has been working at the University of Veszprém, Hungary as a full-professor of mathematics and the head of the Department of Mathematics and Computing. In the period 1995–1998 he was the President of the University of Veszprém.

István Győri has obtained his most significant scientific results in the theory of oscillation and stability of delay differential and difference equations. Among his numerous important results we mention only the well-known oscillation theorem obtained jointly with Professor Ovide Arino. It says that for a general class of linear autonomous retarded functional differential equations the oscillation of all solutions is equivalent to
the nonexistence of a real root of the characteristic equation. István Győri has published more than one hundred scientific papers on this topic including his monograph on oscillation theory written jointly with Professor Gerry Ladas (Oscillation Theory for Delay Differential Equations with Applications, Oxford University Press, Oxford, 1991). Moreover, he has about fifty papers in medical and informatics applications.

István Győri is a well-known and respected researcher in the mathematical community. His open problems and papers have motivated further research, more than a thousand citations to his papers can be counted in the literature. He published papers together with more than 30 coauthors, he has been a supervisor of several PhD dissertations. In Veszprém he has organized a research group in differential and difference equations.

István Győri acts as a member of the editorial boards of international scientific journals, he has been an invited lecturer and a member of the scientific and organizing committees of numerous international conferences. In particular, he organized two conferences in Veszprém in 1995 and 1996. Based on his scientific and teaching activities, he has received several scientific awards.

We have been delighted to edit this issue to celebrate our colleague and friend, István. On behalf of all the contributors of this volume, all the participants of the conference, and his colleagues and students, we wish him many further years of good health.

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SUBEXPONENTIAL SOLUTIONS OF LINEAR ITÔ-VOLTERRA EQUATIONS WITH A DAMPED PERTURBATION

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Abstract. This paper studies the almost sure non-exponential decay rate of solutions of a scalar linear Itô-Volterra equation with state-independent diffusion coefficient. If the kernel is subexponential, and the diffusion term decays sufficiently quickly, the decay rate is subexponential, and the same as in the deterministic case. If the diffusion coefficient is subexponential, there is a subexponential upper bound on the decay rate of solutions.

Key Words. Volterra integro–differential equations, exponential asymptotic stability, subexponential functions.

AMS(MOS) subject classification. 34K20, 34K50, 60H10

1. Introduction. In this paper we study the asymptotic behaviour of the scalar linear convolution Itô-Volterra equation

\[ dX(t) = (-aX(t) + (k \ast X)(t)) \, dt + \sigma(t) \, dB(t), \]

where \( k \) is continuous, positive and integrable on \([0, \infty)\), and \( \sigma \) is continuous on \([0, \infty)\). Here, \( f \ast g \) denotes the convolution of \( f, g \in C(0, \infty) \)

\[ (f \ast g)(t) = \int_0^t f(t-s)g(s) \, ds. \]

We assume that the initial condition \( X(0) = \xi \) is deterministic, and, as conventional, that (1) is shorthand for the integral equation

\[ X(t) = \xi + \int_0^t \{(-aX(s) + (k \ast X)(s)) \, ds + \int_0^t \sigma(s) \, dB(s), \quad t \geq 0. \]

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The probabilistic setting for this equation is a complete filtered probability space $(\Omega, F, (F(t))_{t \geq 0}, P)$, where $B$ is a scalar standard Brownian motion. According to [7], there is a unique continuous process, adapted to the filtration, which is a strong solution of (1). In the following, the abbreviation a.s. is used for “almost sure” or “almost surely”; in each case these refer to almost sure events relative to the objective probability measure $P$.

Let $z$ be the unique solution of

$$z'(t) = -az(t) + (k * z)(t), \quad t \geq 0, \quad z(0) = 1.$$  \hspace{1cm} (3)

In [4], the following necessary and sufficient conditions for the exponential stability of solutions of (1) were established:

(a) The solution of (3) is integrable and

$$\int_0^\infty k(s)e^{\gamma_1 s} \, ds < \infty \text{ for some } \gamma_1 > 0, \hspace{1cm} (4)$$

$$\int_0^\infty e^{2\gamma_2 s} \sigma(s)^2 \, ds < \infty \text{ for some } \gamma_2 > 0. \hspace{1cm} (5)$$

(b) There is $\beta_0 > 0$ such that all solutions of (1) obey

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| \leq -\beta_0, \quad \text{a.s.}$$

Therefore, solutions cannot decay to zero exponentially if $k$ or $\sigma$ do not obey the exponential decay criteria (4), (5). Thus, when solutions of (1) are a.s. asymptotically stable (see, e.g., [1]), they obey

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| = 0, \quad \text{a.s.}$$

In this paper, we find precise estimates on the a.s. non-exponential decay of solutions, under non-exponential hypotheses on $k$ and $\sigma$. Similar hypotheses are used to study non-exponential stability in Itô-Volterra equations in [6, 3].

2. Subexponential functions and deterministic equations. The following class of functions, introduced in [5], derives from a definition in [8].

**Definition 1.** We say $k \in L^1(R^+) \cap C(R^+; R^+)$ is subexponential if

$$\lim_{t \to \infty} \frac{(k * k)(t)}{k(t)} = 2 \int_0^\infty k(s) \, ds, \hspace{1cm} (6)$$

$$\limsup_{t \to \infty} \frac{1}{s \in [0, \tau]} \left| \frac{k(t - s)}{k(t)} - 1 \right| = 0, \quad \forall \tau > 0. \hspace{1cm} (7)$$
If $k$ obeys these properties, we write $k \in \mathcal{U}$.

As pointed out in [5], condition (7) implies that $k$ obeys

\begin{equation}
\lim_{t \to \infty} k(t)e^{\varepsilon t} = \infty, \quad \text{for all } \varepsilon > 0.
\end{equation}

Thus, if $k$ is subexponential, it obeys neither (4) nor (5). The class of subexponential functions is discussed in detail in [5]. It contains, for example, all positive functions which are regularly varying at infinity, with index $\alpha < -1$.

Denote by $BC$ the space of bounded continuous functions on $(0, \infty)$, and $BC_h$ the space of continuous functions $f$ with $f/h \in BC$. For $f \in BC_h$, let

$$\Lambda_h f = \limsup_{t \to \infty} \frac{|f(t)|}{h(t)}.$$

As in [5], we denote by $BC_{h}^{l}$ the space of continuous functions $f \in BC_h$ where $\lim_{t \to \infty} \frac{f(t)}{h(t)}$ exists. For $f \in BC_h$, we write

$$L_h f = \lim_{t \to \infty} \frac{f(t)}{h(t)}.$$

We will frequently use the following lemma, proved in [5].

**Lemma 1.** Let $h \in \mathcal{U}$. If $f, g \in BC_{h}^{l}$, then $f \ast g \in BC_{h}^{l}$ and

$$L_h (f \ast g) = L_h f \int_0^\infty g(s) \, ds + L_h g \int_0^\infty f(s) \, ds.$$

In this paper, we state an analogous result for functions in $BC_h$. As it may be proved in a similar fashion to Lemma 1, its proof is omitted.

**Lemma 2.** Let $h \in \mathcal{U}$. If $f, g \in BC_h$, then $f \ast g \in BC_h$ and

$$\Lambda_h (|f| \ast |g|) \leq \Lambda_h f \int_0^\infty |g(s)| \, ds + \Lambda_h g \int_0^\infty |f(s)| \, ds.$$

These results yield the following perturbation theorem, partly proven in [5].

**Theorem 1.** Let $f \in C(0, \infty) \cap L^1(0, \infty)$, $k \in C([0, \infty); (0, \infty)) \cap L^1(0, \infty)$, $a > \int_0^\infty k(s) \, ds$, and $x$ be the solution of

\begin{equation}
x'(t) = -ax(t) + (k \ast x)(t) + f(t), \quad t \geq 0.
\end{equation}

(a) If $k \in \mathcal{U}$, $f \in BC_{k}^{l}$, then

$$L_k x = \frac{L_k f + \int_0^\infty x(s) \, ds}{a - \int_0^\infty k(s) \, ds}.$$

(b) If $f$ obeys (6), (7), and $L_k k = 0$, then $L_k f = (a - \int_0^\infty k(s) \, ds)^{-1}$.

This result informs our study of the stochastically perturbed equation (1). It suggests that we find a deterministic proxy for the size of the random perturbation, to determine the critical decay rate of $\sigma$ at which there is a transition from “small” to “large perturbation” asymptotics.
3. Asymptotic stability of (1). We now study the a.s. asymptotic stability of solutions of (1) where $\sigma \in L^2(0, \infty)$. This allows for direct comparison with equation (9) when $f \in L^1(0, \infty)$, for when $\sigma \in L^2(0, \infty)$, the Itô integral on the righthand side of (2) has a finite limit as $t \to \infty$, a.s.

**Theorem 2.** If the solution of (3) is in $L^1(0, \infty)$, $k \in C((0, \infty)) \cap L^1(0, \infty)$, and $\sigma \in C(0, \infty) \cap L^2(0, \infty)$, then

(a) $\lim_{t \to \infty} E[X(t)^2] = 0$, $E[X^2] \in L^1(0, \infty)$, and

(b) $\lim_{t \to \infty} X(t) = 0$, $X \in L^2(0, \infty)$, a.s.

**Proof.** With $z$ defined by (3), the solution of (1) is $X(t) = X(0)z(t) + Y(t)$, where $Y(t) = \int_0^t z(t-s)\sigma(s)dB(s)$ (see [1, 9]). In [1] it is noted that $Y(t)$ is a normally distributed random variable with zero mean and variance $\nu(t)^2$, where $\nu(t)^2 = (z^2 \ast \sigma^2)(t)$. Since $z \in L^1(0, \infty)$ implies $z(t) \to 0$ as $t \to \infty$, $z^2 \in L^1(0, \infty)$. But $\sigma^2 \in L^1(0, \infty)$ gives $\nu(t) \to 0$ as $t \to \infty$ and $\nu \in L^2(0, \infty)$. Since $E[X(t)^2] \leq 2z(t)^2X(0)^2 + 2\nu(t)^2$, part (a) holds. Part (b) follows by part (a) and the method of proof of Theorem 1 in [2]. □

As $k$ is positive here, the condition $a > \int_0^\infty k(s) \, ds$ implies $z \in L^1(0, \infty)$.

4. Main Results. The proofs of the main results in this paper rely on rewriting the solution of (1) in terms of the solution of a perturbed Volterra integrodifferential equation whose solution, although random, is in $C^1(0, \infty)$.

**Lemma 3.** If $k \in C([0, \infty); (0, \infty)) \cap L^1(0, \infty)$, $a > \int_0^\infty k(s) \, ds$, then for a.a. $\omega \in \Omega$, the path $X(\omega)$ obeys

\begin{equation}
X(t, \omega) = U(t, \omega) + T(t, \omega)
\end{equation}

where $T(\omega)$ is the function defined by

\begin{equation}
T(t, \omega) = \left( \int_t^\infty \sigma(s) \, dB(s) \right)(\omega),
\end{equation}

and $f(\omega)$, $U(\omega)$ obey

\begin{align}
f(t, \omega) & = -aT(t, \omega) + (k \ast T(\omega))(t), \\
U'(t, \omega) & = -aU(t, \omega) + (k \ast U(\omega))(t) + f(t, \omega).
\end{align}

We refer the reader to a similar result, proven in [3], where a more general result on the asymptotic behaviour of the random variable $T(t)$ also appears.

**Lemma 4.** Let $\sigma \in C([0, \infty); (0, \infty)) \cap L^2(0, \infty)$, and

\begin{equation}
\Sigma(t)^2 = \int_t^\infty \sigma(s)^2 \, ds \log(\int_t^\infty \sigma(s)^2 \, ds)^{-1}.
\end{equation}
Then $T$ defined by (11) obeys
\begin{equation}
\limsup_{t \to \infty} \frac{|T(t)|}{\Sigma(t)} = \sqrt{2}, \quad \text{a.s.}
\end{equation}

This, along with Lemmas 1 and 2 suggests that the random perturbation $f$ in (13) decays at the slower rate between $k$ and $\Sigma$. Viewing Theorem 1 in the light of Lemma 3, the case in which $L_k \Sigma = 0$ seems to correspond to part (a) of the theorem, while that in which $L_k \Sigma = 0$ roughly corresponds to part (b). The following result thus parallels Theorem 1, part (a).

**Theorem 3.** If $k \in C([0, \infty); (0, \infty)) \cap L^1(0, \infty)$, $\sigma \in L^2(0, \infty)$, $\sigma \in L^2(0, \infty)$, $a > \int_0^\infty k(s) ds$, $k$ is subexponential, and $\Sigma$ obeys $L_k \Sigma = 0$, then the unique strong solution of (1) obeys
\begin{equation}
L_k X = \frac{\int_0^\infty X(s)}{a - \int_0^\infty k(s) ds} =: G, \quad \text{a.s.}
\end{equation}

where $G$ is a normally distributed $F$-measurable random variable, which is a.s. nonzero, and has mean $\xi/(a - \int_0^\infty k(s) ds)$, and variance $\int_0^\infty \sigma(s)^2 ds/(a - \int_0^\infty k(s) ds)^2$. Moreover, $X(\omega)$ obeys (6), (7) for almost all $\omega \in \Omega$.

The conclusion of the first part of the result is precisely that of Theorem 1, when $L_k f = 0$. So, although the sample paths of $X$ are nowhere differentiable, a.s., they behave asymptotically like $k$, which can be in $C^\infty(0, \infty)$.

**Proof.** The second statement of the theorem follows from the first by arguments of [5] applied to each $\omega$ in the a.s. set on which (16) holds, and $G$ is non-trivial. As to the proof of the first part, the hypotheses and (15) imply $L_k T = 0$ a.s. Since $k$ is integrable, $T \in L^1(0, \infty)$, a.s. Applying Lemma 1 to $f$ in (12) yields $L_k f = \int_0^\infty T(s) ds$, a.s., so by Theorem 1, $U$ obeys
\begin{equation}
L_k U = \frac{L_k f + \int_0^\infty U(s) ds}{a - \int_0^\infty k(s) ds}.
\end{equation}

(10) now gives (16). To find the distribution of $G$, note that $k \in L^1(0, \infty)$ implies $X \in L^1(0, \infty)$. Therefore, letting $t \to \infty$ on both sides of (2), and noting that $\sigma \in L^2(0, \infty)$ implies $\lim_{t \to \infty} \int_0^t \sigma(s) dB(s)$ exists a.s., gives
\begin{equation}
\int_0^\infty X(s) ds = \frac{\xi + \int_0^\infty \sigma(s) dB(s)}{a - \int_0^\infty k(s) ds}.
\end{equation}

Since $\int_0^\infty \sigma(s) dB(s)$ is normally distributed with variance $\int_0^\infty \sigma(s)^2 ds$, $G$ has the claimed distribution. Since $\sigma \neq 0$, $G$ is not trivial, so $L_k X \neq 0$, a.s. □

We now supply the stochastic analogue of case (b) in Theorem 1.
THEOREM 4. If \( k \in C([0, \infty); (0, \infty)) \cap L^1(0, \infty), \sigma \in L^2(0, \infty) \cap C([0, \infty); (0, \infty)), a > \int_0^\infty k(s) \, ds, \Sigma \) defined by (14) is subexponential, and \( \Lambda_\Sigma k = L \in [0, \infty) \), then the unique strong solution of (1) obeys \( \Lambda_\Sigma X < \infty \), a.s. Moreover, if \( L_\Sigma k = 0 \) then

\[
\Lambda_\Sigma X \leq \frac{2\sqrt{2a}}{a - \int_0^\infty k(s) \, ds}, \quad \text{a.s.}
\]

Proof. We study the case where \( L_\Sigma k = 0 \) only: the case when \( L \neq 0 \) is similar and hence omitted. By (15), \( \Lambda_\Sigma T = \sqrt{2} \), a.s. Thus Lemma 2, \( L_\Sigma k = 0 \), and (12) gives

(17) \[
\Lambda_\Sigma f \leq \sqrt{2}(a + \int_0^\infty k(s) \, ds), \quad \text{a.s.}
\]

Define \( e_a(t) = e^{-at}, h = e_a * k: \) then \( z = e_a + e_a * r \), where \( r \) solves \( r = h + h * r \). Then \( L_\Sigma k = 0 \), Lemma 2 imply \( \Lambda_\Sigma h = 0 \). An argument involving Lemma 2 yields \( L_\Sigma r = 0 \), so \( L_\Sigma z = \Lambda_\Sigma z = 0 \). As \( \Lambda_\Sigma z = 0 \), \( \Lambda_\Sigma U = \Lambda_\Sigma (z * f) \). The result now follows, by Lemma 2, (13), (15), and (17). \( \square \)

Acknowledgements
The author is very grateful for discussions with Kieran Murphy, and also to the anonymous referee for their careful reading of the manuscript.

REFERENCES

SUBEXPONENTIAL SOLUTIONS OF SCALAR LINEAR INTEGRO–DIFFERENTIAL EQUATIONS WITH DELAY

J. A. D. APPLEBY*, I. GYŐRI† AND D. W. REYNOLDS‡

Abstract. This paper considers the asymptotic behaviour of solutions of the scalar linear convolution integro-differential equation with delay

\[ x'(t) = -\sum_{i=1}^{n} a_i x(t - \tau_i) + \int_{0}^{t} k(t-s)x(s) ds, \quad t > 0, \]
\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \]

where \( \tau = \max_{1 \leq i \leq n} \tau_i \). In this problem, \( k \) is a non-negative function in \( L^1(0, \infty) \cap C[0, \infty) \), \( \tau_i \geq 0, a_i > 0 \) and \( \phi \) is a continuous function on \( [-\tau, 0] \). The kernel \( k \) is subexponential in the sense that \( \lim_{t \to \infty} k(t) \alpha(t)^{-1} > 0 \) where \( \alpha \) is a positive subexponential function. A consequence of this is that \( k(t)e^{\epsilon t} \to \infty \) as \( t \to \infty \) for every \( \epsilon > 0 \).

Key Words. Volterra integro–differential equations, subexponential function, exponential asymptotic stability.

AMS(MOS) subject classification. 34K20, 34K25, 34K06, 45D05, 45J05

1. Introduction and Results. This paper examines the asymptotic behaviour of solutions of the scalar linear integrodifferential equation with delay

\[ x'(t) = -\sum_{i=1}^{n} a_i x(t - \tau_i) + \int_{0}^{t} k(t-s)x(s) ds, \quad t > 0, \]

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subject to the initial condition

\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0. \]

The following hypotheses are postulated.

(H1) \( \tau_i \geq 0, a_i \geq 0, \sum_{i=1}^{n} a_i > 0 \) and \( \tau = \max_{1 \leq i \leq n} \tau_i \).

(H2) The characteristic equation

\[ p = \sum_{i=1}^{n} a_i e^{\tau_i p}, \]

has a real root.

(H3) The kernel \( k \) is a non-trivial function in \( L^1(0, \infty) \cap C[0, \infty) \) with \( k(t) \geq 0 \) for all \( t \geq 0 \).

(H4) \( \int_{0}^{\infty} k(s) \, ds < \sum_{i=1}^{n} a_i \).

(H5) \( \lim_{t \to \infty} k(t)/\alpha(t) > 0 \) for some positive subexponential function \( \alpha \).

(H6) The initial function \( \phi \) is in \( C[-\tau, 0] \).

The significance of (H2), and sufficient conditions for it to hold, are discussed in Section 2. The definition of positive subexponential functions and some of their important properties are also reviewed there. The main result of this paper is the following theorem.

**Theorem 1.** Suppose that (H1)–(H6) hold. Then the solution of (1) and (2) satisfies

\[ \lim_{t \to \infty} \frac{x(t)}{k(t)} = \frac{\phi(0) - \sum_{i=1}^{n} a_i \int_{-\tau_i}^{0} \phi(s) \, ds}{\left( \sum_{i=1}^{n} a_i - \int_{0}^{\infty} k(s) \, ds \right)^2}, \]

\[ \lim_{t \to \infty} \frac{x'(t)}{k(t)} = 0. \]

The decay rate given in (3) can also be written as

\[ \lim_{t \to \infty} \frac{x(t)}{k(t)} = \frac{\int_{0}^{\infty} x(s) \, ds}{\sum_{i=1}^{n} a_i - \int_{0}^{\infty} k(s) \, ds}. \]

It is shown in [1, 2] that the decay rate of two classes of stochastic Volterra integrodifferential equations with subexponential kernels, can also be expressed in this form.

In order to prove Theorem 1 we introduce the resolvent for (1), which is the solution of the equation

\[ r'(t) = -\sum_{i=1}^{n} a_i r(t - \tau_i) + \int_{0}^{t} k(t - s) r(s) \, ds, \quad t > 0. \]
which satisfies the initial condition

\[ r(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0. \end{cases} \]

The significance of \( r \) is that the solution of (1) which obeys (2) is given by the variation of parameters formula

\[ x(t) = \phi(0)r(t) + \int_0^t r(t - s)\tilde{\phi}(s) \, ds, \quad t \geq 0, \]

where

\[ \tilde{\phi}(t) = -\sum_{i=1}^n a_i \phi(t - \tau_i)\chi_{[0,\tau_i]}(t), \quad t \geq 0, \]

and \( \chi_J \) denotes the indicator function of a set \( J \). The asymptotic behaviour of the resolvent is described in the next theorem.

**Theorem 2.** Suppose that \((H_1)-(H_4)\) hold. Then the resolvent \( r \), defined by (5) and (6), is in \( \mathcal{L}^1(0,\infty) \), \( r(t) > 0 \) for all \( t \geq 0 \) and \( r(t) \to 0 \) as \( t \to \infty \). If, in addition, \((H_5)\) holds,

\[ \lim_{t \to \infty} \frac{r(t)}{k(t)} = \left( \sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds \right)^{-2}, \quad \lim_{t \to \infty} \frac{r'(t)}{k(t)} = 0. \]

Moreover

\[ \lim_{t \to \infty} \frac{\int_0^t r(t - s)r(s) \, ds}{r(t)} = 2 \int_0^\infty r(s) \, ds. \]

Theorems 1 and 2 are generalisations of [3, Theorem 6.2], which concerns linear scalar convolution integro-differential equations with subexponential kernels but without delays. Theorem 2 has the following converse, which is an extension of [3, Theorem 6.4].

**Theorem 3.** Suppose that \((H_1)-(H_4)\) hold, and that \( k(t) > 0 \) for all \( t \geq 0 \). If the resolvent \( r \) satisfies (9), then \( k \) is a positive subexponential function and (10) is true.

**2. Mathematical Preliminaries.** The convolution of two appropriate functions \( f \) and \( g \) defined on \([0,\infty)\) is denoted, as usual, by

\[ (f * g)(t) = \int_0^t f(t - s)g(s) \, ds, \quad t \geq 0. \]
We recall a definition from [3], based on the hypotheses of Theorem 3 of [5].

**Definition 1.** A positive subexponential function is a continuous integrable function \( \alpha : [0, \infty) \to (0, \infty) \) satisfying

\[
\lim_{t \to \infty} \frac{(\alpha \ast \alpha)(t)}{\alpha(t)} = 2 \int_0^\infty \alpha(s) \, ds,
\]

\[
\lim_{t \to \infty} \sup_{0 \leq s \leq t} \left| \frac{\alpha(t-s)}{\alpha(t)} - 1 \right| = 0, \quad \text{for all } A > 0.
\]

It is noted in [3] that the class of positive subexponential functions includes all positive, continuous, integrable functions which are regularly varying at infinity. It is known that (12) implies for every \( \epsilon > 0 \) that

\[
\alpha(t)e^{\epsilon t} \to \infty \quad \text{as } t \to \infty,
\]

(cf., e.g., [4, Lemma 2.2]), and hence by (H_5) that \( k(t)e^{\epsilon t} \to \infty \) as \( t \to \infty \) for every \( \epsilon > 0 \).

If \( \alpha \) is a positive subexponential function and \( f \) is a function on \((0, \infty)\) such that \( \lim_{t \to \infty} f(t)/\alpha(t) \) exists, we define

\[
L_\alpha f = \lim_{t \to \infty} \frac{f(t)}{\alpha(t)}.
\]

An important result is the following lemma. It is essentially Theorem 4.1 of [3]. Perusal of the proof of this theorem shows that the hypotheses that \( f/\alpha \) and \( g/\alpha \) be bounded continuous functions on \([0, \infty)\) are redundant, and are therefore omitted here.

**Lemma 1.** Suppose that \( \alpha \) is a positive subexponential function. Let \( f \) and \( g \) be integrable functions on \((0, \infty)\) for which \( L_\alpha f \) and \( L_\alpha g \) both exist. Then \( L_\alpha (f \ast g) \) exists and is given by

\[
L_\alpha (f \ast g) = L_\alpha f \int_0^\infty g(s) \, ds + L_\alpha g \int_0^\infty f(s) \, ds.
\]

Next we introduce the resolvent \( z \) associated with the purely point delay part of (1). It satisfies the equation

\[
z'(t) = - \sum_{i=1}^n a_i z(t - \tau_i), \quad t > 0,
\]
and the initial condition

\begin{equation}
  z(t) = \begin{cases} 
    1, & t = 0, \\
    0, & -\tau \leq t < 0. 
  \end{cases}
\end{equation}

We collect in a lemma some salient properties of $z$.

**Lemma 2.** Suppose that (H$_1$) and (H$_2$) hold. Then $z(t) > 0$ for $t \geq 0$ and

\begin{equation}
  \int_0^\infty z(t) \, dt = \frac{1}{\sum_{i=1}^n a_i},
\end{equation}

\begin{equation}
  z(t) \to 0 \quad \text{exponentially as } t \to \infty.
\end{equation}

Thus if $\alpha$ is a subexponential function

\begin{equation}
  L_\alpha z = 0.
\end{equation}

The positivity of $z$ and (17) are part of Proposition 2.1 of [7]; (18) follows from the same proposition and Lemma 6.5.3 of [8]; (19) is a consequence of (13) and (18).

It is shown in [9] that a necessary condition for (H$_2$) to be true is $\sum_{i=1}^n a_i \tau_i \leq e^{-1}$, and that $\tau \sum_{i=1}^n a_i \leq e^{-1}$ is a sufficient condition. In the case of a single delay with $n = 1$, $a_1 = a > 0$, $\tau_1 = \tau$, a necessary and sufficient condition for (H$_2$) to hold is $a\tau \leq e^{-1}$.

The following yields a representation of the solution of

\begin{equation}
  y'(t) = -\sum_{i=1}^n a_i y(t - \tau_i) + f(t), \quad t > 0,
\end{equation}

\begin{equation}
  y(t) = \begin{cases} 
    1, & t = 0, \\
    0, & -\tau \leq t < 0. 
  \end{cases}
\end{equation}

**Lemma 3.** Let $f$ be in $C[0, \infty)$. Then the solution of (20) and (21) can be represented as $y(t) = z(t) + (z * f)(t), \quad t \geq 0$.

3. **Proofs.** Proof. (Theorem 2) The resolvent $r$ of (1) satisfies (5) and (6). It is a consequence of Lemma 3 that $r$ satisfies

\begin{equation}
  r = z + z * (k * r) = z + h * r,
\end{equation}

where $h = z * k$. Due to (H$_3$) and Lemma 2, $h(t) \geq 0$ for all $t \geq 0$. A standard argument shows that $r(t) > 0$ for all $t \geq 0$. By taking the convolution of each term in (22) with $k$, we see that

$$
\rho = h + h * \rho,
$$

...
where $\rho = k * r$. Since by (17) and (H$_4$),

$$\int_0^\infty h(s) \, ds = \int_0^\infty z(s) \, ds \int_0^\infty k(s) \, ds = \frac{1}{\sum_{i=1}^n a_i} \int_0^\infty k(s) \, ds < 1,$$

it can be deduced from a theorem of Paley and Wiener (cf., e.g., [6, Theorem 2.4.1]) that $\rho$ is in $L^1(0, \infty)$. It is then an immediate consequence of Lemma 2 and

$$r = z + \rho * z,$$

that $r$ is in $L^1(0, \infty)$. It then follows from this equation that $r(t) \to 0$ as $t \to \infty$, since $z$ is a bounded continuous function obeying (18).

Integration of (5) gives

$$-1 = -\sum_{i=1}^n a_i \int_0^\infty r(t - \tau_i) \, dt + \int_0^\infty r(s) \, ds \int_0^\infty k(s) \, ds,$$

which the aid of (6) leads to

$$\int_0^\infty r(s) \, ds = \frac{1}{\sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds}.$$

Also we can deduce from Lemma 1, (17) and (19) that

$$L_\alpha h = L_\alpha (z * k) = L_\alpha z \int_0^\infty k(s) \, ds + L_\alpha k \int_0^\infty z(s) \, ds = \frac{L_\alpha k}{\sum_{i=1}^n a_i}.$$

Suppose now that $L_\alpha r$ exists. Then we can infer from Lemma 1, (19), (22) and the above formulae that

$$L_\alpha r = L_\alpha z + L_\alpha h \int_0^\infty r(s) \, ds + L_\alpha r \int_0^\infty h(s) \, ds$$

$$= \frac{L_\alpha k}{\sum_{i=1}^n a_i (\sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds)} + L_\alpha r \frac{1}{\sum_{i=1}^n a_i} \int_0^\infty k(s) \, ds.$$

Rearranging yields the first formula in (9). To obtain the second, note that (12) implies $L_\alpha r(\cdot - \tau_i) = L_\alpha r$, so then, by applying $L_\alpha$ to (5) and using Lemma 1, we get that

$$L_\alpha r' = -L_\alpha r \left( \sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds \right) + L_\alpha k \int_0^\infty r(s) \, ds = 0.$$

We note that by Lemma 1

$$L_\alpha (r * r) = 2L_\alpha r \int_0^\infty r(s) \, ds,$$
which immediately implies (10).

To complete the proof, it only remains to show that $L\alpha r$ exists. For the sake of brevity a proof is indicated here under the additional (and unnecessary) assumption that $k(t) > 0$ for all $t \geq 0$. It follows then that $h(t) > 0$ for all $t > 0$. Then by Lemma 4.3 of [3], $h$ is a subexponential function. By applying Theorem 5.2 of [3] to (22), we conclude that $Lhr$ exists and hence $L\alpha r$. 

Proof. (Theorem 1) First, we observe from (8) that $L\alpha \tilde{\phi} = 0$ and

$$
\int_0^\infty \tilde{\phi}(t) \, dt = -\sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s) \, ds.
$$

Then, by applying $L\alpha$ to (7) and using Lemma 1, we obtain

$$
L\alpha x = \phi(0)L\alpha r + L\alpha r \int_0^\infty \tilde{\phi}(s) \, ds + L\alpha \tilde{\phi} \int_0^\infty r(s) \, ds.
$$

Therefore we can conclude that

$$
L\alpha x = L\alpha k \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s) \, ds}{(\sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds)^2},
$$

and hence that (3) holds.

It also follows easily from (7) that

$$
\int_0^\infty x(t) \, dt = \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s) \, ds}{\sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds}.
$$

By applying Lemma 1 to (1), we then see that

$$
L\alpha x' = -\sum_{i=1}^n a_i L\alpha x(\cdot - \tau_i) + L\alpha x \int_0^\infty k(s) \, ds + L\alpha k \int_0^\infty x(s) \, ds
$$

$$
= \left( \int_0^\infty k(s) \, ds - \sum_{i=1}^n a_i \right) L\alpha x + L\alpha k \int_0^\infty x(s) \, ds
$$

$$
= -L\alpha k \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s) \, ds}{\sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds} + L\alpha k \int_0^\infty x(s) \, ds = 0.
$$

Therefore (4) is true. 

Proof. (Theorem 3) For convenience we introduce

$$
\eta = \sum_{i=1}^n a_i - \int_0^\infty k(s) \, ds.
$$
Dividing (5) by $k(t)$, we see that

$$\frac{r'(t)}{k(t)} = - \sum_{i=1}^{n} a_i \frac{r(t - \tau_i)}{k(t)} + \frac{(k \ast r)(t)}{k(t)}.$$ 

By letting $t \to \infty$ and using (9),

$$\lim_{t \to \infty} \frac{(k \ast r)(t)}{k(t)} = \frac{1}{\eta^2} \sum_{i=1}^{n} a_i.$$ 

Hence $(k \ast r)(t)/r(t) \to \sum_{i=1}^{n} a_i$ as $t \to \infty$. Since $r$ and $k$ are positive and $\lim_{t \to \infty} k(t)/r(t) > 0$, Lemma 3.8 of [3] applies. We can conclude from it that $k$ satisfies (12) and

$$\lim_{t \to \infty} \frac{(k \ast k)(t)}{k(t)} = \sum_{i=1}^{n} a_i + \int_{0}^{\infty} k(s) \, ds - \eta^2 \int_{0}^{\infty} r(s) \, ds = 2 \int_{0}^{\infty} k(s) \, ds.$$ 

Thus $k$ satisfies (11), finishing the proof. ∎

REFERENCES


ON BOHL-PERRON TYPE THEOREMS FOR LINEAR DIFFERENCE EQUATIONS*

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Abstract. For a linear nonhomogeneous difference equation the following result is obtained: if a solution is bounded for any bounded right hand side, then the equation is exponentially stable. As an application, explicit exponential stability conditions are deduced for a scalar equation $x(n+1) - x(n) + \sum_{k=1}^{N} a_k(n)x(g_k(n)) = 0$, with nonconstant bounded delays $g_k(n)$.

Key Words. Difference equations, Bohl-Perron theorems, Exponential stability

AMS(MOS) subject classification. 39A11, 39A12, 39A10

1. Introduction. In this paper we obtain a Bohl-Perron type theorem for linear difference equations. Such results are well known in the theory of ordinary differential equations [1], delay differential equations [2-5], impulsive differential equations [6] and can be described as follows: if for any bounded right hand side a solution $x(t)$ is bounded together with its first derivative, then this equation is exponentially stable. For first order linear difference equations in Banach spaces a Bohl-Perron type result was established in [7]. Here we prove a similar fact for linear difference equations, based on the representation of solution obtained in [9]. Many results in the theory

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* The first author was partially supported by Israel Ministry of Absorption. The second author was partially supported by the NSERC Research Grant and the AIF Research Grant

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of ordinary differential equations and especially delay differential equations have their analogues in the theory of difference equations, see, for example, [8]. We apply the Bohl-Perron type theorem to study an exponential stability of a scalar difference equation with a several variable delays which extends the result of Győri and Pituk [10] for a single constant delay.

2. Preliminaries. Let \( \mathbb{R}^m \) be the space of \( m \)-dimensional column vectors \( x = \text{col}(x_1, x_2, \ldots, x_m) \) with the norm \( \|x\| = \max_{1 \leq i \leq m} |x_i| \), by the same symbol \( \| \cdot \| \) we will denote the corresponding matrix \( m \times m \) matrix or an identity operator, \( I^\infty \) is a space of bounded sequences, \( \|\{a_n\}_{n=0}^\infty\|_\infty = \sup_{n \geq 0} \|a_n\| \).

We consider the linear difference equation

\[
(1) \quad x(n + 1) - x(n) = \sum_{k=0}^{n} B(n, k)x(k) + g(n), \quad n \geq 0,
\]

where \( B(n, k) \) are \( m \times m \) matrices and the initial condition \( x(0) \) is prescribed.

In addition the equation with some prehistory

\[
(2) \quad x(n + 1) - x(n) = \sum_{k=-d}^{n} B(n, k)x(k) + g(n), \quad n \geq 0,
\]

and with the following initial conditions

\[
(3) \quad x_n = \varphi(n), \quad n \leq 0,
\]

will be considered as well as the corresponding homogeneous difference equations

\[
(4) \quad x(n + 1) - x(n) = \sum_{k=0}^{n} B(n, k)x(k), \quad n \geq 0,
\]

\[
(5) \quad x(n + 1) - x(n) = \sum_{k=-d}^{n} B(n, k)x(k), \quad n \geq 0.
\]

We will also consider the following homogeneous equation

\[
(6) \quad x(n + 1) - x(n) = \sum_{k=l}^{n} B(n, k)x(k), \quad n \geq l.
\]

**Definition.** The matrix solution \( X(n, l) \) of the equation (6), with \( X(l, l) = I \), is called the **fundamental matrix** of (1).
We assume \( X(n, l) = 0, n < l \). Let us note that equations (1), (2), (4), (5) have the same fundamental matrix.

**Definition.** The equation (2) is said to be *exponentially stable* if there exist such constants \( N \) and \( \lambda \) that for any solution of the homogeneous equation (5) with the initial conditions (3) the following inequality holds:

\[
\|x(n)\| \leq Ne^{-\lambda n} \max_{-d \leq k \leq 0} \|\varphi(k)\|, \quad n \geq 0.
\]

3. **Representation of Solutions.** We will apply the following result for the equation (1).

**Lemma 1.** [9] Let \( X(n, l) \) be a fundamental matrix of (1). Then the solution of (1) can be presented as

\[
x(n) = X(n, 0)x(0) + \sum_{k=0}^{n-1} X(n, k + 1)g(k).
\]

Now let us proceed to the representation of solutions in the general case.

**Lemma 2.** The solution of (2), (3) can be presented as

\[
x(n) = X(n, 0)x(0) + \sum_{l=0}^{n-1} X(n, l + 1)g(l)
\]

\[
- \sum_{l=0}^{n-1} X(n, l + 1) \sum_{k=-d}^{-1} B(l, k)\varphi(k).
\]

**Proof.** Since for any solution of (2), (3) we have \( \sum_{k=-d}^{n} B(n, k)x(k) \)

\[
= \sum_{k=-d}^{-1} B(n, k)x(k) + \sum_{k=0}^{n} B(n, k)x(k) = \sum_{k=-d}^{-1} B(n, k)\varphi(k) + \sum_{k=0}^{n} B(n, k)x(k),
\]

then (2), (3) can be reduced to the problem with the zero initial conditions up to \( n = -1 \):

\[
x(n + 1) - x(n) = \sum_{k=0}^{n} B(n, k)x(k) + \left[ \sum_{k=-d}^{-1} B(n, k)\varphi(k) + g(n) \right],
\]

\( n \geq 0 \). After substituting the expression in the brackets for \( g(n) \) in the equality (8) we have (9). \( \square \)
DEFINITION. Let us define the following operator in a certain space of sequences (at this step we do not specify the space)

\[(11) \quad \mathcal{C}(\{g(n)\}_{n=0}^{\infty}) = \left\{ y_n = \sum_{l=0}^{n-1} X(n, l + 1)g(l) \right\}_{n=0}^{\infty}, \]

where \(y_0 = 0\). We will call \(\mathcal{C}\) the Cauchy operator.

For the zero initial conditions \(x(n) = 0, \; n \leq 0\), each of equations (1) or (2) describes a linear mapping \(\mathcal{L}:\)

\[(12) \quad \{g(n)\}_{n=0}^{\infty} = \mathcal{L}(\{x(n)\}_{n=1}^{\infty}) = \left\{ x(n + 1) - x(n) - \sum_{k=1}^{n} B(n, k)x(k) \right\}, \]

with \(x(0) = 0\).

**Lemma 3.** Let \(\mathcal{L}, \; \mathcal{M}: \ell^\infty \to \ell^\infty\) be linear bounded operators of type (12), let \(\mathcal{C}_\mathcal{L}, \mathcal{C}_\mathcal{M}\) be the Cauchy operators of the equations \(\mathcal{L}(\{x(n)\}) = \{g(n)\}\) and \(\mathcal{M}(\{x(n)\}) = \{g(n)\}\), respectively. Suppose the Cauchy operator \(\mathcal{C}_\mathcal{L}\) is a bounded operator which maps \(\ell^\infty\) into \(\ell^\infty\) and \(\mathcal{M}\mathcal{C}_\mathcal{L}: \ell^\infty \to \ell^\infty\) is invertible.

Then \(\mathcal{C}_\mathcal{M}\) also maps \(\ell^\infty\) into \(\ell^\infty\) and is bounded.

**Proof.** For any sequence \(\{g(n)\}\) the sequence

\(\{x(n)\} = \mathcal{C}_\mathcal{M}(\mathcal{MC}_\mathcal{L})^{-1}(\{g(n)\})\)

is a solution of \(\mathcal{M}(\{x(n)\}) = \{g(n)\}\) with the zero initial conditions. Consequently \(\mathcal{C}_\mathcal{M} = \mathcal{C}_\mathcal{L}(\mathcal{MC}_\mathcal{L})^{-1}\), thus \(\mathcal{C}_\mathcal{M}: \ell^\infty \to \ell^\infty\) is bounded. \(\square\)

**Lemma 4.** Suppose that for the equation (1) the following condition holds:

\((a1)\) there exists \(M > 0\) such that \(\sup_{n \geq 0} \sup_{l=-d} \|B(n, l)\| \leq M\).

Then (12) is a bounded operator on the space \(\ell^\infty\).

**Proof.** We have \(\mathcal{L}(\{x(n)\}) = \left\{ x(n + 1) - x(n) - \sum_{k=1}^{n} B(n, k)x(k) \right\}\).

Since \(\|\mathcal{L}(\{x(n)\})\|_{\ell^\infty} \leq \sup_{n} \|x(n + 1)\| + \sup_{n} \|x(n)\| + \sup_{n} \|x(n)\| \sup_{n} \sum_{l=1}^{n} \|B(n, l)\| \leq (2 + M) \|\{x(n)\}\|_{\ell^\infty},\) then \(\mathcal{L}\) maps \(\ell^\infty\) into \(\ell^\infty\) and it is bounded with the norm \(\|\mathcal{L}\|_{\ell^\infty \to \ell^\infty} \leq 2 + M\). \(\square\)

4. **Main Results.** Now let us proceed to exponential estimates of the fundamental matrix and to the exponential stability of difference equations in the case when the number of terms in the right hand side of (2) (or (1)) is bounded.
THEOREM 1. Suppose (a1) holds and

(a2) there exists $T > 0$ such that $B(n, l) = 0$ whenever $n - l > T$. Suppose that for every bounded sequence $\{g(n)\} \in l^\infty$ the solution $\{x(n)\}$ of (1) with the zero initial condition is also bounded: $\{x(n)\} \in l^\infty$. Then there exist $N > 0, \lambda > 0$ such that the fundamental matrix $X$ of (1) satisfies

$$\|X(n, l)\| \leq Ne^{-\lambda(n-l)}.$$  

Proof. First let us establish an exponential estimate for $X(n, 0)$. For some positive number $\lambda$ define $y(n) = x(n)e^{\lambda n}$, assume $x(0) = y(0) = 0$ and consider an operator

$$\mathcal{L} (\{ x(n) \}) = \left\{ x(n+1) - x(n) - \sum_{k=1}^{n} B(n, k)x(k) \right\}.$$  

After substituting $x(n) = y(n)e^{-\lambda n}$ we have

$$\mathcal{L} (\{ x(n) \}) = \left\{ y(n+1)e^{-\lambda(n+1)} - y(n)e^{-\lambda n} - \sum_{k=1}^{n} B(n, k)y(k)e^{-\lambda k} \right\}$$

$$= \left\{ e^{-\lambda(n+1)} \left[ y(n+1) - y(n) + y(n) - e^{\lambda}y(n) \right. \right.$$  

$$\left. - \sum_{k=\max\{0,n-T\}}^{n} B(n, k)y(k)e^{\lambda(n+1-k)} \right]\}$$

$$= \left\{ e^{-\lambda(n+1)}\mathcal{L} (\{ y(n) \}) \right\}$$

$$+ \left\{ e^{-\lambda(n+1)} \left[ y(n) (1 - e^{\lambda}) - \sum_{k=\max\{0,n-T\}}^{n} B(n, k)y(k) \left( e^{\lambda(n+1-k)} - 1 \right) \right] \right\}.$$  

Denote

$$\mathcal{G} (\{ y(n) \}) = \left\{ -y(n) (e^{\lambda} - 1) - \sum_{k=\max\{0,n-T\}}^{n} B(n, k)y(k) \left( e^{\lambda(n+1-k)} - 1 \right) \right\},$$

$$\mathcal{M} = \mathcal{L} + \mathcal{G}.$$
Then
\[ \mathcal{L} \{ \{x(n)\} \} = \left\{ e^{-\lambda(n+1)} M \{ \{y(n)\} \} \right\}. \]

Denote
\[ I_0^\infty = \{ \{x(n)\} \in l^\infty, \ x_0 = 0 \} . \]

Lemma 4 implies that operator \( \mathcal{L} \) is bounded and maps \( I_0^\infty \) onto \( l^\infty \). Therefore by the Banach theorem the Cauchy operator \( C_\mathcal{L} : l^\infty \to l^\infty \) is bounded as well as its inverse. Let \( \|C_\mathcal{L}\|_{l^\infty \to l^\infty} = P \). Here \( \|G\|_{l^\infty \to l^\infty} \leq e^\lambda - 1 + M \left( e^{\lambda(T+1)} - 1 \right) \).

Consider \( MC_\mathcal{L} = \mathcal{L}C_\mathcal{L} + GC_\mathcal{L} = I + GC_\mathcal{L} \), where \( I \) is an identity operator.

Since \( \lim_{\lambda \to 0} \left( e^\lambda - 1 \right) = \lim_{\lambda \to 0} \left( e^{\lambda(T+1)} - 1 \right) = 0 \), then for \( \lambda \) small enough we have \( e^\lambda - 1 < \frac{1}{2P} \) and \( M \left( e^{\lambda(T+1)} - 1 \right) < \frac{1}{2P} \), so \( \|G\| < \frac{1}{P} \) which implies
\[ \|GC_\mathcal{L}\|_{l^\infty \to l^\infty} \leq \|G\|_{l^\infty \to l^\infty} \cdot \|C_\mathcal{L}\|_{l^\infty \to l^\infty} < \frac{1}{P}. \]

Hence operator \( MC_\mathcal{L} \) is invertible.

Let us fix such a \( \lambda \). Since \( \mathcal{L} : l^\infty \to l^\infty \) and \( G : l^\infty \to l^\infty \) are continuous, so is \( M = \mathcal{L} + G \). By Lemma 3 the Cauchy operator \( C_M \) of the difference equation \( M(\{y(n)\}) = \{g(n)\} \) maps \( l^\infty \) into \( I_0^\infty \) and is bounded.

Let \( Y(n,l) \) be a fundamental matrix of the equation \( M(\{y(n)\}) = \{g(n)\} \), this equation can be also written as
\[ y(n+1) - y(n) = (e^\lambda - 1) y(n) \]
\[ + \sum_{k=\max\{0,n-T\}}^{n} B(n,k)y(k)e^\lambda(n+1-k) + g(n). \]

Then \( Y(n) = Y(n,0) \) is a solution of (14) with \( Y(0) = I \) and \( g(n) \equiv 0 \).

Denote \( G(n) = I - Y(n) \). Then \( M(\{G(n)\}) = \{F(n)\} \), where
\[ F(n) = - \left( e^\lambda - 1 \right) I - \sum_{k=\max\{0,n-T\}}^{n} B(n,k)e^\lambda(n+1-k). \]

Condition (a1) implies that the columns of \( F(n) \) belong to \( l^\infty \). Since the Cauchy operator \( C_M \) is bounded then the columns of \( G(n) \) also belong to \( l^\infty \).

So the columns of \( Y(n,0) \) are also in \( l^\infty \) which means that for some \( N_0 > 0 \) we have \( \|Y(k,0)\| \leq N_0 \) for any \( k \). The equality \( X(n,0) = e^{-\lambda n} Y(n,0) \) yields
\[ \|X(n,0)\| \leq N_0 e^{-\lambda n}. \]
After making a shift to the initial point \( k > 0 \), denoting
\[ Y(n, k) = e^{\lambda(n-k)}X(n, k) \]
and repeating this argument for an arbitrary positive integer \( k \) one obtains
\[ \|X(n, k)\| \leq N_k e^{-\lambda_k(n-k)}. \]

Finally, we have to prove that \( N_k \) and \( \lambda_k \) can be chosen independently of \( k \). To this end we will show that the constants in the previous estimates can be chosen independently of \( k \). Indeed, let as above \( \|C_\mathcal{L}\|_{1^\infty \to 1^\infty} = P \). Since \( 1^\infty \) contains sequences with \( k \) first vanishing terms (they form a subspace \( 1^\infty(k) \)) then \( \|C_\mathcal{L}\|_{1^\infty(k) \to 1^\infty(k)} \leq P \) for any positive integer \( k \). Further, if \( \lambda \) is such that
\[ |e^\lambda - 1| < \frac{1}{2P}, \quad M \left( e^{\lambda(T+1)} - 1 \right) < \frac{1}{2P}, \]
then
\[ \|G\|_{1^\infty(k) \to 1^\infty(k)} \leq \|G\|_{1^\infty \to 1^\infty} < \frac{1}{P}; \]
thus \( q = \|GC_\mathcal{L}\|_{1^\infty(k) \to 1^\infty(k)} \leq \|G\|\|C_\mathcal{L}\| < \frac{1}{P} P = 1, \)
where \( \lambda \) and \( q < 1 \) do not depend on \( k \). Since the norm of the operator is less than 1 then the inverse \((I + GC_\mathcal{L})^{-1}\) exists and its norm satisfies
\[ \| (I + GC_\mathcal{L})^{-1} \| \leq \frac{1}{1 - q}. \]

We recall \( \mathcal{M} = \mathcal{L} + G, C_\mathcal{L} = \mathcal{L}^{-1} \), so \( \mathcal{M}C_\mathcal{L} = I + GC_\mathcal{L} \) is invertible. Then by Lemma 3
\[ (15) \| C_\mathcal{M} \|_{1^\infty(k) \to 1^\infty(k)} \leq \| C_\mathcal{L} \|_{1^\infty(k) \to 1^\infty(k)} \| (I + GC_\mathcal{L})^{-1} \|_{1^\infty(k) \to 1^\infty(k)} \leq \frac{P}{1 - q} \]

For any fixed \( k \) the fundamental matrix \( Y(n, k) \) of the difference equation \( \mathcal{M}({\{y_n}\}) = 0 \) is a solution of this equation with the initial condition \( y_{k-T} = \ldots = y_{k-1} = 0, y_k = I. \)

Denote \( G(n, k) = I - Y(n, k) \). Then \( \mathcal{M}({\{G(n, k)\}}) = \{F(n, k)\} \), where
\[ F(n, k) = -\left( e^\lambda - 1 \right) I - \sum_{l=\text{max} \{k,n-T\}}^{n} B(n,l) e^{\lambda(n+1-l)}. \]

Hence \( \| \{G(n, k)\} \| \leq \| \mathcal{M} \|_{1^\infty(k) \to 1^\infty(k)} \| \{F(n, k)\} \| \leq N \), where \( N \) does not depend on \( k \). Thus \( Y(n, k) \) is bounded for all \( n, k: \|Y(n, k)\| \leq N. \)

Since \( X(n, k) = Y(n, k)e^{-\lambda(n-k)} \), then \( \|X(n, k)\| \leq Ne^{-\lambda(n-k)}, \)
which completes the proof. \( \Box \)

**Theorem 2.** Under the hypotheses of Theorem 1 the equation (2) is exponentially stable.

**Proof.** Since \( \|X(n,k)\| \leq Ne^{-\lambda(n-k)} \), then by Lemma 2 (see (9) and (a1)) the solution of (5)-(3) satisfies

\[
\|x(n+1)\| \leq \varphi(0)Ne^{-\lambda n} + \left( \sum_{l=-d}^{-1} B(n,l)e^{\lambda l}\varphi(l) \right) Ne^{-\lambda n}
\]

\[
\leq \left( \varphi(0) + Me^{-1} \max_{-d \leq k \leq -1} \|\varphi(k)\| \right) Ne^{-\lambda n}.
\]

Thus \( \|x(n+1)\| \leq Ce^{-\lambda n} \max_{-d \leq k \leq 0} \|\varphi(k)\| \), where \( C \) can be chosen as \( C = (1 + Me^{-1})N \). \( \Box \)

Lemma 3 and Theorem 2 imply

**Corollary 1.** Suppose for the equation \( \mathcal{L}(\{x(n)\}) = 0 \) conditions (a1)-(a2) hold, \( \mathcal{L}, \mathcal{L}_1 \) are operators of type (12), the equation \( \mathcal{L}_1(\{x(n)\}) = 0 \) is exponentially stable and \( \mathcal{C}_1 \) is the Cauchy operator of this equation. If operator \( \mathcal{C}_1\mathcal{L} : \ell^\infty \to \ell^\infty \) has an inverse operator which is bounded, then the equation \( \mathcal{L}(\{x(n)\}) = 0 \) is exponentially stable.

**Corollary 2.** Suppose the conditions of Corollary 1 hold. If

\[
\|\mathcal{T}\|_{\ell^\infty \to \ell^\infty} < 1,
\]

where \( \mathcal{T} = \mathcal{C}_1(\mathcal{L}_1 - \mathcal{L}) = I - \mathcal{C}_1\mathcal{L}, \) then equation \( \mathcal{L}(\{x(n)\}) = 0 \) is exponentially stable.

As an application of Corollary 2 consider the following scalar equation with several bounded delays

\[
(16) \quad x(n+1) - x(n) = -\sum_{k=1}^N a_k(n)x(g_k(n)),
\]

where \( a_k(n) \geq 0, 0 \leq n - g_k(n) \leq K \).

**Theorem 3.** Suppose that the following conditions hold:

\[
\sum_{k=1}^N \sum_{n=0}^{\infty} a_k(n) = \infty, \quad \limsup_{n \to \infty} \sum_{k=1}^N \sum_{l=\min\{g_k(n)\}}^{n-1} a_k(l) < 1.
\]

Then Eq. (16) is exponentially stable.
Proof. Without loss of generality we can assume

\[ A = \sup_{n > 0} \sum_{k=1}^{N} \sum_{l=\min_k\{g_k(n)\}}^{n-1} a_k(l) < 1. \]

Denote

\[ L_1 \{\{x(n)\}\} = \left\{ x(n+1) - x(n) + \sum_{k=1}^{N} a_k(n)x(n) \right\} \]

and \( L_0 = L_1 - L \), where

\[ L\{\{x(n)\}\} = \left\{ x(n+1) - x(n) + \sum_{k=1}^{N} a_k(n)x(g_k(n)) \right\} . \]

Then

\[ L_0\{\{x(n)\}\} = \left\{ \sum_{k=1}^{N} (a_k(n)(x(n) - x(g_k(n))) \right\} . \]

The solution formula

\[ x(n) = \prod_{l=0}^{n-1} \left( 1 - \sum_{k=1}^{N} a_k(l) \right) x(0) \]

for the equation \( L_1\{\{x(n)\}\} = 0 \) and the theorem assumptions imply that this equation is exponentially stable. We will prove that \( \|T\| < 1 \), where \( T = C_1L_0 \). We have

\[ \|T\{\{x(n)\}\}\| \leq \|C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n)|x(n) - x(g_k(n))| \right\} \right)\| \]

\[ \leq \|C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n) \left[ \sum_{l=1}^{N} a_l(n-1)|x(g_l(n-1))| + \ldots + |x(g_k(n)+1) - x((g_k(n))| \right] \right\} \right)\| \]

\[ \leq \|C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n) \left[ \sum_{l=1}^{N} a_l(n-1)|x(g_l(n-1))| + \ldots + \sum_{l=1}^{N} a_l(g_k(n))|x(g_l(n))| \right] \right\} \right)\| \leq A \|C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n) \right\} \right)\| \|\{x(n)\}\|. \]
Denote \( \{y(n)\} = C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n) \right\} \right) \). Then

\[
y(n + 1) - y(n) = - \sum_{k=1}^{N} a_k(n)y(n) + \sum_{k=1}^{N} a_k(n),
\]

which implies

\[
y(n + 1) = \left( 1 - \sum_{k=1}^{N} a_k(n) \right) y(n) + \sum_{k=1}^{N} a_k(n).
\]

By the assumptions of the theorem \( 0 \leq \sum_{k=1}^{N} a_k(n) \leq 1 \). By induction it is easy to see that \( 0 \leq y(n) \leq 1 \). Consequently,

\[
\left\| C_1 \left( \left\{ \sum_{k=1}^{N} a_k(n) \right\} \right) \right\| \leq 1
\]

and therefore \( \|T\| < 1 \).

By Corollary 2 the equation (16) is exponentially stable. \( \Box \)

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ASYMPTOTICS OF SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we give a characterization of the solutions of a scalar linear functional differential equations of the form

\[ y'(t) = \lambda_0(t)y(t) + b(t, y_t), \quad t \geq 0, \]

where \( y_t : [-\tau, 0] \to \mathbb{C} \) is defined by \( y_t(s) = y(t + s) \) for \( t \geq 0 \) and \( s \in [-\tau, 0] \), \( \lambda_0 : [0, +\infty) \to \mathbb{C} \) is a locally integrable function, \( \{b(t, \cdot)\}_{t \geq 0} \) is a family of bounded linear functionals from \( L^\infty([-\tau, 0], \mathbb{C}) \) into \( \mathbb{C} \) with \( \tau > 0 \). Asymptotic formulas under smallness conditions on \( b(t, e^{\int_{t}^{t+} \lambda_0(s)ds}) \) are obtained. Examples are given.


AMS(MOS) subject classification. 34K25, 34E10, 34G10


† Supported partially by FONDECYT 8990013, FONDECYT 1030535 and by Fundación Andes C-13760 for the first author. This research was done when this author was visiting the Departamento de Matemática of the Facultad de Ciencias of the Universidad de Chile by the grant of MECESUP PUC 0103.

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Haddock and Sacker [13] (1980) have given a conjecture on a version of the Hartman-Wintner's theorem for the autonomous diagonal differential system with a linear $L^p$-perturbation ($p = 2$) with delayed argument. This and similar problems were considered by Arino and Győri [1] (1989), Ai [19] (1992) and Cassel and Hou [5] (1993) (here $p \geq 2$ is considered) for a system where the non-perturbed system is diagonal and satisfies the hypotheses of the Hartman-Wintner's asymptotic theorem.


In this paper, we are interested in asymptotic formulas for the linear functional differential equation

\[(1) \quad y'(t) = \lambda_0(t)y(t) + b(t, y_t), \quad t \geq 0,\]

where $\lambda_0 : [0, +\infty[ \to C$ is a locally integrable function, $\{b(t, \cdot)\}_{t \geq 0}$ is a family of linear functionals from $L^\infty([-\tau, 0], C)$ into $C$, and for every function $y : [-\tau, +\infty[ \to C$, $y_t : [-\tau, 0] \to C$ is defined by $y_t(s) = y(t + s)$ for all $t \geq 0$ and $s \in [-\tau, 0]$. As usual, $L^\infty([-\tau, 0], C)$ denotes the space of essentially bounded functions mapping $[-\tau, 0]$ into $C$ equipped with the standard norm. Here we generalize the Haddock-Sacker Conjecture in its scalar version. In this case, we obtain a smallness condition which is weaker than the $L^p$-condition on $b(t, e^{\int_{t-\tau}^{t+} \lambda_0(s)ds})$. According to Example 2 below, it is sufficient that

\[\int_{t-\tau}^{t} b(s, e^{\int_{s}^{t+} \lambda_0(s)ds}) ds_1 \in L^p\]

for some $p \geq 1$ and the given smallness condition is still satisfied although $b(t, e^{\int_{t-\tau}^{t+} \lambda_0(s)ds}) \notin L^r$ for all $r \geq 1$. More restrictive cases can be considered, too, but they will not be shown due to page limitations. Our smallness condition is given in section 3. It is similar to the smallness conditions obtained by Driver [10, see Theorem 3] (1976), Pituk [17, see Corollary 7.6 and Lemma 5] (1993), Arino, Győri and Pituk [2, see Theorem 2.4] (1996) and Arino and Pituk [3] (2001). This will guarantee the convergence of the iterations in the proof of our main result in section 4. Examples are given in section 3.

2. Main Result. The following proposition is our main result.

**Proposition 1.** Consider the linear functional differential equation

\[(1) \quad y'(t) = \lambda_0(t)y(t) + b(t, y_t), \quad t \geq 0,\]

where $\lambda_0 : [0, +\infty[ \to C$ is a locally integrable function and $b : [0, +\infty[ \times L^\infty([-\tau, 0], C) \to C$ is a continuous function, linear in its second variable and such that

\[\sup_{t \geq T} \int_{t-\tau}^{t} |b(s, \exp(\int_{s}^{t+} \lambda_0(\xi)d\xi))| ds < \frac{1}{e},\]
for $T$ large enough. Then every solution of (1) satisfies the following asymptotic formula:

$$\begin{align*}
(2) \quad y(t) &= \exp \left( \int_T^t \left[ \lambda_0(s) + b(s, e^{\int_{T}^{s} \lambda_0(\xi) d\xi}) + \sum_{n=1}^{+\infty} \Delta_n(s) \right] ds \right) (c + o(1)),
\end{align*}$$

as $t \to +\infty$, where

$$\Delta_n(t) = b(t, e^{\int_{T}^{t} \lambda_0(\xi) d\xi} e^{\int_{T}^{t} \mu_n(\xi) d\xi} - e^{\int_{T}^{t} \mu_{n-1}(\xi) d\xi}),$$

and the functions $\mu_n(t)$ are given by

$$\mu_0(t) = 0$$

and

$$\mu_n(t) = \begin{cases} b(t, e^{\int_{T}^{t} \lambda_0(\xi) d\xi} e^{\int_{T}^{t} \mu_n(\xi) d\xi}), & \text{for } t \geq n\tau \\ 0, & \text{for } t \in [0, n\tau[ \end{cases}$$

for $n \in \mathbb{N}$. Conversely, given $c \in \mathbb{C}$, there is a solution $y = y(t)$ of (1) satisfying (2).

To make the asymptotic formula (2) easy to understand, the following remark is given.

**Remark 1.** Let $\lambda_n(t) = \lambda_0(t)$, for $t \in [0, \tau[$ and

$$\lambda_n(t) = \lambda_0(t) + b(t, e^{\int_{T}^{t} \lambda_0(\xi) d\xi} + \sum_{j=1}^{n} \Delta_j(t), \text{ for } t \geq \tau$$

and $n \in \mathbb{N}$. Then, given $n_0 \in \mathbb{N}$, $\lambda_n(t) = \lambda_{n_0}(t)$, for $t \in [0, n_0\tau[ \text{ and } n \geq n_0$.

So, (2) can be written as:

$$y(t) = \exp \left( \int_T^t \left[ \lim_{n \to +\infty} \lambda_n(s) \right] ds \right) (c + o(1)),$$

as $t \to +\infty$.

**Remark 2.** If $\Delta_k \in L^1$ for some $k \in \mathbb{N}$, then $\sum_{m=k}^{+\infty} \Delta_m \in L^1$. In this case (2) can be written as

$$y(t) = \exp \left( \int_T^t \left[ \lambda_0(s) + b(s, e^{\int_{T}^{s} \lambda_0(\xi) d\xi}) + \sum_{j=1}^{n-1} \Delta_j(s) \right] ds \right) (\hat{c} + o(1)),$$

as $n \to +\infty$, for some $\hat{c} \in \mathbb{C}$. Here $\sum_{j=1}^{n-1} \Delta_j(s) = 0$ if $n = 1$. 
3. Applications. One of the immediate applications of Proposition 1 is a scalar extension of the Haddock-Sacker Conjecture [13]:

**Corollary 1.** Consider the linear functional differential equation (1), where \( \lambda_0 : [0, +\infty] \to \mathbb{C} \) is a locally integrable function and

\[
b : [0, +\infty] \times L^\infty([-\tau, 0], \mathbb{C}) \to \mathbb{C}
\]

is a continuous function, linear in its second variable and such that

\[
|b(t, e^{\int_{t+\tau}^{t} \lambda_0(\xi)d\xi})| \in L^p
\]

for some \( p \in [2^{n-1}, 2^n] \) and \( n \in \mathbb{N} \). Then, every solution of (1) satisfies the following asymptotic formula

\[
y(t) = \exp \left( \int_{\tau}^{t} \left[ \lambda_0(s) + b(s, e^{\int_{s+\tau}^{s} \lambda_0(\xi)d\xi}) + \sum_{j=1}^{n-1} \Delta_j(s) \right] ds \right) (c + o(1)),
\]

as \( t \to +\infty \), where \( \Delta_j \) is defined as in Proposition 1. Conversely, given \( c \in \mathbb{C} \), there is a solution \( y = y(t) \) of (1) satisfying (3).

**Proof:** Since \( |b(t, e^{\int_{t+\tau}^{t} \lambda_0(\xi)d\xi})| \in L^p \) for some \( p \in [2^{n-1}, 2^n] \) and \( n \in \mathbb{N} \), we have that \( \Delta_n \in L^1 \). By Remark 2, \( \sum_{m=n}^{+\infty} \Delta_m \in L^1 \). So, (2) can be written as (3).

An example of an \( L^p \)-perturbed equation \( (p = 4) \) is the following

**Example 1.** Consider the delay differential equation

\[
y'(t) = \frac{1}{\sqrt{t+9}} y(t-1), t \geq 0.
\]

Then every solution \( y = y(t) \) of (4) satisfies the following asymptotic formula

\[
y(t) = \exp \left( \int_{3}^{t} \tilde{\lambda}(s) ds \right) (c + o(1)),
\]

as \( t \to +\infty \), where

\[
\tilde{\lambda}(t) = \frac{1}{\sqrt{t+9}} + \frac{1}{\sqrt{t+9}} \int_{t}^{t-1} \frac{ds}{\sqrt{s+9}} + \frac{1}{2\sqrt{t+9}} \left[ \int_{t}^{t-1} \frac{ds}{\sqrt{s+9}} \right]^2
\]

\[
+ \frac{1}{\sqrt{t+9}} \int_{t}^{t-1} \left[ \frac{1}{\sqrt{s+9}} \int_{s}^{s-1} \frac{ds_1}{\sqrt{s_1+9}} \right] ds,
\]

for \( t \geq 3 \).

Indeed, notice that \( \mu_1(t) = \frac{1}{\sqrt{t+9}} \in L^4 \) and

\[
\Delta_1(t) = \frac{1}{\sqrt{t+9}} \left( e^{\int_{t+\tau}^{t} \lambda_0(\xi)d\xi} - 1 \right).
\]
So,
\[ \Delta_1(t) = \frac{1}{\sqrt{t+9}} \int_t^{t-1} \frac{ds}{\sqrt{s+9}} + \frac{1}{2\sqrt{t+9}} \left[ \int_t^{t-1} \frac{ds}{\sqrt{s+9}} \right]^2 + \theta_1(t), \]
where \( \theta_1 \in L^1 \).

\[ \Delta_2(t) = \frac{1}{\sqrt{t+9} e^{t-1}} \int_t^{t-1} \frac{ds}{\sqrt{s+9}} \left( e^{t-1} \frac{1}{\sqrt{s+9}} \int_s^{t-1} \frac{ds}{\sqrt{s+9}} - 1 \right) - 1 \]
\[ = \frac{1}{\sqrt{t+9} e^{t-1}} \left( \int_t^{t-1} \frac{1}{\sqrt{s+9}} \left[ e^{t-1} \frac{1}{\sqrt{s+9}} - 1 \right] ds \right) + \theta_2(t) \]
\[ = \frac{1}{\sqrt{t+9} e^{t-1}} \left[ \int_t^{t-1} \frac{1}{\sqrt{s+9}} \left[ e^{t-1} \frac{1}{\sqrt{s+9}} - 1 \right] ds \right] \]
\[ + \theta_3(t) \]
\[ = \frac{1}{\sqrt{t+9} e^{t-1}} \int_t^{t-1} \frac{1}{\sqrt{s+9}} \left[ e^{t-1} \frac{1}{\sqrt{s+9}} - 1 \right] ds \]
\[ + \theta_4(t), \]
where \( \theta_2, \theta_3, \theta_4 \in L^1 \). \( \Delta_j \in L^1 \) for \( j \geq 3 \). So, (5) is proved.

In the following example we can see that it is enough that \( \int_{t-1}^t \mu_1(s) ds \in L^p \) for some \( p \geq 1 \) and the hypotheses of Remark 2 are still satisfied, although \( \mu_1 \notin L^r \) for all \( r \geq 1 \).

**Example 2.** Consider the delay differential equation
\[ y'(t) = \frac{1}{e} \cos(t^2) y(t-1), \quad t \geq 0. \]

Clearly, \( \int_{t-1}^t \frac{1}{e} \cos(s^2) ds \in L^2 \) although \( \frac{1}{e} \cos(t^2) \notin L^p \) for all \( p \geq 1 \). Every solution \( y = y(t) \) of (6) satisfies the following asymptotic formula:

\[ y(t) = \exp \left( \int_0^t \tilde{\lambda}(s) ds \right) (c + o(1)), \]

as \( t \to +\infty \), where

\[ \tilde{\lambda}(t) = \frac{1}{e} \cos(t^2) + \frac{1}{e^2} \cos(t^2) \int_t^{t-1} \cos(s^2) ds. \]

for \( t \geq 2 \).

Indeed, \( \mu_1(t) = \frac{1}{e} \cos(t^2) \), \( \Delta_1(t) = \frac{1}{e} \cos(t^2) \left( e^{t-1} \frac{1}{e} \cos(s^2) ds - 1 \right) \), i.e.,

\[ \Delta_1(t) = \frac{1}{e^2} \cos(t^2) \int_t^{t-1} \cos(s^2) ds + \theta_1(t), \]

where \( \theta_1 \in L^1 \).

\[ \Delta_2(t) = \frac{1}{e} \cos(t^2) e^{t-1} \frac{1}{e} \cos(s^2) ds \left( e^{t-1} \frac{1}{e} \cos(s^2) ds - 1 \right) \in L^1. \]

By Remark 2, \( \sum_{m=2}^{\infty} \Delta_m \in L^1 \) for \( j \geq 2 \). So, (7) is proved.
4. Proof of the Main Result. In the proof of Proposition 1 we need the following result which is a consequence of a more general convergence theorem due to Atkinson and Haddock [4] (1983) (see [4, Theorem 3.1] and its proof).

**Lemma 1.** Consider the linear functional equation

\[(8)\]
\[x'(t) = \tilde{b}(t, x(t) - x_t), \quad t \geq 0,\]

where \(\tilde{b} : [0, +\infty] \times L^\infty([-\tau, 0], \mathbb{C}) \to \mathbb{C}\) is linear in its second variable and such that

\[\sup_{t \geq T} \int_{t-\tau}^{t} |\tilde{b}(s, \cdot)| ds < 1\]

for \(T\) large enough and \(|\cdot|\) denotes the operator norm. Then all solutions of equation (8) are convergent as \(t \to +\infty\). Clearly, every constant function is a solution of (8).

**Proof of Proposition 1:**

We reduce equation (1) by the change of variables

\[y(t) = \exp \left( \int_{0}^{t} \lambda_0(s) ds \right) x(t),\]

to the equation

\[(9)\]
\[x'(t) = \tilde{b}(t, x_t) = b \left( t, \exp \left( \int_{t}^{t+\tau} \lambda_0(s) ds \right) x_t \right),\]

for \(t \geq 0\). Suppose, without loss of generality, that

\[(10)\]
\[\sup_{t \geq \tau} \int_{t-\tau}^{t} |b(s, \exp \left( \int_{s}^{s+\tau} \lambda_0(\xi) d\xi \right) | ds < \frac{1}{e}.\]

Let \(\mathcal{M}_0\) be the set of the functions \(\nu : [0, +\infty[ \to \mathbb{C}\) such that \(\|\nu\|_0 := \sup_{t \geq \tau} \int_{t-\tau}^{t} |\nu(s)| ds < +\infty\). \((\mathcal{M}_0, \|\cdot\|_0)\) is a Banach space. Let \(T : \mathcal{M}_0 \to \mathcal{M}_0\) be defined by \((T \nu)(t) = 0\) if \(0 \leq t < \tau\) and \((T \nu)(t) = \tilde{b}(t, \exp \left( \int_{t}^{t+\tau} \nu(\xi) d\xi \right))\),

for \(t \geq \tau\). Let

\[B_{01} = \{ \nu \in \mathcal{M}_0 : \|\nu\|_0 \leq 1 \}.\]
By (10), $\mathcal{T}(B_{01}) \subseteq B_{01}$. We have that $\mathcal{T} : B_{01} \to B_{01}$ is a contraction operator. Indeed,

$$|(\mathcal{T}v_1)(t) - (\mathcal{T}v_2)(t)| \leq |\tilde{b}(t, e^{\int_t^t \nu_1(\xi)d\xi} - e^{\int_t^t \nu_2(\xi)d\xi})|,$$

for $t \geq \tau$ and $(\mathcal{T}v_1)(t) - (\mathcal{T}v_2)(t) = 0$ if $0 \leq t < \tau$. Since $B_{01}$ is a convex set, by the Complex Mean Value Theorem and (10), we have that $\mathcal{T}$ is a contraction. By the Banach Fixed Point Theorem, we have a unique function $\mu \in B_{01}$ such that $\mathcal{T} \mu = \mu$.

By the change of variables $x(t) = \exp \left( \int_0^t \mu(\xi)d\xi \right) z(t)$ in (9), we obtain the equation

$$z'(t) = \tilde{b}(t, \exp \left( \int_0^t \mu(\xi)d\xi \right) [z_t - z(t)]) + [(\mathcal{T} \mu)(t) - \mu(t)] z(t),$$

for $t \geq \tau$, i.e., we have that $z = z(t)$ satisfies the equation

$$(11) \quad z'(t) = \tilde{b}(t, \exp \left( \int_0^t \mu(\xi)d\xi \right) [z_t - z(t)]), \quad t \geq \tau.$$

Clearly, every constant function is a solution of (11). By Lemma 1, every solution of (11) is convergent. Therefore, every solution $x = x(t)$ of equation (9) satisfies the asymptotic formula

$$(12) \quad x(t) = \exp \left( \int_0^t \mu(s)ds \right) (c + o(1)),$$

as $t \to +\infty$. Conversely, given any $c \in \mathbb{C}$, there is a solution of (9) which satisfies (12). Since $\mathcal{T} : B_{01} \to B_{01}$ is a contraction,

$$\lim_{n \to +\infty} \|\mathcal{T}^n(0) - \mu\|_0 = 0.$$ 

Since

$$\mathcal{T}^n(0) = \mathcal{T}(0) + \sum_{j=2}^{n} (\mathcal{T}^j(0) - \mathcal{T}^{j-1}(0))$$

and $(\mathcal{T}(0))(t) = b(t, e^{\int_0^t \lambda_0(\xi)d\xi})$, $(\mathcal{T}^{j+1}(0))(t) - (\mathcal{T}^j(0))(t) = \Delta_j(t)$, (2) is proved.

$\square$
Acknowledgement. The authors express their gratitude to the anonymous referee of this paper and to Professor Mihály Pituk for their valuable suggestions.

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FUNCTIONAL
DIFFERENTIAL
EQUATIONS

VOLUME 11
2004, NO 1-2
PP. 37-48

ASYMPTOTIC BEHAVIOR OF SOLUTIONS
OF DISCRETE EQUATIONS

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Abstract. In the present paper a survey of author's latest results concerning asymptotic behavior of solutions of discrete equations achieved (together with his collaborators) by using a retract idea is given. A general principle which gives a guarantee that the graph of at least one solution stays in a prescribed domain is formulated. A version of it suitable for discrete delayed equations is given, too. Applications concerning existence of a positive solution of delayed discrete equations are considered as well.

AMS(MOS) subject classification. 39A10, 39A11

Key Words. Discrete equation, asymptotic behavior of solution, bounded solution, positive solution

Dedicated to Prof. István Győri on the occasion of his 60th birthday.

1. Introduction. In the present paper a survey of author’s latest results achieved (together with his collaborators) in asymptotic theory of discrete equations is given.

A general principle (Theorem 1 below) which gives a guarantee that the graph of at least one solution stays in a prescribed domain is formulated. A version of it suitable for discrete delayed equations is given, too. The corresponding proofs (which are omitted in this survey) use retract type reasoning. Let us note that the application of a retract idea in the theory of ordinary differential equations goes back to T. Ważewski [26].

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This work was supported by the Grant 201/01/0079 of Czech Grant Agency (Prague) and by the Council of Czech Government MSM 2622 000 13.
Applications concerning existence of a positive solution of delayed discrete equations are considered as well. In formulations of results, unlike in those of their original versions, the Lipschitz type conditions (with respect to the second arguments of the corresponding right hand sides) are omitted since for the validity of the corresponding statements only the continuity of right hand sides with respect to the second arguments is sufficient. The author is grateful to prof. M. Kwapisz for the corresponding remark (during the conference “Delay Differential and Difference Equations with Applications”, August 25–29, 2003, Veszprém, Hungary) leading to this improvement.

Let us consider the system of nonlinear discrete equations

\[ \Delta u(k) = F(k, u(k)) \]

with \( F : N(a) \times \mathbb{R}^n \to \mathbb{R}^n \), \( u = (u_1, \ldots, u_n) \), \( \Delta u(k) = u(k + 1) - u(k) \) where \( k \in N(a) = \{a, a + 1, \ldots\}, \ a \in \{0\} \cup \mathcal{N} \) is fixed and \( \mathcal{N} = \{1, 2, \ldots\} \).

The existence and uniqueness of solution of initial problem (1), (2) with

\[ u(a + s) = u^s \in \mathbb{R}^n, \]

\( s \in \{0\} \cup \mathcal{N}, \) (s is fixed) on \( N(a + s) \) is obvious. Let us recall that the solution \( u = u(k), \ k \in N(a + s) \) of initial problem (1), (2) is defined as an infinite sequence of number vectors

\[ \{u(a + s) = u^s, u(a + s + 1), u(a + s + 2), \ldots, u(a + s + n), \ldots\} \]

such that for any \( k \in N(a + s) \), equality (1) holds. The sequence \( \{(k, u(k))\} \), \( k \in N(a + s) \) is called the graph of solution \( u = u(k) \) for \( k \in N(a + s) \) of initial problem (1), (2).

In the following we shall suppose \( F \) to be continuous with respect to the second argument. Then it is easy to see that initial problem (1), (2) depends continuously on the initial data.

**1.1. Description of the Problem Considered.** Let us define a set \( \Omega \subset N(a) \times \mathbb{R}^n \) as

\[ \Omega := \{(k, u) : k \in N(a), u \in \Omega(k)\} \]

where \( \Omega(k) \) is an open bounded and connected set in the set

\[ S_k := \{(k, u) : u \in \mathbb{R}^n\} \]

for every \( k \in N(a) \). The boundary \( \partial \Omega(k) \) of \( \Omega(k) \) is defined in the set \( S_k \) in the usual way, as well as the closure

\[ \overline{\Omega}(k) := \Omega(k) \cup \partial \Omega(k). \]
Define the boundary $\partial \Omega$ and the closure $\overline{\Omega}$ of the set $\Omega$ in the set $\bigcup_{k \in N(a)} S_k$ as

$$\partial \Omega := \bigcup_{k \in N(a)} \partial \Omega(k), \quad \overline{\Omega} := \Omega \cup \partial \Omega.$$ 

Let us describe the problem under consideration.

**Problem 1.** Suppose that the above suppositions hold. The problem is to determine sufficient conditions with respect to the right-hand side of system (1) in order to guarantee the existence of at least one solution $u = u^*(k), \ k \in N(a)$ satisfying $(k, u^*(k)) \subset \Omega(k)$ for every $k \in N(a)$.

1.2. Auxiliary Notions. Let us recall at first notions of retract and retraction.

**Definition 1.** ([22, p. 97]) If $A \subset B$ are any two sets of a topological space and $\pi : B \to A$ is a continuous mapping from $B$ onto $A$ such that $\pi(p) = p$ for every $p \in A$, then $\pi$ is said to be a retraction of $B$ onto $A$. When there exists a retraction of $B$ onto $A$, $A$ is called a retract of $B$.

As a "connection" of discrete sets $\Omega(k), \ k \in N(a)$ we introduce the notion of a connecting function $V$.

**Definition 2 (Connecting function).** A continuous function

$$V(t, u) : [a, \infty) \times \mathbb{R}^n \to \mathbb{R}$$

is called a connecting function for the sets $\Omega(k), \ k \in N(a)$ if for every $k \in N(a)$:

$$\Omega(k) \equiv \{(k, u) : V(k, u) < 0\}.$$

With the aid of the connecting function we define sets that will be used in the following. Let constants $\alpha, \beta \in \mathbb{R}, \ \alpha < \beta$ be given. Then define an auxiliary set

$$\mathcal{V}_{\alpha, \beta} := \{(t, u) : \alpha \leq t \leq \beta, V(t, u) \leq 0\}$$

having (so-called) $u$-boundary

$$\partial_u \mathcal{V}_{\alpha, \beta} := \{(t, u) : \alpha \leq t \leq \beta, V(t, u) = 0\},$$

and an auxiliary set

$$\mathcal{V}_{\alpha, \infty} := \{(t, u) : \alpha \leq t < \infty, V(t, u) \leq 0\}$$

with $u$-boundary

$$\partial_u \mathcal{V}_{\alpha, \infty} := \{(t, u) : \alpha \leq t < \infty, V(t, u) = 0\}.$$
2. Existence Result.

**THEOREM 1 ([9]).** Suppose $F : N(a) \times \overline{\Omega} \to \mathbb{R}^n$ is continuous with respect to the second argument. Suppose, moreover, that the sets $\Omega(k), k \in N(a)$ and a corresponding connecting function $V(t, u)$ are given such that the following properties hold:

1) The set $\mathcal{V}_{k,k+1}$ is convex for every $k \in N(a)$.

2) For every $k \in N(a)$ and $M = (k, u^*) \in \partial \Omega(k)$, the line segment connecting point $M$ and its first consequent point $M'_1 := (k, u^* + F(k, u^*))$ has only one point of intersection with the set $\mathcal{V}_{k,k+1}$, namely, the point $M$ itself.

3) There exists a retraction $\pi$ of the set $\partial_u \mathcal{V}_{a,\infty}$ onto the set $\partial \Omega(a)$.

Then there exists a solution $u = u^*(k), k \in N(a)$ of (1) satisfying the relation

$$(k, u^*(k)) \subset \Omega(k)$$

for every $k \in N(a)$.

2.1. An Application. In this section we give a consequence of Theorem 1 with easily verifiable conditions. Let auxiliary functions

$$b_i(k), c_i(k) : N(a) \to \mathbb{R},$$

$i = 1, \ldots, n$ with $b_i(k) < c_i(k)$ be given. Define functions

$$B_i(k, u) := -u_i + b_i(k), \quad i = 1, \ldots, n,$$

$$C_i(k, u) := u_i - c_i(k), \quad i = 1, \ldots, n$$

and sets (for every $i = 1, \ldots, n$)

$$\Omega_B^i := \{(k, u) : k \in N(a), B_i(k, u) = 0, B_j(k, u) \leq 0, C_s(k, u) \leq 0\}$$

with $j, s = 1, \ldots, n$ and $j \neq i$,

$$\Omega_C^i := \{(k, u) : k \in N(a), C_i(k, u) = 0, B_j(k, u) \leq 0, C_s(k, u) \leq 0\}$$

with $j, s = 1, \ldots, n$ and $s \neq i$.

Suppose that the set $\Omega$ is written in the form

$$\Omega = \{(k, u) : k \in N(a), B_i(k, u) < 0, C_j(k, u) < 0, i, j = 1, \ldots, n\}.$$

**THEOREM 2 ([10]).** Let $b_i(k), c_i(k) (i = 1, \ldots, n)$ be real functions defined on $N(a)$ such that $b_i(k) < c_i(k)$ and suppose that the function $F :$
2.2. Example - Existence of Bounded Solutions of a Nonlinear System. Let us indicate sufficient conditions under which there exists a bounded on \( N(a) \) solution of a system of discrete equations

\[
\begin{align*}
\tag{3} u_i(k + 1) &= \mu_i(k) u_i(k) + \nu_i(k, u(k)), \quad i = 1, \ldots, n \\
\end{align*}
\]

with \( k \in N(a), u = (u_1, u_2, \ldots, u_n) \in R^n, \mu = (\mu_1, \mu_2, \ldots, \mu_n) : N(a) \rightarrow R \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) : N(a) \times \bar{\Omega} \rightarrow R^n \).

**Theorem 3 ([9]).** Let \( b_i(k), c_i(k) \) \( (i = 1, \ldots, n) \) be real functions defined on \( N(a) \) such that \( b_i(k) < c_i(k) \), \( \nu : N(a) \times \bar{\Omega} \rightarrow R^n \) and suppose that \( \nu \) is continuous with respect to the second argument. If, moreover,

\[
\begin{align*}
\tag{4} b_i(k + 1) - \mu_i(k) b_i(k) > \nu_i(k, u) \\
\end{align*}
\]

for every \( i = 1, \ldots, n \) and every \( (k, u) \in \Omega_B^i \) and

\[
\begin{align*}
\tag{5} c_i(k + 1) - \mu_i(k) c_i(k) < \nu_i(k, u) \\
\end{align*}
\]

for every \( i = 1, \ldots, n \) and every \( (k, u) \in \Omega_C^i \), then there exists a solution \( u = u^*(k) \) of system (1) satisfying the inequalities

\[
\begin{align*}
\tag{6} b_i(k) < u^*_i(k) < c_i(k) \\
\end{align*}
\]

for every \( k \in N(a) \) and \( i = 1, \ldots, n \).

**Definition 3 (\( \delta \)-bounded solution).** Let a positive number \( \delta \) be given. We say that a solution \( u = u(k), k \in N(a) \) of system (3) is \( \delta \)-bounded if the inequality \( ||u(k)|| < \delta \) with \( ||u(k)|| = \max_{i=1,\ldots,n} |u_i(k)| \) holds for every \( k \in N(a) \).
The existence of $\delta$-bounded solutions of the system (3) is contained in the previous affirmation in the case when $c_i(k) \equiv -b_i(k) \equiv \delta$. We now reformulate this result.

**Theorem 4 ([9], $\delta$-bounded solution).** Let a positive number $\delta$ be given. Let $\nu : N(a) \times \Omega \to R^n$ be continuous with respect to the second argument. If, moreover,

$$-1 + \mu_i(k) > \nu_i(k, u)/\delta$$

for every $i = 1, \ldots, n$ and every $(k, u) \in \Omega_B^i$ and

$$1 - \mu_i(k) < \nu_i(k, u)/\delta$$

for every $i = 1, \ldots, n$ and every $(k, u) \in \Omega_C^i$, then there exists a solution $u = u(k), k \in N(a)$ of system (3) satisfying $\|u(k)\| < \delta$ for every $k \in N(a)$.

In connection with considered problem of boundedness of solutions of discrete equations note that a lot of investigations is devoted to it. Let us cite, e.g., publications [1, 5, 18].

3. Extension to Discrete Delayed Equations. Let us consider the scalar discrete equation

$$\Delta u(k + n) = f(k, u(k), u(k + 1), \ldots, u(k + n)), \tag{4}$$

where $f(k, u_0, u_1, \ldots, u_n)$ is defined on $N(a) \times R^{n+1}$ with values in $R$ and $n \in N$ is fixed. Together with discrete equation (4) we consider an initial problem. It is posed as follows: for a given $s \in \{0\} \cup N$ we are seeking the solution of (4) satisfying $n + 1$ initial conditions

$$u(a + s + m) = u^{s+m} \in R, \tag{5}$$

$m = 0, 1, \ldots, n$ with prescribed constants $u^{s+m}$. Solution of initial problem (4), (5) is defined as an infinite sequence of numbers

$$\{u(a + s) = u^s, u(a + s + 1) = u^{s+1}, \ldots, u(a + s + n) = u^{s+n},$$

$$u(a + s + n + 1), u(a + s + n + 2), \ldots\}$$

such that for any $k \in N(a + s)$ equality (4) holds. The existence and uniqueness of the solution of initial problem (4), (5) is obvious for every $k \in N(a+s)$. If, moreover, $f$ is continuous with respect to $u_0, u_1, \ldots, u_n$, then initial problem (4), (5) depends continuously on the initial data.
3.1. Formulation of the Problem. Let us define sets \( \omega \subset N(a) \times \mathbb{R} \) and \( \omega(k) \) as

\[
\omega := \{(k, u) : k \in N(a), b(k) < u < c(k)\}
\]

with closure

\[
\overline{\omega} := \{(k, u) : k \in N(a), b(k) \leq u \leq c(k)\}
\]

and

\[
\omega(k) := \{(u) : b(k) < u < c(k)\}
\]

with closure

\[
\overline{\omega}(k) := \{(u) : b(k) \leq u \leq c(k)\},
\]

where \( b(k), c(k) \) are real functions defined on \( N(a) \) such that \( b(k) < c(k) \). Obviously \( (k, \omega(k)) \subset \omega \) for every \( k \in N(a) \). Our aim is to establish a set of sufficient conditions with respect to the right-hand side of equation (4) in order to guarantee the existence of at least one solution \( u = u(k) \) defined on \( N(a) \) such that \( (k, u(k)) \subset (k, \omega(k)) \subset \omega \) for each \( k \in N(a) \). More exactly, we formulate the following

**Problem 2.** Suppose that above formulated conditions with respect to \( f \) hold and that a set \( \omega \) is defined by (6) with the aid of real functions \( b(k), c(k) \) satisfying the inequalities \( b(k) < c(k) \) on \( N(a) \). The problem is posed as follows: Find a set of sufficient conditions which guarantee that there exists an initial problem of (4) with initial data satisfying relations

\[
u_0^* \in \omega(a), u_1^* \in \omega(a+1), \ldots, u_n^* \in \omega(a+n)
\]

such that the corresponding solution \( u = u^*(k) \) of equation (4) satisfies the inequalities

\[
b(k) < u^*(k) < c(k)
\]

for every \( k \in N(a) \) (i.e. \( (k, u^*(k)) \subset (k, \omega(k)) \subset \omega \) for every \( k \in N(a) \)).

**Theorem 5 ([3]).** Let us suppose that \( f(k, u_0, u_1, \ldots, u_n) \) is defined on \( N(a) \times \overline{\omega}(k) \times \overline{\omega}(k+1) \times \ldots \times \overline{\omega}(n) \) with values in \( \mathbb{R} \) and is continuous with respect to arguments \( u_0, u_1, \ldots, u_n \). If, moreover, the inequalities

\[
f(k, u_0, u_1, \ldots, u_{n-1}, b(k+n)) - b(k+n+1) + b(k+n) < 0,
\]

then...
\[ f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) > 0 \]

hold for every \( k \in N(a) \) and every
\[ u_0 \in \omega(k), u_1 \in \omega(k+1), \ldots, u_{n-1} \in \omega(k+n-1), \]
then there exist initial values
\[ u^*(a+m) = u^*_m \in \omega(a+m), \ m = 0, 1, \ldots, n \]
such that the corresponding solution \( u = u^*(k) \) of equation (4) satisfies for every \( k \in N(a) \) the inequalities
\[ b(k) < u^*(k) < c(k). \]

3.2. Positive Solutions. The phenomenon of existence of a positive solution of differential or difference equations often arises when we analyse mathematical models describing various processes. It is an opposite case to the phenomenon of oscillation of all solutions. The existence of positive solutions is very often substantial for a concrete considered model. In biology, e.g., when a model of population dynamics is described by an equation, the positivity of a solution may mean that a concrete biological species can exist in the supposed environment. This is a motivation for intensive study of conditions of existence of positive solutions of differential and difference equations, as well as their properties. Let us note that investigations in this field can be found, e.g., in [1], [8], [12]–[18, 20].

**Theorem 6 ([2], Existence of a Positive Solution).** Let \( a \in N \) and \( n \in N \) be fixed. Let us suppose that \( f(k, u_0, u_1, \ldots, u_n) \) is defined on \( N(a) \times \omega(k) \times \omega(k+1) \times \ldots \times \omega(n) \) with values in \( R \) and is continuous with respect to arguments \( u_0, u_1, \ldots, u_n \). If, moreover, there exists a constant \( \theta \in [0, 1) \) such that
\[
-\sqrt{k} \left( \frac{n}{n+1} \right)^{k+n} \left( \frac{1}{n+1} + \frac{\theta n}{8k^2} \right) < f \left( k, u_0, \ldots, u_{n-1}, \sqrt{k+n} \left( \frac{n}{n+1} \right)^{k+n} \right)
\]
and
\[ f \left( k, u_0, \ldots, u_{n-1}, 0 \right) < 0 \]
for every \( k \in N(a) \) and every
\[ u_0 \in \omega(k), u_1 \in \omega(k+1), \ldots, u_{n-1} \in \omega(k+n-1) \]
with
\[ b(k) := 0, \quad c(k) := \sqrt{k} \cdot \left( \frac{n}{n+1} \right)^k \]
then there exists a positive integer \( a_1 \geq a \) and a solution \( u = u(k), k \in N(a_1) \) of equation (4) such that such that
\[ 0 < u(k) < \sqrt{k} \cdot \left( \frac{n}{n+1} \right)^k \]
hold for every \( k \in N(a_1) \).

**Remark 1.** Let us note that the supposition \( n \in N \) in Theorem 6 cannot be weakened to \( n \in \{0\} \cup N \). Really, if \( n = 0 \), inequalities (7) lose any sense. It means that presented results have a place only in the case of (substantially) delayed discrete equations.

### 3.3. Example - Equation \( \Delta u(k+n) = -p(k)u(k) \)

Let us consider the delayed scalar linear discrete equation
\[ \Delta u(k+n) = -p(k)u(k) \]  
with fixed \( n \in N \) and variable \( k \in N(a) \). The function \( p : N(a) \rightarrow R \) is supposed to be positive. We are interested in the existence of positive solutions of (8).

Equation (8) can be considered as a discrete analogue of the delayed linear differential equation of the form
\[ \dot{x}(t) = -c(t)x(t-\tau) \]
with positive coefficient \( c \) on \( I = [t_0, \infty), t_0 \in R \) which was intensively considered in many works. We refer, e.g., to the above mentioned references in the part 3.2. Remark except this that close problems were investigated, e.g., in [6, 7] and [25].

In the paper [11] it was underlined that if equation (9) admits a positive solution \( \bar{x} \) on an interval \( I \) then it admits on \( I \) two positive solutions \( x_1 \) and \( x_2 \) satisfying
\[ \lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = 0. \]
Moreover, every solution \( x \) of equation (9) on \( I \) is represented by the formula
\[ x(t) = K x_1(t) + O(x_2(t)), \]
where $K \in \mathbb{R}$ depends on $x$ and "$O$" is the Landau order symbol. In the formula (10) the solutions $x_1, x_2$ can be changed by any couple of positive on $I$ solutions $\tilde{x}_1, \tilde{x}_2$ of equation (9) satisfying the property

$$\lim_{t \to \infty} \frac{\tilde{x}_2(t)}{\tilde{x}_1(t)} = 0$$

(see [11, p. 638–639]). This invariance property led to the following terminology - if $(x_1, x_2)$ is a fixed couple of positive solutions (having the above indicated properties) of equation (9) then the solution $x_1$ is called a dominant solution and the solution $x_2$ is called a subdominant solution. Subdominant solutions can serve as an analogy to "small solutions" as they are used, for example, in the book [21], and dominant solutions express an analogy with the notion of "special solution" which is used in many investigations (see, e.g., [23]).

We will give sufficient conditions for the existence of positive solutions of equation (8). We will discuss known sufficient conditions, too, and we will show that our conditions have a more general character than the previous ones. Taking into account the fact that a solution satisfying inequalities (7) tends to zero (if $k \to \infty$) with the speed that is not smaller than the speed characterized by the function $\sqrt{k} \cdot (n/(n+1))^k$ we can conclude that this solution is an analogy to the notion of subdominant solution introduced above, in the case of scalar delayed linear differential equations. Moreover, the motivation for the used terminology is supported by the fact that our result does not hold for non-delayed equations of the type (8), i.e. does not hold if $n = 0$. This is in full accordance with theory of ordinary differential equations again, since obviously the subdominant solution does not appear if we put $\tau = 0$ in (9), i.e. does not appear in the case of ordinary differential equations. Following theorem is a consequence of Theorem 6 with

$$f(k, u(k), u(k+1), \ldots, u(k+n)) := -p(k)u(k).$$

**Theorem 7 ([4], Subdominant positive solution).** Let $a \in \{0\} \cup \mathcal{N}$ and $n \in \mathcal{N}$ be fixed. Suppose that there exists a constant $\theta \in [0, 1)$ such that the function $p : \mathcal{N}(a) \to \mathbb{R}$ satisfies the inequalities

(11) $$0 < p(k) \leq \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right)$$

for every $k \in \mathcal{N}(a)$. Then there exists a positive integer $a_1 \geq a$ and a solution $u = u(k), k \in \mathcal{N}(a_1)$ of equation (8) such that the inequalities (7) hold for every $k \in \mathcal{N}(a_1)$.
Let us remark that analogous (in a sense) problems are discussed, e.g., in [14, 17]–[19, 24]. The following known result (see [18, p. 192]) will be formulated with a notation adapted with respect to our notation.

**Theorem 8.** Assume $n \in \mathbb{N}$, $p(k) > 0$ for $k \geq 0$, and

\[
p(k) \leq \frac{n^n}{(n + 1)^{n+1}}.
\]

Then difference equation (8) where $k = 0, 1, 2, \ldots$ has a positive solution

\[
\{u(0), u(1), u(2), \ldots\}.
\]

Comparing this result with the result given by Theorem 6 we conclude that inequality (11) is a substantial improvement over (12) since the choice $\theta = 0$ in (11) gives inequality (12). Moreover, inequality (11) unlike inequality (12) involves the variable $k$ on the right hand side. As noted in [18, p. 179], for $p(k) \equiv p = \text{const}$, inequality (12) is sharp in a sense since in this case the necessary and sufficient condition for the oscillation of all solutions of (8) is the inequality

\[
p > \frac{n^n}{(n + 1)^{n+1}}.
\]

Inequality (11) can be considered as a discrete analogy of the inequality

\[
c(t) \leq \frac{1}{e} + \frac{1}{8e^2}
\]

($t$ is supposed to be sufficiently large) used in [15, Theorem 3] in order to give a guarantee of existence of a positive solution of equation (9).

**References**


Abstract. We present a survey of the recent results on the positivity and nonnegativity of discrete quadratic functionals corresponding to symplectic difference systems. A new sufficient condition for nonnegativity is proved and a possible direction of further investigation is also discussed.

1. Introduction and motivation. Consider a problem of the discrete calculus of variations

\[ G(x) = \sum_{k=0}^{N} f(k, x_{k+1}, \Delta x_k) \rightarrow \min, \quad x_0 = A, \ x_{N+1} = B. \]

This problem can be viewed as a discrete analogue of the classical problem of the calculus of variations with fixed endpoint. Using the standard procedure of the calculus of variations (first and second variations) or using the methods of the classical mathematical analysis (zeros of the first derivative, the matrix of the second derivative), one finds that if \( x = \{x\}_{k=0}^{N+1} \) with \( x_0 = A, x_{N+1} = B \) is a local extremum of (1), then it solves the discrete Euler-Lagrange equation

\[ \Delta [f_\xi (k, x_{k+1}, \Delta x_k)] = f_x (k, x_{k+1}, \Delta x_k), \quad k = 0, \ldots, N, \]

where \( \Delta x_k = x_{k+1} - x_k \) is the usual forward difference operator, \( f_x \) and \( f_\xi \) refer to the partial derivatives of \( f \) with respect to the second and third variable, respectively. If \( \hat{x} \) is a solution of (2),

\[ r_k = f_\xi (k, \hat{x}_{k+1}, \Delta \hat{x}_k), \quad q_k = f_{xx} (k, \hat{x}_{k+1}, \Delta \hat{x}_k), \quad r_k = f_{xx} (k, \hat{x}_{k+1}, \Delta \hat{x}_k), \]

* Supported by the Grant 201/01/0079 of the Grant Agency of the Czech Republic

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and the discrete quadratic functional

\[ F(y; 0, N) = \sum_{k=0}^{N} \left\{ r_k (\Delta y_k)^2 + 2q_k \Delta y_k y_{k+1} + p_k y_{k+1}^2 \right\}, \]

is positive for every \( 0 \neq y = \{y_k\}_{k=0}^{N+1} \) with \( y_0 = 0 = y_{N+1} \) (such \( y \) is called admissible), then \( \hat{x} \) is a local minimum of (1). Moreover, \( F(y; 0, N) \geq 0 \) for admissible \( y \) is a necessary condition for \( \hat{x} \) to be a local minimum of (1). Consequently, positivity (nonnegativity) of the quadratic functional \( F \) gives a sufficient (necessary) condition for a local minimum of the discrete variational problem (1).

Using a suitable transformation which preserves the value of transformed quadratic functionals (see [7]), functional \( F \) can be transformed into a functional of the same form with \( q_k \equiv 0 \). Hence, in the sequel we consider the functional

\[ F_{SL}(y) := \sum_{k=0}^{N} \left[ r_k (\Delta y_k)^2 + p_k y_{k+1}^2 \right] \]

The subscript \( SL \) indicates that the Euler-Lagrange equation of \( F_{SL} \) is the Sturm-Liouville difference equation

\[ \Delta(r_k \Delta y_k) - p_k y_{k+1} = 0. \]

In the investigation of positivity (nonnegativity) of \( F_{SL} \) one can again use the discretization of continuous methods or methods of linear algebra. As for the continuous approach, \( F_{SL} \) is the discretization of the functional

\[ F(y) = \int_{a}^{b} \left[ r(t)y'^2 + p(t)y^2 \right] dt \]

whose Euler-Lagrange equation is the second order Sturm-Liouville differential equation

\[ (r(t)y')' - p(t)y = 0, \quad r(t) > 0. \]

The condition \( r(t) > 0 \) is usually called the Legendre condition and for its importance in the calculus of variations we refer to classical books on this topics, e.g. to [10].

If \( y \) is a solution of (5) such that \( y(t) \neq 0 \) for \( t \in [a, b] \), then \( w = \frac{ry'}{y} \) solves the Riccati equation

\[ w' + c(t) + \frac{w^2}{r(t)} = 0. \]
In this case one can complete $\mathcal{F}(y)$ to the square

$$\mathcal{F}(y) = \int_a^b \frac{1}{r(t)} (r(t)y' - w(t)y)^2 \, dt$$

for any (sufficiently smooth) $y$ satisfying $y(a) = 0 = y(b)$. Consequently, the existence of a nonvanishing solution of (5) in $[a, b]$ is a sufficient (and also necessary) condition for $\mathcal{F}_{SL} > 0$. Concerning the discretization of this method, the discrete analogue of (6) is the difference equation (related to (4) by the substitution $w = \frac{r_{\Delta y}}{y}$)

$$\Delta w_k + c_k + \frac{w_k^2}{r_k + w_k} = 0$$

and the discrete version of (7) reads

$$\mathcal{F}_{SL}(y) = \sum_{k=0}^N \frac{1}{r_k + w_k} (r_k \Delta y_k - w_k y_k)^2$$

for any $y = \{y_k\}_{k=0}^{N+1}$ with $y_0 = 0 = y_{N+1}$. Consequently, the mere existence of a nonzero solution of (4) (for $k \in \{0, \ldots, N + 1\}$) is not sufficient for $\mathcal{F}_{SL} > 0$, we need also the condition $r_k + w_k > 0$. Substituting for $w_k$, this condition is equivalent to $r_k y_k y_{k+1} > 0$. Therefore, we have the proof of the sufficiency part of the next statement (for the proof of necessity part we refer e.g. to [13]).

**Theorem 1.** We have $\mathcal{F}_{SL}(y) > 0$ for every admissible $y$ if and only if the solution $\tilde{y}$ of (4) given by the initial condition $\tilde{y}_0 = 0$, $\tilde{y}_1 = \frac{1}{r_0}$ satisfies $r_k y_k y_{k+1} > 0$ for $k \in \{1, \ldots, N\}$.

As we have already mentioned before, the positivity of $\mathcal{F}_{SL}$ is equivalent to positive definiteness of the $N \times N$ matrix which represents $\mathcal{F}_{SL}$. After some computations we find that this matrix representation is

$$\mathcal{F}_{SL}(y) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}^T \begin{pmatrix} \beta_0 & -r_1 & 0 & \cdots & 0 \\ -r_1 & \beta_1 & -r_2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \beta_{N-2} & -r_{N-1} \\ 0 & 0 & \cdots & -r_{N-1} & \beta_{N-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix},$$

where $\beta_k = r_k - p_k + r_{k+1}$. Again after some computation we find that the solution $\tilde{y}$ given by $\tilde{y}_0 = 0$, $\tilde{y}_1 = \frac{1}{r_0}$, satisfies $\tilde{y}_{k+1} = \frac{\Delta_k}{r_k \cdots r_1 r_0}$, where $\Delta_k$ is...
the $k$-th principal minor of the matrix representing $F_{SL}$, see [4]. Hence, for 
$\delta_k := \frac{\Delta_{k+1}}{\Delta_k} r_k$ we have $\delta_k = \frac{\Delta_k}{\Delta_{k-1}}$ and this means that there exists a $N \times N$ triangular matrix $U$ such that

$$U^T U = \text{diag}\{\delta_1, \ldots, \delta_N\}. \tag{10}$$

Consequently, we found an alternative proof of Theorem 1 which is based on the algebraic (matrix) approach.

2. Symplectic difference systems. If we denote $x_k = y_k, u_k = r_k \Delta y_k$, equation (4) can be written in the form

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\Delta_k}{r_k} \\ p_k & 1 + \frac{\Delta_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{11}$$

This is a partial motivation to consider the so-called symplectic difference system (further SDS)

$$z_{k+1} = S_k z_k, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{12}$$

where $x, u \in \mathbb{R}^n, A, B, C, D \in \mathbb{R}^{n \times n}$, i.e., $z \in \mathbb{R}^{2n}$ and $S \in \mathbb{R}^{2n \times 2n}$. The crucial assumption is that the matrix $S$ is symplectic, i.e.

$$S^T J S = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{13}$$

The symplecticity of $S$ in terms of $A, B, C, D$ reads as

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I \tag{14}$$

Since (12) is equivalent to $S J S^T = J$, symplecticity of $S$ is also equivalent to

$$A B^T = B A^T, \quad C D^T = D C^T, \quad A D^T - B C^T = I.$$

An important consequence of the symplecticity of $S$ is the Casorati-type identity; if $z = \begin{pmatrix} x \\ u \end{pmatrix}, \tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$ are solutions of (11), then

$$z_k^T J \tilde{z}_k \equiv \text{const} \equiv x_k^T \tilde{u}_k - u_k^T \tilde{x}_k.$$

In particular, if $Z = \begin{pmatrix} X \\ \tilde{X} \\ U \\ \tilde{U} \end{pmatrix}$ is a $2n \times 2n$ matrix solution of (11) which is symplectic for some index $k$, then it is symplectic everywhere, i.e. (11) defines a discrete symplectic flow.
A $2n \times 2n$ matrix solution $(\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix})$ is said to be the \textit{conjoined basis} if $X^T U = U^T X$ and rank $(\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}) = n$. A pair of conjoined bases $(\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix})$, $(\begin{pmatrix} \mathbf{\bar{x}} \\ \mathbf{\bar{u}} \end{pmatrix})$ is called \textit{normalized}, if $X^T \bar{U} - U^T \bar{X} = I$. The solution $(\begin{pmatrix} \mathbf{\bar{x}} \\ \mathbf{\bar{u}} \end{pmatrix})$ given by the initial condition $X_0 = 0$, $U_0 = I$ is called the \textit{natural conjoined basis} (at $k = 0$).

Note that SDS's cover as special cases a variety of difference equations and systems, among them the higher order self-adjoint difference equation

\begin{equation}
\sum_{j=0}^{n} (-1)^j \Delta^j \left( r_{k,j} \Delta^j y_{k+n-j} \right) = 0
\end{equation}

and the linear Hamiltonian difference system

$$
\Delta \mathbf{x}_k = A_k \mathbf{x}_{k+1} + B_k \mathbf{u}_k, \quad \Delta \mathbf{u}_k = C_k \mathbf{x}_{k+1} - A_k^T \mathbf{u}_k
$$

where the matrices $B, C$ are symmetric, i.e. $B^T = B$, $C^T = C$, and $I - A_k$ is invertible, see [1] for details.

The quadratic functional associated with SDS is the functional

\begin{equation}
\mathcal{F}_{\text{SDS}}(z) = \sum_{k=0}^{N} z_k^T \{ S_k^T K S_k - K \} z_k, \quad K = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},
\end{equation}

over the class of \textit{admissible} $z = (\begin{pmatrix} \mathbf{z} \\ \mathbf{\bar{z}} \end{pmatrix})$ satisfying

$$
\mathbf{z} = \{ z_k \}_{k=0}^{N+1}, \quad K z_{k+1} = K S_k z_k, \quad K z_0 = 0 = K z_{N+1}.
$$

The quadratic functional $\mathcal{F}_{\text{SDS}}$ in terms of $x, u$ and $A, B, C, D$ reads as

\begin{equation}
\mathcal{F}_{\text{SDS}}(x, u) = \sum_{k=0}^{N} \{ x_k^T C_k A_k x_k + 2 x_k^T C_k B_k u_k + u_k^T B_k^T D_k u_k \}.
\end{equation}

We say that $x = \{ x_k \}_{k=0}^{N+1}$ with $x_k \in \mathbb{R}^n$ is \textit{admissible} for $\mathcal{F}_{\text{SDS}}$ if there exists $u = \{ u_k \}_{k=0}^{N} \in \mathbb{R}^{m(N+1)}$ such that

$$
x_{k+1} = A_k x_k + B_k u_k \quad \text{and} \quad x_0 = 0 = x_{N+1}
$$

for $k = 0, \ldots, N$.

When investigating positivity (nonnegativity) of $\mathcal{F}_{\text{SDS}}$, the following extension of Picone's identity plays the crucial role. In this statement, $\dagger$, Ker and Im denote the Moore-Penrose generalized inverse, the kernel and the image of the matric indicated, respectively.

**Proposition 1.** ([3, 5]) Let $(\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix})$ be the natural conjoined basis of (11) at $k = 0$, $(x, u)$ be an admissible pair for (16) and let $Q_k$ be symmetric matrices satisfying $Q_k X_k = U_k X_k X_k^\dagger$. 


(i) If
\[ \text{Ker } X_{k+1} \subseteq \text{Ker } X_k, \quad k = 1, \ldots, N, \]
then
\[ F_{\text{SDS}}(x, u) = \sum_{k=0}^{N} (u_k - Q_kx_k)^T P_k (u_k - Q_kx_k), \]
where \( P_k = X_k X_{k+1}^T B_k \).

(ii) If we suppose instead of (18) the condition
\[ x_k \in \text{Im } X_k, \quad k = 1, \ldots, N, \]
then
\[ F_{\text{SDS}}(x, u) = \sum_{k=0}^{N} (u_k - Q_kx_k)^T P_k (u_k - Q_kx_k), \]
where \( P_k = T_k^T P_k T_k, \quad T_k = (I - M_k^T M_k), \quad M_k = (I - X_{k+1} X_{k+1}^T) B_k \).

We finish this section with a statement concerning positivity (nonnegativity) of the continuous quadratic functional
\[ F_{\text{LHS}}(x, u) = \int_a^b [u^T B(t) u + x^T C(t) x] \, dt \]
over the class of \( x, u \) (called again admissible for \( F_{\text{LHS}} \)) satisfying
\[ x' = A(t)x + B(t)u, \quad x(a) = 0 = x(b). \]
It is supposed that \( A, B, C : [a, b] \rightarrow \mathbb{R}^{n \times n} \) are continuous, \( B, C \) are symmetric and \( B \) is nonnegative definite. In this statement, it is supposed that the linear Hamiltonian system (further LHS)
\[ x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u \]
is identically normal on \([a, b]\) (an alternative terminology is controllable) which means that the trivial solution \((x, u) \equiv 0\) is the only solution of (23) for which \( x(t) \equiv 0 \) on a nondegenerate subinterval of \([a, b]\).

**Proposition 2.** Suppose that (23) is identically normal on \([a, b]\) and let \( (X) \) be the \( 2n \times n \) matrix solution given by the initial condition \( X(a) = 0, U(a) = I \). Then \( F_{\text{LHS}}(x, u) > 0 \) \((\geq 0)\) for every admissible \((x, u)\) with \( x \neq 0 \) if and only if \( \det X(t) \neq 0 \) for \( t \in (a, b) \) \((for t \in (a, b))\).
3. Quadratic functional corresponding to SDS. In this section we present main results on positivity and nonnegativity of the functional $F_{\text{SDS}}$.

**Theorem 2.** ([3]) The functional $F_{\text{SDS}}$ is positive for every admissible $(x, u)$ with $x \neq 0$, if and only if the natural conjoined basis at $k = 0$ satisfies (18) and

$$P_k = X_k X_{k+1}^T B_k$$ is nonnegative definite.

Note that if the kernel condition (18) is satisfied, then the matrix $P_k$ in (24) is really symmetric, see [3]. Observe also that if (16) is rewritten functional $F_{\text{SL}}$, i.e. (11) is rewritten (4), then the previous theorem is the same as the “positivity part” of Theorem 1. The main problem in proving Theorem 2 was the fact that no discrete analogue of the controllability assumption is supposed, which means that the first component $X$ of the natural conjoined basis $(\frac{X}{U})$ at $k = 0$ can be identically singular throughout the whole discrete interval $[1, N + 1]$, even if $F_{\text{SDS}} > 0$, as shows e.g. the example of the quadratic functional corresponding to the $2n$ order difference equation $\Delta^{2n} y_k = 0$ (which is a special case of (15) and hence associated quadratic functional can be written in the form $F_{\text{SDS}}$), see [2].

Now we turn our attention to nonnegativity of $F_{\text{SDS}}$. A part of the proof of Theorem 2 is a construction of an admissible pair $(x, u)$ for which $F_{\text{SDS}}(x, u) \leq 0$ if (18) or (24) fails to hold. If (24) does not hold, the constructed $(x, u)$ satisfies $F_{\text{SDS}}(x, u) < 0$, so this condition is necessary also for nonnegativity of $F_{\text{SDS}}$. However, if (18) is violated, the construction in [3] yields only $(x, u)$ for which $F_{\text{SDS}}(x, u) = 0$. So, from this point of view, (18) is not necessary for $F_{\text{SDS}} \geq 0$. In recent years, considerable effort has been made to find a condition which together with (24) (more precisely, together with a modification of this condition since the matrix $P_k$ is no longer symmetric if (18) is violated) is necessary and sufficient for $F_{\text{SDS}} \geq 0$. Some results of this effort can be found in the papers [9, 12]. Finally, it was proved in [5] that the right substitute for (18) and (24) are the following conditions.

**Theorem 3.** Let $(\frac{X}{U})$ be the natural conjoined basis at $k = 0$. The functional $F_{\text{SDS}}(x, u) \geq 0$ for every admissible $(x, u)$ if and only if (20) holds for every admissible $(x, u)$ and

$$P_k := T_k^T P_k T_k$$ is nonnegative definite, $k = 1, \ldots, N$. 
Similarly to Theorem 2, condition (25) has sense since the image condition (20) implies that the matrix \( P_k \) is really symmetric, see [14].

We finish this section with a result which is a "symplectic" analogue of the diagonalization formula (10) and underlines the importance of matrices \( P_k \) in the problem of nonnegativity of \( F_{SDS} \).

**Theorem 4.** ([6]) Suppose that the image condition (20) holds for every admissible \( x = \{x_k\}_{k=0}^{N+1} \). Then there exist matrices \( N, U \in \mathbb{R}^{nN \times nN} \) such that \( x \) is admissible if and only if \( x \in \text{Im} N \), i.e. \( x = Nd \) for some \( d \in \mathbb{R}^{nN} \), and

\[
F_{SDS}(x, u) = d^T N^T U^T \text{diag}\{P_1, \ldots, P_N\} Nu d.
\]

4. Positivity versus nonnegativity. In this concluding section we present some remarks on conditions for \( F_{SDS} > 0 \) and \( F_{SDS} \geq 0 \), and also one new sufficient condition for \( F_{SDS} \geq 0 \).

(i) Let \( (x, u) \) be an admissible pair and let \( \left( \begin{array}{c} X_k \\ U_k \end{array} \right) \) be any conjoined basis of (11). It is shown in [3] that if \( x_k \in \text{Im} X_k \), i.e. \( x_k \in X_k \alpha \) for some \( \alpha \in \mathbb{R}^n \) and \( \text{Ker} X_{k+1} \subseteq \text{Ker} X_k \), then also \( x_{k+1} \in \text{Im} X_{k+1} \). More precisely, if \( x_k = X_k \alpha_k \) for some \( \alpha_k \in \mathbb{R}^n \), \( x_{k+1} = A_k x_k + B_k u_k \), for some \( u_k \in \mathbb{R}^n \), then

\[
x_{k+1} = X_{k+1} (\alpha_k + X_{k+1}^\dagger B_k d_k) + M_k d_k,
\]

where \( d_k = u_k - U_k \alpha_k \) and kernel condition (18) yields that \( B_k = X_{k+1} X_{k+1}^\dagger B_k \) (see [3]), i.e. \( M_k = 0 \) in this case. This observation shows that the image condition (20) is a weaker condition that the kernel condition (18).

(ii) If we compare the "gap" between conditions for \( F_{LHS} > 0 \) and for \( F_{LHS} \geq 0 \), with the gap between conditions (18), (24) (which are necessary and sufficient for \( F_{SDS} > 0 \)), and (20) and (25) (which are necessary and sufficient for \( F_{SDS} \geq 0 \)), we see that this gap is much bigger in the discrete case. A possible explanation of this phenomenon is that we suppose that (23) is identically normal in Proposition 2, while Theorems 2, 3 require no normality assumption on (11).

(iii) The previous remarks suggest to investigate LHS (23) and associated quadratic functionals without normality condition. The first step along this line has been made in the recent paper [15]. Another direction for the next investigation are quadratic functionals corresponding to symplectic dynamic systems on the so-called time scales. A time scale \( T \) is any closed subset of real numbers \( \mathbb{R} \). A calculus on time scales has been developed in such a way that reduces to differential calculus if \( T = \mathbb{R} \) and to the calculus of finite
differences if $\mathbb{T} = \mathbb{Z}$. In particular, if $f : \mathbb{T} \rightarrow \mathbb{R}$, the time scale derivative $f^\Delta$ is defined in such a way that $f^\Delta = f'$ if $\mathbb{T} = \mathbb{R}$ and $f^\Delta = \Delta f$ if $\mathbb{T} = \mathbb{Z}$. The concept of symplectic dynamic system on a time scale was introduced in [8], basic properties of these systems, including equivalent characterization of positivity of associated quadratic functionals are given in [11]. We hope to follow this line of the investigation in subsequent papers.

(iv) Comparing kernel condition (18) and the image condition (20), the image condition is more difficult to verify since it requires an explicit description of the set of admissible $x = \{x_k\}_{k=0}^{N+1} \in \mathbb{R}^{n(N+2)}$ and this is generally rather difficult problem, see e.g. [9]. Here we present a statement which gives a sufficient condition for $F_{\text{SDS}} \geq 0$ when the kernel condition is possibly violated between $N-1$ and $N$.

**Theorem 5.** Suppose that (25) holds, $\text{Ker} X_{k+1} \subseteq \text{Ker} X_k$ for $k = 1, \ldots, N-2,$ and

$$\text{Im} B'_N \subseteq \text{Ker} M'_N.$$

Then $F(x, u) \geq 0$ for every admissible $(x, u)$.

**Proof.** Let $x = \{x_k\}_{k=0}^{N+1}$ be any admissible sequence. According to Theorem 3 it suffices to show that $x_N \in \text{Im} X_N$. Indeed, the kernel condition implies that $x_k \in \text{Im} X_k$, $k = 1, \ldots, N-1$, i.e. $x_k = X_k \alpha_k$ for some $\alpha_k \in \mathbb{R}^n$, see [3], and $0 = x_{N+1} \in \text{Im} X_{N+1}$ trivially holds. We have (no index in the next computation means index $N-1$)

$$x_N = A x + B u = A X \alpha + B u = X_n \alpha + (M + X_N X_N^T B)(u - U \alpha) = X_n \alpha + X_N^T (M + X_N X_N^T B)(u - U \alpha),$$

where $\hat{x}_N \in \text{Im} X_N$, $\bar{x}_N \in \text{Im} M_{N-1}$ and $\hat{x}_N \perp \bar{x}_N$ since by the properties of Moore-Penrose generalized inverses

$$X_N^T M_{N-1} = X_N^T (I - X_N X_N^T) B_{N-1} = X_N^T (I - (X_N^T)^T X_N^T) B_{N-1} = 0.$$ 

Now, any admissible $(x, u)$ satisfies

$$0 = x_{N+1} = A_N x_N + B_N u_N,$$

i.e. $\begin{pmatrix} x_N \\ u_N \end{pmatrix} \in \text{Ker} (A_N \ B_N)$.

The symplecticity conditions (14) imply $\text{Ker} (A_N \ B_N) = \text{Im} \left( -B_N^T A_N^T \right)$, hence

$$x_N = -B_N^T \beta, \quad u_N = A_N^T \beta$$

for some $\beta \in \mathbb{R}^n$. Finally, (26) is equivalent to

$$\left( \text{Ker} M_{N-1}^T \right)^\perp = \text{Im} M_{N-1} \subseteq \left( \text{Im} B_N^T \right)^\perp,$$

i.e. $\hat{x}_N \in \left( \text{Im} B_N^T \right)^\perp$, and hence, taking into account that $\langle \hat{x}_N, \bar{x}_N \rangle = 0$ and $x_N \in \text{Im} B_N^T$, we have

$$0 = \langle x_N, \hat{x}_N \rangle = \langle \hat{x}_N + \bar{x}_N, \hat{x}_N \rangle = ||\hat{x}_N||^2 \Rightarrow \bar{x}_N = 0.$$

This means $x_N = \hat{x}_N \in \text{Im} X_N$ and the proof is complete.
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MONOTONE DELAY EQUATIONS AND RUNGE-KUTTA DISCRETIZATION

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Abstract. We prove that, under natural conditions, monotonicity of delay equations is preserved under small stepsize discretization. Moreover, for nondecreasing initial functions, explicit/implicit Euler approximations are shown to converge from below/above. The respective errors depend on the total variation of the initial function.

Key Words. delay equations, Runge–Kutta discretizations, monotonicity

AMS(MOS) subject classification. 34K28, 34K07

1. Introduction. The basic results of qualitative discretization theory of ordinary differential equations have recently been generalized to delay equations. As for invariant manifolds, periodic orbits, asymptotic stability and (quadratic) Liapunov functions, see [5], [9] and [10], respectively. A more detailed discussion of the relevant literature is given in the last section. For the general theory of discretizing delay equations, see [1].

The aim of the present paper is to investigate if and how Kamke’s concept of monotonicity for delay equations [12] is preserved under Runge–Kutta discretization. In what follows, monotonicity is understood with respect to the standard cone. All results we prove can be combined with coordinate transformations—in particular, they can be reformulated for partial orders induced by other cones as well.

* Supported by the Hungarian National Science Foundation OTKA No. T037491
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2. The basic assumptions. Consider a $C^1$ function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with the properties that $(g_i)'_{x_j}(x, y) > 0$ for $i \neq j$ and $(g_i)'_{y_j}(x, y) > 0$ whenever $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $i, j = 1, 2, \ldots, n$. Here of course $g_i$ stands for the $i$'th coordinate function of $g$, further $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ denote the first $n$ and the last $n$ coordinate variables of $g_i$, $i = 1, 2, \ldots, n$, respectively. For simplicity, we assume that $g, g' = \{(g_i)'_{x_j}\}_{i,j=1}^n$ are uniformly bounded, and, for some $\gamma > 0$ there holds
\[
\begin{cases}
(g_i)'_{x_j}(x, y) \geq \gamma & \text{if } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \ i, j = 1, 2, \ldots, n \text{ and } i \neq j, \\
(g_i)'_{y_j}(x, y) \geq \gamma & \text{if } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \ i, j = 1, 2, \ldots, n.
\end{cases}
\]

The uniformity assumptions are automatically satisfied on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Everything we prove on $\mathbb{R}^n \times \mathbb{R}^n$ for globally defined (semi-)dynamical systems can be proved for the corresponding local (semi-)dynamical systems as well.

Throughout this paper, $\mathbb{R}^n$ will be equipped with the Euclidean norm $|\cdot|$. By letting $x \leq \bar{x}$ for $x, \bar{x} \in \mathbb{R}^n$ if and only if $x_i \leq \bar{x}_i$ for each $i = 1, 2, \ldots, n$, a closed partial order on $\mathbb{R}^n$ is defined. Fix $\tau > 0$ and set $C = C([-\tau, 0], \mathbb{R}^n)$, the Banach space of continuous $\mathbb{R}^n$-valued functions on the interval $[-\tau, 0]$. The maximum norm on $C$ is the denoted by $|||\cdot|||$. The closed partial order $\leq$ on $\mathbb{R}^n$ generates a closed partial order $\preceq$ on $C$. In particular, $\eta \preceq \bar{\eta}$ holds if and only if $\eta(s) \leq \bar{\eta}(s)$ for each $s \in [-\tau, 0]$. Given $\eta \in C$ arbitrarily, consider now the initial value problem
\[
\begin{cases}
\dot{x}(t) = g(x(t), x(t-\tau)) & \text{for } t \geq 0, \\
x(t) = \eta(t) & \text{for } t \in [-\tau, 0].
\end{cases}
\]

The smoothness and boundedness assumptions on $g$ imply that the initial value problem (2) has a unique solution $x = x(\cdot, \eta) : [-\tau, \infty) \to \mathbb{R}^n$. Moreover, formula $(\Phi(t, \eta))(s) = x(t+s, \eta)$ defines a semi-dynamical system $\Phi : \mathbb{R}^+ \times C \to C$. As an easy consequence of assumption (1)—which is slightly stronger than the standard Kamke condition in [12]—the semi-dynamical system $\Phi$ is monotone. In other words, inequality $\Phi(t, \eta) \preceq \Phi(t, \bar{\eta})$ holds true whenever $t \geq 0$ and $\eta, \bar{\eta} \in C$ with $\eta \preceq \bar{\eta}$. The reason is, essentially, that for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t > 0$ sufficiently small,
\[
(I + tg'_x(x, y))p \geq 0 \quad \text{and} \quad g'_y(x, y)p \geq 0 \quad \text{whenever} \quad p \in \mathbb{R}^n, \ p \geq 0.
\]
In particular, $id + tg(\cdot, y) : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone map in the sense that $(id + tg(\cdot, y))(p) \leq (id + tg(\cdot, y))(\bar{p})$ whenever $y, p, \bar{p} \in \mathbb{R}^n$ and $p \leq \bar{p}$. Here $id$ and $I$ stand for the identity map on $\mathbb{R}^n$ and the $n \times n$ identity matrix, respectively. For later use, set $e = \text{col}(1, 1, \ldots, 1) \in \mathbb{R}^n$. 


3. The main result. Monotone discretizations. Fix $h_0 \in (0, \tau]$. A Runge-Kutta method for (2) with stepsize $h \in (0, h_0]$ is a mapping of the form

$$X(\eta) = \eta(0) + h \sum_{i=1}^{m} b_i \, g \left( X^i, \eta(c_i h - \tau) \right)$$

where

$$X^i = \eta(0) + h \sum_{j=1}^{m} a_{ij} \, g \left( X^j, \eta(c_j h - \tau) \right), \quad i = 1, 2, \ldots, m.$$

Here the positive integer $m$ and the real constants $\{a_{ij}\}_{i,j=1}^{m}$, $\{b_i\}_{i=1}^{m}$ and $\{c_i\}_{i=1}^{m}$ are the parameters of the Runge-Kutta method. We leave them unspecified but assume that

$$b_i \geq 0 \quad \text{and} \quad c_i \in [0,1] \quad \text{for} \quad i = 1, 2, \ldots, m.$$

For $h$ sufficiently small, say $h \leq h^*_1 \leq h_0$, the right-hand side of (5) defines a contraction operator on $\mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ ($m$ times). Finally, the stepsize-$h$ Runge-Kutta discretization operator with piecewise linear interpolant is defined as $\varphi_{PL} : (0, h^*_1) \times C \mapsto C$, $(h, \eta) \mapsto \varphi_{PL}(h, \eta)$,

$$\varphi_{PL}(h, \eta)(s) = \begin{cases} \eta(h + s) & \text{if} \quad s \in [-\tau, -h] \\ -\frac{s}{h} \eta(0) \, + \, (1 + \frac{s}{h}) \, X(\eta) & \text{if} \quad s \in [-h, 0]. \end{cases}$$

Now we are in a position to formulate the main result of the present paper.

**Theorem 1.** For sufficiently small $h$, say $0 < h \leq h^*_2 \leq h^*_1$, and for any $\eta \leq \tilde{\eta}$ ($\eta, \tilde{\eta} \in C$), we have that $\varphi_{PL}(h, \eta) \leq \varphi_{PL}(h, \tilde{\eta})$.

**Proof.** It is enough to prove that $X(\eta) \leq X(\tilde{\eta})$ whenever $\eta, \tilde{\eta} \in C$ and $\eta \leq \tilde{\eta}$. The key observation is that the contraction constant of the operator defined by the right-hand side of (5) does not depend on the parameter function $\eta$ and can be chosen as $\kappa = \frac{1}{2}$. By elementary Lipschitz estimates, it follows with some positive constants $\Gamma_1$ and $\Gamma_2$ that inequality

$$|X^i(\tilde{\eta}) - X^i(\eta)| \leq \Gamma_1 |\tilde{\eta}(0) - \eta(0)| + \Gamma_2 \sum_{j=1}^{m} |\tilde{\eta}(c_j h - \tau) - \eta(c_j h - \tau)|$$

holds for $i = 1, 2, \ldots, m$. (Constants $\Gamma_1$, $\Gamma_2$ are of course independent of $\eta, \tilde{\eta}$ and $h$.)

The remainder of the proof is implied by assumption (1) via the Newton-Leibniz formula. In fact with $Q_i = \int_0^1 g'_x \left( X^i + s \left( \tilde{X}^i - X^i \right), \eta(c_i h - \tau) \right) ds$, 

$$\Gamma_2 \sum_{j=1}^{m} |\tilde{\eta}(c_j h - \tau) - \eta(c_j h - \tau)|$$
and \( R_i = \int_0^1 g_y\left( \tilde{X}^i, \eta(c_i h - \tau) + s(\tilde{\eta}(c_i h - \tau) - \eta(c_i h - \tau)) \right) ds, 1 \leq i \leq m, \)
we obtain by using (4) and (5) that
\[
X(\tilde{\eta}) - X(\eta) = \tilde{\eta}(0) - \eta(0) + h\sum_{i=1}^m b_i (Q_i(\tilde{X}^i - X^i) + R_i(\tilde{\eta}(c_i h - \tau) - \eta(c_i h - \tau)))
\]
\[
= \left( I + h\sum_{i=1}^m b_i Q_i \right) (\tilde{\eta}(0) - \eta(0)) + h\sum_{i=1}^m b_i R_i (\tilde{\eta}(c_i h - \tau) - \eta(c_i h - \tau))
\]
\[
+ \sum_{i=1}^m h b_i Q_i h \sum_{j=1}^m a_{ij} \left( g(\tilde{X}^i, \tilde{\eta}(c_j h - \tau)) - g(X^i, \eta(c_j h - \tau)) \right).
\]
As a simple consequence of (8), the \(| \cdot |\) norm of the double sum is bounded by \( h^2 \Gamma_3 |\tilde{\eta}(0) - \eta(0)| + h^2 \Gamma_4 \sum_{i=1}^m |\tilde{\eta}(c_i h - \tau) - \eta(c_i h - \tau)| \) with some positive constants \( \Gamma_3 \) and \( \Gamma_4 \). Note that \( \sum_{i=1}^m b_i = 1 \) by consistency and \( b_i \geq 0 \) by (6). Hence \( X(\tilde{\eta}) \geq X(\eta) \) follows easily from (3) when applied in the equivalent form \( (I + hg_x(x, y)) p \geq h\Gamma_3|p|e \) and \( g_y'(x, y)p \geq \Gamma_6|p|e \), whenever \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n, p \in \mathbb{R}^n, p \geq 0 \) and \( h < h_2^* \). (Note that \( h_2^* \) is a computable constant and that the crucial parameter is \( \gamma \) in (1).)

For \( c_{i_0} \neq 0 \) and \( c_{i_0} \neq 1 \) \((i_0 \in \{1, 2, \ldots, m\})\), it is plausible that the dependence of the stage value \( X_{i_0} \) on parameter \( \eta \) is not necessarily monotone. This is why we restricted our attention to the piecewise linear interpolant \( \varphi_{PL} \) in (7) and did not consider the usual class of interpolants that preserve the intermediate stage values. (For some closer aspects of the relation between monotonicity and interpolants, see [7].) The contraction operator defined by the right-hand side of (5) depends on the parameter in a monotone way but the contraction itself is not monotone.

**Remark 1.** In practice (4) is often replaced by its \( \ell \)-th approximation
\[
X_\ell(\eta) = \eta(0) + h\sum_{i=1}^m b_i g\left( X^i_\ell, \eta(c_i h - \tau) \right)
\]
where \( \ell > 0 \) is an integer, and \( X^i_\ell \) is defined by the iteration \( X^i_0 = \eta(0), \)
\[
(9) \quad X^i_{k+1} = \eta(0) + h\sum_{j=1}^m a_{ij} \eta(c_j h - \tau), \quad i = 1, 2, \ldots, m,
\]
$k = 0, 1, \ldots, \ell - 1$. A twofold application of (9) yields that

$$|\tilde{X}_{k+1}^i - X_{k+1}^i| \leq |\tilde{\eta}(0) - \eta(0)| + h \Gamma_7 \sum_{j=1}^{m} \left( |\tilde{X}_k^i - X_k^i| + |\tilde{\eta}(c_j h - \tau) - \eta(c_j h - \tau)| \right)$$

for $i = 1, 2, \ldots, m$ and $k \in \mathbb{N}$. By passing to a smaller $h$ if necessary, it follows via the discrete Gronwall lemma that (cf. (8))

$$|\tilde{X}_{k+1}^i - X_{k+1}^i| \leq \Gamma_8 |\tilde{\eta}(0) - \eta(0)| + \Gamma_9 \sum_{j=1}^{m} |\tilde{\eta}(c_j h - \tau) - \eta(c_j h - \tau)|.$$ 

Argueing as in the proof of Theorem 1, we arrive at the conclusion that $X_\ell(\eta) \leq X_\ell(\tilde{\eta})$ whenever $\eta \leq \tilde{\eta}$ and $0 < h \leq h_3^* \leq h_2^*$, $\ell = 1, 2, \ldots$.

Theorem 1 can be applied on a constrained mesh consecutively. As a composition of monotone operators, the whole discretization process of monotone delay equations is monotone.

4. A simple proof for ordinary differential equations. The proof of Theorem 1 works if $\dot{x}(t) = g(x(t), x(t - \tau))$ is replaced by $\dot{x}(t) = f(x)$. Here of course $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded $C^1$ function with bounded partial derivatives satisfying the strong Kamke condition $(f_i)'_j(x) \geq \gamma > 0$ whenever $x \in \mathbb{R}^n$ and $i, j = 1, 2, \ldots, n$, $i \neq j$. Solutions to $\dot{x}(t) = f(x)$ define a dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The stepsize-$h$ approximating dynamical system is of the form $\varphi : (0, h_0] \times \mathbb{R}^n \to \mathbb{R}^n$. Under some natural assumptions—which are satisfied e.g. if $f''_{xx}$ (exists and) is bounded and $\varphi$ comes from an explicit or implicit Runge-Kutta method—Taylor expansion with the remainder in the integral form gives that

$$|\Phi(h, \tilde{x}) - \Phi(h, x) - \left[ I + h \int_0^1 f'_x(x + s(\tilde{x} - x)) ds \right](\tilde{x} - x)| \leq \Gamma_{10} h^2 |\tilde{x} - x|$$

and

$$\left| \left[ \Phi(h, \tilde{x}) - \varphi(h, \tilde{x}) \right] - \left[ \Phi(h, x) - \varphi(h, x) \right] \right| \leq \Gamma_{11} h^2 |\tilde{x} - x|$$

whenever $h \in (0, h_0]$ and $x, \tilde{x} \in \mathbb{R}^n$. By the strong Kamke condition, there is a positive constant $h_4^*$ such that, for all $h \leq h_4^*$,

$$(10) \quad \Phi(h, x) \leq \Phi(h, \tilde{x}) , \varphi(h, x) \leq \varphi(h, \tilde{x}) \quad \text{whenever} \quad x, \tilde{x} \in \mathbb{R}^n \text{ and } x \leq \tilde{x}.$$ 

Unfortunately, this simple proof does not carry over to delay differential equations: formulae (4)-(5) can not be differentiated with respect to $h$ because
the initial function is only continuous. Note that inequality $\Phi(h, x) \leq \Phi(h, \bar{x})$ is implied already by the weak Kamke condition (i.e. assuming $(f_i)'_{x_j}(x) \geq 0$ whenever $x \in \mathbb{R}^n$ and $i, j = 1, 2, \ldots, n, i \neq j$).

Recall that the $\Theta$-method is defined as the unique solution $X = \varphi_\Theta(h, x)$ to equation $X = x + h((1 - \Theta)f(x) + \Theta f(X))$ where $h \leq h^*_\Theta, x \in \mathbb{R}^n$ and $\Theta \in [0, 1]$ is a parameter. For $\Theta = 0$ and $\Theta = 1$, $\varphi_\Theta$ simplifies to the explicit and the implicit Euler method, denoted by $\varphi_E$ and $\varphi_I$, respectively.

**Remark 2.** Remarkably, for each $\Theta \in [0, 1]$ (and $h$ sufficiently small), the monotonicity of the $\Theta$-method follows already from the weak Kamke condition. The easy proof combines Newton-Leibniz formula with the observation that $\frac{\partial}{\partial x} \varphi_\Theta(h, x)$ can be expressed as the product of the two nonnegative matrices $[I - h\Theta f'_x(\varphi_\Theta(h, x))]^{-1}$ and $I + h(1 - \Theta)f'_x(x)$. Referring to Remark 1, a simple example shows that inequality $(\varphi_I(h, x))_\ell \leq (\varphi_I(h, \bar{x}))_\ell$ (for $\ell = 2, 3, \ldots$) is not a consequence of the weak Kamke condition. The problem of describing the relation between the weak Kamke condition and the monotonicity of general Runge-Kutta methods requires a much finer technique and is far beyond the level of the present paper.

5. Discretizations as lower/upper approximations to the true solution. This section is devoted to a different type of monotonic behaviour.

**Theorem 2.** Assume that there exists a $\delta > 0$ such that for each $x \in \mathbb{R}^n$

\begin{equation}
(f'_{x}(x)f(x) \geq \delta e.
\end{equation}

Then, for $h$ sufficiently small, say $h \leq h^*_\delta$ and $\chi \in [0, h)$, $x \in \mathbb{R}^n$, we have that

$\varphi_E(h, x) \leq \varphi_E(h - \chi, \varphi_E(\chi, x)) \leq \Phi(h, x) \leq \varphi_I(h - \chi, \varphi_I(\chi, x)) \leq \varphi_I(h, x)$.

**Proof.** Case $\chi = 0$ is a direct consequence of inequality

$|\varphi_\Theta(h, x) - \Phi(h, x) - (\Theta - \frac{1}{2})h^2 f'_x(x)f(x)| \leq \Gamma_1 h^3$.

Also the general case $\chi \in (0, h)$ follows easily via analysing the approximating Taylor polynomials with remainder. (As it is suggested by the previous considerations, indices $E$ and $I$ can be replaced by $\Theta$ with $\Theta \in [0, \frac{1}{2})$ and $\Theta \in (\frac{1}{2}, 1]$, respectively (but then $h^*_\delta = h^*_\Theta(\Theta) \rightarrow 0$ as $\Theta \rightarrow \frac{1}{2}$).)

Geometrically, Theorem 2 means that, under appropriate conditions, both $\varphi_E$ and $\varphi_I$ are monotone with respect to the insertion of a new mesh-point between two old ones. Combining inequality (10) with (case $\chi = 0$
of Theorem 2, we set inductively \( \varphi^0(h, x) = x \), \( \varphi^{k+1}(h, x) = \varphi(h, \varphi^k(h, x)) \) and conclude that \( \varphi^0(h, x) \leq \Phi(kh, x) \leq \varphi^k(h, x) \) whenever \( h \leq h^*_i, x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). With the maximum stepsize \( \leq h^*_i \), the latter inequality can be generalized to variable stepsize sequences. Moreover, the approximating lower and upper point sequences can be replaced by the respective piecewise linear interpolations. Thus—using a self-explanatory notation—we arrive at

\[
\varphi_{E,PL}(t, x) \leq \Phi(t, x) \leq \varphi_{I,PL}(t, x) \quad \text{whenever} \quad x \in \mathbb{R}^n \text{ and } t \geq 0.
\]

6. Back to delay equations. The explicit and the implicit Euler method for (2) with stepsize \( h \in (0, h_0] \) is defined by (cf. (4)-(5))

\[
X_E(\eta) = \eta(0) + h g(\eta(0), \eta(-\tau)) \quad \text{and} \quad X_I(\eta) = \eta(0) + h g(X_I(\eta), \eta(h - \tau)).
\]

With \( X(\eta) \) in (7) replaced by \( X_E(\eta) \) and \( X_I(\eta) \), respectively, \( \varphi_{E,PL} \) and \( \varphi_{I,PL} \) are defined.

Throughout this section, assumption (1) is replaced by assuming that

\[
\begin{align*}
&\text{the initial function } \eta \in C \text{ is nondecreasing and, for } i, j = 1, 2, \ldots, n \quad g_i(x, y) \geq 0, \ (g_i)'_{y_j}(x, y) \geq 0, \ (g_i)'_{y_j}(x, y) \geq 0 \quad \text{if} \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.
\end{align*}
\]

The initial value problem (2) can be rewritten on the interval \([0, \tau]\) as \( \dot{x}(t) = g(x(t), \eta(t - \tau)) \) for \( t \in [0, \tau] \) and \( x(0) = \eta(0) \). For \( \varepsilon \in [0, 1) \) as a parameter, consider the system of algebraic equations

\[
X^\varepsilon(t) = \eta(0) + t g(X^\varepsilon(t), \eta(t - \tau) + \varepsilon e) + \varepsilon te + \varepsilon e
\]
on the time interval \([0, \tau]\). For \( h^*_i \in (0, \tau) \) sufficiently small, existence and uniqueness of \( X^\varepsilon \in C([0, h^*_i], \mathbb{R}^n) \) follow from the contraction mapping principle, \( \varepsilon \in [0, 1) \). By a coordinatewise analysis of the sequence of successive approximations (starting with \( X^{\varepsilon_0}(t) = \eta(0) + \varepsilon e, t \in [0, h^*_i] \)), we infer from (12) that \( X^\varepsilon \) is a nondecreasing function. We claim that \( X^0(t) \geq x(t) \) on the whole interval \([0, h^*_i]\). By letting \( \varepsilon \to 0^+ \), we see that it is enough to prove \( X^\varepsilon_i(t) \geq x_i(t) \) for each \( t \in [0, h^*_i] \) and \( \varepsilon \in (0, 1), i = 1, 2, \ldots, n \). Suppose not. Then there exist an \( \varepsilon_0 \in (0, 1) \) and a \( T_0 \in (0, h^*_i] \) such that \( X^{\varepsilon_0}_i(T_0) = x_{i_0}(T_0) \) for some \( i_0 \in \{1, 2, \ldots, n\} \) but \( X^\varepsilon_i(t) > x_i(t) \) for all \( t \in [0, T_0) \), \( i = 1, 2, \ldots, n \). By a simple geometric argument, for each \( \delta \in (0, T_0] \) we have that

\[
-\frac{1}{\delta} (x_{i_0}(T_0 - \delta) - x_{i_0}(T_0)) \geq -\frac{1}{\delta} (X^{\varepsilon_0}_{i_0}(T_0 - \delta) - X^{\varepsilon_0}_{i_0}(T_0)).
\]

Since partial derivatives of \( g_{i_0} \) are nonnegative and the coordinate functions of both \( X^{\varepsilon_0} \) and \( \eta \) are nondecreasing, the right-hand size of (13) is at least
\[ \varepsilon_0 + g_{i0}(X^0(T_0 - \delta), \eta(T_0 - \delta - \tau) + \varepsilon_0 e). \] By letting \( \delta \to 0^+ \), we conclude that \( g_{i0}(x(T_0), \eta(T_0 - \tau) \geq \varepsilon_0 + g_{i0}(X^0(T_0), \eta(T_0 - \tau) + \varepsilon_0 e), \) a contradiction.

**Theorem 3.** For \( h \) sufficiently small, say \( 0 < h < h^*_0 \), and for any nondecreasing \( \eta \in C \), we have that \( \varphi_{E,PL}(h, \eta) \leq \Phi(h, \eta) \leq \varphi_{I,PL}(h, \eta) \).

**Proof.** By definition, \( (\Phi(h, \eta))(s) = \eta(h + s) = (\varphi_{I,PL}(h, \eta))(s) \) if \( s \in [-\tau, -h] \). If \( s \in [-h, 0] \), our Claim above implies that

\[ (\Phi(h, \eta))(s) = \eta(0) + (h + s)g(X^0(h + s), \eta(h + s - \tau)) \]

and, a fortiori, by using (12) and the monotonicity properties of \( X^0 \) and \( \eta \),

\[ (\Phi(h, \eta))(s) \leq \eta(0) + (h + s)g(X^0(h), \eta(h - \tau)) \]

\[ = \eta(0) + (h + s)g(X_1(\eta), \eta(h - \tau)) = (\varphi_{I,PL}(h, \eta))(s). \]

Thus \( \Phi(h, \eta) \leq \varphi_{I,PL}(h, \eta) \). The proof of the left-hand side inequality is much easier and left to the reader.

7. **Uniform error estimates on subclasses of \( C \).** Parallel to the no-delay case discussed at the end of Section 5, an iterative combination of Theorems 1 and 3 via \( \varphi^0_{PL}(h, \eta) = \eta \), \( \varphi^{k+1}_{PL}(h, \eta) = \varphi_{PL}(h, \varphi^k_{PL}(h, \eta)) \), \( k \in \mathbb{N} \) yields that \( \varphi_{E,PL}(t, \eta) \leq \Phi(t, \eta) \leq \varphi_{I,PL}(t, \eta) \) whenever \( \eta \) is nondecreasing and \( t \geq 0 \).

It is a well-known fact [1], [9] that error estimates between exact and approximating solutions of (2) depend heavily on the initial function \( \eta \). For example, there exist positive constants \( \Gamma_{13}, \Gamma_{14} \), and \( h^*_1 \) (depending only on \( g \)) with the properties as follows. If \( \tau > 0 \), \( 0 \neq N \in \mathbb{N} \), \( h = \frac{\tau}{N} \leq h^*_1 \), \( k \in \mathbb{N} \), index \( M = E \) or \( M = I \), and \( \eta \in C \) is subject to the Lipschitz condition with constant \( \text{Lip}(\eta) \), then

\[ ||\varphi^k_{M,PL}(h, \eta) - \Phi(kh, \eta)|| \leq h\Gamma_{13}e^{\Gamma_{14}kh}(1 + \text{Lip}(\eta)). \]

It is important to note that inequality (14) holds true without any monotonicity assumptions on \( g \) (but it is still assumed that \( g, g'_x, g'_y \) are uniformly bounded on \( \mathbb{R}^n \times \mathbb{R}^m \)). For monotone initial functions, however, we have a better result. Recall that, by the Jordan decomposition theorem, the difference of two monotone functions is of bounded variation and vice-versa.

**Theorem 4.** If \( \eta \in C \) is of bounded variation, then

\[ \text{Tot.Var} \left( \varphi^k_{M,PL}(h, \eta) - \Phi(kh, \eta) \right) \leq h\Gamma_{15}e^{\Gamma_{16}kh}(1 + \text{Tot.Var}(\eta)) \]

where \( \text{Tot.Var}(\cdot) \) stands for the total variation.
**Proof.** Mutatis mutandis—only minor technical changes are needed—the standard proof of inequality (14) works. (We note again that (15) holds true regardless of the possible monotonicity properties of $g$ and of the possible Lipschitz property of $\eta$.)

8. Concluding remarks. We note that ‘monotonicity’ has a variety of different meanings (e.g. w.r. to a Liapunov structure [6], w.r. to a finite arithmetic field [2]) in numerical dynamics. However, methods of finding upper and lower approximating solutions with respect to some partial order play an important role in all branches of numerical analysis. Such two-sided methods [3], [4], [8] are usually based on abstract monotonicity results when combined with iterative and/or discretization techniques. As it was foreseen two decades ago [11], they found a growing number of applications in rigorous computing. The more the original problem is subject to monotonicity assumptions, the easier the results. Our present paper is a collection of some introductory observations on the enclosure dynamics of delay equations.

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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A PERTURBATION OF THE MCKENDRICK-VON FOESTER EQUATION

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Abstract. We consider a perturbation of the classical McKendrick-Von Foester equation originally discussed by Boulanger [1]. As we are dealing with population densities it is more natural to express the equations as integral equations. We establish existence and uniqueness of solutions under weaker conditions than previously.

AMS(MOS) subject classification. 35A05, 45G99, 47H10.

Key Words. McKendrick-Von Foester equation; integral equations; existence and uniqueness.

1. Introduction. Boulanger [1] investigated the following population dynamical model with age interaction:

\[
\frac{\partial \rho_a}{\partial t} + \frac{\partial \rho_a}{\partial a} = -\rho_a \left[ \mu_a(a) + \int_0^\infty \beta_t(a,s)\rho_s(t,s)ds \right]
\]

with boundary condition

\[
\rho_t(t,0) = \int_0^\infty b(a)\rho_t(t,a)da
\]

and initial condition

\[
\rho_t(0,a) = \phi(a).
\]

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The consistency condition

\[ \phi(0) = \int_0^\infty b(a)\phi(a)\,da \]

must also be satisfied. Here \( \mu_\epsilon(a) = \mu_0(a) - \epsilon \), where \( \mu_0(a) \) is such that

\[ R = \int_0^\infty b(a)e^{-\int_0^a \mu_0(\xi)\,d\xi} \, da = 1 \]

and \( \mu_0(a) \geq \epsilon \) for all \( a \). Boulanger also assumed that \( \beta_\epsilon(a, s) = \epsilon\overline{\beta}(a, s) \) where \( \overline{\beta} \) is a function \( \mathbb{R}^+ \times \mathbb{R}^+ \). \( \rho_\epsilon(t, a) \) represents the density with respect to age of the total number of individuals in the population at time \( t \). This means that the total number of individuals between ages \( A_1 \) and \( A_2 \) at time \( t \) is

\[ \int_{A_1}^{A_2} \rho_\epsilon(t, a)\,da. \]

\( \mu_\epsilon(a) \) is the death rate at age \( a \), \( \phi(a) \) is the density of the initial age distribution and \( b(a) \) is the birth rate. \( \beta_\epsilon(a, s) \) is a term corresponding to interaction between generations limiting population growth. When \( \epsilon = 0 \) these equations reduce to the classical linear McKendrick-Von Foester (McK-VF) equation for the growth of a population \([6, 7]\). \( R \) is the reproduction number for the corresponding classical linear McK-VF equation. In his paper Boulanger investigated the approximation of the solutions of the linear McK-VF equation by the solutions of the full equations with interaction term as \( \epsilon \) goes to zero. If the increase in mortality due to crowding is just the total population, one would obtain the nonlinear age-dependent model treated in Gurtin and McCamy \([3]\).

Boulanger gives a set of conditions and asserts that these imply that (1)-(4) have a unique solution for \( \rho_\epsilon(a, t) \). As previously pointed out \([2]\) we disagree with this and claim that much stronger conditions are necessary. However our conditions, based on results in Webb \([8]\), are very strong and as we are dealing with population densities it is more natural to express (1)-(3) as integral equations. By doing this we shall establish existence and uniqueness of solutions under weaker conditions. As \( \rho_\epsilon(t, a) \) is a population density it is natural to look for solutions \( f : [0, \infty) \rightarrow \mathbb{R} \) in the Banach space \( L^1[0, \infty) \) with norm

\[ \| f \| = \int_0^\infty | f(a) | \, da. \]

A similar idea to show existence and uniqueness of solutions was taken for the less general model in Gurtin and McCamy’s paper \([3]\).
2. Results. We shall suppose that \( b(a), \beta_\epsilon(a,s), \mu_\epsilon(s) \) and \( \phi(a) \) satisfy the following conditions:

**Conditions A**

(i) \( b(a) \) is measurable and bounded on finite intervals;

(ii) \( \int_t^\infty b(a)\phi(a - t)e^{-\int_a^t \mu_\epsilon(\xi)d\xi}da \) is bounded above independently of \( t \) by \( M_0 \) say;

(iii) \( \beta_\epsilon(a,s) \) is measurable and bounded absolutely by \( M_1 \) say;

(iv) \( \mu_\epsilon(s) \) is integrable on \([0,a)\) for all \( a > 0 \);

(v) \( \phi \in L^1[0,\infty) \).

In particular Conditions A(i) and A(ii) are automatically satisfied if \( b(a) \) is absolutely bounded and \( \phi \in L^1[0,\infty) \) but they are weaker than assuming that \( b(a) \) is absolutely bounded. In fact as \( b(a) \) represents a birth rate it is biologically reasonable to assume that it is bounded, but we shall prove the existence and uniqueness of solutions to integral equations for \( \rho_\epsilon(t,a) \) corresponding to (1)-(3) under the less restrictive Conditions A.

As in [3] we start the proof by making the transformations

\[
\tilde{\rho}_\epsilon(t,a) = \rho_\epsilon(t,a)e^{\int_0^a \mu_\epsilon(\xi)d\xi}, \quad \tilde{\beta}_\epsilon(a,s) = \beta_\epsilon(a,s)e^{\int_0^a \mu_\epsilon(\xi)d\xi},
\]

\[
\tilde{b}_\epsilon(a) = b(a)e^{-\int_0^a \mu_\epsilon(\xi)d\xi} \quad \text{and} \quad \tilde{\phi}_\epsilon(a) = \phi(a)e^{\int_0^a \mu_\epsilon(\xi)d\xi},
\]
equations (1)-(3) reduce to

\[
\frac{\partial \tilde{\rho}_\epsilon(t,a)}{\partial a} + \frac{\partial \tilde{\rho}_\epsilon(t,a)}{\partial t} = -\tilde{\rho}_\epsilon \int_0^\infty \tilde{\beta}_\epsilon(a,s)\tilde{\rho}_\epsilon(t,s)ds,
\]

\[
\tilde{\rho}_\epsilon(t,0) = \int_0^\infty \tilde{b}_\epsilon(a)\tilde{\rho}_\epsilon(t,a)da,
\]

and

\[
\tilde{\rho}_\epsilon(0,a) = \tilde{\phi}_\epsilon(a).
\]

The consistency condition (4) becomes

\[
\tilde{\phi}_\epsilon(0) = \int_0^\infty \tilde{b}_\epsilon(a)\tilde{\phi}_\epsilon(a)da.
\]

As \( \rho_\epsilon(t,a) \) is a density

\[
\int_0^\infty \tilde{\beta}_\epsilon(a,s)\tilde{\rho}_\epsilon(t,s)ds = \int_0^\infty \beta_\epsilon(a,s)\rho_\epsilon(t,s)ds
\]

exists for each \( a \) and \( t \) using Condition A(iii).
LEMMA 1. Suppose that $\tilde{\rho}_e(t, a)$ satisfies (5)-(7). Define $B_e(t) = \tilde{\rho}_e(t, 0)$.

Then

$$
\tilde{\rho}_e(t, a) = \begin{cases} 
\tilde{\phi}_e(a-t) e^{-\int_0^t \int_0^\infty \tilde{\rho}_e(-t+\sigma, s) \tilde{\rho}_e(\sigma, s) d\sigma ds}, & a \geq t, \\
B_e(t-a) e^{-\int_0^{a} \int_0^\infty \tilde{\rho}_e(\sigma, s) \tilde{\rho}_e(-a+\sigma, s) d\sigma ds}, & t > a,
\end{cases}
$$

and $B_e(t) = \int_0^t \tilde{b}_e(a)B_e(t-a) e^{-\int_0^a \int_0^\infty \tilde{\rho}_e(\sigma, s) \tilde{\rho}_e(-a+\sigma, s) d\sigma ds} da$

Moreover both of the double integrals in (9) and (10) exist.

Proof. Write $x = t - a$ and $y = a$ then

$$
\frac{\partial}{\partial y} \tilde{\rho}_e(t, a) = -\tilde{\rho}_e(t, a) \int_0^\infty \tilde{\beta}_e(a, s) \tilde{\rho}_e(t, s) ds,
$$

and as previously observed the integral in (11) exists. Therefore

$$
\frac{\partial}{\partial y} \left[-\log \tilde{\rho}_e(x, y)\right] = \int_0^\infty \tilde{\beta}_e(y, s) \tilde{\rho}_e(x, y, s) ds.
$$

If $a \geq t$ then integrating (12) between $-x$ and $y$ we deduce that

$$
\tilde{\rho}_e(x + y, y) = \tilde{\rho}_e(0, -x) e^{-\int_0^y \int_0^\infty \tilde{\rho}_e(a', s) \tilde{\rho}_e(t-a'+\sigma, s) d\sigma ds},
$$

and the above double integral exists. Equivalently

$$
\tilde{\rho}_e(t, a) = \tilde{\phi}_e(a-t) e^{-\int_0^t \int_0^\infty \tilde{\beta}_e(a-t+\sigma, s) \tilde{\rho}_e(\sigma, s) d\sigma ds},
$$

using (7). Similarly if $t \geq a$ then integrating (12) between $0$ and $y$

$$
\tilde{\rho}_e(t, a) = B_e(t-a) e^{-\int_0^a \int_0^\infty \tilde{\beta}_e(\sigma, s) \tilde{\rho}_e(t-a+\sigma, s) d\sigma ds}
$$

proving (9). (10) follows immediately, noting that $B_e(t) = \tilde{\rho}_e(t, 0)$ and using (6). The proof is complete. □

Hence if $\tilde{\rho}_e(t, a)$ satisfies (5)-(7) (so in particular is differentiable with respect to $a$ and $t$) then it satisfies (9)-(10). On the other hand if both of the double integrals in (9) exist, $\tilde{\rho}_e(t, a)$ satisfies (9)-(10) and the left-hand side of (5) exists then $\tilde{\rho}_e(t, a)$ satisfies (5)-(7).

LEMMA 2. Consider the renewal equation

$$
B_e(t) = \int_0^t \tilde{b}_e(a)B_e(t-a) da + F(t),
$$

where $B_e(t) = \tilde{\rho}_e(t, 0)$. Then $B_e(t) = \tilde{\rho}_e(t, 0)$ and $\tilde{\rho}_e(t, a)$ satisfies (5)-(7).
where \( F(t) = \int_t^\infty \hat{b}_\varepsilon(a) \hat{\phi}_\varepsilon(a-t) da \) for \( 0 \leq t \leq T \). Fix \( T \) with \( 0 < T < \infty \). Suppose that \( | \hat{b}_\varepsilon(s) | \leq K_T \) and \( | F(t) | \leq M_T \) on \([0, T] \). Then (13) has a unique solution on \([0, T] \).

**Proof.** This is a straightforward modification of the classical Banach fixed point argument for the existence and uniqueness of solutions of a linear Volterra integral equation of the second kind (see Theorem 1.1 on p.87 of [5] and Remark 2.1.5 of [4]). \( \square \).

Consider (9)-(10). Define \( \hat{\phi}(u) = \hat{\phi}_\varepsilon(u) = \phi(u) e^{\int_0^u \mu_\varepsilon(\xi) d\xi} \) and \( \hat{B}(u) \) to be the unique solution of the renewal equation (13). Our first result is to show that if \( 0 \leq T' \leq T_0 \) and \( T' \) is sufficiently small equations (9)-(10) have a unique solution for \( t \in [0, T_0] \) with \( \rho \bigg|_{[0,T'] \times [0, A]} \in L^1(T, A) \) for all \( T' \geq T > 0 \) and \( A > 0 \) and \( B \bigg|_{[0,T]} \in L^1(T) \) for all \( T' \geq T > 0 \). Consider the metric space

\[
\mathcal{M} = \left\{ (\rho, B) : \rho \bigg|_{[0, T] \times [0, A]} \text{ is measurable for all } T > 0 \text{ and } A > 0, \right.
\]

\[
0 \leq B(t) \leq \hat{B}(t), \quad 0 \leq \rho(t, a) \leq \left\{ \begin{array}{ll}
\hat{\phi}(a-t), & a \geq t, \\
\hat{B}(t-a), & t > a,
\end{array} \right.
\]

and \( B \bigg|_{[0, T]} \) is measurable for all \( T > 0 \} \).

\( \mathcal{M} \) is a complete metric space with metric

\[
\| (\rho, B) \| = \max \left[ \sup_{0 \leq t \leq T} B(t), \sup_{0 \leq t \leq T} \int_0^\infty \rho(t, a) e^{-\int_0^a \mu_\varepsilon(\xi) d\xi} da \right].
\]

To show that this metric is well-defined note that if \( (\rho, B) \in \mathcal{M} \) and \( A > t \) then by splitting the first integral at \( t \in [0, A] \) and bounding each part separately

\[
\int_0^A \rho(t, a) e^{-\int_0^a \mu_\varepsilon(\xi) d\xi} da \leq M_2 = \int_0^{T_0} \hat{B}(u) du + \int_0^\infty \phi(u) du < \infty.
\]

Hence for \( 0 \leq t \leq T_0 \), \( \int_0^A \rho(t, a) e^{-\int_0^a \mu_\varepsilon(\xi) d\xi} da \) is positive, monotone increasing in \( A \) and bounded above by \( M_2 \) and \( \| (\rho, B) \| \leq \max(M_2, M_3) \) for \( T \leq T_0 \) where \( M_3 = \sup_{0 \leq t \leq T_0} \hat{B}(t) \).

For \( t \leq T \leq T_0 \) consider \( \int_0^t \int_0^A \tilde{b}_\varepsilon(a - t + \sigma, s) \rho(\sigma, s) ds d\sigma \). This exists for any \( A \) and \( t \) and by splitting the inner integral at \( \sigma \) if \( \sigma \in [0, A] \) it can be bounded above by \( M_4 = M_1 M_2 T_0 \). Similarly by considering \( \int_0^a \int_0^A \tilde{b}_\varepsilon(\sigma, s) \rho(t - a + \sigma, s) ds d\sigma \) with \( T_0 \geq T \geq t \geq a \) and splitting the
inner integral at $t - a + \sigma$ if $t - a + \sigma \in [0, A]$ this integral can be bounded above by $M_4$.

So we can define a map $S : \mathcal{M} \to \mathcal{M}$ by

$$
S \rho(t, a) = \begin{cases} 
\tilde{\phi}(a-t)e^{-\int_0^t \beta(t-s)\rho(s)ds} & a \geq t, \\
B(t-a)e^{-\int_0^a \beta(t-a-s)\rho(s)ds} & t > a,
\end{cases}
$$

and

$$
SB(t) = \int_0^t \tilde{b}(a)B(t-a)e^{-\int_0^a \beta(t-a-s)\rho(s)ds}da + \int_t^\infty \tilde{b}(a)\tilde{\phi}(a-t)e^{-\int_0^t \beta(t-a-s)\rho(s)ds}da.
$$

Both the double integrals in (14) exist and are at most $M_4$.

Clearly

$$
0 \leq S \rho(t, a) \leq \begin{cases} 
\hat{\phi}(a-t), & a \geq t, \\
B(t-a), & t > a,
\end{cases}
$$

and

$$
0 \leq SB(t) \leq \int_0^t \tilde{b}(a)B(t-a)da + \int_t^\infty \tilde{b}(a)\tilde{\phi}(a-t)da = \hat{B}(t).
$$

So $S : \mathcal{M} \to \mathcal{M}$ is well-defined.

Let $d$ be the distance induced on $\mathcal{M}$ by the metric $\| \cdot \|$. We shall show that if $T$ is sufficiently small there is a constant $K < 1$ such that

$$
d((S \rho_1, SB_1), (S \rho_2, SB_2)) \leq K d((\rho_1, B_1), (\rho_2, B_2)) \equiv K d.
$$

For $0 \leq t \leq T$, using the triangle inequality,

$$
\left| SB_1(t) - SB_2(t) \right| \leq \int_0^t \tilde{b}(a) \left| B_1(t-a) - B_2(t-a) \right| e^{-\int_0^a \beta(t-a-s)\rho_1(t-a+s)ds}da
$$

$$
+ \int_t^\infty \tilde{b}(a)\tilde{\phi}(a-t)e^{-\int_0^t \beta(t-a-s)\rho_1(t-a+s)ds}da
$$

$$
+ \int_0^\infty \tilde{b}(a)\tilde{\phi}(a-t)e^{-\int_0^t \beta(t-a-s)\rho_2(t-a+s)ds}da
$$

(16) can be bounded above by $M_5Td$ where $M_5 = \sup_{a \in [0, T]} | b(a) |$. (17) can be bounded above by

$$
M_3M_5 \int_0^t \left| e^{-\int_0^a \beta(t-a-s)\rho_2(t-a+s)ds} - 1 \right| da,
$$
using the inequality $|1 - e^{-x}| \leq e^{x^2} - 1$. Now for $0 \leq \sigma \leq a \leq t \leq T$,

$$
\int_0^\infty \hat{\beta}_c(\sigma, s) \left| \rho_2(t - a + \sigma, s) - \rho_1(t - a + \sigma, s) \right| \, ds \\
= \int_0^\infty \beta(\sigma, s) \left| \rho_2(t - a + \sigma, s) - \rho_1(t - a + \sigma, s) \right| e^{-\int_0^s \mu_c(\xi) \, d\xi} \, ds,
$$

$$
\leq M_1 d.
$$

Hence (17) is at most $M_3 M_5 (e^{M_1 dT} - 1) T \leq (2M_1 M_3 M_5 T) dT$ if $T$ is small enough. A similar argument shows that (18) can be bounded above by $2M_0 M_1 dT$ if $T$ is small enough. So

$$
|SB_1(t) - SB_2(t)| \leq (M_5 + 2M_1 M_3 M_5 T + 2M_0 M_1) dT,
$$

$$
\leq K d, \text{ where } K < 1 \text{ if } T \text{ is small enough.}
$$

Similarly $\int_0^\infty |S\rho_1(t, a) - S\rho_2(t, a)| e^{-\int_0^a \mu_c(\xi) \, d\xi} \, da$

$$
\leq \int_0^t |B_1(t - a) - B_2(t - a)| e^{-\int_0^a \int_0^\infty \hat{\beta}_c(\sigma, s) \rho_1(t - a + \sigma, s) \, d\sigma \, ds} e^{-\int_0^a \mu_c(\xi) \, d\xi} \, da \\
+ \int_0^t B_2(t - a) e^{-\int_0^a \int_0^\infty \hat{\beta}_c(\sigma, s) \rho_1(t - a + \sigma, s) \, d\sigma \, ds} \\
\left| 1 - e^{-\int_0^a \int_0^\infty \hat{\beta}_c(\sigma, s) (\rho_2(t - a + \sigma, s) - \rho_1(t - a + \sigma, s)) \, d\sigma \, ds} \right| e^{-\int_0^a \mu_c(\xi) \, d\xi} \, da \\
+ \int_t^\infty \phi_c(a - t) e^{-\int_t^a \int_0^\infty \hat{\beta}_c(a - t + \sigma, s) \rho_1(\sigma, s) \, d\sigma \, ds} \\
\left| 1 - e^{-\int_t^a \int_0^\infty \hat{\beta}_c(a - t + \sigma, s) (\rho_2(\sigma, s) - \rho_1(\sigma, s)) \, d\sigma \, ds} \right| e^{-\int_t^a \mu_c(\xi) \, d\xi} \, da,
$$

$$
\leq \left[ 1 + 2M_1 M_3 T + 2M_1 \int_0^\infty \phi(u) \, du \right] dT, \text{ if } T \text{ is sufficiently small},
$$

$$
< K d, \text{ where } K < 1, \text{ if } T \text{ is small enough.}
$$

Hence if $T$ is small enough ($T \leq T'$ say where $0 \leq T' \leq T_0$) $S$ is a contraction operator on $\mathcal{M}$ with respect to the metric $d$ and hence has a unique fixed point $(\rho^*, B^*)$ which satisfies (9) - (10) on $0 \leq t \leq T'$. Now define a second metric space

$$
\mathcal{M}_1 = \mathcal{M} \cap \{ (\rho, B) : (\rho, B) = (\rho^*, B^*) \text{ on } [0, T'] \},
$$

a complete metric space with metric

$$
\| (\rho, B) \| = \max \left[ \sup_{T' \leq t \leq 2T'} B(t), \sup_{T' \leq t \leq 2T'} \int_0^\infty \rho(t, a) e^{-\int_0^a \mu_c(\xi) \, d\xi} \, da \right].
$$
This metric is well defined as before and has associated distance $d_1$ on $\mathcal{M}_1$. A similar argument to before shows that $S$ is a contraction operator on $\mathcal{M}_1$ with respect to $d_1$ and has a unique fixed point $(\rho^*_1, B^*_1)$ on $[T', 2T']$. From (9)-(10) $(\rho^*_1, B^*_1) = (\rho^*, B^*)$ at $t = T'$. Define $(\dot{\rho}, \dot{B})$ to be $(\rho^*, B^*)$ on $[0, T']$ and $(\rho^*_1, B^*_1)$ on $[T', 2T']$. $(\dot{\rho}, \dot{B})$ is the unique fixed point of $S$ defined on $[0, 2T']$. Continuing in this way we deduce that $S$ has a unique fixed point $(\rho_{T_0}, B_{T_0})$ defined on $[0, T_0]$ for any $T_0 > 0$. A similar argument to above shows that if $T_{01} \leq T_{02}$ then $(\rho_{T_{02}}, B_{T_{02}})$ restricted to $[0, T_{01}]$ agrees with $(\rho_{T_{01}}, B_{T_{01}})$ there. Hence we can define $(\rho, B)$ on $[0, \infty)$ to be this unique value. $(\rho, B)$ is the unique fixed point of $S$ on $[0, \infty)$.

By making the transformation $x = t - a$, $y = a$ it is straightforward that the linear partial derivative in (5) exists, if the partial differential operator in (1) is understood as

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right]u(t, a) = \frac{d}{dh}u(t + h, a + h).$$

(No differentiability in time and space separately can be inferred.) This completes the proof of the existence and uniqueness of solutions to (1)-(4) under Conditions A.

3. Conclusions. We have looked at existence and uniqueness of solutions of a perturbation of the McKendrick-Von Foester equation [1]. By transforming the variables and working with the corresponding integral equations (9)-(10) we established existence and uniqueness of solutions under much weaker conditions.

REFERENCES

Abstract. In this paper, we considered a new class of stochastic differential equations involving delayed response, where the delay depends on the system’s state. We obtained results for the existence and uniqueness of solutions, and we proved that the Euler discrete-time approximation scheme is convergent with a strong order of convergence. We used the approximation result to simulate the continuous-time GARCH(1,1) model for stochastic volatility with state-dependent delay. The simulation results showed that a choice of state-dependent delay function spans a wide variety of U-shaped implied volatility plots, and the state-dependence can also be used to control the height of the plots.

Key Words. state-dependent delay, Euler discrete-time approximation, implied volatility, continuous-time GARCH model.

AMS(MOS) subject classification. 34K50, 62P05

1. Introduction. In [1], we considered the following stochastic volatility model

\[ dx(t) = r x(t) \, dt + \sqrt{y(t)} x(t) \, dW(t), \]

\[ \frac{dy(t)}{dt} = \gamma V + \alpha \frac{1}{\tau} \left\{ \ln \frac{x(t)}{x(t-\tau)} - \mu \tau + \frac{1}{2} \int_{t-\tau}^{t} y(s) \, ds \right\}^2 - (\alpha + \gamma) y(t), \]

as a continuous-time limit of the well-known GARCH(1,1) model. Here \( x(t) \) represents the stock price and \( \sqrt{y(t)} \) represents its volatility, that is the standard deviation of \( \log x(t) \). The time delay parameter \( \tau \) was considered...
as a constant. An equation for European call option was derived and a numerical scheme was introduced. Some simulation and numerical results showed that the model produces a U-shaped implied volatility smile. On the other hand, via the employed parameter estimation, we showed that the delay varies considerably from year to year. This suggests that the delay involved in the market response may not be a constant but may depend on the stock price.

In this paper, we consider the case where the time delay depends on \( x \) and \( y \). In particular, we assume that \( \tau \) is a decreasing function of \( y \), due to the following empirical observation: since \( y \) represents volatility of the stock, the greater \( y \) the more volatile price, and therefore, the more active trading. On the other hand, it is reasonable to assume that the market’s response to changes in the stock price is faster when the volatility is higher. As a consequence, the delay is a decreasing function of the volatility.

This leads us to consider a general stochastic state-dependent delay differential equation (SSDDE)

\[
\begin{cases}
    dX(t) = F(X(t), X(t-\tau)) \, dt + G(X(t), X(t-\tau)) \, dW(t), \\
    X(t) = \varphi(t), \quad t \in [-\delta, 0],
\end{cases}
\]

where \( \tau \) takes values in \([\delta_0, \delta]\) and \( \tau = \tau(X(t-\kappa)) \) for \( \kappa \in [\delta_0, \delta] \).

We shall establish a result on the existence of a solution to the above SSDDE. We shall also prove that the Euler discrete-time approximation scheme has a strong order of convergence, and this convergence also yields the uniqueness of a solution. We shall apply the approximation result to our continuous-time GARCH model with state-dependent delay, and we shall use the Monte-Carlo method to carry out the simulation and to analyze the implied volatility structure.

2. Existence. Here, we shall establish the existence of a solution to the following multi-dimensional SSDDE:

\[
\begin{cases}
    dX(t) = F(X(t), X(t-\tau)) \, dt + G(X(t), X(t-\tau)) \, dW(t), \\
    X(t) = \varphi(t), \quad t \in [-\delta, 0],
\end{cases}
\]

where \( \tau = \tau(X(t), X(t-\kappa)) \), \( 0 < \delta_0 \leq \tau(s_1, s_2) \leq \delta \) for \( s_1, s_2 \in \mathbb{R}^n \), \( \kappa \in [\delta_0, \delta] \) and \( \{W(t)\} \) is a Wiener process defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with filtration \( \{\mathcal{F}_t\}_{t \geq 0} \).

In what follows, \(|\cdot|\) is the Euclidean norm. For any \( a < b \), \( L^2(\Omega, C[a, b]) \) is the space of \( C[a, b] \)-valued random variables equipped with the norm defined
by
\[ \| \eta \|_L := \sqrt{\mathbb{E}[\| \eta \|^2]}, \]
where \( \| \eta \|_c = \sup_{a \leq t \leq b} |\eta(t)|. \)

For \( \alpha \in (0, 1], \) we define a set \( S_\alpha \subset L_2(\Omega, C[-\delta, 0]) \) by
\[ S_\alpha := \{ \eta \mid \exists M \forall t_1, t_2 \in [-\delta, 0] : E|\eta(t_1) - \eta(t_2)|^2 \leq M|t_1 - t_2|^{2\alpha} \}. \]

**Theorem 1.** Assume \( F, G \) and \( \tau \) are continuous in their arguments. Then for any \( \mathcal{F}_t \)-measurable initial data \( \varphi \in L_2(\Omega, C[-\delta, 0]) \) there exists a solution of SSDDE (1) defined on \([0, +\infty)\).

**Proof.** We use the so-called method of steps to construct a solution to (1). Note that for any \( t \in \left[ n\delta_0, (n + 1)\delta_0 \right], \) \( n \geq 0 \) we have the following.

\begin{align*}
X(t) &= X(n\delta_0) + \int_{n\delta_0}^{t} F(X(s), X(s - \tau(X(s), X(s - \kappa)))) ds \\
&\quad + \int_{n\delta_0}^{t} G(X(s), X(s - \tau(X(s), X(s - \kappa)))) dW(s).
\end{align*}

Since \( \tau(x, y) \geq \delta_0 \) for \( x, y \in \mathbb{R}^n \), we have \( s - \tau(X(s), X(s - \kappa)) \leq s - \delta_0 \) and (2) becomes a stochastic ODE. Note that \( \{ X(u), -\delta \leq u \leq s - \delta_0 \} \) is a.s. continuous, and therefore there is an a.s. continuous solution to (2) defined on \([n\delta_0, (n + 1)\delta_0]\) (see [4]). \( \Box \)

**3. Discrete-time Approximations of SSDDEs.** In this section we prove that the Euler discrete-time scheme for a SSDDE with a special form has \( 1/2 \)-strong order of convergence over \([0, T]\). This extends the result for stochastic ODEs. We also show that the scheme for a slightly more restrictive SSDDE has \( 2^{-\frac{T}{n+2}} \)-strong order of convergence, and we shall derive the uniqueness of solutions as a corollary.

**3.1. SSDDE: Type I.** We consider the following special case of (1).

\begin{align*}
\begin{cases}
    dX_1(t) = f(X(t), X_2(t - \tau)) \, dt + g(X(t), X_2(t - \tau)) \, dW(t), \\
    dX_2(t) = z(X(t), X_2(t - \tau)) \, dt, \\
    X(t) = [X_1(t), X_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0],
\end{cases}
\end{align*}

where \( \tau = \tau(X(t)), X_1(t) \in \mathbb{R}^{n_1}, X_2(t) \in \mathbb{R}^{n_2} \) and \( X(t) \in \mathbb{R}^{n_1+n_2} \). Note that only the \( X_2 \)-component has state-dependent delayed effect.
For a fixed \( h > 0 \) and \( t \in \mathbb{R} \), we denote \([t] = h[t/h]\), where \([\cdot]\) is the integer part. **Strong Euler approximation scheme** for (3) is defined as follows:

\[
\begin{align*}
\frac{d\tilde{X}_1(t)}{dt} &= f(\tilde{X}([t]), \tilde{X}_2([t] - [\tau])), \\
\frac{d\tilde{X}_2(t)}{dt} &= g(\tilde{X}([t]), \tilde{X}_2([t] - [\tau])), \\
\tilde{X}(t) &= [\tilde{X}_1(t), \tilde{X}_2(t)]^T = \varphi(t), \quad t \in [-\delta, 0],
\end{align*}
\]

where \([\tau] = [\tau(\tilde{X}([t]))]\).

**Theorem 2.** Assume that \( f, g, z \) and \( \tau \) are Lipschitz continuous with respect to all of their arguments. Then for any \( \mathcal{F}_t \)-measurable initial data \( \varphi \in S_1 \), there exists a constant \( C(T, \varphi) \) such that

\[
\sup_{t \in [0,T]} E|X(t) - \tilde{X}(t)| \leq C(T, \varphi) h
\]

for sufficiently small \( h \), where \( h \) is the partition’s mesh size, \( X \) and \( \tilde{X} \) satisfy (3) and (4) respectively. Moreover, the solution \( X \) of (3) is pathwise unique.

**Proof.** Using representations (3) and (4) for \( X \) and \( \tilde{X} \), we get

\[
\begin{align*}
&\|X - \tilde{X}\|_{L^2(\Omega, C[0,t])}^2 = \sup_{u \in [0,t]} E|X(u) - \tilde{X}(u)|^2 \\
&\leq 2 \sup_{u \in [0,t]} E|\int_0^u (f(X(s), X_2(s - \tau)) - f(\tilde{X}([s]), \tilde{X}_2([s] - [\tau])))|ds|^2 \\
&+ 2 \sup_{u \in [0,t]} E|\int_0^u (g(X(s), X_2(s - \tau)) - g(\tilde{X}([s]), \tilde{X}_2([s] - [\tau])))|dW(s)|^2 \\
&+ \sup_{u \in [0,t]} E|\int_0^u (z(X(s), X_2(s - \tau)) - z(\tilde{X}([s]), \tilde{X}_2([s] - [\tau])))|ds|^2 \\
&\leq 2t \int_0^t E|f(X(s), X_2(s - \tau)) - f(\tilde{X}([s]), \tilde{X}_2([s] - [\tau]))|^2ds \\
&+ 2 \int_0^t E|g(X(s), X_2(s - \tau)) - g(\tilde{X}([s]), \tilde{X}_2([s] - [\tau]))|^2ds \\
&+ t \int_0^t E|z(X(s), X_2(s - \tau)) - z(\tilde{X}([s]), \tilde{X}_2([s] - [\tau]))|^2ds.
\end{align*}
\]

We now estimate each term in (5). First of all, we have

\[
\begin{align*}
\int_0^t E|f(X(s), X_2(s - \tau)) - f(\tilde{X}([s]), \tilde{X}_2([s] - [\tau]))|^2ds \\
&\leq 5 \left[ J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) \right],
\end{align*}
\]
where

\[
J_1(t) = \int_0^t E|f(X(s), X_2(s - \tau)) - f(X([s]), X_2(s - \tau))|^2 ds,
\]

\[
J_2(t) = \int_0^t E|f(X([s]), X_2(s - \tau)) - f(\bar{X}([s]), X_2(s - \tau))|^2 ds,
\]

\[
J_3(t) = \int_0^t E|f(\bar{X}([s]), X_2(s - \tau)) - f(\bar{X}([s]), X_2([s] - [\tau]))|^2 ds,
\]

\[
J_4(t) = \int_0^t E|f(\bar{X}([s]), X_2([s] - [\tau])) - f(\bar{X}([s]), X_2([s] - [\bar{\tau}]))|^2 ds,
\]

\[
J_5(t) = \int_0^t E|f(\bar{X}([s]), X_2([s] - [\bar{\tau}])) - f(\bar{X}([s]), \bar{X}_2([s] - [\bar{\tau}]))|^2 ds.
\]

Here, \(\tau = \tau(X(s))\), \([\tau] = [\tau(X([s]))]\) and \([\bar{\tau}] = [\tau(\bar{X}([s]))]\). Since \(f\) and \(\tau\) are Lipschitz continuous and \(X\) is 1/2-Lipschitz continuous with the constant \(M(\varphi)\) (as in definition of \(S_{1/2}\)), we obtain

\[
J_1(t) \leq L \int_0^t E|X(s) - X([s])|^2 ds \leq L \int_0^t M(\varphi)(s - [s]) ds
\]

\[
\leq LtM(\varphi) h \equiv C_1(t, \varphi) h,
\]

\[
J_2(t) \leq L \int_0^t E|X([s]) - \bar{X}([s])|^2 ds \leq L \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0, s])} ds,
\]

\[
J_3(t) \leq L \int_0^t E|X_2(s - \tau) - X_2([s] - [\tau])|^2 ds
\]

\[
\leq LM_2(\varphi) \int_0^t E|s - \tau - [s] + [\tau]|^2 ds
\]

\[
\leq 2LM_2(\varphi) \int_0^t (s - [s])^2 ds + 2LM_2(\varphi) \int_0^t E(\tau - [\tau])^2 ds
\]

\[
\leq 6LM_2(\varphi) t \ h^2 + 4LM_2(\varphi) L_\tau \int_0^t E|X(s) - X([s])|^2 ds
\]

\[
\leq 6LM_2(\varphi) t \ h^2 + 4LM_2(\varphi) L_\tau M(\varphi) t h \equiv C_2(t, \varphi) h^2 + C_3(t, \varphi) h.
\]

Here, \(\sqrt{L}\) and \(\sqrt{L_\tau}\) are the Lipschitz constants of \(f\) and \(\tau\), respectively. We also used the fact that \(\varphi\) and \(X_2\) are Lipschitz continuous with constant \(M_2(\varphi)\) because the diffusion coefficient of second equation in (3) is zero. We
also have

\[ J_4(t) \leq LM_2(\varphi) \int_0^t E|\tau| - |\bar{\tau}|^2 \, ds \]

\[ \leq 2LM_2(\varphi)t \, h^2 + 2LM_2(\varphi) L_{t} \int_0^t E|X([s]) - \bar{X}([s])|^2 \, ds \]

\[ \leq 2LM_2(\varphi)t \, h^2 + 2LM_2(\varphi) L_{t} \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0,s])} \, ds \]

\[ \equiv C_4(t, \varphi) \, h^2 + C_5(\varphi) \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0,s])} \, ds, \]

\[ J_5(t) \leq L \int_0^t E|X_2([s] - [\bar{\tau}]) - \bar{X}_2([s] - [\bar{\tau}])|^2 \, ds \]

\[ \leq L \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0,s])} \, ds. \]

Therefore,

\[ \int_0^t E|f(X(s), X_2(s - \tau)) - f(\bar{X}([s]), \bar{X}_2([s] - [\bar{\tau}]))|^2 \, ds \]

\[ \leq C_6(t, \varphi) \, h + C_7(\varphi) \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0,s])} \, ds \]

for sufficiently small \( h \). Carrying out the same analysis for terms in (5) with \( g \) and \( z \), we obtain

\[ \|X - \bar{X}\|^2_{L^2(\Omega, C[0,t])} \leq A(\varphi, T) \int_0^t \|X - \bar{X}\|^2_{L^2(\Omega, C[0,s])} \, ds + B(T, \varphi) \, h \]

for certain constants \( A \) and \( B \). Consequently, an application of the Grownwall inequality yields

\[ \|X - \bar{X}\|^2_{L^2(\Omega, C[0,T])} \leq B(T, \varphi)e^{A(\varphi, T)T} \, h. \]

Uniqueness of the solution follows from this inequality, and the theorem is proved. \( \Box \)

3.2. SSDDE: Type II. In this subsection, we consider SSDDE of the following more general form:

\[ \begin{align*}
    dX(t) &= F(X(t), X(t - \tau)) \, dt + G(X(t), X(t - \tau)) \, dW(t), \\
    X(t) &= \varphi(t), \quad t \in [-\delta, 0],
\end{align*} \]
where $\tau = \tau(X(t-\kappa))$. Note that $\tau$ depends on $\kappa$-delayed value of $X$ only.

The Euler discrete-time scheme for (7) is given by

$$
\begin{align*}
\begin{cases}
    d\tilde{X}(t) = F(\tilde{X}([t]), \tilde{X}([t] - \lfloor \frac{\tau}{h} \rfloor)) dt + G(\tilde{X}([t]), \tilde{X}([t] - \lfloor \frac{\tau}{h} \rfloor)) dW(t), \\
    \tilde{X}(t) = \varphi(t), & t \in [-\delta, 0],
\end{cases}
\end{align*}
$$

where $\lfloor \frac{\tau}{h} \rfloor = \lfloor \tau(\tilde{X}([t-\kappa])) \rfloor$.

**THEOREM 3.** Assume $F$, $G$ and $\tau$ are Lipschitz continuous with respect to all of their arguments. Then for any $F$-measurable initial data $\varphi \in S_{1/2}$, there exists a constant $C(T, \varphi)$ such that

$$
\sup_{t \in [0,T]} E|X(t) - \tilde{X}(t)|^2 \leq C(T, \varphi) h^n
$$

for sufficiently small $h$, where $h$ is the partition’s mesh size, $n = 2^{-\lceil \frac{T}{h} \rceil - 1}$. Moreover, the solution $X$ of (7) is pathwise unique.

**Proof.** We use similar arguments used in the proof of Theorem 2, with necessary modifications. Since $X$ is 1/2-Lipschitz continuous, estimation for $J_3(t)$ and $J_4(t)$ becomes

$$
\begin{align*}
J_3(t) &\leq LM(\varphi) \int_0^t E|\tau - \lfloor \frac{\tau}{h} \rfloor| ds \\
&\leq LM(\varphi)t h + LM(\varphi)L_\tau \int_0^t E|X([s-\kappa]) - \tilde{X}([s-\kappa])| ds \\
&\leq LM(\varphi)t h + LM(\varphi)L_\tau \int_0^t \|X - \tilde{X}\|_{L^2(\Omega, C([-\delta,s-\kappa]))} ds \\
&\equiv C_4(t, \varphi) h + C_5(\varphi) \int_0^t \|X - \tilde{X}\|_{L^2(\Omega, C([-\delta,s-\kappa]))} ds,
\end{align*}
$$

and

$$
J_4(t) \leq L M(\varphi) \int_0^t E|\tau - \lfloor \frac{\tau}{h} \rfloor| ds
$$

$$
\leq LM(\varphi)t h + LM(\varphi)L_\tau \int_0^t E|X([s-\kappa]) - \tilde{X}([s-\kappa])| ds
$$

$$
\leq LM(\varphi)t h + LM(\varphi)L_\tau \int_0^t \|X - \tilde{X}\|_{L^2(\Omega, C([-\delta,s-\kappa]))} ds
$$

$$
\equiv C_4(t, \varphi) h + C_5(\varphi) \int_0^t \|X - \tilde{X}\|_{L^2(\Omega, C([-\delta,s-\kappa]))} ds,
$$

and

$$
J_3(t) \leq C_2(t, \varphi) h + C_3(t, \varphi) \sqrt{h}.
$$

In addition, we should replace (6) by

$$
\varepsilon(t) \leq A(\varphi, T) \int_0^t \left( \varepsilon(s) + \sqrt{\varepsilon(s-\kappa)} \right) ds + B(T, \varphi) \sqrt{h},
$$

where $\varepsilon(t) = \|X - \tilde{X}\|_{L^2(\Omega, C([-\delta, t]))}^2$. Since $X$ and $\tilde{X}$ have the same initial data, $\varepsilon(t) = 0$ for $t \leq 0$. By Gronwall inequality, we then obtain

$$
\varepsilon(t) \leq B e^{A \kappa} \sqrt{h} \quad \text{for } t \in [0, \kappa],
$$

$$
\varepsilon(t) \leq A \int_0^t \varepsilon(s) ds + B_2 \sqrt{h} \quad \text{for } t \in [\kappa, 2\kappa],
$$

$$
\varepsilon(t) \leq B_2 e^{A \kappa} \sqrt{h} \quad \text{for } t \in [\kappa, 2\kappa],
$$

where $A$ and $B_2$ are constants.
where $B_2 = B\sqrt{\delta_0} + \sqrt{B}e^{\frac{\alpha}{2}}$. By iterations, we obtain

$$
\varepsilon(t) \leq B_ne^{\frac{\alpha}{2}} h^{2-n} \quad \text{for } t \in [(n-1)\kappa, n\kappa].
$$

This completes the proof. \(\square\)

4. A Continuous-time GARCH Model with State-dependent Delay. The following model for stock price $x$ and its volatility $\sqrt{y}$ was derived in [1]:

$$
\begin{align*}
\frac{dx(t)}{dt} &= \gamma V + \frac{\alpha}{t} \left\{ \ln \frac{x(t)}{x(t-\tau)} - \mu \tau + \frac{1}{2} \int_{t-\tau}^{t} y(s) \, ds \right\}^2 - (\alpha + \gamma) y(t), \\
\frac{dy(t)}{dt} &= \gamma V + \frac{\alpha}{\tau(y_n-k)} \left\{ \ln \frac{x_n}{x_n-N(y_{n-k})} - \mu \tau(y_{n-k}) + \frac{\Delta t}{2} \sum_{i=0}^{N(y_{n-k})} y_{n-i} \right\}^2 - (\alpha + \gamma) y_n,
\end{align*}
$$

where $\tau > 0$ is a constant. The model is derived from discrete-time GARCH model, and parameter estimation for S&P500 in [1] shows that the delay parameter varies considerably from year to year. This leads us to the assumption that $\tau$ is a function of state values. In this section, we assume $\tau = \tau(y(t-\kappa))$ so Theorem 3 can be applied.

From the previous section, the Euler discrete-time scheme given below is convergent to the unique solution. The scheme is given by

$$
\begin{align*}
x_{n+1} - x_n &= \tau x_n \Delta t + \sqrt{\gamma} \sqrt{\Delta t} \varepsilon_n, \\
\frac{y_{n+1} - y_n}{\Delta t} &= \gamma V + \frac{\alpha}{\tau(y_{n-k})} \left\{ \ln \frac{x_n}{x_n-N(y_{n-k})} - \mu \tau(y_{n-k}) + \frac{\Delta t}{2} \sum_{i=0}^{N(y_{n-k})} y_{n-i} \right\}^2 - (\alpha + \gamma) y_n,
\end{align*}
$$

where $\tau(y_{n-k}) = \delta_0 + \hat{\tau} \exp(\rho y_{n-k})$ for some $-\rho, \hat{\tau}, \delta_0 > 0$, $N = \lceil (\hat{\tau} + \delta_0)/\Delta t \rceil$, $N(y_{n-k}) = \lceil \tau(y_{n-k})/\Delta t \rceil$, $k = \lceil k/\Delta t \rceil$, $\lceil \cdot \rceil$ is the integer part and $\{\varepsilon_n\}_{n \geq 0}$ are i.i.d. Normal(0,1). Here, the initial data $(x_n, y_n)$ is provided for $n = -N, \ldots, 0$.

The particular choice of function $\tau$ is due to the following empirical observation. Since $y$ represents volatility of the stock, the greater $y$ the more volatile price, and therefore, the more active trading. Assuming the market's response to changes in the stock price is faster when the volatility is higher, we then conclude that the delay is a decreasing function of the volatility.

Let us try to find a fair price for the European call option written on the stock with maturity $T$ and strike price $K$. It is known that the option price $C$ is given by the following expectation:

$$
C = E \left[ e^{-rT} \max(x_T - K, 0) \right],
$$
where \( r \) is risk-free interest rate and \( x_T \) is stock price at the time \( T \). This expectation can be found using a Monte Carlo simulation of \( x_T \) approximated by the scheme (10).

Some simulation results are provided in the attached figures for different functions of state-dependent delay \( \tau \). They are presented as plots of implied volatility against strike price \( K \). Note that implied volatility is computed using the inverse of Black-Scholes formula applied to simulated option price \( C \).

It is well-known that the curve of the implied volatility of market option price has a U-shape, this is further confirmed by our plots. Observe also that the curvature of the graph is getting larger and larger when the value of the delay \( \tau \) is increased. A constant delay cannot be used to control the height of the curve independently of the curvature, whereas varying delay can. Moreover, we provide some plots by using \( \tau \) as an increasing or a periodic function to illustrate the variety of curves we can obtain. Solid lines represent 95%-confidence bounds for \( 10^6 \) simulations and dashed lines represent 95%-confidence bounds for \( 10^7 \) simulations.

REFERENCES


\begin{figure}
\centering
\begin{tabular}{ll}
\textbf{Implied Volatility: constant delay 0.028} & \textbf{Implied Volatility: constant delay 0.012} \\
\end{tabular}
\begin{tabular}{l}
\includegraphics[width=0.45\textwidth]{implied_volatility_0028.png} & \includegraphics[width=0.45\textwidth]{implied_volatility_0012.png}
\end{tabular}
\caption{Implied volatility for models with constant delay}
\end{figure}
Implied Volatility: small constant delay

![Graph of Implied Volatility: small constant delay]

Fig. 2. Implied volatility for models with nearly constant delay

Implied Volatility: delay with small variations near 0.028

![Graph of Implied Volatility: delay with small variations near 0.028]

Fig. 2. Implied volatility for models with nearly constant delay

Implied Volatility: decreasing delay with exponent $\rho = 0.25$

![Graph of Implied Volatility: decreasing delay with exponent $\rho = 0.25$]

Fig. 3. Implied volatility for models with decreasing state-dependent delay

Implied Volatility: decreasing delay with exponent $\rho = 0.5$

![Graph of Implied Volatility: decreasing delay with exponent $\rho = 0.5$]

Fig. 3. Implied volatility for models with decreasing state-dependent delay

Implied Volatility: increasing delay

![Graph of Implied Volatility: increasing delay]

Fig. 4. Implied volatility for models with state-dependent delays

Implied Volatility: periodic delay varying in 0.012-0.028

![Graph of Implied Volatility: periodic delay varying in 0.012-0.028]

Fig. 4. Implied volatility for models with state-dependent delays
Abstract. This paper is motivated by mathematical models with bounded distributed delays appearing in the literature. Analytical results on stability and bifurcations of such models can be improved significantly by using numerical methods. We show how computational tools for discrete delay equations can be applied to certain types of equations with distributed delays.

Key Words. Discrete and distributed delay equations, numerical bifurcation analysis.

AMS(MOS) subject classification. 65P30

1. Introduction. We consider integro-differential and integral equations with bounded distributed delays (DIDEs, respectively DIEs),

\( (1) \quad x'(t) = f_1(x(t), x(t - \tau_0), \int_{t - \tau_2}^{t - \tau_1} K(t - s)g(x(s))ds, \eta) \) (DIDE) and

\( (2) \quad x(t) = f_2(\int_{t - \tau_2}^{t - \tau_1} K(t - s)g(x(s))ds, \eta) \) (DIE).

---

* Supported by the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture (IUAP P5/22). The scientific responsibility is assumed by its authors.
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Here \( x \in \mathbb{R}^n, \eta \in \mathbb{R}^p \) is a vector of (time-independent) parameters, the functions \( f_1 : \mathbb{R}^{n+p} \to \mathbb{R}^n, f_2 : \mathbb{R}^{n+p} \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable and \( \tau_0 > 0, \tau_2 > \tau_1 \geq 0 \) denote the bounded time delays. These equations appear e.g. in modelling population dynamics and the spread of infectious diseases [1, 5, 6, 7, 9]. Usually, constant and exponentially decaying kernel (weight) functions \( K \) are used since (i) the type of the kernel is an additional uncertainty for modelers and (ii) no computational tools exist for stability and bifurcation analysis of distributed delay equations, while a (rather limited) analytical analysis is possible for equations with such kernels.

A numerical bifurcation analysis is used to compute and analyze the local stability of steady state and periodic solutions of a given system as well as to study the dependence of these solutions on system parameters via continuation (see [3, 4] for details on analysis of delay equations). Numerical bifurcation analysis of distributed delay equations (1) and (2) can be useful in two directions: to gain new insight into the model dynamics and to choose a more suitable kernel function.

Our goal is to show how bifurcation analysis of DIDEs and DIEs with a constant kernel and a gamma distribution kernel function,

\[
K_m(t) = t^{m-1}a^m e^{-at}/(m-1)!, \quad a \in \mathbb{R}_0^+, \quad m \in \mathbb{N}_0, \quad t \in [\tau_1, \tau_2],
\]

can be done using computational tools for discrete delay equations (DDEs). Note that (3) covers a rather wide range of kernels, e.g. smoothing the effect of a certain moment in the past. In our approach, we replace (1) and (2) by a system of DDEs. We discuss to which extent this system is equivalent to the original equation w.r.t. the main issues of a bifurcation analysis: steady state and periodic solutions and their local stability (the initial value problem is not considered). Using the software package DDE-BIFTOOL [3, 4], developed for bifurcation analysis of DDEs, we present some illustrative results analyzing a DIDE model from [5].

2. Equations with a constant kernel. Let \( u(t) := \int_{t-\tau_2}^{t-\tau_1} g(x(s))ds \in \mathbb{R}^n \). We will compare the system (1) and the system of DDEs,

\[
\begin{align*}
x'(t) &= f_1(x(t), x(t-\tau_0), u(t), \eta), \\
u'(t) &= g(x(t-\tau_1)) - g(x(t-\tau_2)).
\end{align*}
\]

System (4) is not equivalent to (1): every steady state solution \( x^*(t) \equiv x^* \) and \( T \)-periodic solution \( x^*(t + T) = x^*(t), \forall t > 0, \) of (1) give the solution \((x^*, u(x^*))\) of (4) but the reverse is not true. However, system (4), coupled with the following algebraic conditions,

\[
u = g(x)(\tau_2 - \tau_1)
\]

for computing steady state solutions and
(6) \( u(t_0) = \int_{t_0-\tau_1}^{t_0-\tau_2} g(x(s)) \, ds, \quad t_0 \in [0, T], \) for periodic solutions,

has the same steady state and periodic solutions \( x \) as system (1). In DDEBIFTOOL, conditions (5) and (6) can be defined (programmed) by the user in a code specifically designed for algebraic conditions. Thus, steady state and periodic solutions to (1) can be computed using (4)-(6).

System (4) can also be used to study the local (asymptotic) stability of solutions to (1). A precaution should be made in this respect. The local stability of steady state solutions \( x^* \) and \( (x^*, u(x^*)) \) is determined by the rightmost roots of the characteristic equation obtained through the linearization of (1), respectively (4), around the steady state,

(7) \( \det(\lambda I - A_0 - A_1 e^{-\lambda \tau_0} - A_2(e^{-\lambda \tau_1} - e^{-\lambda \tau_2})/\lambda) = 0, \) respectively,

(8) \( \det(\lambda^2 I - (A_0 + A_1 e^{-\lambda \tau_0})\lambda - A_2(e^{-\lambda \tau_1} - e^{-\lambda \tau_2})) = 0, \quad \lambda \in \mathbb{C}. \)

Here, using \( f \equiv f_1(x, x_{\tau_0}, u(x), \eta) \) and \( s := (x^*, x^*, u(x^*), \eta), \)

(9) \( A_0 = \frac{\partial f}{\partial x} \bigg|_{s^*}, \quad A_1 = \frac{\partial f}{\partial x_{\tau_0}} \bigg|_{s^*}, \quad A_2 = \frac{\partial f}{\partial u} \frac{\partial g(x)}{\partial x} \bigg|_{s^*}, \)

and \( I \) is the \((n \times n)\) identity matrix. We observe that the characteristic equation (8) has the same roots \( \lambda \) as (7) except for the (extra) root \( \lambda = 0. \)

It thus follows that although the stability of the solution \( x^* \) to (1) is not equivalent to the stability of \((x^*, u(x^*))\) as the solution to (4), the stability of \( x^* \) can be studied via stability of (4). Additional comments can be found in [2].

When computing the local stability of periodic solutions to (4) coupled with (6), an extra Floquet multiplier at 1 (compared to the multipliers for (1)) appears for the same reason as the extra zero root for steady states.

As an illustration, we (briefly) present some results on bifurcation analysis of an epidemic model [5],

(10) \[ x'(t) = \alpha x(t)(1-x(t)) - \alpha x(t-\tau)(1-x(t-\tau))e^{-(\beta+\gamma)\tau} + \beta \int_{t-\tau}^{t} x(s) \, ds \]
\[ -(\beta + \gamma)x(t) + \beta x^2(t), \]
rewriting it in the form (4)-(6) and using DDE-BIFTOOL.

We fix parameters \( \beta, \gamma, \tau \) as in [5] and compute a branch of steady state solutions of (10) as a function in \( \alpha \), cf. Fig. 1 (left). Computing the rightmost characteristic roots \( \lambda \) with \( \Re(\lambda) > -0.3 \) indicates two Hopf bifurcations.
along the branch. The second Hopf point corresponds to the one obtained analytically for (10) in [5]. Figure 1 (right) shows a branch of Hopf points in the \((\alpha, \beta)\)-plane (passing through these two Hopf points). Steady states are unstable inside the loop formed by this branch.

Starting at the second Hopf point, we computed a branch of periodic solutions of (10) varying \(\alpha\), cf. Fig. 2 (left). Stability of the computed solutions changes at the turning point of the branch and the branch ends at the other Hopf point, indicated in Fig. 1 (left). Solutions corresponding to three points on this branch are shown in Fig. 2 (right).

Fig. 1. Left: stable (--) and unstable (---) solutions \(x\) (top) and real part of the rightmost roots \(\lambda\) of the characteristic equation (bottom) versus \(\alpha\). All roots, except for \(\lambda = 0\), are complex. Hopf points (o) at \(\alpha \approx 728.59\) and \(\alpha \approx 743.34\). Right: branch of Hopf points in the \((\alpha, \beta)\)-plane. \(\beta = 720.81\) (left), \(\gamma = 0.012803\), \(\tau = 1\).

Fig. 2. Left: evolution of maximal and minimal values of \(x\) along the branch of periodic solutions of (10) versus \(\alpha\), \(x_{\text{max}} = \max_{t \in [0,T]} x(t), x_{\text{min}} = \min_{t \in [0,T]} x(t)\). Branches of stable (--) and unstable (---) steady state solutions and stable (bold --) and unstable (bold ---) periodic solutions. Hopf (o) and turning (•) points. Right: stable solutions \(x(t/T), t \in [0,T]\), corresponding to the three points on the branch. \(\beta = 720.81\), \(\gamma = 0.012803\), \(\tau = 1\).

Now consider DIEs. Differentiation of (2) leads to a DDE if \(f_2\) is linear.
and to a DIDE of the type (1) otherwise. In both cases the resulting equation, coupled with the algebraic conditions similar to (5) and (6),

\[(11) x = f_2(g(x)(\tau_2 - \tau_1), \eta), \quad x(t_0) = f_2(\int_{t_0 - \tau_2}^{t_0 - \tau_1} g(x(s))ds, \eta), \quad t_0 \in [0, T],\]

has the same steady state and periodic solutions as (2). For a nonlinear DIE, we further apply the approach described for DIDEs. Due to the differentiation, an extra root \(\lambda = 0\) appears and hence we have more extra zero roots \(\lambda\) in case of a nonlinear DIE. A similar result holds w.r.t. the Floquet multiplier at 1. Although this approach looks rather sophisticated, it can be used in practice.

3. Equations with a gamma distribution kernel. Consider DIDE (1) with \(K(t) = K_m(t), \ t \in [\tau_1, \tau_2]\), cf. (3). Let \(u(t) := \int_{t-\tau_2}^{t-\tau_1} K_j(t-s)g(x(s))ds \in \mathbb{R}^n, \ j = 1, \ldots, m\). The use of a linear chain technique [8], widely applied to equations with infinite delays and kernels (3), leads, in our case, to the system of DDEs,

\[(12) \quad x'(t) = f_1(x(t), x(t-\tau_0), u_m(t), \eta),\]

\[u_1'(t) = K_1(\tau_1)g(x(t-\tau_1)) - K_1(\tau_2)g(x(t-\tau_2)) - au_1(t),\]

\[u_j'(t) = K_j(\tau_1)g(x(t-\tau_1)) - K_j(\tau_2)g(x(t-\tau_2)) + a(u_{j-1}(t) - u_j(t)), \ j = 2, \ldots, m.\]

DIDE (1) and system (12) are equivalent w.r.t. steady state and periodic solutions \(x\). Let \(d_j(\lambda) := a^{m-j}(K_j(\tau_1)e^{-\tau_1\lambda} - K_j(\tau_2)e^{-\tau_2\lambda}), \ j = 1, \ldots, m\). The characteristic equations for (1) and (12) read respectively as

\[(13) \quad \det(\lambda I - A_0 - A_1e^{-\tau_0} - A_2 \sum_{j=1}^m d_j(\lambda)(a + \lambda)^{j-m-1}) = 0, \quad \text{and}\]

\[(14) \quad \det((\lambda I - A_0 - A_1e^{-\tau_0})(a + \lambda)^m - A_2 \sum_{j=1}^m d_j(\lambda)(a + \lambda)^{j-1}) = 0,\]

with \(A_j\) defined by (9). These equations have the same solutions \(\lambda\) except for the root \(\lambda = -a\) with multiplicity \(m\) for (14) (the \((m-1)\)-th derivative of the left hand side of (14) w.r.t. \(\lambda\) equals zero for \(\lambda = -a\)). Hence the local stability analysis of solutions to (1) can be performed using system (12).

Differentiation of DIE (2) with kernel (3) leads to a DIDE of the type (1), except for the case of a linear DIE and \(m = 1\), which is reduced immediately to a DDE. Hence the above approach can be applied. Note that conditions similar to (11), including the corresponding kernel function, have to be used to preserve equivalence of the solutions to (2) and the resulting system, which
is lost due to the differentiation. Also note that, in addition to the root $\lambda = -a$, the extra zero root appears for the characteristic equation of the resulting DDE due to the differentiation.

The above approach can also be used in case of kernels defined by a convex combination of gamma distribution functions.

4. Concluding remarks. DIDEs and DIEs with some other types of the kernel function, e.g. an oscillatory kernel function $K(t) = \sin(t)$, can also be replaced by a system of DDEs using the described approach. As it was shown, two issues have to be considered when using this approach: the preservation of solutions to the original equation and how roots of the corresponding characteristic equations differ.

Replacement of distributed delay equations by equations with discrete delays increases the dimension of the system under study and hence the computational time for its analysis. However, this is not a problem if the dimension of the original equation is low, as is the case for e.g. mathematical models in biosciences, where equations with distributed delays are mostly used nowadays.

REFERENCES

THE METHOD OF SMOOTHING FUNCTIONAL INVERSE CONTROL PROBLEMS FOR DELAY SYSTEMS

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Abstract. A problem of dynamical identification of inputs of systems described by differential equations with time delay is considered. Solving algorithms based on the method of control with a model are suggested.

AMS(MOS) subject classification. 49J25, 49N50

Key Words. Functional differential equations, identification, control

1. Introduction. Problems of determination of some parameters via equation's solutions are often called reconstruction (identification) problems. It is assumed that input information (results of measurements of current phase states of a dynamical system) is forthcoming in the process. As to unknown parameters, they should be reconstructed in the process too. One of the methods of solving similar problems was suggested in [1, 2]. This method based on the ideas of the theory of ill-posed problems actually reduces the identification problem to the control problem for an auxiliary dynamical system-model. Regularization of the problem under consideration is locally realized during the process of choice of positional control in the system-model. The method mentioned above was applied to a number of problems described by some classes of ordinary differential equations as well as by equations with distributed parameters [3–5]. In the present paper, using the methods of dynamical identification worked out earlier (see the cited literature), we indicate algorithms for reconstruction of nonsmooth inputs

* Supported by RFBR under grant # 03-01-00474, by the Program on Basic Research of the Presidium of the Russian Acad. Sci. (project “Control of mechanical systems”) and the Program of supporting leading scientific schools of Russia (project 1846.2003.1).
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acting upon a system described by functional-differential equation. These algorithms are stable with respect to informational noises and computational errors.

Briefly, the essence of the problem under consideration may be formulated in the following way. There is a dynamical system $\Sigma$ functioning during a time period $T = [0, \vartheta]$. Its trajectory $x(t) = x(t; x_0, u_r(\cdot)) \in \mathbb{R}^n$, $t \in T$, depends on an unknown time-varying input $u_r(\cdot) \in P(\cdot)$. Here $P(\cdot) \subset L_2(T; \mathbb{R}^N)$ is the set of admissible controls. On the interval $T$, a uniform net $\Delta = \{\tau_k\}_{k=0}^n$ with a step $\delta$ is taken, $\tau_0 = 0$, $\tau_{k+1} = \tau_k + \delta$, $\tau_n = \vartheta$. An output $y(t) = Cx(t)$ is measured at the moments $\tau_k$ ($C$ is an $r \times q$-dimensional matrix). Sometimes we assume (for the sake of simplicity) that an output $y(t)$ is measured at all time moments $t \in T$. Results of inaccurate measurements are vectors $\xi_k \in \mathbb{R}^r$ satisfying the inequalities

\[
|\xi_k - y(\tau_k)| \leq h, \quad k \in [0 : n - 1],
\]

where $h$ is a value of level of informational noise. It is required to indicate an algorithm which allows to reconstruct an input $u^h_r(\cdot)$ being an approximation to some input $u_r(\cdot)$ generating the output $x(\cdot)$.

2. Solution scheme. Let us describe the scheme of the work of the algorithm. An auxiliary dynamical system $M$ (a model) is introduced. This model functioning on the time interval $T$ has unknown input (control) $U^h(t)$ and output $Y^h(t)$. The process of synchronous feedback control of the systems $\Sigma$ and $M$ is organized on the interval $T$. This process is decomposed into $(n-1)$ identical steps. At the $k$-th step carried out during time interval $\delta_k = [\tau_k, \tau_{k+1})$ the following actions are fulfilled. First, at time moment $\tau_i$ according to the chosen rule $U^h$ the control $U^h(t) = U^h(\tau_k, \xi_0, \ldots, \xi_k, Y^h(\tau_0), \ldots, Y^h(\tau_k)), t \in [\tau_k, \tau_{k+1})$ is calculated. Then (till the moment $\tau_{k+1}$) the control $U^h = U^h(t), \tau_k \leq t < \tau_{k+1}$, is fed to the input of the system $M$. The value $Y^h(\tau_{k+1})$ is the result of work of the algorithm at the $k$-th step. Thus, all complexity of solving the problem under consideration is reduced to the appropriate choice of the model $M$ and the function $U^h$.

The procedure for solving the problem of reconstruction is, in essence, equivalent to the procedure for solving the following two problems: a) the problem of choice of control in the model $M$ and b) the problem of choice of the rule $U^h$ for forming a control in the model. Note that the next two aspects play an important role in the process of solving problems a) and b). The first one is a priori information on the structure of the system $\Sigma$ (the form of equation, the properties of its solution and so on), and the second one is the structure of the set of admissible controls $P(\cdot)$. 
We introduce the following notations: $|\cdot|_r, (\cdot, \cdot)_r$ are Euclidean norm and the scalar product in space $\mathbb{R}^r$ respectively, $\mathcal{N}$ is the set of natural numbers, $\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}; C_1([-\tau, 0]; \mathbb{R}^n)$ is space of continuously differentiable in $[-\tau, 0]$ $n$-dimensional vector-functions, $W^{1,2}(a, b; \mathbb{R}^n) = \{x(t), t \in [a, b] : x(t), \dot{x}(t) \in L_2(T; \mathbb{R}^n)\}$. The prime means transposition. In work [1], one of the ways of realization of this scheme was indicated. It is based on a dynamical variant of the smoothing functional method. The equation

$$\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))u(t), \quad t \in [0, \vartheta],$$

$x(0) = x_0, x \in \mathbb{R}^q, u \in \mathbb{R}^N$, with control $u(t) \in P$ for a.e. $t \in [t_0, \vartheta]$ (where $P \subset \mathbb{R}^N$ is a convex, closed, and bounded set) was considered. Assumed that at frequent enough time moments $\tau_k$ all coordinates $(Y = X)$ were measured, it was recommended to take the equation

$$\dot{Y^h}(t) = f_1(\tau_k, \xi_k) + f_2(\tau_k, \xi_k)U^h(t), \quad t \in \delta_k = [\tau_k, \tau_{k+1}], \quad Y^h(0) = \xi_0$$

as a model. Control $U^h(t)$ in the model was calculated by the feedback principle:

$$U^h(t) = \arg\min\{2(Y^h(\tau_k) - \xi_k, f_2(\tau_k, \xi_k), v)q + \alpha(h)|v|^2_N : v \in P\}, \quad t \in \delta_k.$$

The basic result of the cited work is the following. Under a certain concordance property for regularization parameter $\alpha(h)$, informational error $h$, and partition step $\delta = \delta(h)$, the convergence (in $L_2([t_0, \vartheta]; \mathbb{R}^N)$-metrics) of controls $U^h(\cdot)$ to some control $u_0(\cdot)$ ($u_0(\cdot) \in P$ for a.e. $t \in [t_0, \vartheta]$) generating output $x(\cdot)$ takes place. Here $x(\cdot)$ is an inaccurately measured solution to equation (2). In work [6], this method was modified for a system with hereditary. The goal of this work is to sum up the research carried out in the last years. We will touch on research related to the inverse control problems for delay systems passing over the systems described by ordinary differential equations of mathematical physics and mechanics. We will show the role of a priori information (i.e. information on the system’s structure, properties of set of admissible controls, properties of the values under reconstruction, nature of measurements etc.) in choice of an algorithm. Special attention will be given to the problems of reconstruction of non-smooth and “unbounded” controls; cases of (inaccurate) measurement both whole set and “a part” of coordinates will be considered; some estimates of convergence rates of the corresponding algorithms will be dwelt on.
3. Case of measurement of all coordinates. Let system $\Sigma$ be described by the nonlinear differential equation

$$\dot{x}(t) = f(x(t), x(t-\nu)) + Bu(t), \quad t \in T = [0, \vartheta],$$

$x(s) = x_0(s) \in C_1([-\nu, 0]; \mathbb{R}^q)$, where $x \in \mathbb{R}^q$; $f$ is a nonlinear Lipschitz transformation from $\mathbb{R}^q \times \mathbb{R}^q$ to $\mathbb{R}^q$; $B$ is a $q \times N$-matrix. Consider the case of measurement of all coordinates, i.e., the case when $r = q$, $C = I_q$. Hereinafter $I_q$ is a $q \times q$ identity matrix. We fix time moments $\tau_k^0 = k\vartheta/n$, $0 \leq k \leq n - 1$. Let the following condition be fulfilled.

**Condition 1.** $P(\cdot) = L_2(T; \mathbb{R}^N)$, $u_*(\cdot) \in L_\infty(T; \mathbb{R}^N)$.

We denote by $u_*(\cdot)$ a unique input from $P(\cdot)$ of minimal $L_2(T; \mathbb{R}^N)$-norm that produces the same evolution $x(\cdot)$ as $u_r(\cdot)$. In this case, model $M$ is described by the linear ordinary differential equation

$$\dot{y}(t) = f(\xi_k, \xi_{k-r_k}) + BU_{hn}(t), \quad t \in [\tau_k^n, \tau_{k+1}^n),$$

$Y_{hn}(0) = x_0(0)$, where $\tau_{k-r_k}$ is the unique observation point which belongs to $[\tau_k^n - \nu, \tau_k^n - \nu + \delta)$. We assume (for simplicity) that the initial condition $x_0(s)$ is known. Hence if $k - r_k < 0$ then we put $\xi_{k-r_k} = x_0((k-r_k)\vartheta/n)$. The control in the model is defined by the rule

$$U_{hn}(t) = v^n_k = \arg \min_{|v|_N \leq d_n} \{2(Y_{hn}(\tau_k^n) - \xi_k, Bv)_q + \alpha_n |v|_N^2\},$$

$t \in [\tau_k^n, \tau_{k+1}^n)$.

Consider sequences $\{h_n\}, \{d_n\}$ and $\{\alpha_n\}$ with the following properties:

$$d_n \to +\infty, \quad \alpha_n \to 0, \quad (h_n d_n)/\alpha_n \to 0, \quad h_n \to 0,$$

$$d_n^2 (n\alpha_n)^{-1} \to 0 \quad \text{as} \quad n \to \infty.$$  

**Theorem 1.** [7] Let condition 1 and relations (6) be fulfilled. Then the sequence of functions $\{U_{hn}(\cdot)\}$ defined by (5) converges to $u_*(\cdot)$ in $L_2(T; \mathbb{R}^N)$ as $n \to \infty$.

Let the following condition be fulfilled along with Condition 1.

**Condition 2.** The matrix $BB'$ is positive definite.

In this case, the model $M$ is described by the equation (4). Let sequences $\{h_n\}$ and $\{\alpha_n\}$ be taken with the following properties:

$$h_n \to 0, \quad \alpha_n \to 0, \quad h_n \alpha_n^{-1} \to 0, \quad 1/(n\alpha_n^2) \to 0 \quad \text{as} \quad n \to \infty.$$
The control in the model is defined by the rule

\[ U^{h_n}(t) := \frac{1}{\alpha_n}B'[\xi_k - Y^{h_n}(\tau^n_k)]. \]

**Theorem 2.** [8] Let conditions 1, 2 and relations (7) be fulfilled. The sequence of functions \( \{U^{h_n}(\cdot)\} \) defined by (8) converges to \( u_*(\cdot) \) in \( L_2(T; \mathbb{R}^N) \) as \( n \to \infty \).

Now, let the following condition be fulfilled instead of Condition 1.

**Condition 3.** \( P(\cdot) = L_2(T; \mathbb{R}^N), \ u_*(\cdot) \in L_2(T; \mathbb{R}^N) \).

In this case, the model \( M \) is described by the linear differential equation

\[ \dot{Y}^{h_n}(t) = f(\xi_k, \xi_{k-r}; \xi_k - r) + BU^{h_n}(t) + \nu^{h_n}(t) \text{ for a. a. } t \in [\tau^n_k, \tau^n_{k+1}] \]

with the initial condition \( Y^{h_n}(0) = \xi_0 \). Let sequences \( \{h_n\}, \{\alpha_n\} \) satisfy the following properties:

\[ h_n \to 0, \ \alpha_n \to 0, \ h_n\alpha_n^{-1} \to 0, \ (n\alpha)^{-1} \to 0, \ h_n\alpha \to 0 \text{ as } n \to \infty. \]

We assume that

\[ U^{h_n}(t) = \frac{1}{\alpha_n}B'[\xi_k - Y^{h_n}(\tau^n_k)], \]

\[ \nu^{h_n}(t) = \frac{c}{\alpha_n}[\xi_k - Y^{h_n}(\tau^n_k)], \ t \in [\tau^n_k, \tau^n_{k+1}], \]

where \( c = \text{const} > 0 \).

**Theorem 3.** [9] Let condition 3 and relations (9) be fulfilled. The sequence of functions \( \{U^{h_n}(\cdot)\} \) defined by (10) converges to \( u_*(\cdot) \) in \( L_2(T; \mathbb{R}^N) \) as \( n \to \infty \).

Consider the linear control system

\[ \dot{x}(t) = Lx(t) + Bu(t), \ t \in T, \]

with initial state \( x(s) = x_0(s) \in C_1[-\nu, 0] \). Here map \( L \) is of the form

\[ Lx(t) = \sum_{s=0}^{l}A_s x(t + \theta_s) + \int_{-\nu}^{0} A(\theta)x(t + \theta)d\theta \in \mathbb{R}^q, \ -\nu = \theta_1 < \ldots < \theta_0 = 0, \]

\( A_s, (s = 0, \ldots, l) \) are \( q \times q \) square matrices with real elements; \( s \to A(s) \) is a \( q \times q \) matrix with elements being functions from \( L_2(-\nu, 0; \mathbb{R}) \). Let the following condition be fulfilled.

**Condition 4.** \( P(\cdot) = \{u(\cdot) \in L_2(T; \mathbb{R}^N) : u(t) \in P \text{ for a. a. } t \in T\} \).
Here \( P \subset \mathbb{R}^N \) is a convex, bounded and closed set. Introduce sequences \( \{\alpha_n\}, \{h_n\}, \{N_n\} \in \mathcal{N} \) such that

\[
(12) \quad h_n \to 0, \quad \alpha_n \to 0, \quad N_n \to +\infty, \quad h_n \alpha_n^{-1} \to 0, \quad \alpha_n^{-1} N_n^{-1/2} \to 0 \quad \text{as} \quad n \to \infty.
\]

We choose on the interval \( T \) time moments \( \tau^n_k = k \nu / N_n \). As a model \( M \) we take the control system

\[
\dot{Y}^{h_n}(t) = A^{N_n} Y^{h_n}(t) + C^{N_n} U^{h_n}(t), \quad t \in T
\]

with initial conditions \( Y^{h_n}_i(0) = \frac{N_n}{\nu} \int_{-i \delta_n}^{-i+1 \delta_n} x_0(\tau) \, d\tau, \quad i \in [1 : N_n], \quad \delta_n = \nu / N_n, \)

\[
Y^{h_n}_0(0) = x_0(0), \quad \text{and the phase trajectory} \quad Y^{h_n}(t) \in \mathbb{R}^{q(N_n+1)}. \]

We define the rule of forming a control in the model by

\[
(13) \quad U^{h_n}(t) = \arg \min \left\{ 2(q_k, Bv)_q + \alpha_n |v|_{\mathbb{R}^N}^2 : v \in P \right\}, \quad t \in [\tau^n_k, \tau^n_{k+1}].
\]

Here, vector \( q_k \) is composed from the first \( q \) components of the vector \( S_n = \{\exp(A^{N_n}(T_1 - \tau^n_k))^T \mathbb{R}^N \exp(A^{N_n}(T_1 - \tau^n_k))\} \times \left[ Y^{h_n}(\tau^n_k) - \xi^n_k \right], \quad T_1 > 5 \nu + \vartheta, \)

\[
\xi^n_k = (\xi_k, \xi_{k-1}, \ldots, \xi_{k-N_n}) \in \mathbb{R}^{q(N_n+1)}, \quad \text{and matrices} \quad A^{N_n}, C^{N_n}, \mathbb{R}^{N_n} \text{ are defined in accordance with [10, p. 223].}
\]

**Theorem 4.** [11] Let the condition 4 and relations (12) be fulfilled. The sequence of functions \( \{U^{h_n}(\cdot)\} \) defined by (13) converges to \( u_*(\cdot) \) in \( L_2(T; \mathbb{R}^N) \) as \( n \to \infty \).

4. *Case of measurement of a part of coordinates.* Consider the following system of equations:

\[
(14) \quad \begin{align*}
\dot{x}_1(t) &= L_1 x_1(t) + C_1 x_2(t), \quad t \in T, \\
\dot{x}_2(t) &= L_2 x_2(t) + E(x_1(t)) + u(t) B x_1(t),
\end{align*}
\]

\[
L_j y(t) = \sum_{i=0}^{l_j} A_j^{(i)} y(t - \tau_{l_j}^{(i)}) + \int_{\tau_{l_j}^{(i)}}^{0} A_j^{(i)}(s) y(t + s) \, ds, \quad j = 1, 2,
\]

with the initial conditions \( x_1(s) = x_1^1(s) \) for \( s \in [-\tau_{l_1}^{(1)}, 0] \), \( x_2(s) = x_2^2(s) \) for \( s \in [-\tau_{l_2}^{(2)}, 0] \). Here \( x_1(t) \in \mathbb{R}^N; \quad x_2(t) \in \mathbb{R}^3; \quad u(t) \in \mathbb{R}; \quad x_1^1(s) \in C([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N); \quad x_2^2(s) \in C([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^3); \quad 0 = \tau_0^{(j)} < \tau_1^{(j)} < \ldots < \tau_{l_j}^{(j)} ; \quad x_{1t}(s) : s \to x_1(t + s), \quad s \in [-\tau_{l_1}^{(1)}, 0]; \quad x_{2t}(s) : s \to x_2(t + s), \quad s \in [-\tau_{l_2}^{(2)}, 0]; \)
$A_i^{(j)}, B,$ and $C_1$ are constant matrices of the dimension $N \times N$ (for $j = 1$), $q \times q$ (for $j = 2$), $n \times N$ and $N \times n$, respectively; elements of the matrix functions $s \mapsto A_i^{(j)}(s)$, $s \in [-\tau_i^{(j)}, 0]$, $j = 1, 2$, are square integrable; $E(\cdot) : \mathbb{R}^N \to \mathbb{R}^q$ is a matrix function satisfying the global Lipschitz condition.

Let $P = [\alpha, \beta]$, $-\infty < \alpha < \beta < +\infty$. Let at every time $\tau_k = k\theta/n$, $0 \leq k \leq n - 1$, the first state component, $x_1(\tau_k^n)$, be observed. The observation results are represented by vectors $\xi_k \in \mathbb{R}^N$ such that $|x_1(\tau_k^n) - \xi_k|_N \leq h$, i.e., $C = \begin{bmatrix} I & 0 \end{bmatrix}$. Our goal is to reconstruct the real input $u_r(\cdot) = u_r(\cdot; x_1(\cdot))$ generating the output $x_1(\cdot)$. In this case, model $M$ is described by the delay system

$$\begin{cases}
    \dot{Y}_1^{h_n}(t) = L_1Y_1^{h_n} + C_1v^{1,h_n}(t), & t \in [\tau_k^n, \tau_{k+1}^n) \\
    \dot{Y}_2^{h_n}(t) = L_2Y_2^{h_n} + E(\xi_k) + v^{h_n}(t)B\xi_k,
\end{cases}$$

$w_j^{h_n}(s) = x_j^{(j)}(s)$ for $s \in [-\tau_j^{(j)}, 0]$, $j = 1, 2$, with controls $U^{h_n}(t) = (v^{1,h_n}(t), v^{h_n}(t)) \in \mathbb{R}^q \times \mathbb{R}$. Let the control in the model be defined by the rule

$$v^{1,h_n}(t) = \arg\min\{L_1(\alpha_n, v, s_k^0) : v \in S(d_1)\}$$

$$v^{h_n}(t) = \arg\min\{L_2(\beta_n, v, s_k) : v \in P\}, \quad t \in [\tau_k^n, \tau_{k+1}^n).$$

Here $L_1(\alpha, v, s_k^0) = \alpha_n|v|^2 + 2(s_k^0, C_1v)_N$, $L_2(\beta, v, s_k) = \beta_n|v|^2 + 2(s_k, B\xi_k)v$, $s_k^0 = (Y_1^{h_n}(\tau_k^n) - \xi_k)\exp(-2\omega_1\tau_{k+1}^n)$, $s_k = (Y_2^{h_n}(\tau_k^n) - v^{1,h_n}(\tau_k^n))\exp(-2\omega_2\tau_{k+1}^n)$, $d_1 = \sup\{|x_2(t; x_0, u(\cdot))|_q : u(\cdot) \in P(\cdot), \quad t \in T\}$, $S(d_1) = \{v \in \mathbb{R}^q : |v|_q \leq d_1\}$, $\omega_j = \frac{1 + l_j}{2} + |A_0^{(j)}| + \frac{1}{2}\sum_{i=1}^{l_j} |A_i^{(j)}|^2 + \frac{1}{2}\int_{-\tau_i^{(j)}}^{0} |A_i^{(j)}(\tau)|^2 d\tau$.

Assume that the following relationships between the parameters are valid: $\alpha_n \to 0$, $\beta_n \to 0$, $\{(h_n + n^{-1/2} + \alpha_n)^{1/2} + (h_n + n^{-1/2})\alpha_n^{-1}\} \beta_n^{-1} \to 0$ as $n \to \infty$. For example, we can set $h_n = n^{-1/2}$, $\alpha_n = h_n^{1/2}$, $\beta_n = h_n^\mu$, $\mu = \text{const} \in (0, 1/4)$. Let $s_j(\cdot)$ be a unique solution on $T$ of the functional-differential matrix equation

$$\frac{ds_j(t)}{dt} = A_0^{(j)}s_j(t) + \sum_{i=1}^{l_j} A_i^{(j)}s_j(t+\tau_i) + \int_{-\tau_i^{(j)}}^{0} A_i^{(j)}(s)s_j(t+s) ds \quad \text{for a. a. } t \in T$$

with the initial state $s_j(t) = I$, $t \leq 0$; symbol $V(T; \mathbb{R}^q)$ denotes the space of all functions $t \to x(t) \in \mathbb{R}^q$ with the bounded variation.
THEOREM 5. [12] Let a control \( v^{h_n}(\cdot) \) be determined by (15). Let \( q \leq N \) and the following conditions be fulfilled:
1. \( \inf_{t \in T} |s_1^{-1}(t)x|_N \geq d_1 |x|_N \quad \forall x \in \mathbb{R}^N \quad (d_1 > 0) \),
2. there exist a number \( d_2 > 0 \) and \( q \)-th order minor of matrix \( s_1(t)C_1 \), such that \( q \times q \)-matrix \( s_1(t)C_1 \) corresponding to this minor satisfies inequality \( \inf_{t \in T} |s_1(t)C_1v|_N \geq d_2 |v|_q \quad \forall v \in \mathbb{R}^q \),
3. for any solution \( x_2(\cdot) \) of system (14) inclusion \( (s_1(\vartheta - t)C_1)^{-1}x_2(t) \in V(T; \mathbb{R}^q) \) is true.

Then the convergence \( v^{h_n}(\cdot) \rightarrow u_r(\cdot; x_1(\cdot)) \) in \( L_2(T; \mathbb{R}) \) takes place.

Let the following conditions be also fulfilled:
4. \( \inf_{t \in T} |s_2^{-1}(t)x|_q \geq d^{(1)} |x|_q \quad \forall x \in \mathbb{R}^q \quad (d^{(1)} > 0) \)
5. there exists a coordinate of vector \( s_2(\vartheta - t)Bx_1(t) \) (denote it by \( \{s_2(\vartheta - t)Bx_1(t)\}_* \)) such that \( \inf_{t \in T} |\{s_2(\vartheta - t)Bx_1(t)\}_*| > 0 \).

If \( \{s_2(\vartheta - t)Bx_1(t)\}_*^{-1}u_r(t, x_1(\cdot)) \in V(T; \mathbb{R}) \) then the following estimate of the rate of algorithm convergence takes place
\[
|v^{h_n}(\cdot) - u_r(\cdot; x_1(\cdot))|_{L_2(T; \mathbb{R})} \leq c\{\mu_n + \mu_n\beta_n^{-1}\}.
\]

Here \( \mu_n = (h_n + n^{-1} + \alpha_n)^{1/2} + ((h_n + n^{-1/2})\alpha_n^{-1})^{1/2} \).

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A TROTTER-KATO THEOREM FOR $\alpha$-TIMES INTEGRATED $C$-REGULARIZED SEMIGROUPS

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Abstract. We prove a Trotter-Kato type theorem for the approximation of $\alpha$-times integrated $C$-regularized semigroups.

Key Words. $\alpha$-times integrated $C$-regularized semigroups and Trotter-Kato theorem.

AMS(MOS) subject classification. 41A35, 47D60, 47D62

Dedicated to Professor István Győri on the occasion of his sixtieth birthday.

1. Introduction. A family $\{T(t)\}_{t \geq 0} \in B(X)$ is called a $C_0$-semigroup on $X$ if it satisfies $T(0) = I$, $T(t+s) = T(t)T(s)$ and $\lim_{t \to 0^+} \|T(t)x-x\| = 0$, for all $x \in X$. The generator of a $C_0$-semigroups $\{T(t)\}_{t \geq 0}$ is defined by

$$D(A) = \{x \in X: \lim_{t \to 0^+} \frac{1}{t}(T(t)x-x) \text{ exists} \};$$

and

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(T(t)x-x) \quad x \in D(A).$$

One of the fundamental theorems in the theory of approximation of operator semigroups is the Trotter-Kato theorem (see [9], for instance), which

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tells us that a sequence of $C_0$-semigroups $\{T_m(t); m \in \mathbb{N}\}$ on $X$, generated by $A_m$, converges if only if the semigroups have a uniform exponential bound and the sequence of resolvents $(\lambda - A_m)^{-1}$ converges strongly.

$C_0$-semigroups have been generalized in two directions. Integrated semigroups have been discussed by Arendt, [1], Neubrander [7] and Hieber [3]. Regularized semigroups have been investigated by deLaubenfels [2] and Zheng and Liu [12].

There are some Trotter-Kato type theorems for integrated semigroups and regularized semigroups. The approximation of integrated semigroups was presented by Nicaise [8], Lizama [5] and Xiao and Liang [11]. The approximation of $C$-regularized semigroups was presented by Quan [10].

Combining the two directions of generalization, one arrives at the concept of an $\alpha$-times integrated, $C$-regularized semigroup. In this paper we discuss a Trotter-Kato Theorem on the convergence and approximation of $\alpha$-times integrated $C$-regularized semigroups.

2. Exponentially Bounded $\alpha$-times Integrated $C$-Regularized Semigroup. We start with the definition of an $\alpha$-times integrated $C$-regularized semigroup. The definition reduces to the definition of integrated semigroups for $C = \text{id}$ and of regularized semigroups for $\alpha = 0$ (see : [4] and [6]).

Throughout the paper, let $X$ be a Banach space and $B(X)$ be a space of all bounded linear operators on $X$. By $\hat{f}$ we will denote the Laplace transform of a function $f$.

**Definition 1.** Let $\alpha \geq 0$ and $C \in B(X)$. A strongly continuous family of bounded linear operators $\{S(t)\}_{t \geq 0} \subset B(X)$ is called an $\alpha$-times integrated $C$-regularized semigroup on $X$ if it satisfies:

(a) $S(0)x = \begin{cases} Cx & \text{if } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$

(b) $S(t)C = CS(t)$ for $t \geq 0$.

(c) $S(t)S(s)x = \begin{cases} S(t+s)Cx & \text{if } \alpha = 0, \\ \frac{1}{\Gamma(\alpha)} \left( \int_t^{t+s} - \int_s^t \right) (s + t - r)^{\alpha-1} S(r)Cx \, dr & \text{otherwise.} \end{cases}$

$\{S(t)\}_{t \geq 0}$ is said to be nondegenerate if $S(t)x = 0, \forall t > 0$ implies $x = 0$. Finally, $\{S(t)\}_{t \geq 0}$ is called exponentially bounded if there are constants $M, \omega > 0$ such that $\|S(t)\| \leq Me^{\omega t}, \forall t \geq 0$.

For a nondegenerate exponentially bounded $\alpha$-times integrated $C$-regula-
rized semigroup \( \{S(t)\}_{t \geq 0} \) we define the generator \( A \) by
\[
Ax = y \iff S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)} Cx + \int_0^t S(s)y \, ds \quad \text{for all } s \geq 0.
\]
Moreover we define
\[
R_C(\lambda)x := \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x \, dt, \quad \lambda > \omega, \ x \in X. \tag{1}
\]
It is known that \( A \) is a closed linear operator such that \( A = C^{-1}AC \), and that \( \{R_C(\lambda)\}_{\lambda > \omega} \) form an injective and \( C \)-regularized pseudo-resolvent, i.e.,
\[
(\lambda - \mu)R_C(\lambda)R_C(\mu) = R_C(\mu)C - R_C(\lambda)C
\]
with \( R_C(\lambda) = (\lambda I - A)^{-1}C \).

3. Limits of Integrated Regularized Semigroups. In this section we show that uniform limits of integrated regularized semigroups are again integrated regularized semigroups.

**Theorem 1.** Let \( 0 < \alpha_n < 1 \) with \( \lim_{n \to \infty} \alpha_n = \beta > 0 \), let \( C \) and \( C_n \) be bounded linear operators on \( X \) with \( \lim_{n \to \infty} C_n x = Cx \) for all \( x \in X \). Let \( \{S_n(t)\}_{t \geq 0} \) be \( \alpha_n \)-times integrated \( C_n \)-regularized semigroups on \( X \) such that for each \( x \in X \) the limit \( \lim_{n \to \infty} S_n(t)x = T(t)x \), uniformly for all \( t \geq 0 \). Then \( \{T(t)\}_{t \geq 0} \) is a \( \beta \)-times integrated \( C \)-regularized semigroup on \( X \).

**Proof.** It is clear that \( T(0) = 0 \). For all estimates note that by the uniform boundedness principle the operators \( C_n \) and \( S_n(t) \) are uniformly bounded for \( n \in \mathbb{N} \) and \( t \) in compact intervals. We show that \( T(t) \) commutes with \( C \) for all \( t \geq 0 \)
\[
\| T(t)Cx - CT(t)x \| \\
\leq \| (T(t) - S_n(t))Cx \| + \| S_n(t)(Cx - C_n x) \| + \| S_n(t)C_n x - C_n S_n(t)x \| \\
+ \| C_n(S_n(t)x - T(t)x) \| + \| (C_n - C)T(t)x \|.
\]
Using the strong convergence and uniform boundedness, the latter expression tends to zero as \( n \to \infty \). The strong continuity of \( S(t)x \) follows from the fact
that $S_n(t)x$ converges to $S(t)x$ uniformly on compact intervals. Once again, strong convergence and uniform boundedness imply that

$$T(t)T(s)x = \lim_{n \to \infty} \frac{1}{\Gamma(\alpha_n)} \left( \int_t^{s+t} - \int_0^s \right) (s + t - r)^{\alpha_n-1} S_n(r)C_n x dr$$

for all $s, t \geq 0$ and $x \in X$. Thus $\{T(t)\}_{t \geq 0}$ is a $\beta$-times integrated $C$-regularized semigroup on $X$. □

The case of $\alpha_n > \beta = 0$ is special. The convergence cannot be uniform on intervals $[0, T]$, since $S(0)x = Cx$, but $S_n(0)x = 0$.

**Theorem 2.** Let $\alpha_n > 0$ with $\lim_{n \to \infty} \alpha_n = 0$. Let $C_n$ and $C$ be bounded linear operators on $X$ such that $\lim_{n \to \infty} C_n x = Cx$ for all $x \in X$. Let $\{S_n(t)\}_{t \geq 0}$ be $\alpha_n$-times integrated $C$-regularized semigroups on $X$, uniformly bounded for $t \in [0, T]$ and $n \in \mathbb{N}$. For $x \in X$, let $\lim_{n \to \infty} S_n(t)x = T(t)x$, uniformly for $t \in [\zeta, \eta], 0 < \zeta < \eta < \infty$, and let $T(0)x = \lim_{t \to 0} T(t)x = Cx$. Then $\{T(t)\}_{t \geq 0}$ is a $C$-regularized semigroup on $X$.

**Proof.** It is obvious that $T(0) = C$, $T(t)C = CT(t)$ for all $t \geq 0$, and $T(t)T(0) = T(t + 0)C$. By uniform convergence we can prove that the map $T(\cdot)x : (0, \infty) \to X$ is continuous for all $x \in X$. The strong continuity at $t = 0$ is given by assumption. We prove now that for any $t, s > 0$, $T(t)T(s) = T(t + s)C$. To prove this, take any $t, s \geq 0$ and $x \in X$, and fix $\delta \in (0, s + t)$.

$$\|S_n(t)S_n(s)x - T(t + s)Cx\|$$

\[
\leq \left\| \int_0^{s+t-\delta} \left( \int_t^{s+t} - \int_0^s \right) (s + t - r)^{\alpha_n-1} S_n(r)C_n x dr \right\|
\]

\[
+ \left\| \int_0^{s+t} (s + t - r)^{\alpha_n-1} S_n(r)C_n x - Cx dr \right\|
\]

\[
+ \left\| \int_0^{s+t} (s + t - r)^{\alpha_n-1} S_n(r)Cx - T(t + s)Cx \right\|
\]

\[
\leq \sup_{r \in [0, t+s]} \|S_n(r)C_n x\| \int_0^{s+t-\delta} (s + t - r)^{\alpha_n-1} \frac{dr}{\Gamma(\alpha_n)}
\]

\[
+ \sup_{r \in [t+s-\delta, t+s]} \|S_n(r)\| \|C_n x - Cx\| \int_{t+s-\delta}^{t+s} (t + s - r)^{\alpha_n-1} \frac{dr}{\Gamma(\alpha_n)}
\]

\[
+ \sup_{r \in [t+s-\delta, t+s]} \|S_n(r)Cx - T(r)Cx\| \int_{t+s-\delta}^{s+t} (s + t - r)^{\alpha_n-1} \frac{dr}{\Gamma(\alpha_n)}
\]
\[\int_{s+t-\delta}^{s+t} \frac{(s + t - r)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \|T(r)Cx - T(t + s)Cx\| \, dr + \|T(t + s)Cx\| \leq \left(\frac{(s + t)^{\alpha_n} - \delta^{\alpha_n}}{\Gamma(\alpha_n + 1)} \sup_{n \in \mathbb{N}, r \in [t + s - \delta, t + s]} \|S_n(r)C_n(r)\| + \frac{\delta^{\alpha_n}}{\Gamma(\alpha_n + 1)} \sup_{r \in [t + s - \delta, t + s]} \|S_n(r)\| \|C_n x - Cx\| + \frac{\delta^{\alpha_n}}{\Gamma(\alpha_n + 1)} \sup_{r \in [t + s - \delta, t + s]} \|T(r)Cx - T(t + s)Cx\| + \frac{\delta^{\alpha_n}}{\Gamma(\alpha_n + 1)} - 1 \left\|T(t + s)Cx\right\|.
\]

First we take the limit for \(n \to \infty\) and see that

\[\|T(t)T(s)x - T(t + s)Cx\| \leq \sup_{r \in [t + s - \delta, t + s]} \|T(r)Cx - T(t + s)Cx\|.
\]

With \(\delta \to 0\) we obtain that \(T(t)T(s)x = T(t + s)Cx\). \(\square\)

4. A Trotter-Kato Theorem. In this section we discuss the relation between convergence of \(\alpha\)-times integrated \(C\)-regularized semigroups and the resolvents of their generators. The following lemma is taken from [5] and we denote by \(Rg(\cdot)\) the range.

**Lemma 1.** [5] Let \(M, \omega \geq 0\) and \(f_n : [0, \infty) \to B(X)\) be functions such that \(f_n(0) = 0\) and

\[\|f_n(t + h) - f_n(t)\| \leq Mh e^{\omega(t + h)}, \text{ for } t, h \geq 0, n \in \mathbb{N} \cup \{0\}.
\]

Then the following statements are equivalent:

a) \(\lim_{n \to \infty} f_n(\lambda)x = f(\lambda)x\), for all \(x \in X\) and \(\lambda > \omega\).

b) \(\lim_{n \to \infty} f_n(t)x = f(t)x\), for all \(x \in X\) and \(t > 0\).

The following lemmas will be needed to prove the Theorem 3.

**Lemma 2.** Let \(A_n\) be generator of uniformly exponentially bounded \(\alpha\)-times integrated \(C_0\)-regularized semigroups \(\{S_n(t)\}_{t \geq 0}, n \in \mathbb{N} \cup \{0\}\), (i.e. there exist \(M, \omega \geq 0\) such that \(\|S_n(t)\| \leq M e^{\omega t}\) for all \(t \geq 0\) and \(n \in \mathbb{N} \cup \{0\}\)). Assume that \(Rg(C_0)\) is dense in \(X\) and \(\lim_{n \to \infty} C_n x = C_0 x\), for all \(x \in X\). Then the following statements are equivalent:
a) \( \lim_{n \to \infty} R_{C_n}(\lambda, A_n)x = R_{C_0}(\lambda, A_0)x, \) for all \( \lambda > \omega \) and \( x \in X. \)

b) \( \lim_{n \to \infty} R_{C_n}(\lambda_0, A_n)x = R_{C_0}(\lambda_0, A_0)x, \) for some \( \lambda_0 > \omega \) and all \( x \in X. \)

Proof. a) \( \Rightarrow \) b) It is clear.

b) \( \Rightarrow \) a) Fix \( \lambda > \omega \) and \( x \in C_0 \text{Rg}(\lambda - A_0). \) Then \( y := (\lambda - A)^{-1}x \in \text{Rg}(C_0) \) and \( y_n := R_{C_n}(\lambda_0, A_n)(\lambda_0 - A_0)^{-1}y \to R_{C_0}(\lambda_0, A_0)(\lambda_0 - A_0)^{-1}y = y. \) We also have \( x_n := (\lambda - A_n)y_n = [(\lambda - \lambda_0)R_{C_n}(\lambda, A_n) + C_n](\lambda_0 - A_0)^{-1}y \to [(\lambda - \lambda_0)R_{C_0}(\lambda, A_0) + C_0](\lambda_0 - A_0)^{-1}y = (\lambda - A_0)y = x. \) Since \( ||C_n|| \leq M \) and \( ||R_{C_n}(\lambda, A_n)|| \leq \frac{M\lambda^\alpha}{(\lambda - \omega)}, \) we obtain

\[
||R_{C_n}(\lambda, A_n)x - R_{C_0}(\lambda, A_0)x|| \\
\leq ||R_{C_n}(\lambda, A_n)x - R_{C_n}(\lambda, A_n)x_n|| + ||R_{C_n}(\lambda, A_n)x_n - R_{C_0}(\lambda, A_0)x|| \\
\leq \frac{M\lambda^\alpha}{(\lambda - \omega)}||x_n - x|| + M||y_n - y|| + ||C_n y - C_0 y|| \\
\to 0 \text{ as } n \to \infty.
\]

It means \( \lim_{n \to \infty} R_{C_n}(\lambda, A_n)x = R_{C_0}(\lambda, A_0)x, \) for all \( \lambda > \omega \) and \( x \in C_0 \text{Rg}(\lambda - A_0). \) Furthermore, since \( \text{Rg}(C_0) = X \) and \( \text{Rg}(C_0) \subset \text{Rg}(\lambda - A_0) \) for all \( \lambda > \omega, \) then we get \( \overline{C_0 \text{Rg}(\lambda - A_0)} = X. \) Hence \( \lim_{n \to \infty} R_{C_n}(\lambda, A_n)x = R_{C_0}(\lambda; A_0)x, \) for all \( \lambda > \omega \) and \( x \in X. \)

**Lemma 3.** Let for any \( n \in \mathbb{N}, \) let \( f_n : [0, \infty) \to X \) be a measurable function satisfying

1) \( ||f_n(t)|| \leq Me^{\omega t}, \) for some \( M, \omega \in \mathbb{R} \)

2) \( \forall \delta > 0, \exists \delta > 0, \forall n, 0 < s < t < \gamma, |t - s| < \delta \Rightarrow ||f_n(t) - f_n(s)|| < \epsilon, \)

3) \( f_n(\lambda) \to 0, \) for all \( \lambda > \omega. \)

Then \( f_n(t) \to 0 \) uniformly on compact intervals.

**Proof.** Assume that \( f_n \) does not converge to 0 uniformly on \([0, T].\) Then there exist \( \eta > 0, \) a subsequence \( n_k, \) and points \( t_{n_k} \in [0, T] \) such that \( ||f_{n_k}(t_{n_k})|| > \eta. \) Take \( g_{n_k} \in X^* \) such that \( ||g_{n_k}|| = 1 \) and \( \langle g_{n_k}, f_{n_k}(t_{n_k}) \rangle > \eta. \) We put \( u_{n_k}(t) = \langle g_{n_k}, f_{n_k}(t_{n_k}) \rangle. \) Since \( ||g_{n_k}|| = 1, \) the functions \( u_{n_k} \) satisfy similar conditions:

1* ) \( |u_{n_k}(t)| \leq Me^{\omega t}, \)

2* ) The family \( \{u_{n_k}\}_{k \in \mathbb{N}} \) is equicontinuous on compact intervals,

3* ) \( u_{n_k}(\lambda) \to 0 \) for all \( \lambda > \omega. \)

By the Ascoli-Arzela-Theorem, there exists a subsequence \( u_{n_{k_l}} \) converging uniformly to some limit on \([0, T], \) we will denote this sequence by \( u_n. \) It contains a subsequence \( u_n^2 \) converging uniformly on \([0, 2T], \) and so on. The diagonal sequence \( u_n \) converges uniformly on each compact interval. We
denote the limit by \( v \). Using the fact that \( |u_n(t_n)| \leq Me^{\omega t} \) (independent of \( n \)), we infer that for \( \lambda > \omega \), \( \hat{u}_n(\lambda) \to \hat{v}(\lambda) \). On the other hand \( \hat{u}_n(\lambda) \) converges to 0. Thus \( v = 0 \). Finally we arrive at a contradiction that \( u_n(t_n) > \eta \), \( t_n \in [0, T] \), but \( u_n \to 0 \) uniformly on \( [0, T] \).

**THEOREM 3.** Let \( \{S_n(t)\}_{t \geq 0} \) be uniformly exponentially bounded \( \alpha \)-times integrated \( C_n \)-regularized semigroups with generators \( A_n \), \( n \in \mathbb{N} \). Assume that the limits \( \lim_{n \to \infty} C_n x = C x \), \( \lim_{n \to \infty} R_{C_n}(\lambda_0, A_n) x =: R_C(\lambda_0) x \) exist and that \( \text{Rg}(R_C(\lambda_0)) \) is dense in \( X \). Then \( \lim_{n \to \infty} S_n(t)x = S(t)x \) exists for all \( x \in X \) and uniformly for \( t \geq 0 \) in compact intervals.

Moreover, \( \{S(t)\}_{t \geq 0} \) is an \( \alpha \)-times integrated, \( C \)-regularized semigroup, and its generator \( A \) satisfies \( R_C(\lambda_0) = (\lambda_0 - A)^{-1} C \).

**Proof.** We want to show that \( \{S_n(t)\} \) is a Cauchy sequence for all \( x \in X \) and \( t \geq 0 \). First let \( x \in \text{Rg}(R_C(\lambda_0)) \), i.e. \( x = R_C(\lambda_0)y \).

\[
\|S(t)x - S_m(t)x\| \leq \|S(t)R_C(\lambda_0)y - R_C(\lambda_0)R_C(\lambda, A_n)y\| + \|S(t)R_C(\lambda_0, A_n)y - S_m(t)R_C(\lambda_0, A_m)y\| + \|S_m(t)R_C(\lambda_0, A_m)y - S_n(t)R_C(\lambda_0)y\|
\]

We only need to prove that the second term in the right side converges to zero, since the other terms converge to zero by assumption. Taking Laplace transforms, we obtain

\[
\hat{S}_n(\lambda)R_{C_n}(\lambda_0, A_n)y = \lambda^{-\alpha} R_{C_n}(\lambda, A_n)R_{C_n}(\lambda_0, A_n)y \to \lambda^{-\alpha} R_C(\lambda)R_C(\lambda_0)y.
\]

We also know that

\[
S_n(t)R_{C_n}(\lambda_0, A_n)y = \frac{t^\alpha}{\Gamma(\alpha + 1)} C_n R_{C_n}(\lambda_0, A_n)y + \int_0^t S_n(s) A_n R_{C_n}(\lambda_0, A_n)y ds
\]

is uniformly Hölder continuous. Hence by Lemma 1, we have that

\[
\|S_n(t)S_m(t)R_{C_n}(\lambda_0, A_n)y - S_m(t)R_{C_n}(\lambda_0, A_m)y\| \to 0
\]

uniformly on compact intervals as \( m, n \to \infty \). Finally, since \( R_g(R_C(\lambda_0)) \) is dense in \( X \), and the operators \( S_n(t) \) are uniformly bounded, the convergence holds for all \( x \in X \).

From Theorem 1 we infer that \( \{S(t)\}_{t \geq 0} \) is an \( \alpha \)-times integrated, \( C \)-regularized semigroup. If \( A \) is the generator of \( S(t) \), we have that

\[
(\lambda_0 - A)^{-1} C x = \lambda_0^\alpha \hat{S}(\lambda_0) x = \lim_{n \to \infty} \lambda_0^\alpha \hat{S}_n(\lambda_0) x = \lim_{n \to \infty} R_{C_n}(\lambda_0, A_n) x = R_C(\lambda_0) x.
\]
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Abstract. Nonlinear discrete-time Volterra equations in an Euclidean space are considered. Estimates for the $c_0$-norms and $l^p$-norms of solutions are derived. By virtue of these estimates, conditions for the boundedness of solutions are established. These conditions are formulated in terms of the coefficients of considered equations. The main tool in the paper is an estimate for the norm of the inverse infinite quasi-nilpotent matrix.

Key Words. Volterra difference equations, boundedness, estimates for solutions, quasi-nilpotent matrix.

AMS(MOS) subject classification. 39A10

1. Introduction. Volterra equations with discrete time arise mainly in the process of modeling of some real phenomena or by applying a numerical scheme for solving differential and integral equations, cf. [5,10,11], and are used to construct some models of continuum mechanics, Volterra [17]. At present much of their general quantitative and especially their qualitative theory remains to be developed. There are, at present, only a few papers dealing with their asymptotic behavior. One of the basic methods in the theory of stability and boundedness of Volterra difference equations is the direct Lyapunov method (see [2,6,7]). But finding of Lyapunov functionals for non-stationary Volterra difference equations is a difficult mathematical problem.

In this paper, we derive estimates for the supremum and $l^p$-norms of solutions for a class of matrix Volterra difference equations. These estimates
will allow us to obtain the boundedness conditions. To establish the solution estimates we will interpret the Volterra equations with matrix kernels as operator equations in appropriate spaces. Such an approach for Volterra difference equations have been used by Corduneanu [1], Kolmanovskii and Myshkis [11], Kolmanovskii et al. [12], Kwapisz [13], Medina [14,15,16] and Gil' and Medina [9]. Under some restrictions, these estimates will allow us to generalize the main results from the papers [7,12,14].

Note that our boundedness conditions are formulated in terms of the coefficients of considered equations. Our main tool is an estimate for the norm of the inverse infinite quasi-nilpotent matrix.

2. Preliminary Facts. Let $C^n$ be the $n$-dimensional complex Euclidean space with the Euclidean norm $\| \cdot \|_{C^n}$.

For a positive $r \leq \infty$, put $\omega_r = \{ h \in C^n : \|h\|_{C^n} \leq r \}$.

Let $c_0 = c_0(C^n)$ be the Banach space of sequences of vectors from $C^n$ equipped with the norm

$$\|h\|_{c_0} = \sup_k \|h_k\|_{C^n}, (h = (h_k)_{k=1}^{\infty} \in c_0 \ ; h_k \in C^n, k = 1, 2, \ldots).$$

In addition, $l^p = l^p(C^n)$ ($0 < p < \infty$) is the Banach space of sequences of vectors from $C^n$ equipped with the norm

$$\|h\|_{l^p} = \left[ \sum_{k=1}^{\infty} \|h_k\|_{C^n}^p \right]^{\frac{1}{p}} \ (h = (h_k)_{k=1}^{\infty} \in l^p, h_k \in C^n, k = 1, 2, \ldots).$$

Let $a_{jk}$ ($j, k = 1, 2, \ldots$) be $n \times n$-matrices, $f_j \in C^n$ ($j = 1, 2, \ldots$).

Let us consider the system

$$x_j = f_j + \sum_{k=1}^{j-1} a_{jk} [x_k + G_k(x_k)], (j = 1, 2, \ldots), \quad (1)$$

where the mappings $G_k : C^n \to C^n$ have the property

$$\|G_k(h)\|_{C^n} \leq \beta_k \|h\|_{C^n} \ (h \in \omega_r, k = 1, 2, \ldots \text{ for some } r \leq \infty). \quad (2)$$

Let us introduce the operators $V$ and $F$ defined on the space $l^p$ by

$$[Vx]_j = \sum_{k=1}^{j-1} a_{jk} x_k, \ x \in l^p; \quad [F(x)]_j = \sum_{k=1}^{j-1} a_{jk} G_k(x_k), \ x \in l^p, \ x = (x_1, x_2, \ldots, x_k, \ldots).$$
Here \([h]_j\) means the \(j-th\) coordinate of the element \(h \in l^p\).

In order to deal with (1) in \(l^p (C^n)\), we will also assume that

\[
N_p(V) := \left[ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^q \right)^{\frac{q}{p}} \right]^{\frac{1}{p}} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,
\]

and

\[
\|f\|_{l^p} = \left[ \sum_{j=1}^{\infty} \|f_j\|_{C^n}^p \right]^{\frac{1}{p}} < \infty, \quad f = (f_1, f_2, \ldots, f_k, \ldots),
\]

\[
N_\beta(V) := \left[ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^{q} \beta_k^{q} \right)^{\frac{q}{p}} \right]^{\frac{1}{p}} < \infty.
\]

Under previous assumptions, \(V\) and \(F\) map \(l^p(C^n)\) into itself. Thus, equation (1) can be considered as the following operator equation in the space \(l^p(C^n)\) with respect to \(x\):

\[
x = f + Vx + F(x).
\]

Clearly, if \(\beta_k \leq b_0 < \infty (k = 1, 2, \ldots)\) then (5) implies \(N_\beta(V) \leq b_0 N_p(V)\).

To formulate the result, let us denote

\[
m_p(V) = \sum_{k=0}^{\infty} \frac{N^k_p(V)}{\sqrt[k]{k!}}.
\]

**Lemma 2.1.** We have

\[
m_p(V) \leq b_p \exp(N^p_p(V)),
\]

where \(b_p = (1 - \frac{1}{p^\frac{1}{p}})\). \(\frac{1}{p}\).

Indeed, by Hölder’s inequality,

\[
m_p(V) = \sum_{k=0}^{\infty} \frac{a^k N^k_p(V)}{a^k \sqrt[k]{k!}} \leq \left[ \sum_{k=0}^{\infty} a^{kq} \right]^{\frac{1}{q}} \left[ \sum_{k=0}^{\infty} \frac{N^p_p(V)}{a^{pk} k!} \right]^{\frac{1}{p}},
\]
for any constant $0 < a < 1$. Hence,

$$m_p(V) \leq (1 - a^q)^{-\frac{1}{q}} \exp \left( \frac{N^p_p(V)}{p \alpha^q} \right).$$

In particular, taking $a = \sqrt[3]{\frac{1}{p}}$, we have

$$m_p(V) \leq b_p \exp(N^p_p(V)).$$

3. Main Results. Now, we are in a position to establish our main results.

**Theorem 3.1.** Let the condition (2) hold with $r = \infty$. Then, under the conditions (3)-(5), and

(7) \hspace{1cm} m_p(V)N_\beta(V) < 1,

a solution $x = (x_1, x_2, \ldots, x_k, \ldots)$ of equation (1) is in $l^p(C^m)$ and satisfies the inequality

(8) \hspace{1cm} \|x\|_p \leq m_p(V)(1 - m_p(V)N_\beta(V))^{-1}\|f\|_p.

**Proof.** Rewrite (6) as

$$x = (I - V)^{-1}(f + F(x)).$$

Operator $I - V$ is invertible, since the spectrum of $V$ consists of zero. By Lemma 4.3 from [8],

$$\|V^k\|_p \leq \frac{N^k_p(V)}{\sqrt[k]{k!}}.$$  

Since

$$(I - V)^{-1} = \sum_{k=0}^{\infty} V^k,$$

we have

(9) \hspace{1cm} \|(I - V)^{-1}\|_p \leq \sum_{k=0}^{\infty} \frac{N^k_p(V)}{\sqrt[k]{k!}} = m_p(V).$$
Moreover, by (2) and by Hölder’s inequality, we get

\[
\|F(x)\|_{lp}^p = \sum_{j=1}^{\infty} \left( \sum_{k=1}^{j-1} a_{jk}G_k(x_k) \right)^p_{C^n} \leq \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n} \beta_k \|x_k\|_{C^n} \right]^p \leq \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^\beta \|x_k\|_{C^n}^\beta \right]^p \leq \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^\beta \|x_k\|_{C^n}^\beta \right]^p \leq \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^\beta \|x_k\|_{C^n}^\beta \right]^p \leq \sum_{j=1}^{\infty} \left[ \sum_{k=1}^{j-1} \|a_{jk}\|_{C^n}^\beta \|x_k\|_{C^n}^\beta \right]^p.
\]

This yields,

\[(10) \quad \|F(x)\|_{lp}^p \leq N_\beta(V)\|x\|_{lp}^p.
\]

We get from (9) and (10) that

\[
\|x\|_{lp} \leq m_p(V)[\|f\|_{lp} + N_\beta(V)\|x\|_{lp}].
\]

Thus, from (7) we infer (8), concluding the proof. \(\square\)

Put

\[
M_q(V, \beta) = \sup_{j=1,2,\ldots} \left[ \sum_{k=1}^{j-1} (1 + \beta_k)^q |a_{jk}|_{C^n}^\beta \right]^\frac{1}{q}
\]

and

\[
\sigma(f) = \|f\|_{C^0} + \|f\|_{lp} M_q(V, \beta) m_p(V)(1 - m_p(V) N_\beta(V))^{-1}.
\]

**Theorem 3.2.** Under the hypotheses of Theorem 3.1, the estimate

\[
\|x\|_{C^0} \leq \sigma(f)
\]

is valid.
Proof. It follows from (2) and (6) that
\[
\|x\|_c \leq \|f\|_c + \sup_{j=1,2,\ldots} \left( \sum_{k=1}^{j-1} (1 + \beta_k)\|a_{jk}\|_{c_n}\|x_k\|_{c_n} \right).
\]
Thus, the Hölder's inequality yields
\[
\|x\|_c \leq \|f\|_c + \sup_{j=1,2,\ldots} \left[ \sum_{k=1}^{j-1} (1 + \beta_k)^q\|a_{jk}\|_{c_n}^q \right]^{\frac{1}{p}} \left[ \sum_{k=1}^{j-1} \|x_k\|_{c_n}^p \right]^{\frac{1}{p}}.
\]
From Theorem 3.1, we therefore have
\[
\|x\|_c \leq \|f\|_c + \|f\|_{\ell^p} M_q(V, \beta)\|x\|_{\ell^p}.
\]
concluding the proof. □

In the next theorem, the condition \( r = \infty \) is not assumed.

**Theorem 3.3.** Let the conditions (2)-(5) and (7) hold. Then, a solution \( x = (x_1, x_2, \ldots, x_k, \ldots) \) of equation (1) satisfies the inequalities
\[
\|x\|_{\ell^p} \leq m_p(V)(1 - m_p(V)N_\beta(V))^{-1}\|f\|_{\ell^p}
\]
and
\[
\|x\|_c \leq \|f\|_c + \|f\|_{\ell^p} M_q(V, \beta)m_p(V)(1 - m_p(V)N_\beta(V))^{-1},
\]  
provided \( f \) satisfies the condition
\[
\|f\|_c + \|f\|_{\ell^p} M_q(V, \beta)m_p(V)(1 - m_p(V)N_\beta(V))^{-1} < r.
\]

Proof. If \( r = \infty \), then the required results follow from Theorems 3.1 and 3.2, respectively. It remains to prove the statement in the case \( r < \infty \).

By the Urysohn Theorem [4, page 15], there is a scalar valued function \( \psi_r \) defined on \( C^n \), such that
\[
\psi_r(z) = \begin{cases} 1, & z \in \omega_r, \\ 0, & z \notin \omega_r. \end{cases}
\]
Let us denote
\[
G^r_j(z) = \psi_r(z)G_j(z), \quad z \in C^n.
\]
Now, we consider the equation

\begin{equation}
X_j = f_j + \sum_{k=1}^{j-1} a_{jk}[x_k + G_k(x_k)], \quad (j = 1, 2, \ldots).
\end{equation}

As it was already proved, any solution of equation (13) satisfies (8) and (11), provided (12) holds. Consequently, they belong to \( \omega_r \). But \( G^r_j = G_j \) on \( \omega_r \). This implies that the estimates (8) and (11) are valid if \( r < \infty \), concluding the proof. \( \Box \)

4. Extensions. The main results established in the previous Sections, can be extended to many kind of Volterra difference equations. For example, consider the equation

\begin{equation}
X_j = f_j(x_j) + \sum_{k=1}^{j-1} a_{jk}[x_k + G_k(x_k)], \quad (j = 1, 2, \ldots),
\end{equation}

where the mappings \( G_k : \mathbb{C}^n \to \mathbb{C}^n \) satisfy property (2), and the mappings \( f_j : \mathbb{C}^n \to \mathbb{C}^n \) satisfy

\begin{equation}
\|f_j(h)\|_{\mathbb{C}^n} \leq \sum_{k=1}^{j-1} \alpha_{jk}\|x_k\|_{\mathbb{C}^n}, \quad (h \in \omega_r, \; j = 1, 2, \ldots \text{ for some } r \leq \infty),
\end{equation}

where \( \alpha_{jk} \geq 0 \) are real numbers.

Rewrite equation (14) as

\begin{equation}
x = f(x) + Vx + F(x),
\end{equation}

with \( f(x) = (f_1(x), f_2(x), \ldots, f_k(x), \ldots) \), the operators \( V \) and \( F \) are defined in \( l^p(\mathbb{C}^n) \) by

\begin{align*}
[Vx]_j &= \sum_{k=1}^{j-1} a_{jk}x_k, \quad x \in l^p; \\
[F(x)]_j &= \sum_{k=1}^{j-1} a_{jk}G_k(x_k), \quad x \in l^p, \; x = (x_1, x_2, \ldots, x_k, \ldots).
\end{align*}

Suppose that
THEOREM 4.1. Let conditions (3), (5), (15) and (17) hold. In addition, if
\[ m_p(V)(N_\alpha + N_\beta(V)) < 1, \]
then solution \( x = (x_1, x_2, \ldots, x_k, \ldots) \) of equation (14) is in \( l^p \) and satisfies the estimation
\[ \|x\|_p \leq m_p(V)[1 - m_p(V)(N_\alpha + N_\beta(V))]^{-1}. \]

Proof. Using arguments similar to those used in Theorems 3.1-3.3, the proof follows. Thus, we will omit it. \( \Box \)

5. Conclusion. In this work, some possibilities to use the operator method for investigating the asymptotic behavior of Volterra difference equations have been demonstrated. Thus, we have derived some explicit estimates for the solutions in terms of the coefficients of the considered equations. Furthermore, under some restrictions, the Theorem 3.1 allowed us to generalize the main results from the papers [7,12,14].

REFERENCES


ON THE ADVANCED INTEGRAL AND DIFFERENTIAL EQUATIONS OF THE SIZING PROCEDURE OF STORAGE DEVICES

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Abstract. Advanced integral and differential equations are formulated for underflow probabilities of a finite capacity intermediate storage operated in processing systems under stochastic operation conditions. These equations, developed for infinite operation time, provide the basis for rational design of operation of buffering storages. Analytical solutions to the integral equations for infinite horizons are derived in the cases of constant and exponentially distributed inputs. A numerical method is presented for solving the integral equations in the case of batch sizes described by general distribution functions. Some results of numerical experiments with batch sizes described by random variables the distribution of which is lognormal distribution are also presented.

Key Words. Batch/continuous processing system, intermediate storage, stochastic conditions, advanced integral equations, advanced differential equation, numerical method.

AMS(MOS) subject classification. 60K30, 90B05, 34B05

1. Introduction. The intermediate storage plays an important role in improving the operating efficiency of processing systems, and fitting the subsystems to each other with different operational characteristics. It increases the availability and variability of systems and reduces the process uncertainties. Over a long time interval, it can also buffer the effects of equipment failures and other unexpected events acting as noise of the processing systems. Such buffering action, however, requires a rational sizing of the intermediate storage in the design phase.

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The problem of sizing of an intermediate storage when uncertainties are taken into account has been considered mostly in batch process operations. It often can be treated as a deterministic problem (Ross, 1973; Henley and Hoshino, 1977; Takamatsu et al., 1979; Knopf et al., 1982; Karimi and Reklaitis, 1985; Lee and Reklaitis, 1989), but, when the equipment and batch failures are significantly of random nature, then the sizing problem becomes rather stochastic than a deterministic task.

The aim of this paper is to present a mathematical model developed for determining the appropriate initial amount of material in the storage. For this purpose, integral and differential equations are developed for the underflow probabilities of a finite storage formulated for infinite time interval, that provide the basis for the rational design of such intermediate storages. Analytical solutions to the integral equations are derived in the cases of constant and exponentially distributed inputs. For the batch sizes described by general distribution functions, numerical solution to the integral equation is presented and compared to the results coming from Monte Carlo simulation. The solutions are used for determining the appropriate initial amount of material for operating the system at a given reliability level.

2. The physical model. Let us consider a processing system consisted of a batch and a continuous subsystem which are connected by an intermediate storage, shown schematically in Fig. 1. Here, some batch units form the inputs of the storage, while at the output the material processed is removed continuously with constant volumetric rate \( q(t) = c \) into the continuous subsystem. This storage is aimed to buffer the operational differences between the batch and continuous subsystems, but it is useful also to reduce the operational uncertainties due to equipment failures and other events influencing the system. It assures the material for continuous work of the continuous subsystem.

![Fig. 1. Intermediate storage connecting the batch and continuous subsystems of a processing system](image)

Let us consider the long-term operation of the system allowing that the fill in times can be treated momentarily compared to the time interval of
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operation. Let us assume that the occurrence times of transfers \( N(t) = \# \{ i \geq 1 \mid t_i \in [0, t] \} \) are described by a Poisson process with parameter \( \lambda \), and the Poisson process is independent of the amount of transferred material \( Y_i \) filled in during the \( i^{th} \) transfer. This amount of material can be described by independent identically distributed random variables. Let us denote the common distribution function by \( G \), let its expectation be \( \mu \) and dispersion be \( \sigma \).

In this system, we expect to obey the operating rule: the material transferred into the intermediate storage from the batch processing units must satisfy the demand of the continuously operated subsystem, and any overflow has to be avoided. As a consequence, the following problems can be formulated:

1) Knowing the operational characteristics of the batch and continuous subsystems, determine the initial amount of material in order to assure that the amount of material in the storage will not decrease below some given level. This will be called as the initial amount problem.

2) Determine the volume of the intermediate storage not allowing overfill. This will be called as a sizing problem.

In this paper we study the problem of determining the initial amount of material in the storage requiring that the system operates without failures over a long operation time interval.

3. Mathematical model. Let us consider the mass of material \( S(t) \) being in the storage at time \( t \geq 0 \). The total amount of material filled into the storage by the input units in the time interval \([0, t]\) is \( \sum_{i=1}^{i=N(t)} Y_i \), while \( ct \) denotes the total mass removed by the continuous sub-system during that time interval. Naturally, \( Y_i \geq 0 \). It can be seen that the amount of the material being in the intermediate storage is given by

\[
S(t) = z_0 + \sum_{i=1}^{i=N(t)} Y_i - ct,
\]

where \( z_0 \) denotes the initial amount of material.

In order to have enough material in the intermediate storage the function \( S(t) \) has to satisfy the following inequality for any value of \( t \):

\[
S(t) = z_0 + \sum_{i=1}^{i=N(t)} Y_i - c \geq 0.
\]
Since $S(t)$ depends on random effects, inequality (2) does not hold for all realizations of the process. Hence we define the function

\[ R(z) = P \left( \left\{ 0 \leq \sum_{i=1}^{N(t)} Y_i - ct + z \quad \forall t \geq 0 \right\} \right), \]

which gives the probability that we have enough material for the continuously working subsystem, that is the reliability of the operation. In order to determine the initial amount of material in the intermediate storage which is required for normal operation of the processing system at a given reliability level, we have to derive the function $R(z)$ and its inverse function.

4. Integral and differential equations in the case of constant batch sizes. When the amount of the transferred material is deterministic and constant, then, without loss of generality we can take $Y_i = 1$ for any value of $i$, and we have the following

**Theorem 1.** If $Y_i = 1$ for any $i$, then $R(z)$ is a monotone increasing function of its argument, and $R(z) = 0$ if $z \leq 0$. In the case $\frac{\lambda}{c} > 1$ we have that $R(z) \to 1$ as $z$ tends to infinity. If $\frac{\lambda}{c} \leq 1$, then $R(z) = 0$ for any $z \in \mathbb{R}$. Furthermore, the function $R(z)$ satisfies the following integral equation for any values of parameters:

\[ R(z) = \int_{0}^{z/c} R(z + 1 - ct) \lambda e^{-\lambda r} dr, \quad z > 0. \]

**Proof.** Let $z_1 \leq z_2$. Then

\[ \left\{ z_1 - ct + \sum_{i=1}^{N(t)} Y_i \geq 0 \quad \forall t \geq 0 \right\} \subset \left\{ z_2 - ct + \sum_{i=1}^{N(t)} Y_i \geq 0 \quad \forall t \geq 0 \right\}, \]

hence

\[ P\left( \left\{ z_1 - ct + \sum_{i=1}^{N(t)} Y_i \geq 0 \quad \forall t \geq 0 \right\} \right) \leq P\left( \left\{ z_2 - ct + \sum_{i=1}^{N(t)} Y_i \geq 0 \quad \forall t \geq 0 \right\} \right), \]

which means that $R(z_1) \leq R(z_2)$. 
If \( z = 0 \), then obviously \( R(0) = 0 \), hence taking into account the monotonicity of function \( R(z) \), \( R(z) \leq 0 \) for any negative \( z \). As \( R(z) \) expresses probability, its values are nonnegative, hence \( R(z) = 0 \), if \( z \leq 0 \).

Let us consider the expectation of \( S(t) \). As in our case \( \sum_{i=1}^{N(t)} Y_i = N(t) \), furthermore the expectation of the Poisson process is equal to \( \lambda t \), one can conclude, that

\[
E \left( \sum_{i=1}^{N(t)} Y_i - ct \right) = E(N(t) - ct) = (\lambda - c)t.
\]

If \( \frac{\lambda}{c} > 1 \), then \( \lim_{t \to \infty} E \left( \frac{S(t)}{t} \right) = \lambda - c > 0 \). Applying the law of large numbers we get \( R(z) \to 1 \), if \( z \to \infty \).

If \( \frac{\lambda}{c} < 1 \), then \( \lim_{t \to \infty} E \left( \frac{S(t)}{t} \right) = \lambda - c < 0 \). Applying the law of large numbers we get \( R(z) = 0 \) for any \( z \).

If \( \frac{\lambda}{c} = 1 \), then \( \lim_{t \to \infty} E \left( \frac{S(t)}{t} \right) = \lambda - c = 0 \). As

\[
D^2(S(t)) = D^2(N(t) - ct) = D^2(N(t)) = \lambda t \to \infty,
\]

applying the Chung-Fuchs theorem (Chow and Teicher, 1978), we get that the values of \( S(t) \) are out of any finite interval with probability one, hence \( R(z) = 0 \) for any \( z \).

Let us turn to the integral equation. Let us denote the time of the first input by \( T_1 \). Applying the theorem of total probability for \( T_1 \)

\[
R(z) = \int_0^\infty P(z - ct + N(t) \geq 0 \; \forall t \geq 0 \; | T_1 = \tau) \lambda e^{-\lambda \tau} d\tau
\]

\[
= \int_0^\infty P(z - ct + N(t) \geq 0 \; \forall t < \tau \; \text{and} \; \forall t \geq \tau \; | T_1 = \tau) \lambda e^{-\lambda \tau} d\tau
\]

\[
= \int_0^\infty P \left( z - ct + N(t) \geq 0 \; \forall t \geq \tau \; | z - ct + N(t) \geq 0 \; \forall t < \tau \right.
\]

\[
\text{and} \; T_1 = \tau \left) \cdot P(z - ct + N(t) \geq 0 \; \forall t < \tau \; | T_1 = \tau) \lambda e^{-\lambda \tau} d\tau.\]

If \( z - ct \geq 0 \), then \( z - ct + N(t) \geq 0 \; \forall t < \tau \), as \( N(t) = 0 \) for any \( 0 \leq t < \tau \).

If \( z - ct < 0 \) then \( z - ct + N(t) < 0 \) for some \( t < \tau \), which is close to \( \tau \).

Therefore the right hand side of (9) can be written in the following form:

\[
\int_0^\infty P(z - ct + N(t) \geq 0 \; \forall t > \tau \; | z - ct \geq 0 \; \text{and} \; T_1 = \tau) \cdot 1_{\{z - ct \geq 0\}} \lambda e^{-\lambda \tau} d\tau,
\]
where $1$ denotes the characteristic random variable of the event being in the index. At the moment $\tau$ the process will be renewed but the initial amount will be $z - c\tau + 1$. Therefore

\begin{equation}
P(z - c\tau + N(t) \geq 0 \quad \forall t \geq \tau \mid T_1 = \tau) = R(z - c\tau + 1).
\end{equation}

Hence

\begin{equation}
R(z) = \int_0^{z/c} R(z + 1 - c\tau) \lambda e^{-\lambda\tau} d\tau.
\end{equation}

which is the required formula. \qed

**Remark 1.** It can be proved that Eq. (4) can be transformed into the following advanced differential equation:

\begin{equation}
R'(z) = \frac{\lambda}{c} (R(z + 1) - R(z)), \quad z > 0.
\end{equation}

**Remark 2.** In order to solve the initial amount problem we have to find the solution of Eq. (13) with boundary conditions

\begin{equation}
R(0) = 0
\end{equation}

and

\begin{equation}
\lim_{z \to \infty} R(z) = 1.
\end{equation}

**Theorem 2.** If $\frac{\lambda}{c} > 1$, then Eq. (13) with boundary conditions (13a) and (13b) has a unique solution. This solution is given by

\begin{equation}
R(z) = 1 - e^{-\lambda^* - z}, \quad z > 0,
\end{equation}

where $\lambda^*$ is the unique positive solution of the equation

\begin{equation}
\frac{\lambda \cdot e^{-\lambda^*}}{\lambda - \lambda^* \cdot c} = 1.
\end{equation}

**Proof.** Let $R$ be a solution of (13) satisfying (13a) and (13b). Let us consider the transformation $u(z) = R(-z)$ $z \leq 0$. Now, denoting $\gamma = \frac{\lambda}{c}$, $u$ is a solution of the following linear delay differential equation:

\begin{equation}
u'(z) = \gamma (u(z) - u(z - 1))
\end{equation}
on \((-\infty, 0]\) with boundary conditions \(u(0) = 0\) and \(\lim_{z \to -\infty} u(z) = 1\).

The phase space for (16) is \(C = C([-1, 0], R)\), the space of continuous functions mapping \([-1, 0]\) into \(R\). Clearly the function \(x(z) := u(z) - 1\) is a solution of (16) with boundary conditions

\[(16a) \quad x(0) = -1\]

and

\[(16b) \quad \lim_{z \to -\infty} x(z) = 0.\]

This means that \(\varphi = x \mid_{[-1,0]}\), the restriction of \(x\) onto the interval \([-1, 0]\), belongs to the unstable subspace \(U\) of \(C\) defined by

\[U = \left\{ \varphi \in C \mid u(\varphi) \text{ is defined for all } t \leq 0 \text{ and } \lim_{t \to -\infty} u(\varphi)(t) = 0 \right\},\]

where \(u(\varphi)\) denotes the unique solution of (16) with initial function \(\varphi\). It is well known (see Hale, 1977), that the dimension of \(U\) is finite and is equal to the number of those roots (counting multiplicities) of the characteristic equation

\[(17) \quad v = \gamma(1 - e^{-v})\]

which have positive real parts. It is easy to show that if \(\gamma > 1\), then (17) has exactly one root with positive real part. This root is real, simple and coincides with \(\lambda^*\), the unique positive root of (15). Hence

\[U = \{ cf \mid c \in R \}, \quad \text{where } f(z) = e^{\lambda^*z} \text{ for } z \in [-1, 0],\]

and thus \(x(z) = ce^{\lambda^*z}\) for some \(c \in R\). From (16a) we get that \(c = -1\). Therefore \(R(z) = u(-z) = 1 + x(-z) = 1 - e^{-\lambda^*z}\) for \(z > 0\).

**Remark 3.** Determining the solution of Eq. (13) with boundary conditions (13a) and (13b), the initial amount of material at a given reliability level \(1 - \alpha\) can be derived as \(R^{-1}(1 - \alpha) = -\ln(\alpha) / \lambda^*\), where \(\lambda^*\) is the unique positive solution of Eq. (15).

**5. Integral equations in the case of random batch sizes.** In this part we present the integral equation satisfied by the function describing the reliability in the case when the amount of transferred material varies randomly.
THEOREM 3. (Orbán-Mihálykó and Lakatos, 2003) Suppose that \( Y_i, \) \( i = 1, 2, \ldots \) are independent nonnegative continuous random variables with distribution function \( G(y) \) and expectation \( \mu \). Then, \( R(z) \) is a monotone increasing function of its argument, and \( R(z) = 0 \) if \( z \leq 0 \). If \( \frac{\lambda \mu}{c} \leq 1 \), then \( R(z) = 0 \) for any \( z \in \mathbb{R} \). If \( \frac{\lambda \mu}{c} > 1 \), then \( R(z) \to 1 \), as \( z \to \infty \). Furthermore, the function \( R(z) \) satisfies the following integral equation:

\[
R(z) = \int_0^\infty \int_0^{z/c} R(z-c\tau+y)\lambda e^{-\lambda \tau}d\tau dG(y), \quad z > 0,
\]

or equivalently

\[
R(z) = \frac{\lambda}{c} \int_0^\infty (R(z+y) - R(y))(1-G(y))dy, \quad z > 0.
\]

REMARK 4. Eq. (19) is a Fredholm type integral equation with advanced argument. In order to solve the initial amount problem in the case \( \frac{\lambda \mu}{c} > 1 \), we have to find the solution of Eq. (19) with boundary conditions (13a) and (13b).

In the special case of the exponential distribution of batch sizes, we have the following

PROPOSITION 1. (Orbán-Mihálykó, 2003) If \( G(y) = 1 - \exp\left(-\frac{y}{\mu}\right), \) \( y \geq 0 \), and \( \frac{\lambda \mu}{c} > 1 \), then the unique solution of Eq. (19) with boundary conditions (13a) and (13b) is given by

\[
R(z) = 1 - \exp\left(-\frac{\lambda \mu - c}{\mu c}z\right).
\]

When, however, the batch sizes are governed by some general distribution then we have

THEOREM 4. (Orbán-Mihálykó, 2003) Suppose that \( Y_i \) are independent nonnegative continuous random variables with distribution function \( G(y) \) and expectation \( \mu \), and let \( \frac{\lambda \mu}{c} > 1 \). Then Eq. (19) with boundary conditions (13a) and (13b) has a solution of the form

\[
R(z) = 1 - e^{-\nu z},
\]
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where the parameter $v > 0$ satisfies the equation

$$
\int_0^\infty e^{-vy} g(y) = 1,
$$

and $g(y) = \frac{\lambda}{c} (1 - G(y))$. Eq. (22) possesses a unique positive solution for every continuous distribution function $G(y)$.

6. Numerical solution of the integral equation (19). Since analytical solutions to Eq. (19) have been found only in some special cases, a numerical procedure was developed to solve this integral equation in general case. In this procedure, we utilized the fact that the solution of Eq. (19) is of exponential type (see Theorem 4). Thus the problem can be reduced to find the numerical solution of Eq. (22). Therefore, introducing the notation

$$
f(v) = \int_0^\infty e^{-vy} g(y) dy - 1,
$$

we have to find the solution of the equation $f(v) = 0$. Since $f(v)$ is differentiable and its derivative, $f'(v) = - \int_0^\infty ye^{-vy} g(y) dy$, is known, Newton's method can be applied. The improper integral is computed by means of the Gauss-Laguerre quadrature formulas.

In the computations, Laguerre polynomials of degree 20 were used to compute the improper integrals, and the results were compared with those obtained by the direct Monte Carlo simulation of the process. For the sake of illustration we present the following example. Let $Y_i$ be of lognormal distribution, that is, $G(y) = \int_0^y \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$, where the parameters are $m = 2$ and $\sigma = 1$. Then the expected value becomes $\mu = e^{m+\frac{\sigma^2}{2}} = 12.18$. Let furthermore $\lambda = 0.3$ and $c = 2$. In this case $\frac{\lambda \mu}{c} = 1.827 > 1$. As a consequence, the numerical solution of Eq. (22) gives the value $v \approx 0.0634$. Hence $R(z) \approx 1 - e^{-0.0634z}$.

This solution is compared with the results obtained by using Monte Carlo simulation as are shown in Figs 2a and 2b. In the Monte Carlo method, $N = 10000$ simulation runs were executed. The results obtained by simulation and by using the numerical method are very close and the differences appear to be smaller than the error of simulation.
Consequently, the initial amount of material in the intermediate storage required for operating the system continuously at the reliability level \( 1 - \alpha = 0.99 \) can be determined by the formula \( R^{-1}(1-\alpha) = -\ln \alpha \). In the case of our example, this amount is approximately equal to \( R^{-1}(0.01) \approx -\ln 0.01 \approx 72.6 \) mass units.

7. Conclusions. Advanced integral and differential equations were developed for rational design of operation of intermediate storage connecting batch and continuous subsystems of processing systems under stochastic operation conditions. The problem was considered for infinite operation time interval.

In the case of constant and exponentially distributed inputs, analytical solutions to the integral equations for infinite horizons were derived. In general case, i.e. when the sizes of the inputs vary randomly according to some general probability distribution, the integral equations can be solved only by numerical methods or by using a direct Monte Carlo simulation. The applicability of these methods in the sizing procedure of intermediate storage were illustrated by solving an example problem and comparing the results obtained by both methods.

Acknowledgement. The authors express their thanks to Prof. I. Győri and Prof. M. Pituk for their useful comments.
REFERENCES


A LIMIT SET TRICHOTOMY FOR ABSTRACT 2-PARAMETER SEMIFLOWS ON TIME SCALES

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Abstract. Under certain contractivity conditions, we study the asymptotic behavior of abstract 2-parameter semiflows on normal cones in Banach spaces, and show that there are only three possible scenarios for their limit behavior.

AMS(MOS) subject classification. Primary 37C65; Secondary 37B55, 92D25

Key Words. Limit set trichotomy, 2-parameter semiflow, time scale

Dedicated to Prof. István Győri on the occasion of his 60th birthday.

1. Introduction. In certain relevant situations, e.g. in biological applications from population dynamics, it frequently happens that a dynamical system preserves a (partial) order relation on its state space. Such systems are called order-preserving or monotone and Krasnosel'skii laid the basics for their qualitative theory in [Kra64, Kra68]. Meanwhile many others made important contributions for different types of monotone (semi-)dynamical systems and in this small note we simply refer to [PS03, Chu02] for further references.

The essential property of order-preserving dynamical systems is that they possess a surprisingly simple asymptotic behavior. In fact Krause et al. [KN93, KR92] proved a so-called limit set trichotomy (cf. also [Nes99] for nonautonomous difference equations or [Chu02] for random dynamical systems), describing the only three possible asymptotic scenarios of difference equations under a certain kind of concavity.

* University of Augsburg, Department of Mathematics, D-86135 Augsburg, Germany. Research supported by the “Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik” (GRK 283) financed by the Deutsche Forschungsgemeinschaft and the State of Bavaria.
In an earlier paper (cf. [PS03]), Siegmund and the author proved such a limit set trichotomy for a general model of nonexpansive dynamical processes, namely 2-parameter semiflows in normal cones on time scales. They include, for example, solution operators of dynamic equations on time scales (cf. [BP01]) and in particular of nonautonomous difference and differential equations. Here we impose different contractivity conditions on the 2-parameter semiflows and obtain a stronger limit set trichotomy in this situation, leading to the asymptotic equivalence of all bounded solutions. Despite the more general setting, our arguments follow closely those of [Nes99].

2. Semiflows, Cones, and the Part Metric. Let T be an arbitrary time scale, i.e., a canonically ordered closed subset of the real axis \( \mathbb{R} \). Since we are interested in the asymptotic behavior of evolutionary processes on such sets \( T \), it is reasonable to assume that \( T \) is unbounded above in the whole paper. \((X,d)\) stands for a metric space from now on.

We begin with a very elementary result.

**Lemma 1.** Let \( x_0 \in X, T > 0 \) and \( f : T \to X \). Then \( \lim_{t \to \infty} f(t) = x_0 \) holds, if and only if one has \( \lim_{n \to \infty} f(t_n) = x_0 \) for every sequence \( (t_n)_{n \in \mathbb{N}} \) in \( T \) satisfying \( T \leq t_{n+1} - t_n \) for all \( n \in \mathbb{N} \).

**Proof.** We leave the easy proof to the reader. \( \square \)

Now we are in the position to define an abstract concept to describe nonautonomous evolutionary processes.

**Definition 1.** A mapping \( \varphi : \{(t, \tau) \in T^2 : \tau \leq t\} \times X \to X \) is denoted as a 2-parameter semiflow on \( X \), if the mappings \( \varphi(t, \tau, \cdot) = \varphi(t, \tau) : X \to X \), \( \tau \leq t \), satisfy the following properties:

(i) \( \varphi(\tau, \tau)x = x \) for all \( \tau \in T \), \( x \in X \),

(ii) \( \varphi(t, s)\varphi(s, \tau) = \varphi(t, \tau) \) for all \( \tau, s, t \in T \), \( \tau \leq s \leq t \),

(iii) \( \varphi(\cdot, \cdot)x : \{(t, \tau) \in T^2 : \tau \leq t\} \to X \) is continuous for all \( x \in X \).

For explicit examples of 2-parameter semiflows we only mention strongly-continuous 1-parameter semiflows, as well as solution operators of nonautonomous difference equations (\( T = \mathbb{Z} \)) or ordinary and functional differential equations (\( T = \mathbb{R} \)) under certain canonical assumptions on their right-hand side (cf. [PS03, Example 2.3]).

To provide some concepts from the classical theory of (autonomous) dynamical systems, we denote a point \( x_0 \in X \) as an equilibrium of \( \varphi \), if \( \varphi(t, \tau)x_0 = x_0 \) holds for all \( \tau \leq t \). Moreover, for \( \tau \in T \) and \( x \in X \), the orbit emanating from \( (\tau, x) \) is \( \gamma^{+}_{\tau}(x) := \{\varphi(t, \tau)x \in X : \tau \leq t\} \) and the \( \omega \)-limit set of \( (\tau, x) \) is given by \( \omega^{+}_{\tau}(x) := \bigcap_{t \leq s} \text{cl}\{\varphi(s, \tau)x \in X : t \leq s\} \). Equivalently, \( \omega^{+}_{\tau}(x) \) consists of all points \( x^* \in X \) such that there exists a sequence \( t_n \to \infty \) in \( T \) with \( x^* = \lim_{n \to \infty} \varphi(t_n, \tau)x \).
We say a self-mapping $\Phi : X \to X$ is nonexpansive (on $(X, d)$), if $d(\Phi x, \Phi \bar{x}) \leq d(x, \bar{x})$ for all $x, \bar{x} \in X$. The set of nonexpansive self-mappings is closed under composition. If $P \neq \emptyset$ is a set, then a family of parameter-dependent self-mappings $\Phi(p) : X \to X, p \in P$, is called uniformly contractive, if there exists a continuous function $c : X \times X \to [0, \infty)$, such that the following two conditions are fulfilled (cf. [Nes99]):

(i) $c(x, \bar{x}) < d(x, \bar{x})$ for all $x, \bar{x} \in X, x \neq \bar{x}$,

(ii) $d(\Phi(p)x, \Phi(p)\bar{x}) \leq c(x, \bar{x})$ for all $p \in P, x, \bar{x} \in X$.

In particular, each $\Phi(p)$ is nonexpansive. Moreover, in case, the mappings $\Phi_1(p), \Phi_2(p) : X \to X, p \in P$, are uniformly contractive (with contractivity function $c$) and $\Psi : X \to X$ is nonexpansive, then the compositions $\Phi_1(p) \circ \Phi_2(p)$ and $\Psi \circ \Phi_1(p)$ are uniformly contractive (with the same contractivity function $c$).

Assume from now on that the metric space $X$ is a cone $V_+$ in a real Banach space $(V, \|\cdot\|)$. Recall that a cone is a nonempty closed convex set $V_+ \subset V$ such that $\alpha V_+ \subset V_+$ for $\alpha \geq 0$ and $V_+ \cap (-V_+) = \{0\}$. Moreover, define $V^+_* := V_+ \setminus \{0\}$. Any cone induces a partial order relation on $V$ via $u \leq v$, if $v - u \in V_+$, which is preserved under addition and scalar multiplication with nonnegative reals. A cone $V_+$ is called normal, if there exists an equivalent norm $\|\cdot\'|$ on $V$ such that $\|u\| \leq \|v\|'$, if $u \leq v$.

Although forthcoming results on the boundedness of orbits are stated in the norm topology on $V_+$, our contractivity condition for 2-parameter semiflows will be formulated in a different metric topology:

**Definition 2.** If $\lambda(u, v) := \sup\{\alpha \in [0, \infty) : \alpha u \leq v\}$ for $u, v \in V_+$, then the mapping $p : V^*_+ \times V^*_+ \to [0, \infty), p(u, v) := -\log \min\{\lambda(u, v), \lambda(v, u)\}$ for $u, v \in V^*_+$ defines a quasi-metric on $V^*_+$, called the part metric.

**Remark 1.** (1) One easily sees $p(u, v) = \inf\{\log \alpha : \alpha^{-1}u \leq v \leq \alpha u\}$ for all $u, v \in V^*_+$ and, therefore, the part metric defined in [PS03, Definition 2.4(ii)] coincides with the one from Definition 2.

(2) If the cone $V_+$ is normal, then $\text{int} V_+$ is a complete metric space w.r.t. the part metric $p$ (cf. [Tho63]).

**Lemma 2.** If $V_+ \subset V$ is a normal cone with monotone norm, then

$$\|u - v\| \leq (2e^{p(u,v)} - e^{-p(u,v)} - 1) \min\{\|u\|, \|v\|\} \quad \text{for all } u, v \in V^*_+.$$

**Proof.** See [KN93, Lemma 2.3].

The subsequent result is an adaption from [Nes99, Lemma 4]. Thereto, let $P \neq \emptyset$ be a set again, and we denote $\Phi(p) : V_+ \to V_+, p \in P$, as uniformly ascending on $A \subset V_+$, if there exists a continuous mapping $\phi : [0, 1] \to [0, 1]$
with \( \alpha < \phi(\alpha) \) for all \( \alpha \in (0, 1) \) such that
\[
\alpha v \leq u \implies \phi(\alpha)\Phi(p)v \leq \Phi(p)u \quad \text{for all } \alpha \in [0,1], \ p \in P, \ u, v \in A.
\]
Evidently, each such operator \( \Phi(p) \) is order-preserving and subhomogeneous on \( A \); latter means that \( \alpha \Phi(p)v \leq \Phi(p)\alpha v \) holds for \( \alpha \in (0,1), \ v \in A \) and \( p \in P \). Moreover, if \( \Psi : V_+ \to V_+ \) is a mapping satisfying \( \Psi(A) \subset A \) and
\[
\alpha v \leq u \implies \alpha \Psi v \leq \Psi u \quad \text{for all } \alpha \in [0,1], \ u, v \in A,
\]
then also the composition \( \Phi(p) \circ \Psi : V_+ \to V_+ \), \( p \in P \), is uniformly ascending with \( \phi \). In particular, the composition \( \Phi_1(p) \circ \Phi_2(p) \) of two uniformly ascending mappings \( \Phi_1(p), \Phi_2(p) : V_+ \to V_+ \), \( p \in P \), is uniformly ascending on \( A \), if \( \Phi_2(p)A \subset A \).

**Lemma 3.** Let \( V_+ \subset V \) be a normal cone with \( \text{int} \ V_+ \neq \emptyset \) and assume that the mapping \( \Phi(p) : \text{int} V_+ \to \text{int} V_+ \), \( p \in P \), is uniformly ascending w.r.t. \( \phi \). Then \( \Phi(p) \) is uniformly contractive on \( \text{int} V_+ \) for the part metric, where the contractivity function \( c \) is given by
\[
(1) \quad c(u,v) := -\log(\phi(\min \{\lambda(u,v),\lambda(v,u)\})) \quad \text{for all } u, v \in \text{int} V_+.
\]

**Proof.** Let \( u, v \in \text{int} V_+ \) be given arbitrarily. Since \( p \) is a metric on \( \text{int} V_+ \), one has \( p(u,v) \geq 0 \) and the definition of \( p \) yields \( \min \{\lambda(u,v),\lambda(v,u)\} \leq 1 \), where \( \lambda(u,v) \) is given in Definition 2. Therefore, w.l.o.g. we can assume \( \lambda(u,v) = \min \{\lambda(u,v),\lambda(v,u)\} \leq 1 \). Since \( \lambda(u,v)u \leq v \) and \( \Phi(p) \) is uniformly ascending, it follows that \( \phi(\lambda(u,v))\Phi(p)u \leq \Phi(p)v \), and consequently \( \lambda(\Phi(p)u,\Phi(p)v) \geq \phi(\lambda(u,v)) \) for \( p \in P \). One gets \( \phi(\min \{\lambda(u,v),\lambda(v,u)\}) \leq \lambda(\Phi(p)u,\Phi(p)v) \) and exchanging \( u \) and \( v \) in the proof of the above estimate, yields that \( \Phi(p) \) satisfies property (ii) of a uniformly contractive mapping. On the other hand, due to the metric properties of \( p \), one has \( 0 < \min \{\lambda(u,v),\lambda(v,u)\} < 1 \) for all \( u,v \in V_+^*, \ u \neq v \), and thus we obtain the inequality \( \phi(\min \{\lambda(u,v),\lambda(v,u)\}) > \min \{\lambda(u,v),\lambda(v,u)\} \) for \( u,v \in \text{int} V_+^*, \ u \neq v \). Hence, \( c \) satisfies both conditions in the definition of uniform contractivity w.r.t. the part metric. As in [Nes99, Proof of Lemma 4], one sees that \( c \) is continuous under the part metric, and this implies the assertion. \( \square \)

**3. Limit Set Trichotomies.** The following theorem is a clear manifestation of the intuition that contractivity drastically simplifies the possible long-term behavior of a dynamical system — in fact, only three asymptotic scenarios are possible. In the autonomous discrete time case, these limit set
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trichotomy was discovered (and so named) by Krause and Ranft [KR92] and
generalized in [KN93] to infinite-dimensional autonomous difference equa-
tions; in addition, [Nes99] considers such nonautonomous systems, while
[PS03] prove a limit set trichotomy for general nonexpansive 2-parameter
semiflows. Now the nonexpansiveness of \( \varphi(t, \tau) \) is strengthened to the as-
sumption that \( \varphi(t, \tau) \) is uniformly ascending.

**Theorem 1 (Limit Set Trichotomy).** Let \( V_+ \subset V \) be a normal
cone, \( \text{int} V_+ \neq \emptyset \) and assume that \( \varphi \) is a 2-parameter semiflow on \( V_+ \) with
the following properties:

(i) There exists a real \( T > 0 \) such that for all \( t, \tau \in \mathbb{T} \) with \( T \leq t - \tau \),
one has \( \varphi(t, \tau)V_+^* \subset \text{int} V_+ \) and that \( \varphi(t, \tau) \) is uniformly ascending
on \( \text{int} V_+ \),

(ii) for all \( (\tau, v) \in \mathbb{T} \times V_+ \) every bounded orbit \( \gamma_+^\tau(v) \) is relatively compact
in the norm topology.

Then for every \( \tau \in \mathbb{T} \) the following trichotomy holds, i.e., precisely one of
the following three cases applies:

(a) For all \( v \in V_+^* \) the orbits \( \gamma_+^\tau(v) \) are unbounded in norm,

(b) for all \( v \in V_+ \) the orbits \( \gamma_+^\tau(v) \) are bounded in norm and for all
\( v \in V_+^* \) we have \( \lim_{t \to \infty} \| \varphi(t, \tau)u \| = 0 \),

(c) for all \( v \in V_+^* \) the orbits \( \gamma_+^\tau(v) \) are bounded in norm, for \( v \in V_+^* \) they
have a nontrivial accumulation point, and

\[ \lim_{t \to \infty} \| \varphi(t, \tau)u - \varphi(t, \tau)v \| = 0 \quad \text{for all } u, v \in V_+^*. \]

**Remark 2.** (1) The above limit relation (2) implies that all \( \omega \)-limit sets
\( \omega_+^\tau(v), \ v \in V_+^* \), are identical, and it excludes the existence of two different
equilibria of \( \varphi \) in \( V_+^* \). In fact, if \( \varphi \) possesses an equilibrium \( v_0 \in V_+^* \), then (2)
guarantees \( \omega_+^\tau(v) = \{ v_0 \} \) for all \( v \in V_+^* \).

(2) Let \( T_{\max} \geq T \) and suppose \( \mathbb{T} \) is a time scale such that for all \( t, \tau \in \mathbb{T} \),
\( T \leq t - \tau \), there exist finitely many points \( t_0 := \tau < t_1 < \ldots < t_{N-1} < t_N := t \)
in \( \mathbb{T} \) satisfying \( T \leq t_{n+1} - t_n \leq T_{\max} \) for all \( n \in \{0, \ldots, N - 1 \} \). Then it is
sufficient in hypothesis (i) to assume that \( \varphi(t, \tau) \) is uniformly ascending on
\( \text{int} V_+ \) for all \( t, \tau \in \mathbb{T} \) with \( T \leq t - \tau \leq T_{\max} \). This can be seen as follows:
For arbitrary \( t, \tau \in \mathbb{T}, T \leq t - \tau \), choose \( t_0, \ldots, t_N \) as above. Then, due to the
2-parameter semiflow property, one has that \( \varphi(t, \tau) = \varphi(t_N, t_{n-1}) \ldots \varphi(t_1, t_0) \)
is a composition of uniformly ascending operators \( \varphi(t_n, t_{n-1}), n = 1, \ldots, N \),
with \( \varphi(t_n, t_{n-1}) \text{int} V_+ \subset \text{int} V_+ \) and functions \( \phi \) not depending on \( n \). Hence,
\( \varphi(t, \tau), T \leq t - \tau \), itself is uniformly ascending on \( \text{int} V_+ \).

(3) A remark similar to (2) holds for the nonexpansiveness and uniform
contractivity assumptions from [PS03, Theorem 3.1].
Proof. Let $\tau \in \mathbb{T}$ be fixed. Because of Lemma 3, we know that the mapping $\varphi(t, \tau), T \leq t - \tau$, is nonexpansive, and all assumptions of [PS03, Theorem 3.1] are satisfied. To obtain (2), we show that in case (c) the relation

$$\lim_{t \to \infty} p(\varphi(t, \tau)u, \varphi(t, \tau)v) = 0 \quad \text{for all } u, v \in V_+^*$$

holds. By Lemma 2 this implies (2), since all orbits are bounded in norm.

To verify (3), let $u, v \in V_+^*$ and $(t_n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{T}$ with $t_0 = \tau$ and $\lim_{n \to \infty} t_n = \infty$, where w.l.o.g. we may assume $T \leq t_{n+1} - t_n$ for all $n \in \mathbb{N}_0$ (cf. Lemma 1). Now let neither (a) nor (b) hold. Then the orbits $\gamma_+^+(u), \gamma_+^+(v)$ are norm-bounded (cf. [PS03, Theorem 3.1]), and furthermore, by (i), one has $\varphi(t_n, \tau)u, \varphi(t_n, \tau)v \in \text{int} V_+$ for $n \in \mathbb{N}$. With a view to assumption (ii), this implies that the set $F := \{(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) : n \in \mathbb{N}\} \subset (\text{int} V_+)^2$ is relatively compact.

Due to Lemma 3, we know that $\varphi(t, \tau), T \leq t - \tau$, is uniformly contractive on int $V_+$ and it follows from the 2-parameter semiflow property that there exists a constant $\gamma \geq 0$ with

$$p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) \geq c(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v)$$

$$\geq p(\varphi(t_{n+1}, \tau)u, \varphi(t_{n+1}, \tau)v) \geq \gamma \quad \text{for all } n \in \mathbb{N}.$$ (4)

From now on, we assume that (3) does not hold, which yields $0 < \gamma \leq p(\xi_1, \xi_2) \leq \Gamma$ for $(\xi_1, \xi_2) \in F$, with some real $\Gamma > 0$; note here that (4) implies $\Gamma < \infty$. Setting $\psi(\xi_1, \xi_2) := e^{-p(\xi_1, \xi_2)}$, we therefore obtain $0 < e^{-\Gamma} \leq \psi(\xi_1, \xi_2) \leq e^{-\gamma} < 1$ for $(\xi_1, \xi_2) \in F$, and hence the continuity of $\phi$ and $\alpha < \phi(\alpha)$ for $\alpha \in (0, 1)$ implies the existence of a $C > 0$ with $\frac{\phi(\psi(\xi_1, \xi_2))}{\psi(\xi_1, \xi_2)} \geq C > 1$ for $(\xi_1, \xi_2) \in F$. Thus, using $\psi(\xi_1, \xi_2) = \min \{\lambda(\xi_1, \xi_2), \lambda(\xi_2, \xi_1)\}$ we arrive at the estimate

$$\gamma \leq p(\varphi(t_{n+1}, \tau)u, \varphi(t_{n+1}, \tau)v) \stackrel{(4)}{=} p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v)$$

$$\leq - \log \phi(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v))$$

$$\leq - \log (C\psi(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v))$$

$$= p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) - \log C$$

$$\ldots$$

$$\leq p(\varphi(t_1, \tau)u, \varphi(t_1, \tau)v) - n \log C \quad \text{for all } n \in \mathbb{N},$$

yielding a contradiction for $n \to \infty$, since $C > 1$. So we must have $\gamma = 0$ and in the light of Lemma 1, the limit relation (3) holds true. □

Now we switch to a finite-dimensional situation.
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**Theorem 2 (Limit Set Trichotomy).** Let $V_+ \subset [0, \infty)^d$ be a normal cone, $\text{int} V_+ \neq \emptyset$ and assume that $\varphi$ is a 2-parameter semiflow on $V_+$ with the following properties:

(i) There exists a real $T > 0$ such that one has $\varphi(t, \tau)V_+^* \subset \text{int} V_+$ for all $t, \tau \in \mathbb{T}$ with $T \leq t - \tau$,

(ii) $\varphi(t, \tau)|_{\text{int} V_+}, T \leq t - \tau$, is continuously differentiable, and

\[ \sum_{k=1}^{d} u_k \left| \frac{\partial \varphi_j(t, \tau, v)}{\partial v_k} \right| \leq a(v)\varphi_j(t, \tau, v) \text{ for all } T \leq t - \tau, \; v \in \text{int} V_+ \]

and $j = 1, \ldots, d$, where $a : V_+ \to [0, 1)$ is a continuous mapping.

Then for every $\tau \in \mathbb{T}$ the following trichotomy holds, i.e., precisely one of the following three cases applies:

(a) For all $v \in V_+^*$ the orbits $\gamma_\tau^+(v)$ are unbounded in norm,

(b) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm and for all $v \in V_+^*$ we have $\lim_{t \to \infty} \|\varphi(t, \tau)v\| = 0$,

(c) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm, for $v \in V_+^*$ they have a nontrivial accumulation point, and

\[ \lim_{t \to \infty} \|\varphi(t, \tau)u - \varphi(t, \tau)v\| = 0 \text{ for all } u, v \in V_+^*. \]

**Proof.** Let $\tau \in \mathbb{T}$ be fixed. Now we define the mapping

\[ c(u, v) := \sup_{\theta \in [0, 1]} a(u^\theta v^{1-\theta})p(u, v) \]

for all $u, v \in V_+^*$ with $p(u, v) < \infty$, where $u^\theta v^{1-\theta} \in [0, \infty)^d$ abbreviates the vector with components $u_i^\theta v_i^{1-\theta} \in [0, \infty)$, $i = 1, \ldots, d$. By assumption we have $c(u, v) < p(u, v)$ for $u, v \in V_+^*, u \neq v$, with $p(u, v) < \infty$, and [Nes99, Lemma 6] applied to $\varphi(t, \tau), T \leq t - \tau$, gives us $p(\varphi(t, \tau)u, \varphi(t, \tau)v) \leq c(u, v)$ for all $t, \tau \in \mathbb{T}, T \leq t - \tau$, and $u, v \in \text{int} V_+$. The definition of $c$ readily implies its continuity w.r.t. the part metric and, therefore, $\varphi(t, \tau), T \leq t - \tau$, is a uniform contraction on $\text{int} V_+$. Since we are in a finite-dimensional setting, each bounded orbit of $\varphi$ is relatively compact and the limit set trichotomy from [PS03, Theorem 3.1] applies. It remains to strengthen the assertion in case (c) of this trichotomy, by showing the limit relation (5).

Thereto, let $u, v \in V_+^*$ and assume that neither (a) nor (b) of the limit set trichotomy in [PS03, Theorem 3.1] holds. Then the orbits $\gamma_\tau^+(u), \gamma_\tau^+(v)$ are bounded in norm and one has $\varphi(t, \tau)u, \varphi(t, \tau)v \in \text{int} V_+$ for $T \leq t - \tau$. Now choose an arbitrary sequence $(t_n)_{n \in \mathbb{N}_0}$ in $\mathbb{T}$ with $t_0 := \tau, \lim_{n \to \infty} t_n = \infty$, and w.l.o.g. we suppose $T \leq t_{n+1} - t_n$ for $n \in \mathbb{N}_0$ (cf. Lemma 1). In case, the
sequence \((\varphi(t_n, \tau)u)_{n \in \mathbb{N}}\) has a trivial accumulation point, then there exists an infinite set \(N \subset \mathbb{N}\) with \(\lim_{n \to \infty, n \in N} \|\varphi(t_n, \tau)u\| = 0\); otherwise we set \(N := \emptyset\). Using mathematical induction, one obtains from (4) the inequality
\[p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) \leq p(\varphi(t_1, \tau)u, \varphi(t_1, \tau)v) \quad \text{for all } n \in \mathbb{N}.\]
Hence we can find some real \(\lambda > 0\) with \(0 \leq \varphi(t_n, \tau)v \leq \lambda \varphi(t_n, \tau)u\) for all \(n \in \mathbb{N}\) and consequently one has \(\lim_{n \to \infty, n \in N} \|\varphi(t_n, \tau)v\| = 0\), which immediately implies \(\lim_{n \to \infty, n \in N} \|\varphi(t_n, \tau)u - \varphi(t_n, \tau)v\| = 0\). It remains to prove
\[
(7) \quad \lim_{n \to \infty, n \notin N} \|\varphi(t_n, \tau)u - \varphi(t_n, \tau)v\| = 0.
\]
Due to the construction of \(N\), the set \(F := \{\varphi(t_n, \tau)u, \varphi(t_n, \tau)v \colon n \in N\} \subset (\text{int} V^+)^2\) has compact closure in \((V^+)^2\). Consequently, there exists an \(\alpha < 1\) with \(\sup_{\theta \in [0,1]} a(u^\theta v^{1-\theta}) \leq \alpha\) for \((u, v) \in F\), we obtain from the definition of \(c\) and the 2-parameter semiflow property
\[
p(\varphi(t_{n+1}, \tau)u, \varphi(t_{n+1}, \tau)v) \leq c(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) \leq \alpha p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v)
\]
and inductively
\[
0 \leq p(\varphi(t_n, \tau)u, \varphi(t_n, \tau)v) \leq \alpha^{n-1} p(\varphi(t_1, \tau)u, \varphi(t_1, \tau)v) \quad \xrightarrow{\text{n \to \infty, n \notin N}} 0.
\]
Finally, Lemma 2 and the norm-compactness of \(\text{cl} F\) implies the limit relation (7), which concludes our present proof. \(\Box\)

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NONOSCILLATION PROPERTIES OF CERTAIN FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS

EWA SCHMEIDEL

Abstract. We study the following equation $\Delta(a_n \Delta(b_n \Delta(c_n \Delta y_n))) = f(n, y_{n+2})$. The classification of nonoscillatory solutions of above equation is given. We prove that above equation has not quickly oscillatory solutions.

Next, we divide the set of solutions of this equations into two subsets: $F_+$ and $F_-$-solutions. Relations between these types of solutions and their nonoscillatory behavior are the main purpose of this paper. The results are illustrated by examples.

AMS(MOS) subject classification. 39A10

Key Words. Difference equations, Fourth order difference equations, Oscillatory solutions, Nonoscillatory solutions

1. Introduction. We will study fourth order difference equations of the form

$$\Delta(a_n \Delta(b_n \Delta(c_n \Delta y_n))) = f(n, y_{n+2}), \quad n \in N. \quad (E)$$

We denote by $N$ the set of positive integers, by $R$ the set of real numbers, by $R_+$ the set of real positive numbers and by $R_-$ the set of negative numbers. For a function $x : N \to R$, the forward difference operators are defined as follows: $\Delta x_n = x_{n+1} - x_n$, $n \in N$ and $\Delta^k x_n = \Delta(\Delta^{k-1} x_n)$, for $k = 2, 3, \ldots$.

The sequence $y$ is the trivial sequence if there exists $n_0 \in N$, such that $y_n = 0$, for all $n \geq n_0$. By a solution of $(E)$ we mean any nontrivial sequence $y$ satisfying equation $(E)$ for all $n \in N$.

A solution is oscillatory if for every $m \in N$, there exists $n \geq m$, such that $y_n y_{n+1} \leq 0$. A sequence $y$ is termed quickly oscillatory if and only if $y_n = (-1)^n a_n$, where $a$ is a sequence of positive numbers or negative numbers. In this paper we assume that the function $f : N \times R \to R$ satisfies the condition

$$xf(n, x) < 0, \text{ for } n \in N, x \in R \setminus \{0\}.$$

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We assume also that $a, b, c : \mathbb{N} \to \mathbb{R}^+$ and
\[
\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} \frac{1}{c_n} = \infty.
\]

We will start with the following, known Lemma.

**LEMMA 1.** Every eventually positive solution of equation $(E)$ is exactly one of the following types:

\[
y_n > 0, \quad c_n \Delta y_n > 0, \quad b_n \Delta (c_n \Delta y_n) > 0, \quad a_n \Delta (b_n \Delta (c_n \Delta y_n)) > 0 \quad (I)
\]
\[
y_n > 0, \quad c_n \Delta y_n > 0, \quad b_n \Delta (c_n \Delta y_n) < 0, \quad a_n \Delta (b_n \Delta (c_n \Delta y_n)) > 0 \quad (II)
\]

for all sufficiently large $n$.

**Proof.** See proof of Lemma 2 in [7]. □

**THEOREM 1.** Equation $(E)$ does not have a quickly oscillatory solution.

**Proof.** Assume that $y_n = (-1)^n p_n, p : \mathbb{N} \to \mathbb{R}^+$, is a solution of $(E)$. Then $(E)$ can be written in the form:

\[
(-1)^{n+1} \left\{ a_{n+1} [b_{n+2} c_{n+3} (p_{n+4} + p_{n+3}) + b_{n+2} c_{n+2} (p_{n+3} + p_{n+2})]
\right.
\]
\[
+ b_{n+1} c_{n+2} (p_{n+3} + p_{n+2}) + n_{n+1} c_{n+1} (p_{n+2} + p_{n+1})
\]
\[
+ a_n [b_{n+1} c_{n+2} (p_{n+3} + p_{n+2}) + b_{n+1} c_{n+1} (p_{n+2} + p_{n+1})
\]
\[
+ b_n c_{n+1} (p_{n+2} + p_{n+1} + b_n c_n (p_{n+1} + p_n))] = f(n, (-1)^{n+2} p_{n+2}).
\]

Hence, for $n$ even, and for $n$ odd, we have a contradiction. This proves our Theorem. □

2. **Main results.** Fourth order linear difference equations were investigated by Smith and Taylor in [12], [13] and [14]. Following them in 1995, Popenda and Schmeidel considered the equation $\Delta^4 y_n = f(n, y_{n+2})$, and to classify solutions of the above equation, they used the operator $F_n = x_{n+1} \Delta^3 x_n - \Delta x_n \Delta^2 x_n$ (see [7]). Next, in 2002, Schmeidel studied the equation $\Delta^2 (r_n \Delta^2 y_n) = f(n, y_{n+2})$ (see [8]). The operator used there is $F_n = x_{n+1} \Delta (r_n \Delta^2 x_n) - (\Delta x_n) (r_n \Delta^2 x_n)$. Fourth order difference equation was studied by many others including Cheng [1], Graef and Thandapani [2], Hooker and Patula [3], Liu and Yan [4], Schmeidel and Szmanda [11], Taylor and Sun [15], Thandapani and Arockiasamy [16], and Zhang and Cheng [17].

From now on, we assume that $c_n = a_{n-1}$ in equation $(E)$. Then equation $(E)$ takes the form:

\[
\Delta (a_n \Delta (b_n \Delta (a_{n-1} \Delta y_n))) = f(n, y_{n+2}). \quad (E1)
\]
We introduce an operator as follows:

\[ F_n = x_{n+1}(a_n \Delta(b_n \Delta(a_{n-1} \Delta x_n))) - (a_{n-1} \Delta x_n)(b_n \Delta(a_{n-1} \Delta x_n)). \]

**Lemma 2.**

\[ \Delta F_n = x_{n+2} \Delta(a_n \Delta(b_n \Delta(a_{n-1} \Delta x_n))) - b_n[\Delta(a_{n-1} \Delta x_n)]^2. \]

*Proof.* The proof is obvious and will be omitted. \( \square \)

**Lemma 3.** *Operator F is nonincreasing for every solution y of (E1).*

*Proof.* Lemma 3 follows directly from Lemma 2. \( \square \)

If \( F_n \geq 0 \) for all \( n \in \mathbb{N} \), then a solution \( y \) of equation (E1) is called an \( F_+ \)-solution. If \( F_n < 0 \) for some \( n \), then \( y \) is called an \( F_- \)-solution.

We use the operator \( F \) to classify the solutions of equation (E1). The operator \( F \) divides the set of solutions into two disjoint subsets: \( F_+ \) and \( F_- \)-solutions.

**Theorem 2.** *If y is a solution of equation (E1) then there exists at most one value of n such that \( y_n = y_{n+1} = 0 \) (double zero).*

*Proof.* This Theorem follows directly from Lemma 3. \( \square \)

**Theorem 3.** *Assume that \( b \) is a bounded sequence and let \( y \) be an \( F_+ \)-solution of (E1). Then for \( j = 0, 1, 2, ... \)

\[ (1) \quad \sum_{n=1}^{\infty} b_n[\Delta^j(a_{n-1} \Delta y_n)]^2 < \infty, \]

and

\[ (2) \quad \lim_{n \to \infty} b_n \Delta^j(a_{n-1} \Delta y_n) = 0. \]

*Proof.* Let \( y \) be an \( F_+ \)-solution of (E1).

Let \( j = 0 \). Then, from Lemma 2 and equation (E1) we obtain

\[ \Delta F_k = y_{k+2} f(k, y_{k+2}) - b_k[\Delta(a_{k-1} \Delta y_k)]^2. \]

By summation, we get

\[ F_n = F_1 + \sum_{k=1}^{n-1} y_{k+2} f(k, y_{k+2}) - \sum_{k=1}^{n-1} b_k[\Delta(a_{k-1} \Delta y_k)]^2. \]

Since \( F_n \geq 0 \), we have

\[ \sum_{k=1}^{n-1} b_k[\Delta(a_{k-1} \Delta y_k)]^2 \leq F_1. \]
Therefore \( \sum_{k=1}^{\infty} b_k[(\Delta(a_{k-1}\Delta y_k))^2] \leq F_1 < \infty. \) We have established Condition (1) for \( j = 0. \)

For \( j > 0 \) we will prove Theorem 3 by induction.

\[
\sum_{k=1}^{\infty} b_n[\Delta^j(a_{n-1}\Delta y_n)]^2 \leq \sum_{k=1}^{\infty} 2b_n[\Delta^{j-1}(a_n\Delta y_{n+1})]^2 + 2b_n[\Delta^{j-1}(a_{n-1}\Delta y_n)]^2
\leq 2 \sum_{k=1}^{\infty} [b_{k+1}(\Delta^{j-1}(a_k\Delta y_{k+1}))^2 + b_k(\Delta^{j-1}(a_{k-1}\Delta y_k))^2]
\leq 4 \sum_{k=1}^{\infty} b_k[\Delta^{j-1}(a_{k-1}\Delta y_k)^2] < \infty
\]

We proved that (1) holds for all \( j \geq 0. \)

From (1) we get \( \lim_{n \to \infty} b_n[\Delta^j(a_{n-1}\Delta y_n)]^2 = 0. \) Hence

\[
\lim_{n \to \infty} \frac{1}{b_n}[\Delta^j(a_{n-1}\Delta y_n)]^2 = 0.
\]

Since \( b \) is bounded, \( \frac{1}{b} \) is bounded away from zero. Thus Condition (2) holds.

**THEOREM 4.** Assume that \( b \) is a bounded sequence. Then every nonoscillatory solution \( y \) of equation \((E1)\) is an \( F_+ \)-solution if and only if \( y \) is type (II)-solution.

**Proof.** We prove this Theorem for an eventually positive solution.

Let \( y \) be an eventually positive \( F_+ \)-solution. Suppose for the sake of contradiction that it is type (I)-solution. Let \( M \) be sufficiently large that \( \Delta(b_n\Delta(a_{n-1}\Delta y_n)) > 0. \) Then we get \( b_n\Delta(a_{n-1}\Delta y_n) > b_M\Delta(a_{M-1}\Delta y_M) > 0, \) for \( n > M. \) This inequality contradicts Condition (2) of Theorem 3. So, \( y \) is type (II)-solution.

Let \( y \) be type (II)-solution. We will show the positivity of the operator \( F \) on the whole sequence. Choose \( m \) sufficiently large. Then from the definition of a type (II)-solution, we have \( F_n > 0 \) for \( n \geq m. \) By Lemma 3, the operator \( F \) is nonincreasing. Hence \( F_j \geq F_m > 0 \) for all \( j < m. \) Since \( m \) is arbitrary, \( F_n > 0 \) for all \( n \in \mathbb{N}. \) So, \( y \) is an \( F_+ \)-solution.

**REMARK 1.** Assume that \( b \) is a bounded sequence. Then every nonoscillatory solution \( y \) of equation \((E1)\) is an \( F_- \)-solution if and only if \( y \) is type (I)-solution.

**EXAMPLE 1.** We consider equation

\[
\Delta((n+1)\Delta((n+1)\Delta(n\Delta y_n))) = \frac{-2n^2}{(n-1)(n+2)(n+3)(n+4)} y_n.
\]

Here \( a_n = n+1, \) \( b_n = n+1, \) and \( n > 1. \) The sequence \( y_n = 1 - \frac{1}{n} \) is a solution of the above equation. It is easy to see that \( y \) is a type (II)-solution. The
operator $F$ takes the form:

$$F_n = 2 \frac{n^2 + 2n + 3}{(n + 1)(n + 2)(n + 3)} > 0 \text{ for each } n \in N,$$

then

$$\Delta F_n = \frac{-2n(n + 1)^2 - 4(n + 3)(n + 4)}{(n + 1)(n + 2)^2(n + 3)(n + 4)} < 0$$

and $y_n = (1 - \frac{1}{n})$ is an $F_+$-solution of the above equation.

**Example 2.** The equation

$$\Delta \left( \frac{1}{4n(n + 1)} \Delta \left( \frac{1}{(n - 1)n \Delta y_n} \right) \right)$$

$$= -\frac{1}{2} \frac{y_n}{(n - 3)(n - 2)(n - 1)n^2(n + 1)(n + 2)},$$

where $a_n = \frac{1}{4n(n + 1)}$, $b_n = n$, $n > 3$ has solution $y_n = n(n - 1)(n - 2)(n - 3) = n^{(4)}$. It is easy to see that $y_n = n^{(4)}$ is a type (I)-solution.

The operator $F$ takes the form:

$$F_n = -4(n + 1)(3n + 1) < 0, \text{ and } \Delta F_n = -8(3n + 5).$$

We see that $y_n = n^{(4)}$ is an $F_-$-solution.

**Theorem 5.** Assume that $b$ is bounded sequence. Then every nonoscillatory solution $y$ of equation (E1) such that

$$\sum_{n=1}^{\infty} b_n |\Delta(a_{n-1}\Delta y_n)| < \infty$$

is an $F_+$-solution.

**Proof.** Let $y$ be a nonoscillatory solution of (E1) such that

$$\sum_{n=1}^{\infty} b_n |\Delta(a_{n-1}\Delta y_n)| < \infty.$$

Suppose that $y$ is an $F_-$-solution. Then there exists $m \in N$ such that $F_m < 0$. Hence, by Lemma 3, $F_n \leq F_m < 0$ for $n \geq m$. Since $y$ is nonoscillatory, then from Lemma 1, $y$ can be type (II) or type (I)-solution. We will exclude both of these cases. For type (II)-solution there exists $M \in N$ such that $F_n > 0$ for all $n \geq M$. This is a contradiction. Next, let $y$ be type (I)-solution. From Lemma 1, we have $\Delta(b_n \Delta(a_{n-1}\Delta y_n)) > 0$. So we get $\sum_{n=1}^{\infty} b_n \Delta(a_{n-1}\Delta y_n) = \infty$. This contradiction completes the proof. □
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SOME ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We consider a second order nonlinear difference equation

\begin{equation}
\Delta^2 y_n = a_n y_{n+1} + f(n, y_n, y_{n+1}).
\end{equation}

The necessary conditions under which every solution of equation (E) can be written in the following form

\[ y_{n+1} = \alpha_n u_n + \beta_n v_n, \]

are given.

Here \(u\) and \(v\) are two linearly independent solutions of the equation

\[ \Delta^2 y_n = a_{n+1} y_{n+1}, \quad (\lim_{n \to \infty} \alpha_n = \alpha < \infty \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \beta < \infty). \]

Key Words. Nonlinear difference equation, nonoscillatory solution, second order

AMS(MOS) subject classification. 39A10

1. Introduction. Consider the difference equation (E) where \(N\) denotes the set of positive integers, \(R\) the set of real numbers and \(R_+\) the set of nonnegative real numbers. By a solution of equation (E) we mean a sequence \((y_n)\) which satisfies (E) for sufficiently large \(n\).

In the last few years there has been an increasing interest in the study of asymptotic behavior of solutions of difference equations, in particular second order difference equations (see for example [1]-[9]).

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2. Main result. We start with the following modification of Gronwall’s Lemma.

**Lemma 1.** Assume that $F : R \to R$ is continuous, nondecreasing function, such that $F(x) \neq 0$ for $x \neq 0$ and

$$\int_{\varepsilon}^{\infty} \frac{ds}{F(s)} = \infty$$

where $\varepsilon$ is a positive constant. Moreover, let the function $B : N \times R_{+}^{2} \to R_{+}$ be continuous on $R_{+}^{2}$ for each $n \in N$ and such that

$$B(n, z_{1}, z_{2}) \leq B(n, y_{1}, y_{2}) \quad \text{for} \quad 0 \leq z_{k} \leq y_{k}, \quad k = 1, 2,$$

and

$$B(n, a_{n}z_{1}, a_{n}z_{2}) \leq F(a_{n})B(n, z_{1}, z_{2}) \quad \text{for} \quad a : N \to R_{+}.$$  

If positive sequences $(\mu_{n})$ and $(\rho_{n})$ satisfy the following inequality

$$\mu_{n} \leq \mu_{n_{0}} + c \sum_{j=n_{0}}^{n-1} \rho_{j}B(j, \rho_{j-1}\mu_{j-1}, \rho_{j}\mu_{j}),$$

for $n \geq n_{0}, n_{0} \in N$ and some positive constant $c$, and the series

$$\sum_{j=n_{0}}^{\infty} \rho_{j}B(j, \rho_{j-1}, \rho_{j})$$

is convergent, then there exists a constant $M > 0$ such that $\mu_{n} \leq M$ for $n \in N$.

**Lemma 2.** The equation

$$(E1) \quad \Delta^{2}z_{n} = a_{n+1}z_{n+1}, \quad n \in N$$

where $a : N \to R$, has linearly independent solutions $u, v : N \to R$ such that

$$\begin{vmatrix} u_{n} & v_{n} \\ \Delta u_{n} & \Delta v_{n} \end{vmatrix} = -1, \quad \text{for all} \quad n \in N.$$  

The main result of this paper is the following Theorem.
THEOREM 1. Let \((u_n)\) and \((v_n)\) are linearly independent solutions of equation \((E1)\). Assume also that (1) holds and

\[ |f(n, x_1, x_2)| \leq B(n, |x_1|, |x_2|), \]

for all \(x_1, x_2 \in \mathbb{R}\), and any fixed \(n \in \mathbb{N}\), where \(f : \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}\) and function \(B\) fulfil conditions (2) and (3). Let us denote

\[ U_j = \max \{|u_{j-1}|, |v_{j-1}|, |u_j|, |v_j|, |u_{j+1}|, |v_{j+1}|\}. \]

If

\[ \sum_{j=2}^{\infty} U_j B(j, U_{j-1}, U_j) = K < \infty \]

for some positive constant \(K\), then every solution \((y_n)\) of the equation

\[ \Delta^2 y_n = a_n y_{n+1} + f(n, y_n, y_{n+1}), \quad n \in \mathbb{N} \]

can be written in the form

\[ y_{n+1} = \alpha_n u_n + \beta_n v_n, \]

where \(\lim_{n \to \infty} \alpha_n = \alpha\) and \(\lim_{n \to \infty} \beta_n = \beta\), \((\alpha, \beta)\) - constants.

Proof. First we prove the theorem for two linearly independent solutions \((u_n)\) and \((v_n)\) of equation \((E1)\) which fulfil condition (4). Let \((y_n)\) be an arbitrary solution of equation \((E)\). We denote

\[ A_n = v_n \Delta y_n - y_{n+1} \Delta v_{n-1} \]

\[ B_n = -u_n \Delta y_n + y_{n+1} \Delta u_{n-1}. \]

From (4) we get

\[ y_{n+1} = u_n A_n + v_n B_n. \]

Applying the difference operator \(\Delta\) to (8) and (9) we obtain

\[ \Delta A_n = v_n \Delta^2 y_n - y_{n+1} \Delta^2 v_{n-1} \]

\[ \Delta B_n = -u_n \Delta^2 y_n + y_{n+1} \Delta^2 u_{n-1}. \]
Using \((E1)\) and \((E)\) we have

\[
\Delta A_n = v_n f(n, y_n, y_{n+1})
\]
\[
\Delta B_n = -u_n f(n, y_n, y_{n+1}).
\]

From (10) we obtain

\[
\Delta A_j = v_j f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j)
\]
\[
\Delta B_j = -u_j f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j), \quad j > 1.
\]

By summation the above equalities, we get

\[
A_n = A_2 + \sum_{j=2}^{n-1} v_j f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j)
\]
(11)
\[
B_n = B_2 - \sum_{j=2}^{n-1} u_j f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j).
\]

Then

\[
|A_n| \leq |A_2| + \sum_{j=2}^{n-1} |v_j||f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j)|
\]
\[
|B_n| \leq |B_2| + \sum_{j=2}^{n-1} |u_j||f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j)|.
\]

Therefore, we have

\[
|A_n| + |B_n| \leq |A_2| + |B_2|
\]
(12)
\[
+ \sum_{j=2}^{n-1} (|v_j| + |u_j|)|f(j, u_{j-1}A_{j-1} + v_{j-1}B_{j-1}, u_j A_j + v_j B_j)|.
\]

Let us denote

(13) \quad h_n = |A_n| + |B_n|, \quad n \in N.

By (6) we see that

\[
|v_{j-1}| \leq U_j, \quad |u_{j-1}| \leq U_j, \quad |v_j| \leq U_j, \quad |u_j| \leq U_j, \quad |v_{j+1}| \leq U_j, \quad |u_{j+1}| \leq U_j.
\]

It is clear that

\[
|A_j u_j + B_j v_j| \leq |A_j||u_j| + |B_j||v_j| \leq U_j(|A_j| + |B_j|) \leq U_j h_j.
\]
Hence, by (5) we get

$$|f(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_ju_j + B_jv_j)| \leq B(j, U_{j-1}h_{j-1}, U_jh_j).$$

Therefore, (12) and (13) yields

$$h_n \leq h_2 + 2 \sum_{j=2}^{n-1} U_jB(j, U_{j-1}h_{j-1}, U_jh_j).$$

By Lemma 1, we obtain $h_n \leq M < \infty$.

Properties of function $B$ and (5) give the following inequalities

$$|v_jf(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_ju_j + B_jv_j)|$$

$$\leq U_jB(j, |A_{j-1}u_{j-1} + B_{j-1}v_{j-1}|, |A_ju_j + B_jv_j|)$$

$$\leq U_jB(j, U_{j-1}h_{j-1}, U_jh_j) \leq U_jB(j, U_{j-1}M, U_jM)$$

$$\leq F(M)U_jB(j, U_{j-1}, U_j).$$

This means by (7) that the series

$$\sum_{j=2}^{\infty} v_jf(j, A_{j-1}u_{j-1} + B_{j-1}v_{j-1}, A_ju_j + B_jv_j)$$

is absolutely convergent. Therefore, by (11) the finite limits $\lim_{n \to \infty} A_n = \alpha$ and $\lim_{n \to \infty} B_n = \beta$ exist. Hence, in this case we get the thesis.

Now, we will prove this theorem for any two linearly independent solutions $(u_n)$ and $(v_n)$ of equation (E1). Let $(u_n)$ and $(v_n)$ be two linearly independent solutions of equation (E1) fulfilling condition (4). Then for some constants $c_1$, $c_2$, $c_3$ and $c_4$ we have

$$u_n = c_1\tilde{u}_n + c_2\tilde{v}_n, \quad v_n = c_3\tilde{u}_n + c_4\tilde{v}_n.$$

Now,

$$\tilde{U}_j = \max \{|\tilde{u}_{j-1}|, |\tilde{v}_{j-1}|, |\tilde{u}_j|, |\tilde{v}_j|, |\tilde{u}_{j+1}|, |\tilde{v}_{j+1}|\}.$$

We will show that the condition (7) holds.

Let $\tilde{c} = \max\{|c_1|, |c_2|, |c_3|, |c_4|\}$. Hence

$$U_j \leq \tilde{c} \max \{|\tilde{u}_{j-1}| + |\tilde{v}_{j-1}|, |\tilde{u}_j| + |\tilde{v}_j|, |\tilde{u}_{j+1}| + |\tilde{v}_{j+1}|\} \leq 2\tilde{c}\tilde{U}_j.$$
Therefore, we obtain inequalities
\[ U_j B(j, U_{j-1}, U_j) \leq 2c \tilde{U}_j B(j, 2c \tilde{U}_{j-1}, 2c \tilde{U}_j) \leq 2c \tilde{U}_j F(2c) B(j, \tilde{U}_{j-1}, \tilde{U}_j), \]
and
\[ \sum_{j=2}^{\infty} U_j B(j, U_{j-1}, U_j) < \infty. \]

We see that assumptions of the Theorem 1 hold for solutions \((u_n)\) and \((v_n)\), also. Then a solution of equation \((E)\) can be written in the form
\[
y_{n+1} = A_n (c_1 \tilde{u}_n + c_2 \tilde{v}_n) + B_n (c_3 \tilde{u}_n + c_4 \tilde{v}_n) \\
= (c_1 A_n + c_3 B_n) \tilde{u}_n + (c_2 A_n + c_4 B_n) \tilde{v}_n \\
= \alpha_n \tilde{u}_n + \beta \tilde{v}_n,
\]
where \(\alpha_n = c_1 A_n + c_3 B_n\), \(\beta_n = c_2 A_n + c_4 B_n\), and \(\lim_{n \to \infty} \alpha_n = \alpha\), \(\lim_{n \to \infty} \beta_n = \beta\) (\(\alpha\), \(\beta\)-constants). This completes the proof of this Theorem. \(\square\)

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ON NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL LINEAR CONTACT CONDITIONS

LÁSZLÓ SIMON*

Abstract. The results of [3] by W. Jäger and N. Kutev on a nonlinear elliptic transmission problem are extended (in a modified way) to nonlinear parabolic problems with linear and nonlocal contact conditions.

Key Words. parabolic functional differential equation, nonlocal boundary condition, contact condition

AMS(MOS) subject classification. 35R10

1. Introduction. In [3] W. Jäger and N. Kutev considered the following nonlinear transmission (contact) problem for nonlinear elliptic equations:

\[
\begin{align*}
(1) \quad & \sum_{i=1}^{n} D_i[a_i(x, u, Du)] + b(x, u, Du) = 0 \text{ in } \Omega \\
(2) \quad & u = g \text{ on } \partial \Omega \\
(3) \quad & \left[ \sum_{i=1}^{n} a_i(x, u, Du) \nu_i \right]_{S} = 0 \\
(4) \quad & u_1 = \Phi(u_2) \text{ on } S
\end{align*}
\]

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This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T 031807.

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where \( \Omega \subset R^n \) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \) which is divided into two subdomains \( \Omega_1, \Omega_2 \) by means of a smooth surface \( S \) which has no intersection point with \( \partial \Omega \), the boundary of \( \Omega_1 \) is \( S \) and the boundary of \( \Omega_2 \) is \( S \cup \partial \Omega \). Further, \([f]_S\) denotes the jump of \( f \) on \( S \) in the direction of the normal \( \nu \), \( \Phi \) is a smooth strictly increasing function and \( u_j \) denotes the restriction of \( u \) to \( \Omega_j \) \( (j = 1, 2) \). The coefficients of the equation are smooth in \( \Omega_j \) and satisfy standard conditions but they have jump on the surface \( S \). The problem was motivated, e.g., by reaction-diffusion phenomena in porous medium. The authors formulated conditions which implied comparison principles, existence and uniqueness of the weak and the classical solution, respectively.

The aim of this paper is to consider nonlinear parabolic functional differential equations with a modified contact condition on \( S \): with Dirichlet type nonlocal linear boundary condition, (possibly) containing delay. In [8] we studied parabolic differential equations with contact conditions, considered in [3]. In Section 1 we shall prove existence and uniqueness theorems and in Section 2 we shall formulate a theorem on boundedness of the solutions.

2. Existence and uniqueness of solutions. We shall consider weak solutions of the problem

\[
D_t u_j - \sum_{i=1}^{n} D_i[a_i^j(t, x, u^j, Dw^j)] + b^j(t, x, u^j, Dw^j) + G^j(u_1, u^2) = F(t, x),
\]

\((t, x) \in Q_{T_j}^j = (0, T_j) \times \Omega_j, \quad j = 1, 2\)

\(u = 0 \) on \( \Gamma_{T_j} = [0, T_j] \times \partial \Omega \)

\(u(0, x) = u_0(x), \quad x \in \Omega_1 \cup \Omega_2,\)

where \( u_j = u|_{Q_{T_j}^j} \), \( G^1, G^2 \) are operators (which will be defined below), \( a_i^j, b^j \) have certain polynomial growth in \( u^j, Dw^j \). The (contact) boundary condition on \( S_{T_j} = [0, T_j] \times S \) will be formulated later.

Let \( p \geq 2 \) be a real number. For any domain \( \Omega_0 \subset R^n \) denote by \( W^{1,p}(\Omega_0) \) the usual Sobolev space of real valued functions with the norm

\[ ||u|| = \left[ \int_{\Omega_0} (|Du|^p + |u|^p) \right]^{1/p}. \]
Let $V_1 = W^{1,p}(\Omega_1)$, $V_2 = \{ w \in W^{1,p}(\Omega_2) : w|_{\partial \Omega} = 0 \}$ and $V = V_1 \times V_2$. Further, let $V_0$ be a closed linear subspace of $V$, containing $W^{1,p}_0(\Omega_1) \times W^{1,p}_0(\Omega_2)$ where $W^{1,p}_0(\Omega_j)$ is the closure of $C_c^\infty(\Omega_j)$ with respect to the norm of $W^{1,p}(\Omega_j)$. Denote by $X = X_T = L^p(0,T;V_0)$ the Banach space of the set of measurable functions $u = (u^1, u^2) : (0,T) \to V_0$ such that $\| u \|^p$ is integrable and define the norm by

$$\| u \|^p_{L^p(0,T;V_0)} = \int_0^T \| u(t) \|^p_{V_0} \, dt.$$ 

The dual space of $L^p(0,T;V_0)$ is $X_T^* = L^q(0,T;V_0^*)$ where $1/p + 1/q = 1$ and $V_0^*$ is the dual space of $V_0$ (see, e.g., [4], [9]). Since for $p \geq 2$ $L^p(\Omega_j) \subset L^2(\Omega_j)$, $u \in X$, $u' = D_t u \in X^*$ imply $u \in C([0,T];H)$ with $H = L^2(\Omega_1) \times L^2(\Omega_2)$ (see, e.g., [9]), so $u(0)$ makes sense.

Define operator $L$ by

$$D(L) = \{ v \in X_T : v' \in X_T^*, \ v(0) = 0 \}, \quad Lv = v'.$$

Then $L$ is a maximal monotone, continuous linear operator from $X = X_T$ into $X^* = X_T^*$. Assume that $A : X \to X^*$ is a bounded (nonlinear) operator which is demicontinuous, i.e. if $(u_k) \to u$ in the norm of $X$ then $(A(u_k)) \to A(u)$ weakly in $X^*$ and pseudomonotone with respect to $D(L)$, i.e. if $(u_k) \to u$ weakly in $X$, $u_k, u \in D(L)$ and

$$(Lu_k) \to Lu \text{ weakly in } X^*, \quad \limsup_{k \to \infty} [A(u_k), u_k - u] \leq 0$$

then

$$\lim_{k \to \infty} [A(u_k), u_k - u] = 0 \text{ and } (Au_k) \to Au \text{ weakly in } X^*.$$ 

Further, assume that $A$ is coercive:

$$\lim_{\| u \| \to \infty} \frac{[A(u), u]}{\| u \|_X} = +\infty.$$ 

According to [2] the following theorem holds:

**Theorem 1.** Assume that $A : X \to X^*$ satisfies the above conditions. Then for any $F \in X^*$ there exists $u \in X$ such that

$$[Lu, v] + [A(u), v] = [F, v] \text{ for each } v \in X.$$
Now we formulate the contact condition on $S_{T_2}$. Let $T_1 \leq T_2$, $\psi : [0, T_2] \to [0, T_1]$ be a $C^1$ function satisfying:

$$\psi' > 0, \quad 0 \leq \psi(t) \leq t, \quad \psi(0) = 0, \quad \psi(T_2) = T_1$$

and define

$$\tilde{u}^1(\tau, x) = u^1_\psi(\tau, x) = u^1(\psi(\tau), x), \quad \tau \in [0, T_2], \quad x \in \Omega_j, \quad \tilde{u}^2(\tau, x) = u^2(\tau, x).$$

The contact boundary condition on $S_{T_2}$ is given by

$$(9) \quad (u^1_\psi, u^2) \in L^p(0, T_2; V_0).$$

(Later we shall formulate examples for (9).) By using the notation $t = \psi(\tau)$, we obtain that $u$ satisfies (5) in $Q^j_{T_1}$ (in classical sense) iff $\tilde{u}$ satisfies

$$(10) \quad D_\tau \tilde{u}^j(\tau, x) - \sum_{i=1}^n D_i[\tilde{a}^1_i(\tau, x, \tilde{u}^j(\tau, x), D\tilde{u}^j(\tau, x))] + \tilde{b}^j(\tau, x, \tilde{u}^j(\tau, x), D\tilde{u}^j(\tau, x)) + \tilde{G}^j(\tilde{u}^1, \tilde{u}^2) = \tilde{F}(\tau, x), \quad (\tau, x) \in (0, T_2) \times \Omega_j$$

where

$$\tilde{a}^1_i(\tau, x, \tilde{u}^1(\tau, x), D\tilde{u}^1(\tau, x)) = \psi'(\tau)a^1_i(\psi(\tau), x, \tilde{u}^1(\tau, x), D\tilde{u}^1(\tau, x)),$$

$$\tilde{b}^j(\tau, x, \tilde{u}^j(\tau, x), D\tilde{u}^j(\tau, x)) = \psi'(\tau)b^j(\psi(\tau), x, \tilde{u}^1(\tau, x), D\tilde{u}^1(\tau, x)),$$

and $\tilde{a}^2_i = a^1_i$, $\tilde{b}^2 = b^2$,

$$[\tilde{G}^j(\tilde{u}^1, \tilde{u}^2)](\tau, x) = \psi'(\tau)[G^1(u^1, u^2)](\psi(\tau), x), \quad \tilde{G}^2 = G^2.$$

Now we formulate the conditions with respect to the problem (5) - (7), (9) and the existence theorem on the weak solutions of this problem.

Assume that

(I) The functions $a^j_i, b^j : Q^j_{T_2} \times R^{n+1} \to R$ satisfy the Carathéodory conditions, i.e. $a^j_i(t, x, \eta, \zeta)$, $b^j(t, x, \eta, \zeta)$ are measurable in $(t, x) \in Q^j_{T_2} = (0, T) \times \Omega_j$ for each fixed $(\eta, \zeta) \in R^{n+1}$ and they are continuous in $(\eta, \zeta) \in R^{n+1}$ for a.e. $(t, x) \in Q^j_{T_2}$. 

(II) \(|\alpha_i^j(t, x, \eta, \zeta)| \leq c_1[|\eta|^{p-1} + |\zeta|^{p-1}] + k_1^j(x), \) for a.e. \((t, x) \in Q_T^j, \) each \((\eta, \zeta) \in R^{n+1}\) with some constant \(c_1\) and a function \(k_1^j \in L^q(\Omega_j),\)

\(|b^j(t, x, \eta, \zeta)| \leq c_1[|\eta|^{p-1} + |\zeta|^{p-1}] + k_1^j(x).\)

(III) \(\sum_{i=1}^{n}[a_i^j(t, x, \eta, \zeta) - a_i^j(t, x, \eta, \zeta^*)](\zeta_i - \zeta_i^*) > 0\) if \(\zeta \neq \zeta^*.\)

(IV) \(\sum_{i=1}^{n}[a_i^j(t, x, \eta, \zeta)\zeta_i + b_i^j(t, x, \eta, \zeta)\eta \geq c_2[|\zeta|^{p} + |\eta|^{p}] - k_2^j(x), (t, x) \in Q_T^j,\)

\(\sum_{i=1}^{n} a_i^j(t, x, \eta, \zeta)\zeta_i + b_i^j(t, x, \eta, \zeta)\eta \geq c_2|\zeta|^{p} - k_2^j(x), (t, x) \in Q_T^{j_2}\)

with some constant \(c_2 > 0, k_2^j \in L^1(\Omega_j).\)

(V) \(G^j : L^p(Q_{T_1}^1) \times L^p(Q_{T_2}^2) \to L^q(Q_T)\) are bounded (nonlinear) operators which are demicontinuous, i.e. \((u_k) \to u\) with respect to the norm \(L^p(Q_{T_1}^1) \times L^p(Q_{T_2}^2)\) implies that \(G^j(u_k) \to G^j(u)\) weakly in \(L^q(Q_T).\)

(VI) \(\lim_{||u|| \to \infty} \frac{||G^j(u)||}{||u||_{L^p(Q_{T_1}^1)} + ||u||_{L^p(Q_{T_1}^1)}} = 0\) for any \(u \in L^p(Q_{T_1}^1) \times L^p(Q_{T_2}^2).\)

Clearly, \(\tilde{\alpha}_i^j, \tilde{b}_i^j, \tilde{G}_i^j\) also satisfy conditions, analogous to (I) - (VI).

Thus we may define the operators \(A^j : L^p(0, T_2; V) \to L^q(0, T_2; V^*)\) by

\([A^j(\bar{u}^j), v^j] = \int_{Q_{T_2}^j} \sum_{i=1}^{n} \tilde{\alpha}_i^j(t, x, \bar{u}^j, D\bar{u}^j)D_i v^j dt dx +\]

\[\int_{Q_{T_2}^j} \tilde{b}_i^j(t, x, \bar{u}^j, D\bar{u}^j) v^j dt dx = \int_{0}^{T_2} \langle A^j(\bar{u}^j)(t), v^j(t) \rangle dt,\]

where \(Q_{T_2}^j = (0, T_2) \times \Omega_j,\)

\[A = (A^1, A^2) : L^p(0, T_2; V) \to L^q(0, T_2; V^*)\]

by

\([A(\bar{u}), v] = [A^1(\bar{u}^1), v^1] + [A^2(\bar{u}^2), v^2],\]

and the operators \(B^j : L^p(0, T_2; V) \to L^q(0, T_2; V^*_j)\) by

\([B^j(\tilde{u}), v^j] = \int_{Q_{T_2}^j} \tilde{G}_i^j(\tilde{u}) v^j dt dx, \quad j = 1, 2,\]

\(\tilde{u} = (\tilde{u}^1, \tilde{u}^2) \in L^p(0, T_2; V),\quad (v^1, v^2) \in L^p(0, T_2; V).\)
By (I), (II), (V), Hölder’s inequality and Vitali’s theorem the operator

\[ A + B = (A^1, A^2) + (B^1, B^2) : L^p(0, T_2; V) \to L^q(0, T_2; V^*) \]

is bounded and demicontinuous. Further, \( A + B \) is pseudomonotone with respect to \( D(L) \). Because, it is well known (see, e.g. [2], [5]) that (I) – (IV) imply: \( A \) is pseudomonotone with respect to \( D(L) \). By using (V) and a well known compact imbedding theorem (see, e.g. [4]), \( A + B \) is pseudomonotone with respect to \( D(L) \), too.

Finally, it is easy to derive from (IV) and (VI) that \( A + B \) is coercive (see, e.g., [5]). Thus we can apply Theorem 1 to the operator \( A + B \) instead of \( A \) and we obtain that there exists a solution \( \tilde{u} \in L^p(0, T_2; V_0) \) of

\[ (11) \quad [L\tilde{u}, v] + [(A + B)(\tilde{u}), v] = [\tilde{F}, v], \quad v \in L^p(0, T_2; V_0). \]

Since \( W^{1,p}_0(\Omega_1) \times W^{1,p}_0(\Omega_2) \subset V_0 \), the solution \( \tilde{u} \) of (11), is obviously a distributional solution of the differential equation (10) and \( u \) is a distributional solution of (5) which satisfies (6) and (7) in (usual) weak sense. Further, it is easy to show that if the solution \( \tilde{u} \) of (11) is sufficiently smooth then

\[ (12) \quad \sum_{i=1}^{n} \int_{S_{T_2}} \tilde{a}_i^2(t, x, \tilde{u}^2, D\tilde{u}^2) \nu_i v_2 d\sigma - \sum_{i=1}^{n} \int_{S_{T_2}} \tilde{a}_i^1(t, x, \tilde{u}^1, D\tilde{u}^1) \nu_i v_1 d\sigma = 0 \]

for each \((v_1, v_2) \in V_0 \) where \( \nu = (\nu_1, ..., \nu_n) \) is the normal vector to \( S_{T_2} \), i.e.

\[ \left( \sum_{i=1}^{n} \tilde{a}_i^2(t, x, \tilde{u}^2, D\tilde{u}^2) \nu_i, - \sum_{i=1}^{n} \tilde{a}_i^1(t, x, \tilde{u}^1, D\tilde{u}^1) \nu_i \right) \]

is orthogonal in \( L^2(S_{T_2}) \) to \((v_1, v_2) \in V_0 \). Therefore, it is natural

**DEFINITION 1.** If \( \tilde{u} \in L^p(0, T_2; V_0) \) is a solution of (11) then \( u \) is called a weak solution of the problem (5) - (7), with the contact condition (9), (12).

So we have

**THEOREM 2.** Assume (I) – (VI). Then for any

\[ F = (F^1, F^2) \in L^q(0, T; V^*) \]

there exists a weak solution \( u \) of (5) - (7), (9), (12).

**Examples for \( V_0 \)**

(a)

\[ V_0 = \{(w_1, w_2) \in V : w_1(x) = \int_S a(x, z) w_2(z) dz, \quad x \in S\} \]
where $a$ is a given $L^\infty$ function and the traces of $w_1, w_2$ on $S$ are considered.

(b)  
$$V_0 = \{(w_1, w_2) \in V : w_1(x) = a(x)w_2(\varphi(x)), \quad x \in S\}$$

where $a \in L^\infty$, $\varphi : S \to S$ is a sufficiently smooth bijection.

Examples for $G^j$ satisfying (V) and (VI)

(c) Let

$$[G^1(u)](t, x) = \gamma^1(t, x, u^1(\chi_1(t), x), \int_{\Omega_1} d^2(y)u^2(\chi_2(t), y)dy), \quad (t, x) \in Q^1_{T_1},$$

$$[G^2(u)](t, x) = \gamma^2(t, x, \int_{\Omega_1} d^1(y)u^1(\chi_1(t), y)dy, u^2(\chi_2(t), x)), \quad (t, x) \in Q^2_{T_2},$$

where $\chi_1, \chi_2$ are $C^1$ functions satisfying $\chi'_j > 0$, $0 \leq \chi_1(t) \leq t$; $d^1, d^2$ are $L^\infty$ functions; the functions $\gamma^j$ satisfy the Carathéodory conditions and

$$|\gamma_j(t, x, \theta_1, \theta_2)| \leq c^j(\theta_1, \theta_2)|(|\theta_1, \theta_2)|^{p-1} + k^j_1(x)$$

with continuous functions $c^j$ having the property

$$\lim_{|(\theta_1, \theta_2)| \to \infty} c^j = 0, \quad k^j_1 \in L^q(\Omega_j).$$

By using Hölder’s inequality and Vitali’s theorem it is not difficult to prove that condition (V) and (VI) are fulfilled (see [5], [6]).

(d) Similarly operators

$$[G^1(u)](t, x) = \int_0^t \gamma^1(t, \tau, x, u^1(\tau, x), \int_{\Omega_2} d^1(y)u^2(\tau, y)dy) \, d\tau, \quad (t, x) \in Q^1_{T_1},$$

$$[G^2(u)](t, x) = \int_0^t \gamma^2(t, \tau, x, \int_{\Omega_1} d^1(y)u^1(\tau, y)dy, u^2(\tau, x)) \, d\tau, \quad (t, x) \in Q^2_{T_2},$$

can be considered, where $\gamma^j$ satisfy analogous conditions to that of example (c).

By using monotonicity arguments one can prove uniqueness of the solution.

**Theorem 3.** Assume that

$$\sum_{i=1}^n [a^i(t, x, \eta, \zeta) - a^i(t, x, \eta^*, \zeta^*)](\zeta_i - \zeta_i^*) +$$

(13)
\[ b^j(t, x, \eta, \zeta) - b^j(t, x, \eta^*, \zeta^*) \langle \eta - \eta^* \rangle \geq -c_0(\eta - \eta^*)^2 \]

with some constant \( c_0 \). Further, for the operators

\[ [\tilde{G}^j(\tilde{u})](t, x) = e^{-\alpha t}[G^j(e^{\alpha t}\tilde{u})](t, x), \]

the inequality

\[ (14) \quad \| \tilde{G}^j(\tilde{u}) - \tilde{G}^j(\tilde{v}) \|_{L^2(Q_T^j)} \leq \tilde{c} \| \tilde{u} - \tilde{v} \|_{L^2(Q_T^j) \times L^2(Q_T^j)} \]

holds where the constant \( \tilde{c} \) does not depend on the positive number \( \alpha \) and \( \tilde{u}, \tilde{v} \). Then the problem (5) - (7), (9), (12) may have at most one solution.

**Remark 1.** It is easy to show that (14) holds for the above examples (c) and (d) if functions \( \tilde{\gamma}^j \) satisfy (global) Lipschitz condition with respect to \( \theta_1 \) and \( \theta_2 \).

### 3. Boundedness of the solutions

One can prove an existence theorem also for the interval \((0, \infty)\). Denote by \( X_\infty \) and \( X_\infty^* \) the set of functions

\[ u : (0, \infty) \to V_0, \quad w : (0, \infty) \to V_0^*, \]

respectively, such that (for their restrictions to \((0, T)\))

\[ u \in L^p(0, T; V_0), \quad w \in L^q(0, T; V_0^*) \]

for any finite \( T > 0 \). Further, let \( Q_T^j = (0, \infty) \times \Omega_j \). \( L^p_{\text{loc}}(Q_T^j) \) will denote the set of functions \( v^j : Q_T^j \to \mathbb{R} \) such that \( v^j|_{Q_T^j} \in L^p(Q_T^j) \) for any finite \( T \).

**Theorem 4.** Let \( \psi : [0, \infty) \to [0, \infty) \) be a \( C^1 \) function satisfying

\[ \psi' > 0, \quad 0 \leq \psi(t) \leq t, \quad \psi(0) = 0, \quad \lim_{\infty} \psi = +\infty. \]

Assume that we have functions \( a_t^j, b^j : Q_T^j \times \mathbb{R}^{n+1} \to \mathbb{R} \) such that assumptions (I) - (IV) are satisfied for any finite \( T_2, T_1 = \psi(T_2) \) with the same constants \( c_j \) and functions \( k_t^j \). Further, operators \( G^j : L^p_{\text{loc}}(Q_T^j) \times L^p_{\text{loc}}(Q_T^j) \to L^q_{\text{loc}}(Q_T^j) \) are such that their restrictions to \( Q_T^j \) satisfy (V), (VI). Assume that \( G^j \) are of Volterra type, which means that \([G^j(u)](t, x)\), depends only on the restrictions of \( u^j \) to \((0, t) \times \Omega_j \) \((j = 1, 2)\). Then for any \( F \in X_\infty^* \) there exists \( u \in X_\infty \) such that the statement of Theorem 2 holds for the restrictions of \( u^j \) to \( Q_T^j \).
Remark 2. The assumptions of Theorem 4 imply that if \( u \) is a solution in \( [0, T_1], [0, T_2] \) with \( T_1 = \psi(T_2) \) and \( T'_2 < T_2 \) then \( u \) is a solution in \( [0, T'_1], [0, T'_2] \) with \( T'_1 = \psi(T'_2) \). By using this fact, the proof is standard (see, e.g., [6]).

Define function \( y \) by

\[
y(t) = \| \hat{u}(t) \|^2_{L^2(\Omega_1) \times L^2(\Omega_2)}.
\]

If some additional conditions are satisfied then one can prove that \( y \) is bounded in \( (0, \infty) \) for a solution \( u \).

**Theorem 5.** Let the assumptions of Theorem 4 be satisfied and assume that \( p > 2 \),

\[
\| F(t) \|_{V^*} \text{ is bounded}, t \in [0, \infty),
\]

for arbitrary \( \hat{u} \in X_\infty \)

\[
\int_{\Omega_i} |\hat{G}^j(\hat{u})(t, x)|^q \, dx \leq c_4 \sup_{[0, t]} y + c_5(t) \sup_{[0, t]} y^{p/2} + c_6
\]

where \( c_4, c_6 \) are constants and \( c_5 \) is a continuous function with \( \lim_{\infty} c_5 = 0 \). Then \( y \) is bounded in \( [0, \infty) \) for a solution \( u \). Further,

\[
\int_{t_1}^{t_2} \| \hat{u}(t) \|_{V_0}^p \, dt \leq c'(t_1 - t_2) + c'' \quad 0 < t_1 < t_2
\]

with some constants \( c', c'' \), not depending on \( t_1, t_2 \).

**Proof.** Applying (11) to \( v = \hat{u} = (\hat{u}^1, \hat{u}^2) \) with arbitrary \( t_1 < t_2 \) we obtain

\[
\int_{t_1}^{t_2} \langle D_t \hat{u}^j(t), \hat{u}^j(t) \rangle \, dt + \int_{t_1}^{t_2} \langle [A^j(\hat{u}^j)](t), \hat{u}^j(t) \rangle \, dt + \\
\int_{t_1}^{t_2} \langle [B^j(\hat{u}^1, \hat{u}^2)](t), \hat{u}^j(t) \rangle \, dt = \int_{t_1}^{t_2} \langle F^j(t), \hat{u}^j(t) \rangle \, dt.
\]

Since \( y \) is absolutely continuous and

\[
y'(t) = 2\langle D_t \hat{u}^1(t), \hat{u}^1(t) \rangle + 2\langle D_t \hat{u}^2(t), \hat{u}^2(t) \rangle
\]

(see, e.g., [9]), by using assumption IV, (15), (16), Young’s inequality and Hölder’s inequality, we obtain from (18) the inequality

\[
y(t_2) - y(t_1) + c_3^* \int_{t_1}^{t_2} [y(t)]^{p/2} \, dt \leq c_4^* \int_{t_1}^{t_2} \left[ \sup_{[0, t]} y + c_5(t) \sup_{[0, t]} y^{p/2} + 1 \right] \, dt
\]
where \( c_3^* > 0, c_4^* \) are constants. It is not difficult to show that (19) and \( p > 2 \) imply the boundedness of \( y \) and (17).

**Remark 3.** The estimation (16) is fulfilled for \( G^j \), e.g. if \( G^j \) is given in examples (c) or (d) and the functions \( \gamma^j \) satisfy

\[
|\gamma^1(t, x, \theta_1, \theta_2)|^q, \quad |\gamma^1(t, \tau, x, \theta_1, \theta_2)|^q \leq c_5^*(\theta_1^2 + \theta_2^2) + c_6^*(t)|\theta_2|^p + c_7^*,
\]

\[
|\gamma^2(t, x, \theta_1, \theta_2)|^q, \quad |\gamma^2(t, \tau, x, \theta_1, \theta_2)|^q \leq c_5^*(\theta_1^2 + \theta_2^2) + c_6^*(t)|\theta_1|^p + c_7^*,
\]

respectively, with some constants \( c_5^*, c_7^* \), \( \lim_{\infty} c_6^* = 0 \) and there is a positive number \( \rho \) such that

\[
\gamma^j(t, \tau, x, \theta_1, \theta_2) = 0 \text{ if } \tau \leq t - \rho,
\]

\( \psi' \) is bounded in \([0, \infty)\).

**Remark 4.** By using monotonicity arguments, similarly to Theorem 5, one can prove a stabilization result. (See [7].)

**REFERENCES**


Abstract. Consider the first-order linear delay differential equation
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1) \]
the (discrete analogue) difference equation
\[ x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \quad (1)' \]
and the functional equation
\[ x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq t_0. \quad (1)'' \]

The most interesting oscillation criteria for Eq.(1) especially in the case where
\[ 0 < \liminf_{t \to \infty} \int_{r(t)}^{t} p(s)ds < \frac{1}{e} \quad \text{and} \quad \limsup_{t \to \infty} \int_{t-r}^{t} p(s)ds < 1, \]
(the discrete analogues) for Eq.(1)' when
\[ \liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n \to \infty} \sum_{i=n-k}^{n} p_i < 1, \]
and (the analogues) for Eq.(1)'' when
\[ 0 < \liminf_{t \to \infty} \{Q(t)P(g(t))\} \leq \frac{1}{4} \quad \text{and} \quad \limsup_{t \to \infty} \{Q(t)P(g(t))\} < 1, \]
are presented.

Key Words. Oscillation; delay differential equations, difference, functional equations.

AMS(MOS) subject classification. Primary 34K11; Secondary 39A11, 39A12, 39B22, 34C10.
1. **Introduction.** The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1) \]

where the functions \( p, \tau \in C([t_0, \infty), \mathbb{R}^+ ) \) (here \( \mathbb{R}^+ = [0, \infty) \)), \( \tau(t) \) is non-decreasing , \( \tau(t) < t \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} \tau(t) = \infty \), has been the subject of many investigations. See, for example, [7, 11, 13, 17-22, 24, 26-29, 30-39, 41, 44-49, 51, 52, 55, 57, 58, 67, 74-80, 84-86, 94] and the references cited therein.

By a solution of Eq.(1) we understand a continuously differentiable function defined on \([\tau(T_0), \infty)\) for some \( T_0 \geq t_0 \) and such that (1) is satisfied for \( t \geq T_0 \). Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

The oscillation theory of the (discrete analogue) delay difference equation

\[ x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \quad (1)' \]

where \( \{p_n\} \) is a sequence of nonnegative real numbers and \( k \) is a positive integer, has also attracted growing attention in the last decades. The reader is referred to [2-6, 8 ,9, 12, 14-16, 40, 43, 50, 53, 54, 60, 61, 64-66, 68-73, 81, 82, 87-91] and the references cited therein.

By a solution of Eq.(1)' we mean a sequence \( \{x_n\} \) which is defined for \( n \geq -k \) and which satisfies (1)' for \( n \geq 0 \). A solution \( \{x_n\} \) of (1)' is said to be oscillatory if the terms \( x_n \) of the solution are not eventually positive or eventually negative. Otherwise the solution is called nonoscillatory.

The oscillation theory of functional equations of the form

\[ x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq t_0, \quad (1)'' \]

where \( P, Q, g \in C([t_0, \infty), \mathbb{R}^+) \) are given real valued functions, \( x \) is an unknown real valued function, \( g(t) \neq t \) for \( t \geq t_0 \), \( \lim_{t \to \infty} g(t) = \infty \), and \( g^m \) denotes the m-th iterate of the function \( g \), i.e.,

\[ g^0(t) = t, \quad g^m(t) = g(g^{m-1}(t)), \quad t \geq t_0, \quad m = 1, 2, ..., \]

has also been developed in the recent few years. We refer to [10, 25, 42, 56, 59, 83, 92, 93].

By a solution of Eq.(1)'' we mean a real valued function \( x : [t_0, \infty) \to \mathbb{R} \) such that \( \sup \{|x(s)| : s \geq t^*\} > 0 \) for any \( t^* \geq t_0 \) and \( x \) satisfies (1)'' on \([t_0, \infty)\).
In this paper our main purpose is to present the state of the art on the oscillation of all solutions to Eq.(1) especially in the case where

$$0 < \liminf_{t \to \infty} \int_{\tau(t)}^t p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t \to \infty} \int_{t-\tau}^t p(s)ds < 1,$$

(the discrete analogues) for Eq.(1)' when

$$\liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n \to \infty} \sum_{i=n-k}^{n} p_i < 1,$$

and (the analogues) for Eq.(1)'' when

$$0 < \liminf_{t \to \infty} \left\{Q(t)P(g(t))\right\} \leq \frac{1}{4} \quad \text{and} \quad \limsup_{t \to \infty} \left\{Q(t)P(g(t))\right\} < 1.$$

2. Oscillation Criteria for Eq. (1). In this section we study the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \quad (1)$$

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [55] he proved that every solution of Eq.(1) oscillates if

$$\limsup_{t \to \infty} [t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t \to \infty} [t - \tau(t)] \liminf_{t \to \infty} p(t) > \frac{1}{e}. \quad (C_1)$$

In 1972, Ladas, Lakshmikantham and Papadakis [41] proved that the same conclusion holds if

$$A := \limsup_{t \to \infty} \int_{\tau(t)}^t p(s)ds > 1. \quad (C_2)$$

In 1979, Ladas [39] established integral conditions for the oscillation of Eq.(1) with constant delay. Tomaras [78-80] extended this result to Eq.(1) with variable delay. For related results see Ladde [46-48]. The following most general result is due to Koplatadze and Canturija [34].

If

$$\alpha := \liminf_{t \to \infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{e}, \quad (C_3)$$
then all solutions of Eq. (1) oscillate; If
\[
\limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) ds < \frac{1}{e}, \quad (N_1)
\]
then Eq. (1) has a nonoscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [45] and in 1984 Fukagai and Kusano [24] established oscillation criteria (of the type of conditions \((C_2)\) and \((C_3)\)) for Eq. (1) with oscillating coefficient \(p(t)\).

It is obvious that there is a gap between the conditions \((C_2)\) and \((C_3)\) when the limit \(\lim_{t \to \infty} \int_{\tau(t)}^{t} p(s) ds\) does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [22] developed new oscillation criteria by employing the upper bound of the ratio \(x(\tau(t))/x(t)\) for possible nonoscillatory solutions \(x(t)\) of Eq. (1). Their result says that all the solutions of Eq. (1) are oscillatory, if \(0 < \alpha \leq \frac{1}{e}\) and
\[
A > 1 - \frac{\alpha^2}{4}. \quad (C_4)
\]
Since then several authors tried to obtain better results by improving the upper bound for \(x(\tau(t))/x(t)\).

In 1991, Jian [32] derived the condition
\[
A > 1 - \frac{\alpha^2}{2(1 - \alpha)}, \quad (C_5)
\]
while in 1992, Yu and Wang [85] and Yu, Wang, Zhang and Qian [86] obtained the condition
\[
A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \quad (C_6)
\]

In 1990, Elbert and Stavroulakis [19] and in 1991 Kwong [38], using different techniques, improved \((C_4)\), in the case where \(0 < \alpha \leq \frac{1}{e}\), to the conditions
\[
A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \quad (C_7)
\]
and
\[
A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \quad (C_8)
\]
respectively, where \( \lambda_1 \) is the smaller real root of the equation \( \lambda = e^{\alpha \lambda} \).

In 1994, Koplatadze and Kvinikadze [35] improved (C6), while in 1998, Philos and Sficas [57] and in 1999, Zhou and Yu [93] and Jaros and Stavroulakis [31] derived the conditions

\[
A > 1 - \frac{\alpha^2}{2(1 - \alpha)} - \frac{\alpha^2}{2} \lambda_1, \quad (C_9)
\]

\[
A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \quad (C_{10})
\]

and

\[
A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \quad (C_{11})
\]

respectively.

Consider Eq.(1) and assume that \( \tau(t) \) is continuously differentiable and that there exists \( \theta > 0 \) such that \( p(\tau(t))\tau'(t) \geq \theta p(t) \) eventually for all \( t \). Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [33] and in 2003, Sficas and Stavroulakis [58] established the conditions

\[
A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4\Theta}}{2} \quad (2.1)
\]

and

\[
A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta \lambda_1 M}}{\theta \lambda_1} \quad (2.2)
\]

respectively, where \( \Theta = \frac{e^{\lambda_1 \theta\alpha - \lambda_1 \alpha - 1}}{(\lambda_1 \theta)^2} \) and \( M = \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4\Theta}}{2} \).

**Remark 2.1.** ([33], [58]) Observe that when \( \theta = 1 \), then \( \Theta = \frac{\lambda_1 - \lambda_1 \alpha - 1}{\lambda_1^2} \), and (2.1) reduces to

\[
A > 2\alpha + \frac{2}{\lambda_1} - 1, \quad (C_{12})
\]

while in this case it follows that \( M = 1 - \alpha - \frac{1}{\lambda_1} \) and (2.2) reduces to

\[
A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha \lambda_1}}{\lambda_1}, \quad (C_{13})
\]
In the case where $\alpha = \frac{1}{e}$, then $\lambda_1 = e$, and $(C_{13})$ leads to

$$A > \frac{\sqrt{7} - 2e}{e} \approx 0.459987065.$$  

It is to be noted that as $\alpha \to 0$, then all the previous conditions $(C_4) - (C_{12})$ reduce to the condition $(C_2)$, i.e.

$$A > 1.$$  

However, the condition $(C_{13})$ leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover $(C_{13})$ improves all the above conditions when $0 < \alpha \leq \frac{1}{e}$ as well. Note that the value of the lower bound on $A$ can not be less than $1 \approx 0.367879441$. Thus the aim is to establish a condition which leads to a value as close as possible to $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on $A$ under these conditions when $\alpha = \frac{1}{e}$.

\begin{align*}
(C_4): & \quad 0.966166179 \\
(C_5): & \quad 0.892951367 \\
(C_6): & \quad 0.863457014 \\
(C_7): & \quad 0.845181878 \\
(C_8): & \quad 0.735758882 \\
(C_9): & \quad 0.709011646 \\
(C_{10}): & \quad 0.708638892 \\
(C_{11}): & \quad 0.599215896 \\
(C_{12}): & \quad 0.471517764 \\
(C_{13}): & \quad 0.459987065
\end{align*}

We see that the condition $(C_{13})$ essentially improves all the known results in the literature.

Example 2.1 ([58]) Consider the delay differential equation

$$x'(t) + px(t - q \sin^2 \sqrt{t} - \frac{1}{pe}) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \lim \inf_{t \to \infty} \int_{\tau(t)}^{t} pds = \lim \inf_{t \to \infty} p(q \sin^2 \sqrt{t} + \frac{1}{pe}) = \frac{1}{e}.$$
and
\[ A = \limsup_{t \to \infty} \int_{\tau(t)}^t p(s) ds = \limsup_{t \to \infty} p(q \sin^2 \sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46. \]

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions \((C_4)-(C_{12})\) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where
\[ \int_{\tau(t)}^t p(s) ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t \to \infty} \int_{\tau(t)}^t p(s) ds = \frac{1}{e} \]
this problem has been studied in 1995, by Elbert and Stavroulakis [20], by Kozakiewicz [36], Li [51,52] and in 1996, by Domshlak and Stavroulakis [18].

3. Oscillation Criteria for Eq. \((1)'\). In this section we study the difference equation
\[ x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \ldots \quad (1)' \]

In 1981, Domshlak [8] was the first who studied this problem in the case where \(k = 1\). Then, in 1989, Erbe and Zhang [23] established the following oscillation criteria for Eq.\((1)'\). Either one of the conditions
\[ \beta := \liminf_{n \to \infty} p_n > 0 \quad \text{and} \quad \limsup_{n \to \infty} p_n > 1 - \beta, \quad (D_1) \]
\[ \liminf_{n \to \infty} p_n > \frac{k^k}{(k + 1)^{k+1}}, \quad (D_2) \]
or
\[ A := \limsup_{n \to \infty} \sum_{i=n-k}^n p_i > 1, \quad (C_2)' \]
implies that all solutions of Eq.\((1)'\) oscillate.

In the same year 1989, Ladas, Philos and Sficas [43] derived the condition
\[ \liminf_{n \to \infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k + 1)^{k+1}}, \quad (C_3)' \]
Therefore they improved condition (D2) by replacing the $p_n$ of (D2) by the arithmetic mean of the terms $p_{n-k}, \ldots, p_{n-1}$ in (C3)'.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (D2) and (C3)' it should be emphasized that, as it is shown in [23], if

$$\sup p_n < \frac{k^k}{(k+1)^{k+1}} \quad (N_1)'$$

then Eq.(1)' has a nonoscillatory solution. Note that (C2)' is the discrete analogue of (C2) and that $\frac{k^{k+1}}{(k+1)^{k+1}} \to \frac{1}{e}$ as $k \to \infty$ and therefore (C3)' may also be seen as the discrete analogue of (C3).

In 1990, Ladas [40] conjectured that Eq.(1)' has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \leq \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Yu, Zhang and Wang [88].

It is interesting to establish sufficient conditions for the oscillation of all solutions of Eq.(1)' when (C2)' and (C3)' are not satisfied. (See previous section 2).

In 1993, Yu, Zhang and Qian [87] and Lalli and Zhang [50], trying to improve (C2)', established the following (false) sufficient oscillation conditions for Eq.(1)'

$$0 < a := \liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i \leq \left( \frac{k}{k+1} \right)^{k+1} \quad \text{and} \quad A > 1 - \frac{a^2}{4} \quad (F_1)$$

and

$$\sum_{i=n-k}^{n} p_i \geq d > 0 \text{ for large } n \quad \text{and} \quad A > 1 - \frac{d^4}{8} \left( 1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}} \right)^{-1} \quad (F_2)$$

respectively.

Unfortunately, the above conditions (F_1) and (F_2) are not correct. This is due to the fact that they are based on the following (false) discrete version of Koplatadze-Chanturija Lemma. (See [14] and [6]).
Lemma A (False). Assume that \( \{x_n\} \) is an eventually positive solution of Eq. (1)' and

\[
\sum_{i=n-k}^{n} p_i \geq M > 0 \quad \text{for large } n. \tag{3.1}
\]

Then

\[
x_n > \frac{M^2}{4} x_{n-k} \quad \text{for large } n.
\]

As one can see, the erroneous proof of Lemma A is based on the following (false) statement. (See [14] and [6]).

Statement A (False). If (3.1) holds, then for any large \( N \), there exists a positive integer \( n \) such that \( n - k \leq N \leq n \) and

\[
\sum_{i=n-k}^{N} p_i \geq \frac{M}{2}, \quad \sum_{i=N}^{n} p_i \geq \frac{M}{2}. 
\]

It is obvious that all the oscillation results which have made use of the above Lemma A or Statement A are incorrect. For details on this problem see the paper by Cheng and Zhang [6].

Here it should be pointed out that the following statement (see [43], [64]) is correct and it should not be confused with the Statement A.

Statement 3.1. ([43], [64]) If

\[
\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \text{for large } n, \tag{3.2}
\]

then for any large \( n \), there exists a positive integer \( n^* \) with \( n - k \leq n^* \leq n \) such that

\[
\sum_{i=n-k}^{n^*} p_i \geq \frac{M}{2}, \quad \sum_{i=n^*}^{n} p_i \geq \frac{M}{2}. 
\]

In 1995, Stavroulakis [64], based on this correct Statement 3.1, proved the following theorem.

Theorem 3.1. ([64]) Assume that \( 0 < a \leq \left( \frac{k}{k+1} \right)^{k+1} \) and

\[
\limsup_{n \to \infty} p_n > 1 - \frac{a^2}{4}. \tag{D_3}
\]
Then all solutions of Eq. (1)' oscillate.

In 1999, Domshlak [14] and in 2000, Cheng and Zhang [6] established the following lemmas, respectively, which may be looked upon as (exact) discrete versions of Koplatadze-Chanturija Lemma.

**Lemma 3.1.** ([14]) Assume that \( \{x_n\} \) is an eventually positive solution of Eq.(1)' and condition (3.2) holds. Then

\[
x_n > \frac{M^2}{4} x_{n-k} \quad \text{for large } n.
\]

**Lemma 3.2.** ([6]) Under the assumptions of Lemma 3.1

\[
x_n > M^k x_{n-k} \quad \text{for large } n.
\]

Based on these lemmas the following theorem was established in [65].

**Theorem 3.2.** ([65]) Assume that \( 0 < a \leq \left(\frac{k}{k+1}\right)^{k+1} \). Then either one of the conditions

\[
\limsup_{n \to \infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{a^2}{4} \quad (C_4)'
\]

or

\[
\limsup_{n \to \infty} \sum_{i=n-k}^{n-1} p_i > 1 - a^k \quad (C_4)'
\]

implies that all solutions of Eq. (1)' oscillate.

**Remark 3.1.** ([65]) From the above theorem it is now clear that

\[
0 < a := \liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n \to \infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{a^2}{4}
\]

is the correct oscillation condition (the discrete analogue of \((C_4)\)), by which the (false) condition \((F_1)\) should be replaced.

**Remark 3.2.** ([65]) Observe the following:

(i) When \( k = 1, 2, \)

\[
a^k > \frac{a^2}{4},
\]
(since, from the above mentioned conditions, it makes sense to investigate
the case when $a < \left( \frac{k}{k+1} \right)^{k+1}$) and therefore condition $(C_4)'$ implies $(C_4)'$.

(ii) When $k = 3$,
\[ a^3 > \frac{a^2}{4} \text{ when } a > \frac{1}{4} \text{ while } a^3 < \frac{a^2}{4} \text{ when } a < \frac{1}{4}. \]
So in this case the conditions $(C_4)'$ and $(C_4)'$ are independent.

(iii) When $k \geq 4$,
\[ a^k < \frac{a^2}{4}, \]
and therefore condition $(C_4)'$ implies $(C_4)'$.

(iv) When $k < 12$ condition $(C_4)'$ or $(C_4)'$ implies $(C_2)'$.

(v) When $k \geq 12$ condition $(C_4)'$ may hold but condition $(C_2)'$ may not
hold.

We illustrate these by the following examples.

Example 3.1. ([65]) Consider the equation

\[ x_{n+1} - x_n + p_n x_{n-3} = 0, \quad n = 0, 1, 2, \ldots, \]

where
\[ p_{2n} = \frac{1}{10}, \quad p_{2n+1} = \frac{1}{10} + \frac{64}{95} \sin^2 \frac{n\pi}{2}, \quad n = 0, 1, 2, \ldots. \]

Here $k = 3$ and it is easy to see that
\[ a = \lim \inf_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} < \left( \frac{3}{4} \right)^4 \]
and
\[ \lim \sup_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} + \frac{64}{95} > 1 - a^3. \]

Thus, condition $(C_4)'$ is satisfied and therefore all solutions oscillate. Observe, however, that condition $(C_4)'$ is not satisfied.

If, on the other hand, in the above equation
\[ p_{2n} = \frac{8}{100}, \quad p_{2n+1} = \frac{8}{100} + \frac{746}{1000} \sin^2 \frac{n\pi}{2}, \quad n = 0, 1, 2, \ldots, \]
then it is easy to see that

\[ a = \liminf_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < \left(\frac{3}{4}\right)^4 \]

and

\[ \limsup_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{746}{1000} > 1 - \frac{a^2}{4}. \]

In this case condition \((C_4)'\) is satisfied and therefore all solutions oscillate. Observe, however, that condition \((C_4)'\) is not satisfied.

**Example 3.2.** ([65]) Consider the equation

\[ x_{n+1} - x_n + p_n x_{n-16} = 0, \quad n = 0, 1, 2, \ldots, \]

where

\[ p_{17n} = p_{17n+1} = \ldots = p_{17n+15} = \frac{2}{100}, \quad p_{17n+16} = \frac{2}{100} + \frac{655}{1000}, \quad n = 0, 1, 2, \ldots. \]

Here \(k = 16\) and it is easy to see that

\[ a = \liminf_{n \to \infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} < \left(\frac{16}{17}\right)^{17} \]

and

\[ \limsup_{n \to \infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} + \frac{655}{1000} = 0.975 > 1 - \frac{a^2}{4}. \]

We see that condition \((C_4)'\) is satisfied and therefore all solutions oscillate. Observe, however, that

\[ A = \limsup_{n \to \infty} \sum_{i=n-16}^{n} p_i = \frac{34}{100} + \frac{655}{1000} = 0.995 < 1; \]

that is, condition \((C_2)'\) is not satisfied.

In 1995, Chen and Yu [2], derived the condition

\[ A > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}. \]

\((C_6)'\)
In 1998, Domshlak [13], studied the oscillation of all solutions and the existence of nonoscillatory solution of Eq.(1)' with \( r \)-periodic positive coefficients \( \{ p_n \} \), \( p_{n+r} = p_n \). It is very important that in the following cases where \( \{ r = k \}, \{ r = k + 1 \}, \{ r = 2 \}, \{ k = 1, r = 3 \} \) and \( \{ k = 1, r = 4 \} \) the results obtained are stated in terms of necessary and sufficient conditions and it is very easy to check them.

In 2000, Tang and Yu [72] improved condition (C6)' to the condition

\[
A > \lambda_2^k(1 - k \ln \lambda_2) - \frac{1 - a - \sqrt{1 - a - a^2}}{2},
\]

where \( \lambda_2 \) is the greater root of the algebraic equation \( k \lambda^k(1 - \lambda) = a \).

In 2000, Shen and Stavroulakis [61], using new techniques, improved the previous results.

**Theorem 3.3.** ([61]) Assume that \( 0 \leq a \leq \left( \frac{k}{k+1} \right)^{k+1} \) and that there exists an integer \( l \geq 1 \) such that

\[
\limsup_{n \to \infty} \left\{ \sum_{i=1}^{k} p_{n-i} + [\bar{d}(a)]^{-k} \sum_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \right. \\
+ \sum_{m=0}^{l-1} [d(a/k)]^{-(m+1)k} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \} > 1,
\]

where \( \bar{d}(a) \) and \( d(a/k) \) are the greater real roots of the equations

\[
d^{k+1} - d^k + a^k = 0 \quad \text{and} \quad d^{k+1} - d^k + a/k = 0,
\]

respectively. Then all solutions of Eq. (1)' oscillate.

When \( k = 1, \ d(a) = \bar{d}(a) = (1 + \sqrt{1 - 4a})/2 \) (see [61]), and so (D5) reduces to

\[
\limsup_{n \to \infty} \left\{ C p_n + p_{n-1} + \sum_{m=0}^{l-1} C^{m+1} \prod_{j=0}^{m+1} p_{n-j-1} \right\} > 1,
\]

where \( C = 2/(1 + \sqrt{1 - 4a}), \ a = \liminf_{n \to \infty} p_n \). Therefore, from Theorem 3.3, we have the following corollary.

**Corollary 3.1.** ([61]) Assume that \( 0 \leq a \leq 1/4 \) and that (D6) holds. Then all solutions of the equation

\[
x_{n+1} - x_n + p_n x_{n-1} = 0
\]

(3.3)
oscillate.

A condition derived from \((D_6)\) and which can be easier verified, is given in the next corollary.

**Corollary 3.2.** ([61]) Assume that \(0 \leq a \leq 1/4\) and that

\[
\limsup_{n \to \infty} p_n > \left(\frac{1 + \sqrt{1 - 4a}}{2}\right)^2.
\]

Then all solutions of (3.3) oscillate.

**Remark 3.3.** ([61]) Observe that when \(a = 1/4\), condition (3.4) reduces to

\[
\limsup_{n \to \infty} p_n > 1/4,
\]

which can not be improved in the sense that the lower bound 1/4 can not be replaced by a smaller number. Indeed, by condition \((N_1)'\), we see that Eq. (3.3) has a nonoscillatory solution if 

\[
\sup_p n < 1/4.
\]

Note, however, that even in the critical state where

\[
\lim_{n \to \infty} p_n = 1/4,
\]

Eq. (3.3) can be either oscillatory or nonoscillatory. For example, if \(p_n = \frac{1}{4} + \frac{c}{n^2}\) then Eq. (3.3) will be oscillatory in case \(c > 1/4\) and nonoscillatory in case \(c < 1/4\) (the Kneser-like theorem, [11]).

**Example 3.3.** ([61]) Consider the equation

\[
x_{n+1} - x_n + \left(\frac{1}{4} + p \sin^4 \frac{n\pi}{8}\right) x_{n-1} = 0,
\]

where \(p > 0\) is a constant. It is easy to see that

\[
\liminf_{n \to \infty} p_n = \liminf_{n \to \infty} \left(\frac{1}{4} + p \sin^4 \frac{n\pi}{8}\right) = \frac{1}{4},
\]

and

\[
\limsup_{n \to \infty} p_n = \limsup_{n \to \infty} \left(\frac{1}{4} + p \sin^4 \frac{n\pi}{8}\right) = \frac{1}{4} + p.
\]
Therefore, by Corollary 3.2, all solutions oscillate. However, none of the conditions $(D_1) - (D_4), (C_3)' - (C_6)'$ is satisfied.

Following this historical (and chronological) review we also mention that in the case where
\[
\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \geq \frac{k^k}{(k + 1)^{k+1}} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i = \frac{k^k}{(k + 1)^{k+1}},
\]
the oscillation of $(1)'$ has been studied in 1994 by Domshlak [11] and in 1998 by Tang [66] (see also Tang and Yu [69]). In a case when $p_n$ is asymptotically close to one of the periodic critical states, unimprovable results about oscillation properties of the equation $x_{n+1} - x_n + p_n x_{n-1} = 0$, were obtained by Domshlak in 1999 [15] and in 2000 [16].

4. Oscillation Criteria for Eq. $(1)''$. In this section we study the functional equation
\[
x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \quad t \geq t_0. \quad (1)''
\]

In 1993 Domshlak [10] studied the oscillatory behaviour of equations of this type. In 1994, Golda and Werbowski [25] proved that all solutions of Eq.$(1)''$ oscillate if
\[
A := \limsup_{t \to \infty} \{Q(t)P(g(t))\} > 1, \quad (C_2)''
\]
or
\[
a := \liminf_{t \to \infty} \{Q(t)P(g(t))\} > \frac{1}{4}. \quad (C_3)''
\]

In the same paper they also improved condition $(C_2)''$ to
\[
\limsup_{t \to \infty} \left\{ Q(t)P(g(t)) + \sum_{i=0}^{k} \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\} > 1, \quad (4.1)
\]
where $k \geq 0$ is some integer.

It should be noted that conditions \((C_2)'\) and \((C_3)'\) may be seen as the analogues of the oscillation conditions \((C_2)\) and \((C_3)\) for Eq. (1) and the conditions \((C_2)'\) and \((C_3)'\) for Eq. (1)'.

As far as the lower bound \(1/4\) in the condition \((C_3)'\) is concerned, as it was pointed out in [25], it cannot be replaced by a smaller number. Recently, in [62], this fact was generalized by proving that

\[
Q(t)P(g(t)) \leq 1/4 \quad \text{for large } t, \quad (N_1)'
\]

implies that Eq. (1)'' has a nonoscillatory solution.

It is obvious that there is a gap between the conditions \((C_2)''\) and \((C_3)''\) when the limit \(\lim_{t \to \infty} \{Q(t)P(g(t))\}\) does not exist. How to fill this gap is an interesting problem. Here we should mention that condition (4.1) is an attempt in this direction. In fact, from condition (4.1) we can obtain (see [62]) that all solutions of (1)'' oscillate if \(0 \leq a \leq 1/4\) and

\[
A > \frac{1 - 2a}{1 - a}. \quad (4.2)
\]

In 2002, Shen and Stavroulakis ([63]) proved the following.

**Theorem 4.1.** ([63]) Assume that \(0 \leq a \leq 1/4\) and that for some integer \(k \geq 0\)

\[
\limsup_{t \to \infty} \left\{ \bar{a}Q(t)P(g(t)) + \sum_{i=0}^{k} \bar{a}^i \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)) \right\} > 1, \quad (4.3)
\]

where \(\bar{a} = \left(\frac{1 + \sqrt{1 - 4a}}{2}\right)^{-1}\). Then all solutions of Eq. (1)'' oscillate.

**Corollary 4.1.** ([63]) Assume that \(0 \leq a \leq 1/4\) and

\[
A > \left(\frac{1 + \sqrt{1 - 4a}}{2}\right)^2. \quad (4.4)
\]

Then all solutions of Eq. (1)'' oscillate.

**Remark 4.1.** ([63]) It is to be noted that as \(a \to 0\), the condition (4.3) reduces to the condition (4.1) and the conditions (4.4) and (4.2) reduce to the condition \((C_2)''\). However the improvement is clear as \(0 < a \leq 1/4\) because

\[
1 > \frac{1 - 2a}{1 - a} > \left(\frac{1 + \sqrt{1 - 4a}}{2}\right)^2.
\]
It is interesting to observe that when $a \rightarrow 1/4$ condition (4.4) reduces to $A > 1/4$, which cannot be improved in the sense that the lower bound $1/4$ cannot be replaced by a smaller number. Observe also that Eq.(3.3) is a first order delay difference equation with delay $k = 1$ or a second order difference equation; that is the exact analogue of the second order functional Eq.(1)". Therefore the exact analogy between (3.4) and (4.4). (See also Remark 3.3).

**Example 4.1.** ([63]) Consider the equation

$$x(t - 2 \sin^2 t) = x(t) + \left(\frac{1}{4} + q \cos^2 t\right)x(t - 2 \sin^2 t - 2 \sin^2(t - 2 \sin^2 t)), \quad (4.5)$$

where $g(t) = t - 2 \sin^2 t$, $P(t) \equiv 1$, $Q(t) = \frac{1}{4} + q \cos^2 t$, and $q > 0$ is a constant. It is easy to see that

$$a = \liminf_{t \to \infty} \left(\frac{1}{4} + q \cos^2 t\right) = \frac{1}{4},$$

$$A = \limsup_{t \to \infty} \left(\frac{1}{4} + q \cos^2 t\right) = \frac{1}{4} + q > \frac{1}{4}.$$ 

Thus, by Corollary 4.1 all solutions of (4.5) oscillate. However, the condition $(C_2)"$ is satisfied only for $q > 3/4$ and the condition (4.2) is satisfied only for $q > 5/12$.

**REFERENCES**


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DIFFERENCE AND ITERATIVE EQUATIONS

JACEK TABOR*

Abstract. Let \( \{A_k\}_{k \in \mathbb{N}} \) be a sequence of bounded linear operators on a Banach space \( X \). We investigate the difference equation

\[
\sum_{k=0}^{\infty} A_k x_{k+n} = 0 \quad \text{for } n \in \mathbb{N},
\]

and its application to iterative functional equation.

Key Words. difference equation, oscillation, iterative equation.

AMS(MOS) subject classification. 39B12, 58F08.

1. Introduction. Let us consider the difference equation

\[
x_{n+k} = \sum_{i=0}^{k-1} a_i x_{n+i} \quad \text{for } n \in \mathbb{N}.
\]

Given initial values \( x_0, \ldots, x_{n-1} \) equation (1) gives us a recurrence formula to compute \( x_k \) for \( k \geq n \). For results on oscillation of solutions to this difference equation we refer the reader to [2], [6], [7].

In this paper we investigate the following difference equation

\[
\sum_{i=0}^{\infty} a_i x_{n+i} = 0 \quad \text{for } n \in \mathbb{N}.
\]

This is a direct generalization of (1) as every solution to (1) is a solution to (2) (with respective coefficients). We are especially interested in the behaviour of the oscillatory solutions to (2).

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A motivation to consider such difference equations comes also from iterative functional equations (see [4]). Probably the most basic iterative functional equation is

$$\sum_{i=0}^{\infty} a_i f^i(x) = 0,$$

where $f^i$ denotes the $i$-th iterate of $f$ (see for example [3], [5]). One of the possible ways to investigate this equation follows from the observation that $f$ is a solution iff for every $x$ the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ is a solution to (2).

2. Difference equation. Let $X$ be a Banach space. By $\mathcal{L}(X)$ we denote the space of bounded linear operators on $X$. We investigate the difference equation

$$\sum_{k=0}^{\infty} A_k x_{k+n} = 0 \quad \text{for } n \in \mathbb{N},$$

where $A_k \in \mathcal{L}(X)$. The first question which we address is whether all solutions to (3) are exponentially bounded, as is the case for (1). Not surprisingly this is not the case.

**Example 1.** We consider the following difference equation:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_{k+n} = 0 \quad \text{for } n \in \mathbb{N}.$$ 

Let us first notice that for every $l \in \mathbb{N}$ the sequence $\{x(l)_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ defined by

$$x(l)_k := (\pi l)^{2k} \quad \text{for } k \in \mathbb{N}$$

is the solution to (4)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x(l)_{k+n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\pi l)^{2k+2n} = (\pi l)^{2n} \cos(l\pi) = 0.$$ 

Now we define the sequence $\{u_k\}_{k \in \mathbb{N}}$

$$u_k := \sum_{l=0}^{\infty} \frac{x(l)_k}{e^{4l}} \quad \text{for } k \in \mathbb{N}.$$
By the definition to show that the sequence \( \{u_k\}_{k \in \mathbb{N}} \) satisfies (4) it is enough to show that the series in (4) is absolutely convergent. For \( n \in \mathbb{N} \) we have

\[
\sum_{k=0}^{\infty} \frac{1}{(2k)!} u_{k+n} = \sum_{l=0}^{\infty} \frac{l^{2n}}{e^{4l}} \sum_{k=0}^{\infty} \frac{(\pi l)^{2k}}{(2k)!} \\
\leq \sum_{l=0}^{\infty} \frac{l^{2n}}{e^{4l} e^{\pi l}} = \sum_{l=0}^{\infty} \frac{l^{2n}}{e^{(4-\pi)l}} < \infty.
\]

One can easily notice that the sequence \( \{u_k\} \) is not exponentially bounded.

This is the reason why we assume from now on that there exists \( K > 0 \) such that

\[
R := 1/ \limsup_{n \to \infty} \sqrt[n]{\|A_n\|} > K,
\]

and restrict to sequences \( x = \{x_n\}_{n \in \mathbb{N}} \subset X \) such that

\[
\|x\|_K := \sup_{n \in \mathbb{N}} \frac{\|x_n\|}{K^n} < \infty.
\]

In other words we consider only solutions in the Banach space \( l^\infty_K(X) := \{x : \|x\|_K < \infty\} \). We will need its the subspace consisting of solutions to (3):

\[
S := \{x \in l^\infty_K(X) : x \text{ satisfies (3)}\}.
\]

The behavior of solutions will be described with the use of the shift operator

\[
P : l^\infty_K(X) \ni (x_0, x_1, x_2, \ldots) \to (x_1, x_2, x_3, \ldots) \in l^\infty_K(X).
\]

By \( P_S \) we denote the restriction of \( P \) to \( S \).

In our considerations the crucial role will play the characteristic function of (3)

\[
W : B(0, R) \ni z \to \sum_{k=0}^{\infty} A_k z^k \in \mathcal{L}(X),
\]

where \( B(0, R) \subset \mathbb{C} \) denotes the ball centered at zero and with radius \( R \).

For an operator valued function \( G : \Omega \to \mathcal{L}(X) \) we define the spectrum

\[
\sigma(G) := \{z \in \Omega : G(z) \text{ is invertible in } \mathcal{L}(X)\}.
\]
Now we are ready to proceed to our main result where we compute the formula for the resolvent of $P_S$.

**Theorem 1.** There exists a holomorphic function $E : B(0, R) \to l^\infty_K(X)$ such that
\[
(\lambda I_S - P_S)^{-1} = W(\lambda)^{-1} \circ E(\lambda)|_S \quad \text{for } \lambda \in B(0, R) \setminus \sigma(W),
\]
where $I_S$ denotes the identity operator on $S$.

**Proof.** We define $E$ by
\[
E(\lambda) := \left\{ \sum_{i=0}^{\infty} A_{i+1}(\sum_{j=0}^{i} \lambda^{i-j}u_{j+n}) \right\}_{n \in \mathbb{N}},
\]
where $u = \{u_n\}_{n \in \mathbb{N}} \in l^\infty_K(X)$, $\lambda \in B(0, R)$. One can easily verify that $E(\lambda)$ is a well-defined bounded operator on $l^\infty_K(X)$.

To prove (6) we have to show that for every $\lambda \in \sigma(W) \cap B(0, R)$
\[
(\lambda I_S - P_S) \circ (W(\lambda)^{-1}E(\lambda)) = I_S,
\]
(7)
\[
(W(\lambda)^{-1}E(\lambda)) \circ (\lambda I_S - P_S) = I_S.
\]
(8)
\[
W(\lambda)^{-1}E(\lambda)(S) \subset S
\]
(9)
We prove (7). Let $u = \{u_k\}_{k \in \mathbb{N}} \in S$ be arbitrary and let $c = W(\lambda)^{-1}E(\lambda)u$. It is enough to prove that
\[
u_k = \lambda c_k - c_{k+1} \quad \text{for } k \in \mathbb{N},
\]
(10)
Let us check (10). Since $u \in S$, we have $\sum_{i=0}^{\infty} A_iu_{i+k} = 0$, and therefore
\[
\lambda c_k - c_{k+1} = W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1}(\sum_{j=0}^{i} \lambda^{i+1-j}u_{j+k} - \sum_{j=0}^{i} \lambda^{i-j}u_{j+k+1}) \right)
\]
\[= W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1}(\lambda^{i+1}u_k - u_{i+k+1}) \right)
\]
\[= W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1}\lambda^{i+1}u_k - \sum_{i=0}^{\infty} A_{i+1}u_{i+k+1} \right)
\]
\[= W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1}\lambda^{i+1}u_k + Au_k - \sum_{i=0}^{\infty} A_iu_{i+k} \right)
\]
\[= W(\lambda)^{-1}W(\lambda)u_k = u_k.
\]
Let us proceed to (8). Let \( x = \{x_n\}_{n \in \mathbb{N}} \in S \) and \( k \in \mathbb{N} \) be fixed. Since \( x \in S \), 
\[- \sum_{i=0}^{\infty} A_{i+1} x_{i+k+1} = A_0 x_k \]
and consequently

\[
W(\lambda)^{-1} \sum_{i=0}^{\infty} A_{i+1} \left( \sum_{j=0}^{i} \lambda^{i-j} (\lambda x_{j+k} - x_{j+k+1}) \right) \\
= W(\lambda)^{-1} \sum_{i=0}^{\infty} A_{i+1} (\lambda^{i+1} x_k - x_{j+k+1}) \\
= W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1} \lambda^{i+1} x_k - \sum_{i=0}^{\infty} A_{i+1} x_{j+k+1} \right) \\
= W(\lambda)^{-1} \left( \sum_{i=0}^{\infty} A_{i+1} \lambda^{i+1} x_k + A_0 x_k \right) = x_k
\]

Now we prove (9). Let \( c \in S \) and \( k \in \mathbb{N} \) be fixed. By (10) we have

\[c_{k+l+1} = \lambda c_{k+l} - u_{k+l} \quad \text{for } l \in \mathbb{N}.
\]

By induction we obtain that

\[c_{k+n} = \lambda^n c_k - \sum_{j=0}^{n-1} \lambda^{n-1-j} u_{j+k} \quad \text{for } n \in \mathbb{N}.
\]

Then by the definition of function \( E \) we get

\[
\sum_{i=0}^{\infty} A_i u_{k+i} = \sum_{i=0}^{\infty} A_i (\lambda^n c_k - \sum_{j=0}^{n-1} \lambda^{n-1-j} u_{j+k}) \\
= W(\lambda) c_k - \sum_{i=0}^{\infty} A_{i+1} (\sum_{j=0}^{i} \lambda^{i-j} u_{j+k}) = 0.
\]

As a direct corollary we have

**Corollary 1.**

\[\sigma(P_S) \subset \sigma(W) \cap B(0, K).
\]

**Proof.** One can easily notice that \( \|P\| = K \). This yields that \( \sigma(P_S) \subset B(0, K) \). The rest follows from Theorem 1. \( \square \)
3. Oscillation. We say that a sequence \( \{ r_n \}_{n \in \mathbb{N}} \subset \mathbb{R} \) oscillates if for every \( k \in \mathbb{N} \) such that \( r_k \neq 0 \), there exists \( l > k \) such that \( r_k r_l < 0 \). Let \( \xi \in X^* \) be arbitrary. A sequence \( \{ x_n \}_{n \in \mathbb{N}} \subset X \) \( \xi \)-oscillates if the sequence \( \{ \xi(x_n) \}_{n \in \mathbb{N}} \) oscillates. The sequence \( \{ x_k \} \) oscillates if it \( \xi \)-oscillates for every \( \xi \in X^* \).

Let us mention (see Theorem 7.1.1, citeGL) that every solution to (1) oscillates iff the characteristic equation of (1) has no real nonnegative roots. As is common for the systems in infinite dimensional Banach spaces we have only the if part.

**Proposition 1.** We assume that

\[
\sigma(W) \cap [0, K] = \emptyset.
\]

Then every solution \( x \in l_\infty^R(X) \) to equation (3) oscillates.

**Proof.** By Theorem 1 \( \sigma(P_S) \cap \mathbb{R}_+ = \emptyset \). Let \( \xi \in X^* \) be arbitrary. We define \( \xi_S \in S^* \) by

\[
\xi_S(x_0, x_1, \ldots) = \xi(x_0) \quad \text{for} \ x = (x_0, x_1, \ldots) \in S.
\]

By Theorem 3.1 from [7] \( \xi_S \)-oscillates, that is the sequence \( \{ \xi_S(P_S^k x) \}_{k \in \mathbb{N}} \) oscillates. This means that the sequence \( \{ \xi(x_k) \}_{k \in \mathbb{N}} \) oscillates. \( \square \)

Now we will show a more complicated result, which shows that in some cases we can obtain information on nonoscillating solutions. We will need the following lemma.

**Lemma 1.** Let \( \lambda_0 \in B(0, K) \) be such that

\[
W(\lambda_0) = 0, \quad W'(\lambda_0) \text{ is invertible}.
\]

Then \( \lambda_0 \) is a pole of order one of the resolvent of \( P_S \).

**Proof.** From Theorem 1 and assumptions of the lemma we directly obtain that

\[
z \to (z - \lambda_0) \cdot (z1_S - P_S)^{-1}
\]

is bounded in the neighborhood of \( \lambda_0 \). This implies that \( \lambda_0 \) is either a pole of order one or a regular point of the resolvent of \( P_S \).

We prove that \( \lambda_0 \) is not a regular point of the resolvent of \( P_S \). Let \( x_0 \) be an arbitrary nonzero element of \( X \). Let \( x = \{ x_k \}_{k \in \mathbb{N}} \in l_\infty^R(X) \) be given by \( x_k = \lambda_0^k x_0 \) for \( k \in \mathbb{N} \). Then \( x \in S \) and

\[
P_S(x) = \lambda_0 x.
\]
Theorem 2. We assume that there exist $\lambda_0 \in [0, K]$ such that

\begin{align}
(11) \quad W(\lambda_0) &= 0, \quad W'(\lambda_0) \text{ is invertible}, \\
(12) \quad \sigma(W) \cap B(0, K) \setminus \{\lambda_0\} &\subset \{z \in \mathbb{C} : |z| > \lambda_0, z \notin \mathbb{R}_+\}.
\end{align}

Let $x = \{x_k\} \in S$. Then either the sequence $\{x_k\}$ oscillates or

$$x_k = \lambda_0^k x_0 \quad \text{for } k \in \mathbb{N}.$$ 

Proof. Let $S_0$ denote the eigenspace corresponding to $\lambda_0$ with respect to the operator $P_S$.

We first discuss the case when $x$ belongs to $S_0$. Since $\lambda_0$ is a pole of order one of resolvent of $P_S$,

$$P_S|_{S_0} = \lambda_0 I|_{S_0},$$

and consequently $P_S x = \lambda_0 x$. This means that

$$x_k = \lambda_0^k x_0 \quad \text{for } k \in \mathbb{N}.$$ 

Now we consider the case when $x$ does not belong to $S_0$. Let $\xi \in X^*$ be arbitrary. We define $\xi_S \in S^*$ by

$$\xi_S(x_0, x_1, \ldots) = \xi(x_0) \quad \text{for } x = (x_0, x_1, \ldots) \in S.$$ 

By Theorem 5.1 from [8] we obtain that $x \xi_S$-oscillates, which yields that the sequence $\{x_k\}$ $\xi$-oscillates. □

4. Iterative equations. In this section we present application of the previous result for iterative functional equations. Let us now quote the main result of W. Jarczyk from [3].

Theorem J. We assume that $\{a_k\} \subset \mathbb{R}_+$ is a sequence such that

$$\gcd\{i \in \mathbb{N} : a_i > 0\} = 1,$$

where $\gcd$ stands for the greatest common divisor. Let $D \subset (0, \infty)$ be nonempty and let $f : D \to D$ be a solution to

$$\sum_{i=1}^{\infty} a_i f^i(r) = r \quad \text{for } r \in D.$$
Then the characteristic function

\[ W(z) := \sum_{i=1}^{\infty} a_i z^i - 1 \]

has a unique nonnegative real root \( \lambda_0 \) and

\[ f(r) = \lambda_0 r \quad \text{for} \quad r \in D. \]

We show that although in the proof of W. Jarczyk there does not appear the notion of oscillation, it can partially explain what mechanism "stands" behind.

Proof of Theorem J under some additional assumptions. We assume additionally that \( K > 0 \) is given such that

\[ R := \frac{1}{\limsup_{k \to \infty} k \sqrt{|a_k|}} > K. \]

We restrict also only to functions for which there exists \( A > 0 \) with

\[ |f(r)| \leq A + K|r| \quad \text{for} \quad r \in D. \tag{14} \]

Let \( r \in D \) be arbitrary and let \( x = \{f^k(r)\}_{k \in \mathbb{N}} \). Then by (14) \( x \in l^\infty_K(\mathbb{R}) \). By (13) we obtain that \( x \in S \).

There are two possible cases. Either the characteristic function \( W \) has a root in \([0, K]\) or not. If it does not have a root, then by Proposition 1 every solution oscillates, and therefore we obtain contradiction with the assumption that \( D \subset \mathbb{R}_+ \).

In the case when \( W \) has a root \( \lambda_0 \in [0, R] \), one can easily check that it is a single root and other roots have absolute values greater than \( \lambda_0 \). Now by Theorem 2 we obtain that either the sequence \( \{f^k(r)\}_{k \in \mathbb{N}} \) oscillates or \( f(r) = \lambda_0 r \), which makes the proof complete. \( \square \)

Remark 1. The proof gives us in fact a partial generalization of Theorem J to Banach spaces. Assume that \( D \) is a nonempty subset of a cone in a Banach space \( X \) and that \( f : D \to D \) satisfies

\[ |f(d)| \leq A + K|d| \quad \text{for} \quad d \in D, \]

with certain \( A > 0 \), and

\[ \sum_{i=0}^{\infty} A_i f^i(d) = 0 \quad \text{for} \quad d \in D. \]
Then under the assumptions on the characteristic polynomial stated in Theorem 2 we obtain that

\[ f(d) = \lambda_0 d \quad \text{for} \ d \in D. \]

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SIMULATION RESULTS ON THE ASYMPTOTIC PERIODICITY OF COMPARTMENTAL SYSTEMS WITH LAGS*

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Abstract. In this paper, we study the asymptotic behavior of solutions for a pharmacokinetic application, where the drug level is controlled between two boundary levels by repeated impulsive doses.

Key Words. asymptotic periodicity, impulsive state-dependent input, compartment, numerical simulation, estimation of the periodic solution

AMS(MOS) subject classification. 34K25, 34K28, 34K13

1. Introduction. More than fifty years ago, compartmental systems were originally introduced as the dynamic models of biological systems [1],[6]. Since then, this type of modeling have become an essential theory for medical sciences, because with it, the qualitative analysis of drugs in living organisms can be implemented. In many pharmacokinetic applications the biological systems are described by models consisting of three or two compartments, but in case of a very simplified system, the whole behavior can be described by one compartment. Although, one compartment seems to be a very sketchy representation of the original system, the presence of an inner delayed feedback in the model can make this level of resolution useful [2]. For a dosage model, impulsive input is very common, by which we would like to control the drug level, or the state of the system between an efficiency drug level c and an overdose level a + c. By considering this, a basic drug dosage model

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has the following form:

(1) \[ \dot{x}(t) = -\alpha x(t) + \beta x(t - \tau) \quad t > \sigma, \]

\[ x(t) = \varphi(t - \sigma), \quad \text{if } \sigma - \tau \leq t \leq \sigma, \]

\[ \varphi(0) > c, \quad \text{and if } x(t-) = c \text{ then } x(t+) = a + c, \quad \text{when } t \geq \sigma, \]

where \( \alpha, \beta, \tau \in \mathbb{R}_0^+ \), the starting time is denoted by \( \sigma \in \mathbb{R}_0^+ \), \( \varphi : [-\tau, 0] \rightarrow \mathbb{R}_0^+ \) is integrable, and \( \mathbb{R}_0^+ = [0, \infty) \). Because of the strong dissipativity of the original system, we assume \( \alpha > \beta \). In this short note, our main goal is to verify and also to extend a conjecture (see [3]) on the asymptotic periodicity of the solutions of problem (1). We also give a brief explanation of the numerical methods which have been used to develop our computer programs.

2. Asymptotic periodicity of solutions. The fundamental solution \( v : [-\tau, \infty) \rightarrow \mathbb{R}_0^+ \) of (1) is defined as follows:

(2) \[ \dot{v}(t) = -\alpha v(t) + \beta v(t - \tau), \quad t > 0, \]

\[ v(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \leq t < 0, \end{cases} \]

where \( v(t) \) exists, is unique on \([-\tau, \infty)\), and continuous on \([0, \infty)\). Because \( \alpha > \beta \), it follows \( \lim_{t \to \infty} v(t) = 0 \), and the zero solution is asymptotically stable (see [3]). Moreover, we show

**Proposition 1.** Let \( v \) be the fundamental solution of (1), and assume \( \alpha > \beta \). Then

(3) \[ e^{-\alpha t} \leq v(t) \leq e^{-\lambda_0 t}, \quad t \geq 0, \]

where \( \lambda_0 \) is the positive root of \( \lambda_0 = \alpha - \beta e^{\lambda_0 \tau} \).

**Proof.** Clearly, such unique \( \lambda_0 > 0 \) exists. Equation (2) and the positivity of \( \beta \) and \( v(t) \) yield \( \dot{v}(t) \geq -\alpha v(t) \). Therefore

\[ \log(v(t)) - \log(v(0)) = \int_0^t \frac{\dot{v}(s)}{v(s)} ds \geq \int_0^t -\alpha ds = -\alpha t, \]

which clearly leads to \( v(t) \geq e^{-\alpha t} \). To get the upper estimation of \( v(t) \) we introduce the function \( w(t) = v(t)e^{\lambda_0 t} \), and consider

\[ \dot{w}(t) = \lambda_0 v(t)e^{\lambda_0 t} + e^{\lambda_0 t}(-\alpha v(t) + \beta v(t - \tau)) \]

\[ = (\lambda_0 - \alpha)w(t) + \beta e^{\lambda_0 \tau}w(t - \tau) \]

\[ = -\beta e^{\lambda_0 \tau}(w(t) - w(t - \tau)), \quad t \geq 0. \]
Therefore
\[ \frac{d}{dt} \left( w(t) + \beta e^{\lambda_0 \tau} \int_{t-\tau}^{t} w(s) \, ds \right) = 0, \quad t \geq 0, \]
and thus, using the initial condition of \( v \), we get
\[ w(t) = 1 - \beta e^{\lambda_0 \tau} \int_{t-\tau}^{t} w(s) \, ds, \quad t \geq 0. \]
This concludes the proof, since \( w(t) > 0 \) for \( t \geq 0 \) implies \( w(t) \leq 1, \ t \geq 0. \)

The exact solution of problem (1) has been constructed in [3]:

**Theorem 1.** There exist a sequence of constants \( \{t_k\}, \sigma = t_0 < \ldots < t_k < \ldots \) called injection times, such that the original boundary value problem (1) has a unique solution \( x(t) = x(\sigma, \varphi, \alpha, \beta, \tau, a, c)(t) \) which can be written into the following form:

\[ x(t) = v(t-\tau) \varphi(0) + \beta \int_{-\tau}^{0} v(t-\sigma-s-\tau) \varphi(s) \, ds + a \sum_{i=1}^{\infty} v(t-t_i), \]

where \( t \geq \sigma, \ t \notin \{t_k\}_{k \geq 1} \).

To show the asymptotic periodicity of solutions we consider a special, related problem:

\[ y(t) = -\alpha x(t) + \beta y(t-\tau), \quad t > \sigma, \]
\[ y(t) = \varphi(t-\sigma), \quad \text{if} \ \sigma - \tau \leq t \leq \sigma, \]
\[ y(\sigma + iT+) = y(\sigma + iT-), \quad i = 0, 1, 2, \ldots, \]

where \( T, \alpha, \beta, \tau, \sigma \in \mathbb{R}_0^+ \) and \( \alpha > \beta \). This related problem is very similar to (1) except of that the input of this model is a \( T \) periodic constant impulsive input sequence. For (5) the unique solution is (see [4]):

\[ y(\sigma, \varphi, \alpha, \beta, \tau, a)(t) = y(\sigma, \varphi, \alpha, \beta, \tau, 0)(t) + a \sum_{i=0}^{[t/T]} v(t-\sigma-iT), \quad t \geq \sigma, \]

where \( [.] \) is the greatest integer part function. Based on [4], we know that:

\[ \lim_{t \to \infty} |y(\sigma, \varphi, \alpha, \beta, \tau, a)(t) - aW_{\sigma,T}(t)| = 0, \]

where \( W_{\sigma,T}(t) = \sum_{i=0}^{\infty} v(t-\sigma-[t/T] + iT) \) is a \( T \) periodic, piecewise continuous function and

\[ W_{\sigma,T}(T+) = \sum_{i=0}^{\infty} v(iT) = 1 + W_{\sigma,T}(T-) = (a + c)/a. \]
Relation (7) motivated the following conjecture in [3] for problem (1).

**Conjecture 1.** (see [3]) For (1) if \( \lim_{t \to \infty} |t_{k+1} - t_k| = T \), then there exists \( a \in \mathbb{R} \) such that \( \lim_{t \to \infty} |x(\sigma, \varphi, \alpha, \beta, \tau, a, c)(t) - aW_{\kappa,T}(t)| = 0 \) holds.

If this result is true, then asymptotic periodicity exists in (1). Our numerical experiments suggest the following extension of the above conjecture:

**Conjecture 2.** For (1) there exists constants \( l \in \mathbb{N} \) and \( a \in \mathbb{R} \) such that \( \lim_{t \to \infty} |t_{k+1} - t_k| = T \), and \( \lim_{t \to \infty} |x(\sigma, \varphi, \alpha, \beta, \tau, a, c)(t) - aW_{\kappa,T}(t)| = 0 \).

In addition to numerically verifying Conjecture 2, one of our main goals is to give an approximation on the fundamental time period \( T \) of the limiting periodic function \( aW_{\kappa,T} \). These questions are important in constructing optimized drug dosing strategies.

We can derive form (3)

\[
\sum_{i=1}^{\infty} e^{-\alpha i T} \leq \sum_{i=1}^{\infty} v(iT) \leq \sum_{i=1}^{\infty} e^{-\lambda_0 i T}.
\]

Using relation \( W_{\sigma,T}(T^+) - 1 = \sum_{i=1}^{\infty} v(iT) = c/a \) we get

\[
\frac{e^{-\alpha T}}{1 - e^{-\alpha T}} \leq \frac{c}{a} \leq \frac{e^{-\lambda_0 T}}{1 - e^{-\lambda_0 T}},
\]

since \( e^{-\alpha T}, e^{-\lambda_0 T} < 1 \) by the positivity of \( \alpha \) and \( \lambda_0 \). From (8) simple calculation implies

\[
\frac{1}{\alpha} \ln \frac{c + a}{c} \leq T \leq \frac{1}{\lambda_0} \ln \frac{c + a}{c}.
\]

The efficiency of estimation (9) will be investigated in the next section (see Fig. 7–10).

### 3. Numerical simulations.**

To investigate the problem numerically, we developed a software package in Matlab. We used a version of the explicit Euler method and the chain method, described in the next section, in our simulations. In all of our experiments we found that always exists an \( l \), such that \( \lim_{t \to \infty} |t_{k+1} - t_k| = T \), showing that with this input strategy, periodicity always reachable. Based on these investigations, we classified the solutions by speed of convergence and by order of periodicity:
Quickly convergent, \( l = 1 \) solutions: These solutions (Fig. 1) have a short transient and their limiting periodic function has an exponential decay. In this case \( l = 1 \), thus the difference between the consecutive injection points tends to a constant value (Fig. 2), which equals to the time period \( T \) of the limiting solution, as stated in Conjecture 1.

Slowly convergent, \( l = 1 \) solutions: These solutions (Fig. 3) have a much longer transient and their periodic function is the multiplication of an exponential function and a polynomial. We analytically computed these polynomials in several cases. For these systems, the relative value of \( \alpha \) and \( \beta \) is much closer to each other than for quickly convergent systems. As in the previous case, Conjecture 1 holds (see Fig. 4).

Solutions with \( l > 1 \): As the relative value of \( \alpha \) and \( \beta \) gets close enough to each other and the time delay is larger than an unknown value related to \( a \), then \( |t_{k+1} - t_k| \) will be asymptotically periodic, but not convergent (see Fig. 5), and, at the same time, the solution is also asymptotically periodic, and Conjecture 2 holds. In this case during an interval with length \( T \) there will be more than one injection points.

In Fig. 7-10 we investigate the effect of the parameters on the values of \( T \) on examples, where all parameters except one are fixed. We found that the time delay has no effect on the value of \( T \). We observed that for small \( a \) or distant \( \alpha \) and \( \beta \), the lower and upper estimations given by (9) are efficient but if \( a \) increases and \( \alpha, \beta \) get closer to each other, our knowledge about the true behavior of the system fades away. On Fig. 6 the efficiency of our estimation is presented when \( l = 6 \). We found numerically that our estimation (9) is also valid in the case, when we replace \( T \) in (9) by \( |t_{k+1} - t_k| \) (see Fig. 6, where we plotted out the upper and lower estimates in (9) together with \( T \) and \( |t_{k+1} - t_k| \)).

4. Estimation of the periodic solution. In the following, we present a numerical method in case of \( l = 1 \), to approximate the periodic function \( W \). With this method, we would like to give an easy way to describe the behaviour of (1) after a large number of inputs. To do this, we will use the chain method from [5]. By this method, any channel with time delay is substituted by the series connection of infinite number of compartments with no time lags. If we use only finite number \((r-1) \in N\) of substituting compartments, then our system is approximated with a first order ODE system:

\[
\dot{x} = Ax(t) + bu(t),
\]
\[ A = \begin{pmatrix} -\alpha & 0 & \cdots & 0 & r/\tau \\ \beta & -r/\tau & 0 & \cdots & 0 \\ 0 & r/\tau & -r/\tau & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & r/\tau & -r/\tau \end{pmatrix}_{r \times r}, \quad \bar{x}(\sigma) = \begin{pmatrix} \varphi(\sigma) \\ \int_0^{-\tau/r} \varphi(s)ds \\ \vdots \\ \int_{-\tau}^{-(r-1)/r} \varphi(s)ds \end{pmatrix}_{r \times 1} \]

and \( b = (a, 0, \cdots, 0)^T_{1 \times r} \), \( u(t) = \sum_{k=1}^{\infty} \delta(t - t_k) \), where \( t_k \) denotes the injection points and \( \delta \) is the Dirac-delta function. By solving (10), the estimation of the unique solution of (1) can be obtained between each injection points:

\[
\bar{x}(\alpha, \beta, \tau, a, c)(t, r) = e^{A(t-\sigma)}\bar{x}(\sigma) + \sum_{i=0}^{n-1} \int_{t_i}^{t} e^{A(t-s-t_i)} q(s-t_i)ds,
\]

where \( q(s) = (a\delta(s), 0, \cdots, 0)^T_{1 \times r} \) and \( t \in [\sigma, t_n] \). After large number of inputs the solution of (11) gets close enough to its periodic state, which can be described by its fundamental time period \( \tilde{T} \), and initial values \( z_1, \ldots, z_{r-1} \in \mathbb{R}_{0}^+ \) of the substituting compartments at the beginning of the period. These parameters can be obtained by solving

\[
z_0 = e^{A\tilde{T}}z_1,
\]

where \( z_0 = (c, z_1, \cdots, z_{r-1})^T \) and \( z_1 = (a + c, z_1, \cdots, z_{r-1})^T \). If we assumes that \( \tilde{T} = T \), then from these parameters, the estimated periodic solution of (1) can be constructed, which one period is as follows:

\[
\bar{x}(\alpha, \beta, \tau, a, c)(t, r) = e^{At}z_1, \quad t \in (0+, T-).
\]

**Conjecture 3.** There exists a \( \kappa \in \mathbb{R} \) such that the unique solution of (1) satisfies:

\[
\lim_{t, r \to \infty} |x(\sigma, \varphi, \alpha, \beta, \tau, a, c)(t) - p\bar{x}(\alpha, \beta, \tau, a, c)(t - \kappa, r)| = 0,
\]

where \( p = (1, 0, \cdots, 0) \).

In Fig. 11 an example is given which affirms that Conjecture 3 is valid. By analyzing Fig. 12, it can be concluded that the estimated solution is a bit faster due to the finite resolution, but in between consecutive injection points the difference of the two solutions is relatively small.
COMPARTMENTAL SYSTEMS WITH LAGS

Fig. 1: $x(0, 2, 1, 0.5, 0.2, 1, 3)(t)$

Fig. 2: $|t_{k+1} - t_k|$ for $x(0, 2, 1, 0.5, 0.2, 1, 3)(t)$

Fig. 3: $x(0, 6, 4, 3.5, 0.4, 4, 5)(t)$

Fig. 4: $|t_{k+1} - t_k|$ for $x(0, 6, 4, 3.5, 0.4, 4, 5)(t)$

Fig. 5: $|t_{k+1} - t_k|$ for $x(0, 14, 3, 2.75, 4, 9, 5)(t)$

Fig. 6: Estimations for $T_{l=6}$, $x(0, 14, 3, 2.75, 4, 9, 5)(t)$

Fig. 7: Effect of $\alpha$ on $T$ for $x(0, 1, \alpha, 0.1, 1, 1, 1)(t)$

Fig. 8: Effect of $\beta$ on $T$ for $x(0, 1, 1, \beta, 1, 1, 1)(t)$

Fig. 9: Effect of $\alpha$ on $T$ for $x(0, 1, 1, 0.1, 1, 1, \alpha)(t)$

Fig. 10: Effect of $c$ on $T$ for $x(0, 1, 1, 0.1, 1, c, 1)(t)$

Fig. 11: $x(t)$ and $\hat{x}(t, 25)$, $x(0, 14, 3, 2.75, 4, 9, 5)(t)$

Fig. 12: $|x(t) - \hat{x}(t, 25)|$, $x(0, 14, 3, 2.75, 4, 9, 5)(t)$
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Abstract. In this paper, we shall study a very general class of functional equations with infinite delays of the form

\[ x(t) = f(t, x_t), \]

and develop the basic theory. Easily verifiable sufficiency criteria are established for the existence, uniqueness, continuation and continuous dependence of solutions. The paper concludes with some interesting examples and discussions.

Key Words. Functional equations with infinite delays, basic theory, existence, uniqueness, continuation, continuous dependence, phase space.

AMS(MOS) subject classification. 39B05, 39B52, 39B72.

1. Introduction. Functional equations are relations in which the unknown (or unknowns) are functions. Broadly, functional equations can be classified into two groups: functional equations in single variable (also known as iterative functional equations) and functional equations in several variables [4].

The history of the study of functional equations can be traced back to Euler (1768), D’Alembert (1769), Poisson (1804), Cauchy (1821), Abel (1823),
Darboux (1875), to name a few among the most famous, who were associated with indepth research on functional equations. In connection with the fifth of his famous problems, Hilbert remarked that, while the theory of differential equations provides elegant and powerful techniques for solving functional equations, the differentiability assumptions are not inherently required. Motivated by Hilbert’s suggestion, many researchers have treated various functional equations without any (or mild) regularity assumptions. This effort has given rise to the modern theory of functional equations. The bibliography in [1, 4] is a testimony to the continuous and attractive development in the field of functional equations during the 20th century. The theory of functional equations has developed rapidly in the last four decades, mainly due to the efforts of Aczel [1, 2, 3, 4] and has become an independent mathematical discipline. Solutions of functional equations have become important tools in an increasing number of problems in social sciences such as economics and psychology [1, 2, 3, 4, 7, 14]. Recently, there has been an increasing interest in functional equations [1-4, 7-10, 13, 16]. Although much progress has been made in the study of the properties of solutions of functional equations, very little is known about the basic theory, such as, existence, uniqueness, continuation and continuous dependence [8, 9]. It is worth mentioning that the results on functional equations with delays, particularly with infinite delays are rare.

In this paper, we focus our attention on a very general type of functional equations with infinite delay and develop their basic theory. In Section 2, we present some preliminary results, notations which will be used throughout this paper, some basic definitions, and a general phase space $B$ for functional equations with infinite delays. In the succeeding four sections, we establish sufficiency criteria for the existence, uniqueness, continuation and continuous dependence respectively. Finally, in Section 7, we present some examples of functionals satisfying the quasi-Lipschitzian assumption and a phase space $C_h$ satisfying the general assumption for the phase space $B$.

2. Preliminary Results. As usual, throughout this paper, let $R^+ = [0, \infty)$, $R^- = (-\infty, 0]$, $R = (-\infty, \infty)$ and $B$ denote a real normed linear vector space of functions mapping $R^-$ into $R^n$ endowed with some norm $\|\cdot\|$ and its elements written as $\varphi, \psi, \cdots$.

**Definition 2.1.** Let $\sigma \in R$, $A > 0$ and $x : (-\infty, \sigma + A) \mapsto R^n$ be a given function. For any $t \in [\sigma, \sigma + A]$, we define $x_t : R^- \mapsto R^n$ by $x_t(\theta) = x(t + \theta)$, $\theta \in R^-$. For any $\varphi \in B$ and $A$ with $0 < A \leq +\infty$, $\sigma \in R$, let $x^\varphi(\sigma, \varphi)$ be a $R^n-$valued
function defined on \((-\infty, A]\) such that \(x_p^\sigma(\sigma, \varphi) = \varphi\) and \(x_p^\sigma(\sigma, \varphi)(t) = p(t-\sigma)\) for \(\sigma < t \leq \sigma + A\), where \(p \in C([0, A] \rightarrow \mathbb{R}^n) := C_A\). For any \(p \in C_A\) define the norm of \(p\) by \(|p| = \sup_{0 \leq s \leq A} |p(s)|\).

Let \(\Omega\) be an open set in \(\mathbb{R} \times B\). Consider the functional equation with infinite delay of the form

\[ x(t) = f(t, x_t), \]

where \(f(t, \varphi) : \Omega \mapsto \mathbb{R}^n\) is continuous, i.e., \(f \in C(\Omega, \mathbb{R}^n)\). We define

\[ \|f\| = \sup_{(t, \varphi) \in \Omega} |f(t, \varphi)|. \]

The development of the theory of functional differential equations with infinite delay FDE(ID) primarily depends on the choice of a phase space. Many authors have actively studied, and various phase spaces have been proposed \([6, 12, 15, 16]\). However, no attempts to construct a phase space for FDE(ID) have been made. So we shall first consider an abstract phase space \(B\) satisfying some general assumptions, and then in Section 7, we provide a concrete interesting phase space \(C_h\).

The phase space \(B\) is assumed to have the following properties:

(\(A_1\)) For any \(\varphi \in B\), \(p \in C_A\), \(x_p^\sigma(\sigma, \varphi) \in B\) for all \(t \in [\sigma, \sigma + A]\). Moreover, \(x_p^\sigma(\sigma, \varphi)\) is continuous in \(t \in [\sigma, \sigma + A]\) for any fixed \(\varphi \in B\).

(\(A_2\)) There are continuous functions \(K : R^+ \mapsto R^+\) and \(M : R^+ \mapsto R^+\) such that, for any \((t, p, \varphi), (t, q, \psi) \in [\sigma, \sigma + A] \times C_A \times B\),

\[ \|x_p^\sigma(\sigma, \varphi) - x_q^\sigma(\sigma, \psi)\| \leq K(t-\sigma)|p-q| + M(t-\sigma)||\varphi-\psi||, \quad (2.2) \]

where \(\cdot ||\cdot||^{[\alpha, \beta]}\) denotes the supremum norm on the interval \([\alpha, \beta]\).

(\(A_3\)) There is a constant \(J > 0\) such that \(||\varphi|| \geq J |\varphi(0)|\) for \(\varphi \in B\).

(\(A_4\)) There is a decreasing continuous function \(q(\tau) : R^- \rightarrow R^+\) with \(q(0) = 1\) such that

\[ ||\varphi\tau|| \leq q(\tau)||\varphi||, \text{ for } \tau \in R^-, \varphi \in B. \]

(\(A_5\)) There is a continuous function \(Q(\tau) : R^- \rightarrow R^+\) such that

\[ ||\psi - \varphi||^{[r,0]} \leq Q(\tau)||\psi - \varphi||, \text{ for any } \varphi, \psi \in B, \tau \in R^-\.

**Definition 2.2.** A function \(x : (-\infty, \sigma + A) \mapsto \mathbb{R}^n\), where \(A > 0\), is said to be a solution of (2.1) on \((-\infty, \sigma + A)\) if \((t, x_t) \in \Omega\) and \(x(t)\) satisfies
(2.1) for $t \in [\sigma, \sigma + A]$. For a given $\sigma \in \mathbb{R}$ and $\varphi \in B$, we say $x(\sigma, \varphi)(t)$ is a solution of (2.1) with initial value $\varphi$ at $\sigma$, i.e.,

$$x_\sigma = \varphi,$$  

(2.3)

or simply a solution through $(\sigma, \varphi)$, if there is an $A > 0$ such that $x(\sigma, \varphi)(t)$ is a solution of (2.1) on $(-\infty, \sigma + A)$ and satisfies $x_\sigma(\sigma, \varphi) = \varphi$.

**Remark 2.1.** In general, a solution $y(t)$ of IVP (2.1) - (2.3) (if it exists on $(-\infty, \sigma + A)$) may not be continuous at $t = \sigma$ even though $\varphi(s)$ is continuous on $R^-$. For a solution $y(t)$ to be continuous at $t = \sigma$, the condition, $f(\sigma, \varphi) = \varphi(0)$ must be satisfied.

3. **Existence.** In this section, we establish basic existence theorems for the IVP (2.1) - (2.3) using Schauder fixed-point theorem. First, we need to introduce some definitions. Let

$$C_A(L) = \{ p \in C_A : |p(t) - p(s)| \leq L|t - s| \text{ for any } t, s \in [0, A] \}.$$ 

**Definition 3.3.** Suppose that $\Omega$ is an open subset in $\mathbb{R} \times B$. A functional $f(t, \varphi) : \Omega \mapsto R^n$ is said to be locally quasi-Lipschitzian at the point $(\sigma, \varphi) \in \Omega$, if there exists an $\epsilon > 0$ and a positive constant $L > 0$ such that

$$|f(t, x_t^p(\sigma, \varphi)) - f(s, x_s^p(\sigma, \varphi))| \leq L|t - s|,$$

for any $t, s \in [\sigma, \sigma + \epsilon]$ and $p \in C_A(L)$ with $p(0) = f(\sigma, \varphi)$. Further, if $f(t, \varphi)$ is locally quasi-Lipschitzian at every point in $\Omega$, then $f(t, \varphi)$ is called locally quasi-Lipschitzian on $\Omega$.

**Definition 3.4.** Suppose that $\Omega$ is an open subset in $\mathbb{R} \times B$ and $E$ is a subset of $\Omega$. A functional $f(t, \varphi) : \Omega \mapsto R^n$ is said to be locally quasi-Lipschitzian on $E$ if $f(t, \varphi)$ is locally quasi-Lipschitzian at every point in $E$. Further, if there is a constant $L(E) > 0$ and an $\epsilon > 0$ such that

$$|f(t_1, x_{t_1}^p(\sigma, \varphi)) - f(t_2, x_{t_2}^p(\sigma, \varphi))| \leq L(E)|t_1 - t_2|,$$

provided $(t, x_t^p(\sigma, \varphi) \in E$ for $t \in [\sigma, t_1] \cup [\sigma, t_2]$, then $f(t, \varphi)$ is called quasi-Lipschitzian on $E$.

We now state the Schauder fixed-point theorem.

**Lemma 3.1.** If $D$ is a convex, compact subset of a Banach space $X$ and $g : D \mapsto D$ is continuous, then $g$ has a fixed point in $D$.

We will carry out the discussion in two cases. First, let $\Omega = \mathbb{R} \times B$. 
**Theorem 3.1.** Assume that the phase space $B$ satisfies $(A_1)$ and $(A_2)$. For any $(\sigma, \varphi) \in \mathbb{R} \times B$, if $f(t, \varphi)$ is locally quasi-Lipschitzian at $(\sigma, \varphi)$, then there exists a positive constant $b > 0$ such that the initial value problem (IVP) (2.1) - (2.3) has a solution defined on $(-\infty, \sigma + b]$.

**Proof.** From Definition 3.1, it follows that there exists a $b > 0$ and a positive constant $L > 0$ such that

$$|f(t, x^\sigma_t(\sigma, \varphi)) - f(s, x^\sigma_s(\sigma, \varphi))| \leq L|t - s|$$

for any $t, s \in [\sigma, \sigma + b]$ and $p \in S$, where

$$S = \{p \in C_b(L) : p(0) = f(\sigma, \varphi)\}.$$ 

It is trivial to show that $C_b$ is a Banach space and $S$ is a convex compact set in $C_b$. Define a mapping $F : S \mapsto C_b$ as follows:

$$F(p)(t) = f(\sigma + t, x^p_{\sigma+t}(\sigma, \varphi)), \text{ for } t \in [0, b].$$

By $(A_1)$ and the continuity of $f(t, \varphi)$, we can conclude that the mapping $F$ is continuous in $p$, and for a given $p \in S$, $F(p)(t)$ is continuous in $t$. Clearly, we have

$$F(p)(0) = f(\sigma, x^p_0(\sigma, \varphi)) = f(\sigma, \varphi)$$

for $p \in S$. By Definitions 3.1 we get

$$|F(p)(t_1) - F(p)(t_2)| = |f(\sigma + t_1, x^p_{\sigma+t_1}(\sigma, \varphi)) - f(\sigma + t_2, x^p_{\sigma+t_2}(\sigma, \varphi))| \leq L|t_1 - t_2|$$

for any $t_1, t_2 \in [0, b]$. Now we can claim that $F$ maps $S$ into itself. The Schauder's fixed point theorem asserts that $F$ has a fixed point $\bar{p} \in S$. Let

$$x^*(t) = \begin{cases} 
\varphi(t + \sigma) & \text{for } t \leq \sigma, \\
\bar{p}(t - \sigma) & \text{for } t \in [\sigma, \sigma + b].
\end{cases}$$

One can easily show that $x^*(t)$ is a solution of the IVP (2.1) - (2.3). The proof is complete. □

Now we consider the case when $\Omega$ is a subset of $\mathbb{R} \times B$. Let

$$S_d(\varphi) := \{\psi \in \Omega : \|\psi - \varphi\| \leq d\}, \text{ } d > 0.$$ 

**Lemma 3.2.** For any $p \in S := \{\psi \in C_A(L) : \psi(0) = f(\sigma, \varphi)\}$, there is a $t^* = t^*(L, \sigma, \varphi) > \sigma$ such that $x^\sigma_t(\sigma, \varphi) \in S_d(\varphi)$ for $t \in [\sigma, t^*]$. 

Proof. Since $x_0^p(\sigma, \varphi) = x_0^p(\sigma, \varphi), \text{there is a } t(p) > \sigma \text{ such that } x_t^p(\sigma, \varphi) \in S_d(\varphi) \text{ for any } t \in [\sigma, t(p)]. \text{ Let } t^*(p) = \sup t(p) \text{. We now prove that}

$$t^* := \inf_{p \in S} t^*(p) > \sigma. \quad (3.1)$$

If not, then there exist $p_k \in S$ and $t_k \geq \sigma, \ k = 1, 2, \cdots$, such that $t_k \to \sigma$ and

$$x_{t_k}^{p_k}(\sigma, \varphi) \in S_d(\varphi).$$

Since $S$ is compact, we can find a subsequence $\{p_{k_m}\}$ and a $p^* \in S$ such that $p_{k_m} \to p^*$ as $m \to \infty$. Without loss of generality, for convenience, we can assume that $p_k \to p^*$ as $k \to \infty$. By $(A_1)$ and $(A_2)$, we have

\[
|x_{t_k}^{p_k}(\sigma, \varphi) - \varphi| &= |x_{t_k}^{p_k}(\sigma, \varphi) - x_{t_k}^{p^*}(\sigma, \varphi)| \\
&\leq |x_{t_k}^{p_k}(\sigma, \varphi) - x_{t_k}^{p^*}(\sigma, \varphi)| + |x_{t_k}^{p^*}(\sigma, \varphi) - x_{t_k}^{p^*}(\sigma, \varphi)| \\
&\leq K(t_k - \sigma)|p_k - p^*| + |x_{t_k}^{p^*}(\sigma, \varphi) - x_{t_k}^{p^*}(\sigma, \varphi)| \to 0
\]

as $k \to \infty$. Therefore, $x_{t_k}^{p_k}(\sigma, \varphi) \in S_d(\varphi)$ for sufficiently large $k$. This contradiction proves (3.1). The proof is complete. \hfill \square

Lemma 3.3. For any $t \geq \sigma, \varphi \in B, \ p \in C_A$ and $\tau \in \mathbb{R}$,

$$x_t^p(\sigma, \varphi) = x_{t-\sigma-\tau}(\tau, \varphi).$$

Proof. From the definition we have

$$x_t^p(\sigma, \varphi)(\theta) = \begin{cases} 
\varphi(\sigma - t + \theta) & \text{for } t \leq \sigma, \\
\varphi(t - \sigma + \theta) & \text{for } t > \sigma.
\end{cases}$$

In addition, we also have

$$x_{t-\sigma+\tau}(\tau, \varphi) = \varphi(\tau - (t - \sigma + \tau) + \theta) = \varphi(\sigma - t + \theta)$$

for $t - \sigma + \tau \leq \tau$, i.e., $t \leq \sigma$ and

$$x_{t-\sigma+\tau}(\tau, \varphi) = \varphi(t - \sigma + \tau - \tau + \theta) = \varphi(t - \sigma + \theta)$$

for $t - \sigma + \tau > \tau$, i.e., $t > \sigma$. The proof is complete. \hfill \square

Theorem 3.2. Assume that the phase space $B$ satisfies $(A_1)$ and $(A_2)$. Further, for any given $(\sigma, \varphi) \in \Omega$, assume that $f(t, \varphi)$ is locally quasi-Lipschitzian at $(\sigma, \varphi)$. Then, there is a $b > 0$ such that the IVP (2.1) - (2.3) has a solution defined on $(-\infty, \sigma + b]$.  

Proof. As in the proof of Theorem 3.1, by Definition 3.2, there exists a $c > 0$, and an $L > 0$ such that

$$|f(\sigma + t_1, x_{\sigma + t_1}^p(\sigma, \varphi)) - f(\sigma + t_2, x_{\sigma + t_2}^p(\sigma, \varphi))| \leq L|t_1 - t_2|$$

for any $t_1, t_2 \in [0, c]$ and $q \in S$, where

$$S = \{p \in C_c(L) : p(0) = f(\sigma, \varphi)\}.$$  

Since $(\sigma, \varphi) \in \Omega$ and $\Omega$ is open, there is a $d : 0 < d < c$ such that $(\sigma - d, \sigma + d) \times S_d(\varphi) \subset \Omega$, where

$$S_d(\varphi) := \{\psi \in \Omega : \|\psi - \varphi\| \leq d\}.$$  

From Lemma 3.2, there exists a $t^* > \sigma$ such that

$$x_{t^*}^p(\sigma, \varphi) \in S_d(\varphi)$$  

for any $p \in S$ and $t \in [\sigma, b^*]$. Take $b = \min\{t^*, d\}$. For any fixed $p \in S$, we have

$$\|x_{t^*}^p(\sigma, \varphi) - \varphi\| \leq d.$$  

Then the closed set $[\sigma, \sigma + b] \times S_b(\varphi) \subset \Omega$. By Tietze's extension theorem there exists a continuous functional $F(t, \psi) : R \times B \mapsto R^n$ such that $F(t, \psi) = f(t, \psi)$ for any $(t, \psi) \in [\sigma, \sigma + b] \times S_b(\varphi)$. It is obvious that $F(t, \psi)$ is locally Lipschitzian at $(\sigma, \varphi)$. By Theorem 3.1, the IVP (2.1) - (2.3) has a solution $y(t)$ defined on $(-\infty, \sigma + b]$, and $|y(t) - y(s)| \leq L|t - s|$ for $t, s \in [\sigma, \sigma + b]$. By the definition of $F(t, \varphi)$, we assert that $y(t), t \in (-\infty, \sigma + b]$, is a solution of the IVP (2.1) - (2.3). The proof is complete. \hfill $\Box$

4. Uniqueness. To prove uniqueness of solutions of the IVP (2.1) - (2.3), we need to further assume that

\begin{itemize}
  \item[(A_6)] there is a continuous function $n(t) \geq 0$ such that $\lim_{t \to 0} n(t) = 0$ and
  \[|f(t, x_t^\varphi(\sigma, \varphi)) - f(t, x_t^\xi(\sigma, \varphi))| \leq n(t - \sigma)\|p - q\|^{0, t - \sigma}\]
  \end{itemize}

for any $\varphi \in B$, $p, q \in C_A$, $t \in [\sigma, \sigma + A]$.

Theorem 4.3. Suppose that the phase space $B$ satisfies the assumption (A_1) and that assumption (A_6) holds. Then, the IVP (2.1) - (2.3) has at most one solution.
Proof. Let $x(\sigma, \varphi)(t)$ and $y(\sigma, \varphi)(t)$, $t \in (-\infty, \sigma + b]$, $0 < b \leq A$ be two solutions of the IVP (2.1) - (2.3). Let $p(t) = x(\sigma, \varphi)(t)$, $q(t) = y(\sigma, \varphi)(t)$, $t \in [\sigma, \sigma + b]$. Then, $p, q \in C_b$. By (A6), we have

$$|p(t) - q(t)| \leq |f(t, x_0^0(\sigma, \varphi)) - f(t, x_0^0(\sigma, \varphi))|$$
$$\leq n(t - \sigma)|p - q|^{[0,t-\sigma]}, \quad t \in [\sigma, \sigma + b].$$

It follows that

$$|p - q|^{[0,t-\sigma]} \leq n(t - \sigma)|p - q|^{[0,t-\sigma]} \quad \text{for} \quad t \in [\sigma, \sigma + b].$$

Since \( \lim_{t \to 0} n(t) = 0 \), there is an $s \in (\sigma, \sigma + b]$ such that $n(s - \sigma) < 1$, and thus we have

$$|p - q|^{[0,s-\sigma]} \leq n(s - \sigma)|p - q|^{[0,s-\sigma]} < |p - q|^{[0,s-\sigma]},$$

Therefore, we conclude that $p(t) \equiv q(t)$ for $t \in [0, s - \sigma]$. Taking $s$ as a new $\sigma$ and repeating the argument above, we conclude that $p(t) \equiv q(t)$ for $t \in [0, b]$. The proof is complete.

**Corollary 4.1.** Assume that (A6) holds. In addition, assume that the conditions in Theorem 3.1 or in Theorem 3.2 are satisfied. Then, there exists a $b > 0$ such that the IVP (2.1) - (2.3) has a unique solution defined on $(-\infty, \sigma + b]$.

**5. Continuation.**

**Definition 5.5.** Let $x$ be a solution of (2.1) - (2.3) on an interval $[\sigma, b]$, $b > \sigma$. We say that $\tilde{x}$ is a continuation of $x$, if there is exists a $c > b$ such that $\tilde{x}$ is defined on $[\sigma, c]$, it coincides with $x$ on $(-\infty, b)$, and satisfies (2.1) - (2.3) on $[\sigma, c]$. A solution $x(t)$, $t \in (-\infty, b)$ of the IVP (2.1) - (2.3) is said to be noncontinuable, if no such continuation exists: that is, the interval $[\sigma, b]$ is the maximal interval of existence of the solution $x$.

The existence of a noncontinuable solution follows from the famous Zorn's lemma. Further, the maximal interval of existence must be open. Our first observation is the following lemma.

**Lemma 5.4.** Assume that the phase space $B$ satisfies (A1), (A2) and (A5). Let $y(t)$ be a solution of the IVP (2.1) - (2.3). If there exist $t_n \to b - 0$ and $\psi \in B$ such that $y_{t_n} \to \psi$ as $n \to \infty$, then

$$y(t_n) \to \varphi(0) \quad \text{as} \quad n \to +\infty. \quad (5.1)$$
Proof. We can assume that \( t_1 < t_2 < \cdots < t_n < \cdots < b \). Take \( \tau < t_1 - b < t_n - b < 0 \). By \((A_5)\), we have
\[
\|y_{t_n} - \psi\|_{[\tau,0]} \to 0 \quad \text{as} \quad n \to +\infty.
\] (5.2)
Thus \( \psi(\theta) \) is continuous in \( \theta \) for \( \theta \in [\tau,0] \). We shall prove (5.2) by contradiction. If (5.1) does not hold, then there exist an \( \epsilon > 0 \) and \( \tau_n \in [t_{n-1}, t_n] \), \( n = 2, \cdots \), such that
\[
|y(\tau_n) - \psi(0)| = |y_{\tau_n}(0) - \psi(0)| > \epsilon > 0.
\]
There is an \( N > 0 \) such that
\[
|\psi(\tau_n - t_n) - \psi(0)| \leq \frac{\epsilon}{2}. \tag{5.3}
\]
Further from (5.3), it follows that
\[
\|y_{t_n} - \psi\|_{[\tau,0]} \geq |y_{t_n}(\tau_n - t_n) - \psi(\tau_n - t_n)|
= |y(\tau_n) - \psi(\tau_n - t_n)|
= |y(\tau_n) - \psi(0) + \psi(0) - \psi(\tau_n - t_n)|
\geq |y(\tau_n) - \psi(0)| - |\psi(0) - \psi(\tau_n - t_n)|
\geq |y(\tau_n) - \psi(0)| - |\psi(0) - \psi(\tau_n - t_n)| \geq \frac{\epsilon}{2},
\]
which contradicts (5.2). The proof is hence complete.

**Theorem 5.4.** Assume that the phase space \( B \) satisfies \((A_1), (A_2), (A_5)\) and \( f(t,\varphi) \) is locally quasi-Lipschitzian on \( \Omega \), where \( \Omega \) is an open set in \( R \times B \). Then a solution \( y(t), t \in (-\infty,b), b > \sigma, \) of the IVP (2.1) \(- (2.8)\) is noncontinuable if and only if
\[
\begin{align*}
&\text{there exist } t_n \to b - 0 \text{ and } \psi \in B \\
&\text{such that } y_{t_n} \to \psi \text{ as } n \to \infty \\
&\quad \implies (b, \psi) \in \partial \Omega.
\end{align*}
\]

Proof. **Necessity:** If there are \( t_n \to b - 0 \) and \( \psi \in B \) such that \( (t_n, y_{t_n}) \to (b, \psi) \) as \( n \to +\infty \), and \( (b, \psi) \in \Omega \), then from Lemma 5.1, we have \( y(t) \to \psi(0) \), as \( t \to b - 0 \). Thus, \( y(t) \) may be extended continuously on \((-\infty,b] \). This is a contradiction, since \( y(t) \) is noncontinuable.

**Sufficiency:** Let \( x(t), t \in (-\infty,b) \) be a solution of the IVP (2.1) \(- (2.3)\). If it were continuable, then we would have \( \lim_{t \to b^-} (t, x_t) = (b, x_b) \in \Omega \), which contradicts the fact that \( (t, x_b) \in \partial \Omega \). The proof is complete.

**Theorem 5.5.** Assume that the phase space \( B \) satisfies \((A_1), (A_2)\) and \((A_5)\). Further, assume that \( W \) is a compact set in \([\sigma, \sigma + A] \times B \), \( (\sigma, \varphi) \in \partial \Omega \).
$W$, $f(t, \varphi)$ is locally quasi-Lipschitzian on $W$, and $x(t)$ is a noncontinuable solution of the IVP (2.1) - (2.3) defined on $(-\infty, b)$, $\sigma < b \leq \sigma + \Delta$. Then there is a $T > \sigma$ such that $(T, x_T) \in B - W$.

Proof. If $b = +\infty$ and $(t, x_t) \in W$ for $t \geq \sigma$, then the sequence \( \{ (\sigma + n, x_{\sigma+n}) \} \) has no convergent subsequence. This contradicts the compactness of $W$.

Suppose that $b < +\infty$, and $(t, x_t) \in W$ for $t \geq \sigma$. We can choose $t_n \to b$. Since $W$ is a compact set, there is a $\psi \in W$ such that $x_{t_n} \to \psi$ as $n \to +\infty$.

From Lemma 5.1 we have
\[
x(t) \to \psi(0) \quad \text{as} \quad t \to b.
\]

Thus, $x(t)$ may be extended continuously on $(-\infty, b]$. However, $x(t)$ is non-continuable at $t = b$. So we reach a contradiction, which proves the claim. The proof is complete. \( \square \)

**Theorem 5.6.** Assume that the phase space $B$ satisfies $(A_1)$ and $(A_2)$; $f(t, \varphi)$ is quasi-Lipschitzian on any bounded subset of $\Omega$, and $x(t)$ is a non-continuable solution of the IVP (2.1) - (2.3) defined on $[\sigma, b)$. If $x(t)$ is bounded on $[\sigma, b)$, then $b = +\infty$.

Proof. Suppose that $|x|^{[\sigma, t]} \leq \alpha < +\infty$ for any $t \in [\sigma, b)$. If $b < +\infty$, then from $(A_2)$, we have
\[
\|x_t\| \leq K(t - \sigma)|x|^{[\sigma, t]} + M(t - \sigma)\|x_\sigma\| \leq K^* \alpha + M^*\|x_\sigma\| := \beta,
\]
for any $t \geq \sigma$, where $K^* = \max_{0 \leq s \leq b-\sigma} K(s)$, $M^* = \max_{0 \leq s \leq b-\sigma} M(s)$. Then the set $E = \{(t, x_t) : t \in [\sigma, b)\}$ is bounded subset of $R \times B$. There is a constant $L(E)$ such that
\[
|x(t) - x(s)| = |f(t, x_t) - f(s, x_s)| \leq L(E)|t - s|, \text{ for any } s, t \in [\sigma, b).
\]

Therefore the limit $x(b - 0) = \lim_{t \to b-0} x(t)$ will exist, and $x(t)$ be extended continuously on $(-\infty, b]$. This is a contradiction, which proves the theorem. \( \square \)

6. Continuous Dependence. Throughout this section, we assume that $\Omega \subset R \times B$, $(\sigma, \varphi) \in \Omega$, and $x(t)$ is a solution of the IVP (2.1) - (2.3) defined on $(-\infty, b]$, $b > \sigma$. Let $W = \{(t, x_t) : t \in [\sigma, b]\}$ and $V$ be a neighborhood of $W$. Assume that $f(t, \psi)$ is quasi-Lipschitzian on $V$. Assume that the distance $d(W, B - V)$ between $W$ and $B - V$ is $2d > 0$. Define
\[
U = \{ p \in C_{b-\sigma} : |p(0) - f(\sigma, \varphi)| < d, \quad |p(t) - p(s)| \leq L(V)|t - s| \text{ for } t, s \in [0, b - \sigma] \},
\]
where $L(V)$ is $f$’s quasi-Lipschitzian constant on $V$ and

$$K^* := \max_{0 \leq t \leq b-\sigma} K(t), \quad M^* := \max_{0 \leq t \leq b-\sigma} M(t).$$

**Lemma 6.5.** Assume that the phase space $B$ satisfies $(A_1) - (A_3)$ and $(A_6)$ holds. If $p \in U$, $s \in [\sigma, b]$ and

$$|\bar{x}_s^p(\sigma, \varphi) - x_s| < \min \left( \frac{d}{2M^*}, \frac{Jd}{2K^*} \right) := \beta,$$

then,

$$\boxed{(\tau, \bar{x}_\tau^p(\sigma, \varphi)) \in V, \text{ for } \tau \in [s, s + \lambda],}$$

where $\lambda = \frac{d}{5K^* L(V)}$.

**Proof.** First, we note that

$$\|x_\tau^p(\sigma, \varphi) - x_\tau\| \leq \|x_\tau^{p_1}(s, \bar{x}_s^p(\sigma, \varphi)) - x_\tau^{x_1}(s, x_s)\|, \quad \tau \in [s, s + \lambda], \quad (6.1)$$

where $p_1(t) = p(t + s)$, $x_1(t) = x(t + s)$. From (6.1) and $(A_2)$, we get

$$\|x_\tau^p(\sigma, \varphi) - x_\tau\| \leq K(\tau - s)\|p_1 - x_1\|^{[0, \tau-s]} + M(\tau - s)\|x_\tau^p(\sigma, \varphi) - x_s\| \leq K^*\|p_1 - x_1\|^{[0, \lambda]} + M^*\|x_\tau^p(\sigma, \varphi) - x_s\|. \quad (6.2)$$

By $(A_4)$, we have

$$|p_1(0) - x_1(0)| = |x_\tau^p(\sigma, \varphi)(0) - x_s(0)| \leq \frac{1}{J}\|x_\tau^p(\sigma, \varphi) - x_s\| \leq \frac{1}{J} \min \left( \frac{d}{2M^*}, \frac{Jd}{2K^*} \right) \leq \frac{d}{2K^*}. \quad (6.3)$$

Then it follows that

$$|p_1 - x_1|^{[0, \lambda]} \leq \frac{d}{2K^*} + 2L(V)\lambda \leq \frac{d}{K^*}. \quad (6.3)$$

From (6.2) and (6.3), we have

$$\|x_\tau^p(\sigma, \varphi) - x_\tau\| \leq K^*\frac{d}{K^*} + M^*\frac{d}{2M^*} < 2d.$$
Therefore
\[ d((\tau, x^\#_\sigma), (\tau, x_\tau)) < d, \]
i.e., \((\tau, x^\#_\sigma, \sigma) \in V\). The lemma is proved. \(\square\)

**Theorem 6.7.** Assume that the phase space \(B\) satisfies \((A_1)-(A_3)\) and \((A_6)\) holds. Let \(\Omega \subset R \times B\), \((\sigma, \phi, \varphi) \in \Omega, x(t)\) be the solution of IVP (2.1) - (2.3) defined on \((-\infty, b]\), \(b > \sigma\) and \(f\) defined on \(V\) (\(V\) is defined as above) be quasi-Lipschitzian in \(t\). If \(\varphi_i \in B\) with \(\|\varphi - \varphi_i\| \to 0\) as \(i \to +\infty\), then there is an \(N > 0\) such that the IVP

\[ x(t) = f(t, x_t), \quad x_\sigma = \varphi_i, \quad (6.4) \]

has a solution \(x^i(t)\) defined on \((-\infty, b]\), provided that \(i \geq N\), and \(x^i(t)\) uniformly converges to \(x(t)\) on the interval \([\sigma, b]\).

**Proof.** Define the numbers \(\beta, \lambda, K^*, M^*, d\) as in Lemma 6.1. Let \(n(t)\) be the function in \((A_6)\). Then there is a \(\lambda_1 > 0\) such that

\[ \max_{0 \leq t \leq \lambda_1} n(t) = n^* < \frac{1}{K^*}. \]

Let \(\lambda^* = \min(\lambda, \lambda_1)\). Since \(\|\phi - \phi_1\| \to 0\) as \(i \to +\infty\), there is an \(N_1 > 0\) such that \(\|\varphi_i - \varphi\| < \beta\) for \(i > N_1\). By Theorem 5.2 and Lemma 6.1 we conclude that the IVP (6.4) has a solution \(x^i(t)\) defined on \([\sigma, \sigma + \lambda^*] := I_1\). For \(i > N_1\) and \(t \in I_1\), we have

\[ |x^i(t) - x(t)| = |f(t, x^i_t) - f(t, x_t)| \leq n(t - \sigma)||x^i_t - x_t|| \leq n(t - \sigma)[K(t - \sigma)||x^i - x||_{[\sigma, t]} + M(t - \sigma)||\varphi - \varphi_i||] \leq n^*K^*||x^i - x||_{[\sigma, \sigma + \lambda^*]} + n^*M^*||\varphi - \varphi_i||. \]

It follows that,

\[ |x^i - x||_{[\sigma, \sigma + \lambda^*]} \leq \frac{n^*M^*}{1 - n^*K^*}||\varphi - \varphi_i||. \]

Therefore, \(x^i(t)\) uniformly converges to \(x(t)\) on \(I_1\) as \(i \to +\infty\). Let \(a_1 = \sigma + \lambda^*\). Since

\[ ||x^i_{a_1} - x_{a_1}|| \leq n^*K^*||x_i - x||_{[\sigma, \sigma + \lambda_1]} + n^*M^*||\varphi - \varphi_i||, \]

there is an \(N_2 > N_1\) such that \(||x^i_{a_1} - x_{a_1}|| < \beta\) provided \(i > N_2\). By Theorem 5.2 and Lemma 6.1, we conclude that the solution \(x^i(t)\) of (6.4) is defined on
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$[\sigma, \sigma + 2\lambda^*] := I_2$. Carrying out the same argument as above we can prove that $x_i(t)$ uniformly converges to $x(t)$ on $I_2$ as $i \to +\infty$. Similarly, we can show that there is an $N_3 > N_2$ such that $\|x_{a_2}^i - x_{a_2}\| < \beta$ provided $i > N_3$, where $a_2 := \sigma + 2\lambda^*$. There is an integer $L > 0$ such that $L\lambda^* > b - \sigma$. Therefore, after repeating above argument $L$ times, the theorem will be proved. □

**Theorem 6.8.** Assume that the conditions of Theorem 6.1 hold. Then, there is an $N > 0$ such that the IVP (6.4) has a solution $x_i(t)$ defined on $(-\infty, b]$ for $i > N$, and

$$\|x_i^i - x_i\| \to 0 \text{ as } i \to +\infty,$$

for any $t \in [\sigma, b]$.

It is easy to obtain the proof from Theorem 6.1 and $(A_2)$. So we omit the details.

**7. Examples.** In this section, we first illustrate some functionals satisfying the quasi-Lipschitzian assumptions. We then, propose a concrete phase space $C_h$ for the functional equation with infinite delays.

**7.1 Functionals that satisfy the quasi-Lipschitzian assumptions.**

Let us consider the functional integral equation of the form

$$x(t) = a(t) + \int_0^t f(t, s, x(s))ds,$$

(7.1)

which is clearly a special case of (2.1), where $a(t) : R \to R$ is locally Lipschitzian in $t$; $f(t, s, \varphi) : R \times R \times B$ is continuous in $\varphi$ and $s$, and is continuously differentiable with respect to $t$. Further, $f'_i(t, s, \varphi)$ maps bounded set to bounded set.

Equation (7.1) is very general in the sense that, it includes several other types of equations as special cases. For example:

- If $a(t) \equiv a$, $f(t, s, \varphi) \equiv f(s, \varphi(0))$, then (7.1) is equivalent to a nonautonomous ordinary differential equation $x' = f(t, x(t))$.
- If $a(t) \equiv a$, $f(t, s, \varphi) \equiv f(s, \varphi(-r))$, then (7.1) is a differential difference equation $x' = f(t, x(t - r))$.
- If $a(t) \equiv a$, $f(t, s, \varphi) \equiv f(s, \varphi)$, then (7.1) reduces to a functional differential equation with infinity delay $x' = f(t, x_i)$, where $x_i(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$.
- If $f(t, s, \varphi) \equiv f(t, s, \varphi(0))$, then (7.1) becomes an integral equation

$$x(t) = a(t) + \int_0^t f(t, s, x(s))ds.$$
• The IVP of Volterra type integral equation with infinite delay of the form

\[ x(t) = a(t) + \int_{-\infty}^{t} f(t, s, x(s))ds, \quad x(s) = \varphi(s), \quad s \in (-\infty, 0]. \]

Let

\[ A(t) = a(t) + \int_{-\infty}^{0} f(t, s, \varphi(s))ds. \]

We then obtain the following equation

\[ x(t) = A(t) + \int_{0}^{t} f(t, s, x(s))ds, \]

which is of form (7.1).

• The following integral equation involving double integrals

\[ x(t) = a(t) + \int_{0}^{t} \int_{-\infty}^{s} C(t, \tau, x(\tau))d\tau ds \]

is obviously a special case of (7.1).

Now, let

\[ F(t, \varphi) = \int_{0}^{t} f(t, s, \varphi_{s-t})ds. \]

**Lemma 7.6.** Assume that the phase space \( B \) satisfies (A1), (A2) and (A4). Then, for any \((\sigma, \varphi) \in \mathbb{R} \times B \) with \( \sigma \geq 0 \), the functional \( g(t, \varphi) := a(t) + F(t, \varphi) \) is locally quasi-Lipschitzian at \((\sigma, \varphi)\).

**Proof.** For any given \( p \in C_{A}(L) \), we have

\[ \frac{d}{dt} F(t, x_{t}^{p}(\sigma, \varphi)) = \frac{d}{dt} \int_{0}^{t} f(t, s, x_{s}^{p}(\sigma, \varphi))ds \]

\[ = f(t, t, x_{t}^{p}(\sigma, \varphi)) + \int_{0}^{t} f_{s}(t, s, x_{s}^{p}(\sigma, \varphi))ds. \]

(7.2)

From (A1) and (A2), it follows that,

\[ \frac{d}{dt} F(t, x_{t}^{p}(\sigma, \varphi)) := F'(t, x_{t}^{p}(\sigma, \varphi)) \]
is continuous in \((t, \sigma, \varphi, p)\). By \((A_4)\), we have
\[
\max_{0 \leq s \leq \sigma} \|\varphi_{s-\sigma}\| \leq q(-\sigma)\|\varphi\|.
\]
Thus, there is an \(M > 0\) such that \(|f'_t(\sigma, s, \varphi_{s-\sigma})| \leq M\). Clearly \(M\) is independent of \(p\). From
\[
F'(\sigma, x_\sigma^p(\sigma, \varphi)) = f(\sigma, \sigma, \varphi) + \int_0^\sigma f'_t(\sigma, s, \varphi_{s-\sigma})\,ds,
\]
it follows that,
\[
|F'(\sigma, x_\sigma^p(\sigma, \varphi))| \leq |f(\sigma, \sigma, \varphi)| + M\sigma.
\]
Since \(a(t)\) is locally Lipschitzian in \(t\), there is a \(\delta > 0\) and an \(L_1 > 0\) such that,
\[
|a(t) - a(s)| \leq L_1|t - s|, \text{ for any } t, s \in [\sigma, \sigma + \delta].
\]
From the continuity of \(f(t, s, \varphi)\), there is an \(\epsilon_1 > 0\) such that,
\[
|f(t, t, \psi)| \leq 2|f(\sigma, \sigma, \varphi)|, \text{ } t \in [\sigma, \sigma + \epsilon_1], \psi \in S_{\epsilon_1}(\varphi).
\]
Further, there is an \(\epsilon_2 > 0\) such that,
\[
|f'_t(t, s, \psi)| \leq 2M \text{ for } t, s \in [\sigma, \sigma + \epsilon_2], \psi \in S_{\epsilon_2}(\varphi).
\]
Take \(L = L_1 + 2|f(\sigma, \sigma, \varphi)| + 2M(\sigma + 1)\). For any \(p \in C_A(L)\) with \(p(0) = f(\sigma, \varphi)\), by Lemma 3.1, there is a \(t^* > 0\) such that,
\[
x_t^p(\sigma, \varphi) \in S_{\epsilon_1}(\varphi) \bigcup S_{\epsilon_2}(\varphi),
\]
for \(t \in [\sigma, \sigma + t^*], \text{ } p \in C_A(L)\). Choose \(\epsilon = \min(\epsilon_1, \epsilon_2, t^*, 1)\). We have
\[
|g(t, x_t^p(\sigma, \varphi)) - g(s, x_s^p(\sigma, \varphi))| \leq L|t - s|,
\]
for \(p \in C_A(L), \text{ } t, s \in [\sigma, \sigma + \epsilon]\). Hence, the lemma is proved. \(\square\)

**Lemma 7.7.** Assume that the phase space \(B\) satisfies \((A_1), (A_2), (A_4)\) and

(i) for any bounded subinterval \(I \subset R\), there is an \(L(I) > 0\) such that,
\[
\|a(t) - a(s)\| \leq L(I)|t - s|, \text{ for } t, s \in I;
\]

(ii) \(f(t, s, \varphi)\) maps bounded set into bounded set.
Then, the functional \( g(t, \varphi) \) is quasi-Lipschitzian on any bounded set \( E \subset \mathbb{R} \times B \).

Proof. Let \( E \subset \mathbb{R} \times B \) be bounded. It follows that there is a bounded subinterval \( I \subset \mathbb{R} \) and a bounded subset \( B_1 \subset B \) such that \( (t, \varphi) \in E \) implies \( t \in I \) and \( \varphi \in B_1 \). Let

\[
A = \sup_{t, s \in I, \varphi \in B_1} \| f(t, s, \varphi) \|, \quad B = \sup_{t, s \in I, \varphi \in B_1} \| f'(t, s, \varphi) \|.
\]

If \( (t, x^p_t(\sigma, \varphi)) \in E \) for \( t \in I \), then from (7.2), we have

\[
\left\| \frac{d}{dt} F(t, x^p_t(\sigma, \varphi)) \right\| \leq A + m(I)B,
\]

where \( m(I) \) is the measure of \( I \). It follows that

\[
\| f(t, x^p_t(\sigma, \varphi)) - f(s, x^p_s(\sigma, \varphi)) \| \leq [L(I) + A + m(I)B]\|t - s\|.
\]

The proof is complete. \( \square \)

### 7.2 Phase space for functional equations with infinite delays.

Let \( h : \mathbb{R}^- \to \text{int} \mathbb{R}^+ \) be continuous satisfying \( \int_{-\infty}^0 h(s) \text{d} s < \infty \). Define

\[
C_h = \left\{ \varphi \in C(\mathbb{R}^-, \mathbb{R}^n) : \int_{-\infty}^0 h(s)\|\varphi\|[s,0] \text{d} s < \infty \right\}
\]

and

\[
\|\varphi\|_h = \int_{-\infty}^0 h(s)\|\varphi\|[s,0] \text{d} s,
\]

where

\[
\|\varphi\|[s,0] = \sup_{s \leq u \leq 0} |\varphi(u)|.
\]

It has been proved that \( \| \cdot \| \) is a norm and \((C_h, \| \cdot \|)\) is a Banach space \([15, 16]\).

As in \([15, 16]\), one can easily prove that the phase space \( C_h \) satisfies the assumptions \((A_1) - (A_3)\) for the general phase space \( B \).

**Lemma 7.8.** If there is a continuous decreasing function \( q : \mathbb{R}^- \to \mathbb{R}^+ \) with \( q(0) = 1 \) such that,

\[
h(s) \leq h(s + \tau)q(\tau), \text{ for } t, s \in \mathbb{R}^-,
\]

then...
then the phase space $C_h$ satisfies assumption ($A_4$).

Proof. In fact,

\[
\|\varphi - \psi\|_h = \int_{-\infty}^{0} h(s)\|\varphi\|_{[s,0]}\,ds
\]

\[
\leq \int_{-\infty}^{0} h(s)\|\varphi\|_{[s+\tau,0]}\,ds
\]

\[
\leq q(\tau) \int_{-\infty}^{0} h(s + \tau)\|\varphi\|_{[s+\tau,0]}\,ds
\]

\[
= q(\tau) \int_{-\infty}^{\tau} h(s)\|\varphi\|_{[s,0]}\,ds < q(\tau)\|\varphi\|_h.
\]

The proof is complete. \qed

REMARK 7.2. Take $h(s) = e^s$ and $q(\tau) = e^{-\tau}$, then the assumption in Lemma 7.3 is satisfied, i.e., $h(s) = h(s + \tau)q(\tau)$.

LEMMA 7.9. The phase space $C_h$ satisfies ($A_5$).

Proof. Note that

\[
\|\varphi - \psi\|_h = \int_{-\infty}^{0} \|\varphi - \psi\|_{[s,0]}\,ds
\]

\[
\geq \int_{-\infty}^{\tau} h(s)\|\varphi - \psi\|_{[s,0]}\,ds
\]

\[
\geq \int_{-\infty}^{\tau} h(s)\|\varphi - \psi\|_{[\tau,0]}\,ds.
\]

Thus

\[
\|\varphi - \psi\|_{[\tau,0]} \leq Q(\tau)\|\varphi - \psi\|_h,
\]

where $Q(t) = [\int_{-\infty}^{t} h(s)\,ds]^{-1}$. The proof is complete. \qed

To conclude, we have established a concrete phase space $C_h$ for functional equations with infinite delay of the form (2.1).

REMARK 7.3. In [12], the authors have established the relationship between the phase space $C_h$ [15, 16] and the phase space $C_g$ [6]. Following the ideas presented in this paper and in [6, 12, 15, 16], one can easily establish another phase space $C_g$ for functional equations with infinite delay.

REFERENCES


