FUNCTIONAL DIFFERENTIAL EQUATIONS

Guest Editor
A.L. SKUBACHEVSKII

VOLUME 12, 2005
No. 3-4

DEDICATED TO A.D. MYSHKIS
ON THE OCCASION OF HIS
85th BIRTHDAY

THE COLLEGE OF JUDEA & SAMARIA
ARIEL, ISRAEL
# TABLE OF CONTENTS

G. Kamenskii. On boundary value problems connected with variational problems for nonlocal functionals. 245

I. Kiguradze and B. Půža. On two-point boundary value problems for second order singular functional differential equations. 271

V. Kolmanovskii. Boundedness in average for Volterra nonlinear difference equations. 295

S. Kornev and V. Obukhovskii. On some developments of the method of integral guiding functions. 303

V.G. Kurbatov and V.I. Kuznetsova. Ryabov’s theorem on small lag: the dichotomy of solutions point of view. 311


I.L. Oynas and Z.B. Tsalyuk. The structure of the resolvent of the discrete renewal equation with nonsummable kernel. 365

N. Roytvarf and Y. Yomdin. Analytic continuation of Cauchy-type integrals. 375

B. Shklyar and V. Marchenko. Approximate null-controllability of distributed systems with unbounded input operator by distributed controller. 389

I.V. Shragin. On structure of some $\sigma$-algebras related to measurability of superpositions. 407

A. Spivak and Z. Schuss. The short time asymptotic expansion for the trace of the heat kernel by ray method. 415

V.V. Vlasov and Jianhong Wu. Sharp estimates of solutions to neutral equations in Sobolev spaces. 437

M.N. Zubova and T.A. Shaposhnikova. On homogenization of variational inequalities with obstacles on $\varepsilon$ - periodically situated inclusions. 463
ON BOUNDARY VALUE PROBLEMS CONNECTED WITH VARIATIONAL PROBLEMS FOR NONLOCAL FUNCTIONALS *

G. KAMENSKII †

Abstract. It is a survey of the theory of boundary value problems arising in the variational problems for nonlocal functionals. There are described the statements of different variational problems for nonlocal functionals and the arising here boundary value problems, the problems of existence and uniqueness of solutions to symmetrical and asymmetrical boundary value problems for functional differential equations and for mixed functional differential equations, problems of smoothness of solutions and the approximate methods of solution of these problems.

Key Words. Boundary value problems, functional differential equations, variational problems, nonlocal functionals.

34K10

Introduction. There are different statements of boundary value problems for functional differential equations. See, for example, [5],[14],[37],[38]. In this paper we consider only such kind of boundary value problems that arise in the theory of extrema of nonlocal functionals and of mixed nonlocal functionals. There are considered the generalized solutions to FDE. They appear necessarily by investigation of boundary value problems for functional differential equations connected with nonlocal functionals. In the first part of this paper we describe the problems of extrema for four types of nonlocal functionals and the arising here the boundary value problems for FDE. The second part of the paper is a survey of the theory of these boundary value problems. It contains the theorems of existence and uniqueness of solutions, the theory of linear boundary value problems, problems of smoothness of

* Supported by the Russian Foundation of Basic Research (Grant 03-01-00174)
† Department of Differential Equations, Moscow Institute of Aviation, Moscow, 125871, Russia
solutions to these problems. The third part of the paper contains the description of the approximate methods of these boundary value problems.

1. Variational problems for nonlocal functionals. There are considered symmetrical and asymmetrical problems of extrema for nonlocal functionals and for mixed nonlocal functionals. The most developed are the symmetrical problems of extrema for nonlocal functionals and corresponding boundary value problems (BVP). These problems will be described in the most detailed way. Other problems will be described by the comparison with the first ones.

1.1. Symmetrical variational problems for nonlocal functionals. Consider the problem of extremum of the functional

$$J(y) = \int_a^b F(t, y(t), y(\alpha(t)), y'(t), y'(\alpha(t))) \, dt.$$  

It is supposed that $y \in \mathbb{R}^1$, $\alpha(t)$ is a given scalar continuously differentiable function, $\alpha(t)$ is monotonic increasing on $[a, b]$, $\alpha(t) \leq t$ and has a continuously differentiable inverse function $\gamma$. Denote $A = \alpha(a)$, $B = \alpha(b)$ and $E_a = [A, a]$, $E_b = [B, b]$. Suppose that $A < a$ and $B > b$. It is given on $E_a$ the left boundary function $\varphi(t)$ and it is given on $E_b$ the right boundary function $\psi(t)$. Functional (1.1) is considered under boundary conditions

$$y(t) = \varphi(t) \text{ for } t \in E_a; \quad y(t) = \psi(t) \text{ for } t \in E_b.$$  

Many examples, one of which is given below, show that by all smooth $F, \varphi, \psi,$ and $\alpha$, as a rule, the solutions of problem (1.1), (1.2) are not smooth and therefore it is natural to assume that admissible functions belong to the space $\mathcal{AC}[a, b]$ of absolutely continuous functions satisfying the boundary conditions (1.2). Usually it is supposed additionally that the admissible functions belong to the Sobolev’s space $W^1_2$ (or to $W^1_{\infty}$). The function $F \in \mathbb{R}^1$ is assumed to be continuous and twice continuously differentiable with respect to the aggregate of all arguments, except $t$. It is supposed that there are fulfilled some conditions on $F$ that guarantee the existence of all encountered integrals. Consider the function $F(u_1, u_2, u_3, u_4, u_5)$ as function of arguments $u_1, \ldots, u_5$ and put

$$u_1 = t, u_2 = y(t), u_3 = y(\alpha(t)), u_4 = y'(t), u_5 = y'(\alpha(t)).$$

Denote the partial derivatives of $F$ with respect to $u_2, u_3, u_4, u_5$ at the values (1.3) as $F_{y(t)}, F_{y(\alpha(t))}, F_{y'(t)}, F_{y'(\alpha(t))}$ or, for brevity, $F_y, F_{y\alpha}, F'_{y}, F'_{y\alpha}$, correspondingly. Analogous notation is used in similar cases and later. Introduce the notation

$$\tilde{F} = F(\gamma(t), y(\gamma(t)), y(t), y'(\gamma(t)), y'(t)).$$
Denote \( \tilde{F}_{y(t)} \) and \( \tilde{F}'_{y(t)} \) the partial derivatives of \( \tilde{F} \) with respect to the third and the fifth arguments. Denote

\begin{align}
\Phi &= F(t, y(t), y(\alpha(t)), y'(\alpha(t))) + \\
&\quad + F(\gamma(t), y(\gamma(t), y(t), y'(\gamma(t), y'(t)))y'(t).
\end{align}

It is proved in [21] the following theorem

**Theorem 1.1.** If functional (1.1) under boundary conditions (1.2) attains on \( y(t) \) an extremum in \( W^1_2 \) (or in \( W^1_\infty \)), then there is such a constant \( C \) that \( y(t) \) satisfies almost everywhere on \([a, B]\) the equation

\begin{align}
\int_a^t \Phi_{y(s)} \, ds - \Phi_{y'(t)} &= C.
\end{align}

The function \( \Phi_{y'(t)} \) in (1.6) is the sum of an indefinite integral and a constant and is an absolutely continuous function. Therefore it can be differentiated and \( y(t) \) must satisfy the equation

\begin{align}
\Phi_{y(t)} - \frac{d}{dt} \Phi_{y'(t)} &= 0.
\end{align}

This equation is an analog to the Euler equation for functional (1.1). The function \( y \) that is the solution to problem (1.1),(1.2) is the generalized solution to equation (1.7) and must satisfy this equation almost everywhere on \([a, B]\). The second derivative, may be, does not exist, but the composite function \( \Phi_{y'(t)} \) must be absolutely continuous and its derivative must almost everywhere coincide with \( \Phi_{y(t)} \).

**Theorem 1.2.** Suppose that all conditions of Theorem 1.1 are fulfilled and \( \xi \) is a corner point of the solutions to problem (1.1), (1.2). Then the equality

\begin{align}
\Phi_{y'}|_{\xi-0} = \Phi_{y'}|_{\xi+0}.
\end{align}

holds.

Equation (1.8) is an analog to the Weierstrass-Erdman condition. The analog to the second Weierstrass-Erdman condition has the form

\begin{align}
(\Phi - y'\Phi_y')|_{\xi-0} = (\Phi - y'\Phi_y')|_{\xi+0}
\end{align}

and was derived in [27].

**Example.** Find an extremum of the functional

\begin{align}
J(y) = \int_0^4 (y'^2(t) + y'(t)y'(t - 1) + y(t)) \, dt
\end{align}

with boundary conditions

\begin{align}
y(t) = 0 \quad \text{at} \quad t \in [-1, 0] \cup [3, 4].
\end{align}

Equation (1.7) for this functional takes the form

\begin{align}
1 - \frac{d}{dt}(y'(t - 1) + 2y'(t) + y'(t + 1)) = 0.
\end{align}

This equation must be solved under boundary conditions (1.11). It was solved in [47] and proved that under conditions of continuity of \( y(t) \) at the
points 1, 2, 3, 4 and the continuity of function $y'(t - 1) + 2y'(t) + y'(t + 1)$ at the points 2, 3 the solution is

$$y_0(t) = \begin{cases} \frac{t^2}{4} - \frac{3}{4}t & \text{at } t \in [0, 1], \\ -\frac{1}{2} & \text{at } t \in [1, 2], \\ \frac{t^2}{4} - \frac{3}{4}t & \text{at } t \in [2, 3]. \end{cases}$$

(1.13) and that functional (1.10) attains the absolute minimum on $y_0(t)$. But we can find a more "smooth" solution of BVP (1.10), (1.11). Instead of continuity of $y'(t - 1) + 2y'(t) + y'(t + 1)$ at the points 1, 2 (remark, that this is a necessary condition of the extremum of the considered functional) demand the continuity of $y'(t)$ at the points 1, 2. In [39] it was found the function

$$y_1(t) = \begin{cases} \frac{1}{4}t^2 - \frac{1}{2}t, & \text{at } t \in [0, 1]; \\ \frac{1}{4} & \text{at } t \in [1, 2]; \\ \frac{1}{4}t^2 - t - \frac{3}{4} & \text{at } t \in [2, 3]. \end{cases}$$

(1.14) which satisfies equation (1.10) under these boundary conditions. This function belongs to $C^1(0, 3)$, but not to $C^1(-1, 4)$, and it does not solve the corresponding variational problem. Theory of this type BVP was studied in [37], [38] and some other papers. The analysis of these two kinds of BVP and properties of their solutions was made in [39].

We describe here two important applications of variational problems for nonlocal functionals. In paper the of J.A.Wheeler and R.Ph. Feynman [65] the study of the direct interparticle action in a moving field based on a variational principle. The mathematical interpretation of a particular case of this problem was made by L.S.Schulman [63]. He considered the problem of minimum of the functional

$$S(x) = \int_a^b [(x'(t))^2 - \omega^2 x^2(t) + \alpha x(t + \frac{T}{2})x(t - \frac{T}{2})] \, dt$$

(1.15) with symmetrical boundary conditions. He derived an analog to the Euler equation for this functional. N.N.Krasovskii in his book [56] considered the problem of damping of a control system with retardation. Suppose that it is described by the scalar equation

$$y'(t) = Ay(t) + By(t - \tau) + Cu(t) \quad (\tau = \text{const > 0})$$

(1.16) with initial condition

$$y(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0],$$

(1.17) where $\varphi(t)$ is a given initial function. The problem is to determine the control function $u(t)$ such that the trajectory $y(t)$ on an interval $[t_1, t_1 + \tau]$ stays equal to zero. If we take $t_1 + \tau$ for the new initial point and put $u(t) \equiv 0$ for $t \geq t_1 + \tau$, then it will be $y(t) \equiv 0$ for $t \geq t_1 + \tau$. N.N. Krasovskii solved
this problem with the additional condition that

$$\int_{t_0}^{t_1+\tau} u^2(t) \, dt \rightarrow \text{min}.$$  

This problem can be reduced to a symmetrical variational problem for a non-local functional. It is evident that the problem of solution of equation (1.16) under boundary conditions

$$y(t) = \varphi(t), \quad t \in [t_0 - \tau]; \quad y(t) = 0, \quad t \in [t_1, t_1 + \tau]$$

with the additional condition

$$\int_{t_0}^{t_1+\tau} u^2(t) \, dt \rightarrow \text{min}$$

is equivalent to the symmetrical variational problem for the functional

$$\int_{t_0}^{t_1+\tau} [y'(t) - Ay(t) - By(t - \tau)]^2 \, dt \rightarrow \text{min}$$

This problem can be solved with the use of theories developed in [21], [27]. In the similar way can be solved the damping problem for the general control nonlinear system of the neutral type

$$G(t, y(t), y(t- \tau), y'(t), y'(t- \tau), u(t)) = 0$$

if this equation can be solved with regard to $u(t)$. The case, when this solution is difficult or impossible to achieve, is considered in [25] and solved with the use of the theory of conditional extremum of nonlocal functionals. Another approach to the damping problem for linear control systems was proposed in [10].

1.2. Comments and Bibliography. The first statement and investigation of the symmetrical variational problem for nonlocal functionals was made in the book [12]. To the development of the theory of symmetrical variational problems for nonlocal functionals is dedicated the monograph [47], the necessary conditions of the Lagrange type and of the Weierstrass type were received in [27], conditional extremum of nonlocal functionals with symmetric boundary conditions was studied in [23], some sufficient conditions for nonlocal functional with symmetric boundary conditions were received in [24], the Weierstrass-Erdman type conditions were described in [21, 27]. Another approach to these problems see in papers [7, 9].

1.3. Asymmetrical variational problems for nonlocal functionals. In this section we consider the problem of extremum of the functional

$$(1.19) \quad J(y) = \int_{a}^{b} F(t, y(t), y(\alpha(t)), y'(t), y'(\alpha(t))) \, dt \rightarrow \text{extr.}$$

It is supposed here that $y, \alpha, F$ satisfy the same conditions and $\tilde{F}$ and $\Phi$ are defined in the same way as in the case of symmetrical boundary conditions. Denote $A = \alpha(a), \quad B = \alpha(b)$ and $E_a = [A, a]$. It is given on $E_a$ the
left boundary function \( \varphi(t) \). Functional (1.19) is considered under boundary conditions

\[
(1.20) \quad y(t) = \varphi(t) \quad \text{for} \quad t \in E_a; \quad y(b) = y_b.
\]

It is supposed that admissible functions belong to the space \( W^1_2(a, b) \) or to \( W'_\infty(a, b) \). As before, denote

\[
\tilde{F} = \gamma'(t)F(\gamma(t), y(\gamma(t)), y(t), y'(\gamma(t)), y'(t)))
\]

and \( \Phi = F + \tilde{F} \). Introduce now the notation

\[
(1.21) \quad \Psi = \begin{cases} 
\Phi, & \text{if} \quad t \in [a, B] \\
F, & \text{if} \quad t \in (B, b)
\end{cases}
\]

**Theorem 1.3.** Let an extremum of functional (1.19) under boundary conditions (1.20) be attained at \( y \) in the space \( W^1_2 \). Then there exists such a constant \( C \) that \( y \) almost everywhere on \( [a, b] \) satisfies the equation

\[
(1.22) \quad \int_a^t \Psi_y dt - \Psi_y = C.
\]

We can differentiate this equation and write it in the form

\[
(1.23) \quad \Psi_y - \frac{d}{dt} \Psi_y = 0.
\]

Equation (1.23) is the generalized Euler equation and must be solved with boundary conditions (1.20).

**Theorem 1.4.** Let \( y \) be a solution of problem (1.19), (1.20). Then at the relation

\[
(1.24) \quad \Psi_y|_{\xi^{-}} = \Psi_y|_{\xi^{+}}
\]

holds at every corner point of the solution \( y \) of problem (1.19), (1.20).

Equality (1.24) is the generalized Weierstrass - Erdman's condition.

1.4. **Comments and Bibliography.** First investigations of asymmetrical variational problems for nonlocal functional were made by L.E. El'sgol'ts and the results were described in the Chapter 6 of his book [16]. L.E. El' sgol'ts derived the Euler type equation and the Lagrange type condition for asymmetrical problem for nonlocal functionals. He considered also the moving boundary problem and the conditional extremum for these functionals. L.E. El'sgol'ts understood that classical solutions of these variational problems, may be, do not exist. He wrote: "Remark that the methods of solution or, at least, of qualitative investigation of the boundary value problems for the derived Euler type equation are almost wholly unknown." ([16], Chapter 6, §5.) The development of this theory, where the generalized solutions of this problem were introduced, was made in [26]. There were also considered the cases of many unknown functions, of functionals depending on many unknown functions and their higher derivatives, there were derived the necessary conditions of the Legendre type and the Bolza problem was
investigated. The moving boundary problem for nonlocal functionals with asymmetric boundary conditions was studied in [32]. Consider now the problem of damping of the control system

\[(1.25) \quad G(t, y(t), y(t - \tau), y'(t), y'(t - \tau), u(t)) = 0\]

in the case, when we are not interested in the behavior of the trajectory after the moment \(t_1\), but with the condition

\[\int_{t_0}^{t_1} u^2(t) \, dt \rightarrow \min.\]

If equation (1.25) can be solved with respect to \(u(t)\), then this problem can be reduced to the asymmetrical variational problem and can be solved with the use of the theory developed in [26] In the case, when it is difficult or impossible to solve this equation with respect to \(u\), it would be useful to apply a theorem of conditional extremum for asymmetrical variational problem analogous to the theorem proved in [23]. It would be interesting to derive the second generalized Weierstrass-Erdman condition for the asymmetrical case.

1.5. Symmetrical variational problem for the mixed type non-local functionals. There is considered the problem of extremum of the nonlocal functional

\[(1.26) \quad J(u) = \int_{t_0}^{t_1} \int_{s_0}^{s_1} F[t, s, u(t, s - r_1), \ldots, u(t, s), \ldots, u(t, s + r_2), u'(t, s - r_1), \ldots, u'(t, s), \ldots, u'(t, s + r_2)] \, ds,\]

where \(u : \mathbb{R}^2 \rightarrow \mathbb{R}^1, s_1 - s_0 > r_1 + r_2, r_1, r_2\) are some integers; \(u'(t, s)\) denotes the partial derivative of \(u(t, s)\) with respect to \(t\). On the set \(E_0 = \{(t, s) | t \in [t_0, t_1], s \in (s_0 - r_1, s_0 + r_2)\}\) the boundary function \(\varphi(t, s)\) is given; on the set \(E_1 = \{(t, s) | t \in [t_0, t_1], s \in (s_1 - r_1, s_1 + r_2)\}\) the boundary function \(\psi(t, s)\) is given. On the intervals

\[G_0 = \{(t, s) | t = t_0, s \in (s_0 + r_2, s_1 - r_1)\},\]
\[G_1 = \{(t, s) | t = t_1, s \in (s_0 + r_2, s_1 - r_1)\}\]

there are given functions \(\mu(s)\) and \(\nu(s)\). The problem of extremum of the functional (1.26) is considered under boundary conditions

\[(1.27) \quad u(t, s) = \varphi(t, s), \quad (t, s) \in E_0; \]
\[u(t, s) = \psi(t, s), \quad (t, s) \in E_1;\]
\[u(t, s) = \mu(s), \quad (t, s) \in G_0;\]
\[u(t, s) = \nu(s), \quad (t, s) \in G_1;\]

Define here the set \(Q = (t_0, t_1) \times (s_0 + r_2, s_1 - r_1)\). Denote \(L_2(Q)\) the space of square integrable in \(t\) at almost all \(s \in (s_0 + r_2, s_1 - r_1)\) and piecewise continuous in \(s\) functions \(u(t, s)\) and \(H(Q)\) the space of absolutely continuous with respect to \(t\) at almost all \(s \in (s_0 + r_1, s_1 - r_2)\) functions \(u(t, s)\) such that
u ∈ L₂(Q) and u' ∈ L₂(Q) with the norm

\[ ||u||_H = \left( ||u||_{L₂(Q)}^2 + ||u'||_{L₂(Q)}^2 \right)^{\frac{1}{2}} \]

It is assumed that admissible functions \( u(t, s) \) ∈ \( \mathbb{H}(Q) \) and \( \varphi \in \mathbb{H}(E₀) \), \( \psi \in \mathbb{H}(E₁) \); the functions \( \mu \) and \( \nu \) are integrable and belong to \( L₂(G₀) \) and \( L₂(G₁) \), correspondingly. Introduce the notation

\[ \Phi(t, s, u(t, s - r₁ - r₂), \ldots, u(t, s), \ldots, u(t, s + r₁ + r₂), \]

\( u'(t, s - r₁ - r₂), \ldots, u'(t, s), \ldots, u'(t, s + r₁ + r₂)) \]

(1.29)

\[ = \sum_{i=-r₂}^{r₁} F(t, s + i, u(t, s - r₁ + i), \ldots, u(t, s + r₂ + i), \]

\( u'(t, s - r₁ + i), \ldots, u'(t, s + r₂ + i)). \]

**Theorem 1.5.** If functional (1.26) attains an extremum on \( u(t, s) \), then there exists a constant \( C \) such that \( u(t, s) \) satisfies almost everywhere on \( Q \) the equation

(1.30) \[ \Phi_{u'(t,s)} = \int_{t₀}^{t} \Phi_{u(t,s)} dt + C, \]

**Theorem 1.6.** Suppose that functional (1.26) attains an extremum on \( u(t, s) \). Then \( u(t, s) \) satisfies almost everywhere on \( Q \) the equation

(1.31) \[ \Phi_{u(t,s)} - \frac{d}{dt} \Phi_{u'(t,s)} = 0. \]

Equation (1.31) is a generalized Euler’s equation for the considered problem.

**1.6. Asymmetrical variational problem for the mixed type nonlocal functionals.** There is considered the problem of extremum of the nonlocal functional

(1.32) \[ J(u) = \int_{t₀}^{t₁} dt \int_{s₀}^{s₁} F[t, s, u(t, s - h), u(t, s), u'(t, s - h), u'(t, s)] ds, \]

where \( h > 0, s₁ - s₀ > h \) are some integers. Here as before \( u'(t, s) \) denotes the partial derivative of \( u(t, s) \) with respect to \( t \). Suppose that on the set \( E₀ = \{(t, s)|t \in [t₀, t₁], s \in [s₀ - h, s₁]\} \) there is given the boundary value function \( \varphi(t, s) \); On the intervals

\( G₀ = \{(t, s)|t = t₀, s \in (s₀, s₁)\}, \)

\( G₁ = \{(t, s)|t = t₁, s \in (s₀, s₁ - h)\} \)

there are given functions \( \mu(s) \) and \( \nu(s) \).

On the interval \( P = \{t \in [t₀, t₁], s = s₁\} \) there is given the function \( \chi(t) \). Functional (1.32) is considered with boundary conditions
(1.33) \[ u(t, s) = \varphi(t, s), \quad (t, s) \in E_0; \]
\[ u(t, s_1) = \chi(t), \quad t \in [t_0, t_1]; \]
(1.34) \[ u(t, s) = \mu(s), \quad (t, s) \in G_0; \]
\[ u(t, s) = \nu(s), \quad (t, s) \in G_1. \]

Define here the set
\[ Q = (t_0, t_1) \times (s_0, s_1). \]

Denote \( L_2(Q) \) the space of square integrable functions \( u(t, s) \) and \( \mathcal{H}(Q) \) the space of absolutely continuous with respect to \( t \) almost everywhere on \( Q \) functions \( u(t, s) \) such that \( u \in L_2(Q) \) and \( u' \in L_2(Q) \) with the norm
\[ ||u||_{\mathcal{H}} = \left( ||u||_{L_2(Q)} + ||u'||_{L_2(Q)} \right)^{\frac{1}{2}}. \]

Define now \( \tilde{F} := F(t, s + h, u(t, s), u(t, s + h), u'(t, s), u'(t, s + h)) \) and \( \tilde{F}_u \) and \( \tilde{F}_{u'} \) denote the partial derivatives of \( \tilde{F} \) with respect to the third and to the fifth arguments, respectively. Introduce the notation

(1.35) \[ \Phi = F + \tilde{F}; \]
\[ \Psi = \Phi \quad \text{if} \quad s_0 \leq s \leq s_1 - h; \]
\[ \Psi = F \quad \text{if} \quad s_1 - h < s \leq s_1. \]

**Theorem 1.7.** Let an extremum of functional (1.32) be attained at \( u \) under boundary conditions (1.33),(1.34) in the space \( \mathcal{H}(Q) \). Then there is such a constant \( C \) that \( u \) almost everywhere on \( Q \) satisfies the equation

(1.36) \[ \int_{t_0}^{t} \Psi_u(t, s) \, dt - \Psi_{u'}(t, s) = C. \]

The equation (1.36) can be written in the differential form

(1.37) \[ \Psi_u(t, s) - \frac{d}{dt} \Psi_{u'}(t, s) = 0, \]

which is also satisfied by \( u(t, s) \) almost everywhere on \( Q \), and is an analog of the Euler equation for the considered problem.

**1.7. Comments and Bibliography.** The problem of extremum of functional (1.26) was studied in [28], and in [49],[50]. The sufficient conditions for the weak extremum of the mixed type nonlocal functionals are published in [51]. The conditional extremum of the functionals of this type is considered in [49]. The problem (1.32),(1.33),(1.34) was studied in [34]. In this paper there were considered also the necessary condition of the Legendre type and the case of functionals of this type depending on many unknown functions.

2.1. Statements of problems. There are different statements of boundary value problems (BVP) for functional differential equations. See, for example, [5],[6],[14]. We consider here only such kinds of BVP that arise in the theory of extrema of nonlocal functionals and for which there is developed the theory of generalized solutions. They appear necessarily by investigation of BVP arising in the theory of extrema of nonlocal functionals. Consider the functional differential equation

\[ F(y(a_1(t)), \ldots, y(a_m(t)), y'(a_1(t)), \ldots, y'(a_m(t))) + \Phi(t, y(a_1(t)), \ldots, y(a_m(t)), y'(a_1(t)), \ldots, y'(a_m(t))) = 0. \]

We shall use the following short form of this equation:

\[ f(t, \{y(t)\}, \{y'(t)\}) + \Phi(t, \{y(t)\}, \{y'(t)\}) = 0. \]

Here \( y, F, \Phi \in \mathbb{R}^n \), \( t \in [a, b] \) and \( \sigma_j(t), j = 1, \ldots, m \) are given scalar continuous functions on \([a, b]\). Denote

\[ \alpha = \min_{1 \leq j \leq m} \sigma_j(t), \quad \beta = \max_{1 \leq j \leq m} \sigma_j(t). \]

There are given a boundary function \( \varphi(t) \) on \([\alpha, a]\) and a boundary function \( \psi(t) \) on \([b, \beta]\) and equation (2.2) is considered under boundary conditions

\[ y(t) = \varphi(t) \quad (t \in [\alpha, a]), \quad y(t) = \psi(t) \quad (t \in [b, \beta]). \]

We seek solution to BVP (2.2),(2.3) in the space \( AC[a, b] \) of absolutely continuous on \([a, b]\) functions. The boundary functions also belong to the spaces \( AC[a, a] \) and \( AC[b, b] \), correspondingly. If \( \sigma_j(t) \leq a \), then \( y(\sigma_j(t)) = \varphi(\sigma_j(t)) \) and if \( \sigma_j(t) \geq b \), then \( y(\sigma_j(t)) = \psi(\sigma_j(t)) \) (\( j = 1, \ldots, m \)). The solution of problem (2.2),(2.3) we understand in the following generalized sense. The composite function \( F(t, \{y(t)\}, \{y'(t)\}) \) must belong to the space \( AC[a, b] \) and its derivative has to be almost everywhere on \([a, b]\) equal to \( -\Phi(t, \{y(t)\}, \{y'(t)\}) \).

It is possible that \( y''(\sigma_j(t)) \) do not exist and it is impossible to apply to \( \frac{d}{dt} F(t, \{y(t)\}, \{y'(t)\}) \) the rule of differentiating a composite function, but only the composite function \( F \) has to belong to \( AC[a, b] \) without any additional requirement to its arguments. Besides, we suppose that solutions \( y \) to BVP (2.2),(2.3) are elements of the Sobolev space \( W_1^1(a, b) \) of absolutely continuous functions with square integrable derivatives or of the Sobolev space \( W_{\infty}^1(a, b) \) of absolutely continuous functions with derivatives \( y' \in L_{\infty}(a, b) \).

2.2. Theorems of existence and uniqueness of solutions to boundary value problems for functional differential equations. After the use of well known change of the unknown function equation (2.2)
will be transformed to the equation of the same form, but with zero boundary functions. Suppose that this change of the unknown function is already made and going to previous notation we shall consider equation (2.2) with zero boundary conditions

\[ y(t) = 0 \quad (t \in [\alpha, \alpha]) \quad y(t) = 0 \quad (t \in [b, \beta]). \]

Denote

\[ G(t, [y(t)], [y'(t)]) = y'(t) + F(t, [y(t)], [y'(t)]). \]

In the following theorem solutions \( y \) to BVP (2.2),(2.4) are the elements of the Sobolev space \( W^1_\infty(a, b) \). We assume that the function \( G(t, [u], [v]) \) satisfies the Lipschitz conditions on arguments \( u_i \) and \( v_i \) with constants \( p_i \) and \( q_i \) and the function \( \Phi(t, [u], [v]) \) satisfies the Lipschitz conditions on arguments \( u_i \) and \( v_i \) with constants \( r_i \) and \( s_i \), correspondingly. Denote

\[ P = \sum_{i=1}^{m} p_i, \quad Q = \sum_{i=1}^{m} q_i, \quad R = \sum_{i=1}^{m} r_i, \quad S = \sum_{i=1}^{m} s_i. \]

**Theorem 2.8.** Suppose that functions \( \sigma_i(t) \) are continuous and \( G \) and \( \Phi \) satisfy the above mentioned Lipschitz conditions, are integrable in \( t \) at arbitrary values of other arguments and

\[ (b - a)^2 R + 2(b - a)(P + S) + 2Q < 1 \]

Then the BVP (2.2),(2.4) has a unique solution \( y \in W^1_\infty(a, b) \).

Assume now that instead of the space \( W^1_\infty \) solutions belong to the Sobolev space \( W^2_1 \). Introduce some notations. Let \([c_i, d_i]\) be the image of \([a, b]\) by mapping \( \sigma_i(t) \) and \( \sigma_i^{-1}(E) \) be the preimage of the set \( E \subseteq [c_i, d_i] \) by this mapping. It is assumed that if \( z(t) \) is defined on \([a, b]\), then and \( z(\sigma_i(t)) \) is also defined on \([a, b]\) if we put it equal to zero on the set \([a, b] \setminus (\sigma_i^{-1}([a, b]) \cap [c, d])\). Let exist such numbers \( \theta_i > 0 \) that for any measurable subset \( E \subseteq [a, b] \cap [c_i, d_i] \) there are true the inequalities

\[ \text{mes} \sigma_i^{-1}(E) \leq \theta_i^2 \text{mes} E. \]

Condition (2.6) is necessary and sufficient that for from \( z(t) \in L^2[a, b] \) would follow that \( z(\sigma_i(t)) \in L^2[a, b] \). (Remark that if \( \sigma_i(t) \) is continuous and monotonic, then for validity of (2.6) it is necessary and sufficient that the function \( \gamma_i(t) \), inverse to \( \sigma_i(t) \), would satisfy the Lipschitz condition with the Lipschitz constant equal to \( \theta_i^2 \).) Denote now

\[ P = \sum_{i=1}^{m} p_i \theta_i, \quad Q = \sum_{i=1}^{m} q_i \theta_i, \quad R = \sum_{i=1}^{m} r_i \theta_i, \quad S = \sum_{i=1}^{m} s_i \theta_i. \]

**Theorem 2.9.** Suppose that functions \( \sigma_i(t) \) are continuous and satisfy (2.6), \( G \) and \( \Phi \) satisfy the above mentioned Lipschitz conditions, are integrable in \( t \) at arbitrary values of other arguments and

\[ \frac{1}{2} (b - a)^2 R + \frac{b - a}{\sqrt{2}} (P + S) + Q < 1. \]
Then BVP (2.2), (2.4) has a unique solution \( y \in W^1_2(a, b) \).

In Theorem 2.8 and Theorem 2.9 it was applied the Banach's fixed point theorem. In the following theorem it will be applied another theorem of the fixed point of an operator due to M.A. Krasnosel'skii ([54], ch.3, §1). We again state our BVP in \( W^1_2(a, b) \). In the following theorem we shall not require the Lipschitz condition on \( \Psi \) but shall impose the following conditions of growth on \( G \) and \( D \):

\[
|G(t, [u], [v])| \leq \sum_{i=1}^{m} [\alpha_i |u_i| + \beta_i |v_i|] + k_1
\]

and

\[
|D(y, [u], [v])| \leq \sum_{i=1}^{m} [|\gamma_i |u_i| + \delta_i |v_i|] + k_2.
\]

Denote

\[
\alpha = \sum_{i=1}^{m} \alpha_i, \beta = \sum_{i=1}^{m} \beta_i, \gamma = \sum_{i=1}^{m} \gamma_i, \delta = \sum_{i=1}^{m} \delta_i.
\]

**Theorem 2.10.** Suppose that functions \( \sigma_i, i = 1, \ldots, m \) satisfy condition (2.6), functions \( G \) and \( D \) are square integrable in \( t \) at fixed values of other arguments, function \( G \) satisfies the Lipschitz condition with the above mentioned constants,

\[
\frac{b-a}{\sqrt{2}} P + Q < 1,
\]

the inequalities (2.8) and (2.9) hold and

\[
\frac{(b-a)^2}{2} \gamma + \frac{b-a}{\sqrt{2}} (\alpha + \delta) + \beta < 1,
\]

then there exists at least one solution to BVP (2.2), (2.4) in \( W^1_2(a, b) \).

Assume now that all functions \( \sigma_i(t), i = 1, \ldots, m \) map the interval \([a, b]\) into itself. In this special case boundary conditions (2.4) turn into

\[
y(a) = 0, \quad y(b) = 0.
\]

In previous cases the choice of spaces \( W^1_\infty \) or \( W^1_2 \) was necessary because the discontinuities of derivatives of \( y \) at the points \( a \) and \( b \) are shifted by \( \sigma_i(t) \) inside of the interval \((a, b)\) and by all smooth functions \( G, F, \varphi, \psi \) the derivatives of \( y \) may have discontinuities inside of \((a, b)\). In this special case it is impossible and we can prove the following theorem of existence and uniqueness of smooth solutions of BVP (2.2),(2.13).

**Theorem 2.11.** Suppose that functions \( \sigma_i(t), i = 1 \ldots, m \) map the interval \([a, b]\) in itself, functions \( G \) and \( \Phi \) are continuous in \( t \) at fixed values of other arguments and satisfy the Lipschitz conditions on \( u_i \) and \( v_i \) with the above mentioned constants and these constants satisfy inequality (2.5). Then there exists and is unique the solution \( y \in C^1(a, b) \) of BVP (2.2),(2.13).
2.3. Boundary value problems for linear functional differential equations. Consider second order linear functional differential equation

\[\sum_{i=1}^{m} \left[\frac{d}{dt} (A_i y'(\sigma_i(t)) + C_i y'\sigma_i(t)) + D_i y(\sigma_i(t))\right] = r(t)\]

Equation (2.14) is of the type of equations considered in previous section and arise in the problem of minimum of a nonlocal quadratic functional. It is considered under boundary conditions

\[y(t) = 0 (t \in [\alpha, \beta]) \quad y(t) = 0 (t \in [\beta, \gamma]).\]

Here \(A_i(t), C_i(t), D_i(t)\) are quadratic \(n \times n\) matrices, \(r(t)\) is a vector-function. We suppose in this section that functions \(\sigma_i(t), i-1, \ldots, m\) are absolutely continuous and satisfy condition (2.6). Elements of matrices \(C_i, D_i\) and vector-function \(r\) are continuous on \([\alpha, \beta]\). Elements of matrices \(A_i\) are absolutely continuous on \([\alpha, \beta]\). The generalized solution of problem (2.14),(2.15) must belong to \(W_2^1(a, b)\). Introduce operators

\[(B\psi)(t) = \psi(t) - \frac{1}{b-a} \int_a^b \psi(t) \, dt,\]
\[(A\psi)(t) = \sum_{i=1}^{m} A_i(t) \psi(\sigma_i(t)),\]
\[(C\psi)(t) = \sum_{i=1}^{m} \int_a^t C_i(s) \psi(\sigma_i(s)) \, ds,\]
\[(D\psi)(t) = \sum_{i=1}^{m} \int_a^t D_i(s) \int_a^{\sigma_i(s)} z(v) \, dv \, ds.\]

Equation (2.14) can be written in the operator form

\[B(A + C + D)z = q,\]

where \(q = Br\). Denote

\[A = BA, \quad C = BC, \quad D = BD.\]

Then equation (2.14) has the form

\[(A + C + D)z = q\]

All these operators act in \(L_2[a, b]\). For equation (2.21) the Fredholm’s alternative is true. Consider together with (2.21) equations

\[(A + C + D)z = 0\]

and

\[(A^* + C^* + D^*)w = 0,\]

where \(A^*, C^*, D^*\) are the operators conjugated to \(A, C, D\), correspondingly. If the operator \(A\) has a linear inverse operator, then it is true the following

Fredholm’s alternative. There is true one and only one of following assertions. I. Equation (2.21) has a solution for every \(q\). Equation (2.22) has only trivial (zero) solution. II. Equations (2.22) and (2.23) have equal number
μ > 0 of linear independent solutions \( z_1, z_2, \ldots, z_\mu \) and \( w_1, w_2, \ldots, w_\mu \). It is necessary and sufficient for solvability of equation (2.21) that the equalities 
\[ (q, w_k) = 0, \quad k = 1, \ldots, \mu \]
hold. The general solution of equation (2.21) has the form 
\[ z = \hat{z} + \sum_{k=1}^\mu C_k z_k, \]
where \( \hat{z} \) is a particular solution of equation (2.21) and \( C_k \) are arbitrary constants.

### 2.4. Smoothness of solutions to symmetrical boundary value problems for functional differential equations.

The problem of smoothness of solutions of BVP for functional differential equations is closely connected with notions of classical and generalized solutions. In these equations the values of unknown function and its derivatives are nonlocally connected with their values at different values of arguments. If there are discontinuities at the boundary of the domain of definition of solutions, then these discontinuities can be shifted to inner points of the domain and it appears the necessity to consider the generalized solutions. But one can consider the boundary functions and the solution as parts of description of one smooth process and assume that the solution together with the boundary functions have to be a smooth function. In this case we speak about classical solution that satisfies the equation everywhere. Consider the functional differential equation

\[
(2.24) \quad -(Ru)(t) + A_1 u(t) = f(t) \quad (t \in (0, d))
\]

under boundary conditions

\[
(2.25) \quad u(t) = \psi_1(t) \quad (t \in [-N, 0]) \quad u(t) = \psi_2(t) \quad (t \in [d, d+n]),
\]

where the operator \( R \) is defined by the formula

\[
(2.26) \quad (Ru)(t) = \sum_{j=-N}^{N} b_j u(t+j)
\]

where \( b_j \) are real numbers; \( d = N + \theta, \quad 0 < \theta \leq 1, \) \( N \) is a natural number; \( f \in L_2(0, d) \) and \( A_1 : W_2^2(-N, d+N) \rightarrow l_2(0, d) \) is a linear bounded operator, \( \psi_1 \in (-N, 0), \) \( \psi_2 \in (d, d+N). \) Denote \( B_{ij} \) the cofactor of the element \( r_{ij} \) of the matrix \( R_1 \) of order \( (N+1, N+1) \) with the elements \( r_{ij} = b_{j-i}. \) Introduce the space of vector functions \( \tilde{W} = (L_2(0, d) \times W_2^2(-N, 0) \times W_2^2(d, d+N)). \)

Define the linear bounded operator \( D : (-N, d+N) \rightarrow \tilde{W} \) by the formula 
\[
(Au, u|_{(-N,0)}, u|_{(d,d+N)}), \quad \text{where} \quad Au = -(Ru)'' + A_1 u.
\]

**Definition 1.** The function \( u \in W_2^2(-N, d+N) \) is called the smooth solution of boundary value problem (2.24),(2.25), if \( Du = F, \) where \( F = (f, \psi_1, \psi_2). \)

**Theorem 2.12.** Suppose that \( \det R_1 \neq 0, \) \( B_{11} \neq 0. \) Then the operator \( D : W_2^2(-N, d+N) \rightarrow \tilde{W} \) is a Noether operator and \( \chi(D) = 2, \) where 
\[ \chi(D) = \dim N(D) - \text{codim} R(D) \]
is the index of the Noether operator \( D. \)
It follows from this theorem that for the existence of a smooth solution of boundary value problem (2.24), (2.25) it is necessary to put on the right hand part \( f \) of equation (2.24) and on the boundary functions \( \psi_1, \psi_2 \) not less then two and no more then a finite number of the orthogonality conditions.

In [46] it is considered the problem of smoothness of solutions of the linear equation
\[
\sum_{i=1}^{n} a_i(t) y(\omega_i(t))' = F[y](t) \quad (\alpha < t < \beta),
\]
where \( F \) is a given operator, \( a_i \in W^1_1(\alpha, \beta), \omega_i \in W^2_1(\alpha, \beta) \) \((i = 1, \ldots, n)\) and \( A < \alpha < \beta < B, \omega_i([\alpha, \beta]) \subset [A, B] \), under boundary conditions
\[
y(t) = \varphi(t), \quad t \in [A, \alpha]; \quad y(t) = \psi(t), \quad t \in [\beta, B].
\]
It was proved the following conditional theorem.

**Assumption A.** Suppose that the solution \( z \) of the equation
\[
\sum_{i=1}^{n} a_i(t) (\text{sgn} \omega_i^t) z(\omega_i(t)) = 0 \quad (\alpha < t < \beta)
\]
satisfying zero boundary conditions and the condition
\[
\forall \varepsilon > 0 \quad \text{the set} \quad \{t | t \in (\alpha, \beta); |z(t)| \geq \varepsilon\} \quad \text{is finite}.\]
is identically equal to zero.

**Theorem 2.13.** Let Assumption A hold, \( \varphi \in C^1[A, \alpha], \psi \in C^1[\beta, B] \) and the solution of BVP (2.27), (2.28) satisfy the conditions
\[
y'(\alpha^+) = \varphi'(\alpha^-); \quad y'(\beta^-) = \psi'(\beta^+).
\]
Then \( y \in C^1[A, B] \).

### 2.5. Boundary value problems for mixed functional differential equations.

In this section we consider the mixed functional differential equation
\[
\Psi(t, s, u(t, s - r), \ldots, u(t, s + r), u'(t, s - r), \ldots, u'(t, s + r))
\]
\[
\frac{d}{dt} \Theta(t, s, u(t, s - r), \ldots, u(t, s + r), u'(t, s - r), \ldots, u'(t, s + r)) = 0
\]
or using the abbreviated form of notation (see Section 5.2), equation
\[
\Psi(t, s, [u(t, s)], [u'(t, s)]) + \frac{d}{dt} \Theta(t, s, [u(t, s)], [u'(t, s)]) = 0.
\]

Here \( u'(t, s) := \frac{\partial u(t, s)}{\partial t} \). On the set \( E_0 = \{(t, s) | t \in [t_0, t_1], s \in (s_0 - r, s_0)\} \) the boundary function \( \varphi(t, s) \) is given; on the set \( E_1 = \{(t, s) | t \in [t_0, t_1], s \in (s_1, s_1 + r)\} \) the boundary function \( \psi(t, s) \) is given. On the intervals \( G_0 = \ldots \)
\[(t,s) \mid t = t_0, s \in (s_0, s_1) \}\) and \(G_1 = \{(t,s) \mid t = t_1, s \in (s_0, s_1)\}\) there are given functions \(\mu(s)\) and \(\nu(s)\). Equation (2.33) is considered with boundary conditions
\[
\begin{align*}
u(t,s) &= \phi(t,s), \quad (t,s) \in E_0; \\
u(t,s) &= \psi(t,s), \quad (t,s) \in E_1; \\
u(t,s) &= \mu(s), \quad (t,s) \in G_0; \\
u(t,s) &= \nu(s), \quad (t,s) \in G_1.
\end{align*}
\]
Define the set
\[
Q = (t_0, t_1) \times (s_0 + r, s_1 - r).
\]
Denote \(L_2(Q)\) the space of square integrable with respect to \(t\) for almost all \(s \in (s_0, s_1)\) functions \(u(t,s)\) and \(H(Q)\) the space of absolutely continuous with respect to \(t\) for almost all \(s \in (s_0, s_1)\) functions \(u(t,s)\) such that \(u \in L_2(Q)\) and \(u' \in L_2(Q)\) with the norm
\[
||u||_H = (||u||_{L_2(Q)} + ||u'||_{L_2(Q)})^{\frac{1}{2}}
\]
We shall suppose that the functions \(\Theta\) and \(\Theta\) are continuous and have continuous first and second derivatives with respect to all of their arguments. If we change the unknown function \(u(t,s)\) by
\[
v(t,s) = u(t,s) - \frac{1}{t - t_0} (\mu(s)(t_1 - t) + \nu(s)(t - t_0)), \quad (t,s) \in Q,
\]
then equation (2.33) transforms to an equation of the same type and the new boundary conditions will be homogeneous. Suppose that such a change of variables has been made previously and instead of boundary conditions (2.35) the conditions
\[
\begin{align*}
\nu(t,s) &= 0, \quad (t,s) \in G_0, \nu(t,s) = 0, \quad (t,s) \in G_1,
\end{align*}
\]
are fulfilled. We describe now an operator equation equivalent to the boundary value problem (2.33),(2.34),(2.37). Define the function
\[
G(t,s,[u(t,s)],[u'(t,s)]) = u'(t,s) + \Theta(t,s,[u(t,s)],[u'(t,s)]).
\]
Introduce now operators \(B\) and \(I\) by formulae
\[
\begin{align*}(Bg)(t,s) &= g(t,s) - \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} g(\tau,s) \, d\tau, \\
(Ig)(t,s) &= \int_{t_0}^{t} g(\tau,s) \, d\tau.
\end{align*}
\]
Equation (2.32) is equivalent to the operator equation
\[
(2.38) \quad v = Av,
\]
where
\[
A = B(G + I\psi),
\]
\[
(2.39) \quad (Gv)(t,s) = G \left( t, s, \left[ \int_{t_0}^{t} v(\tau,s) \, d\tau \right], [v(t,s)] \right),
\]
(2.40) \( (\Psi v)(t, s) = \Psi \left( t, s, \left[ \int_{t_0}^{t} \psi(\tau, s) \, d\tau \right], [v(t, s)] \right) \).

The operators \( G \) and \( \Psi \) are determined by (2.39),(2.40) and boundary conditions (2.34),(2.37). We suppose that the function \( G \) satisfies the Lipschitz conditions on \( u(t, s + i) \) with constants \( z_j \), on \( u'(t, s + i) \) with constants \( w_i \); the function \( \Psi \) satisfies the Lipschitz conditions on \( u(t, s + i) \) with constants \( p_i \), on \( u'(t, s + i) \) with constants \( q_i \). Denote

\[
Z = \sum_{i=-r}^{r} z_i, \quad W = \sum_{i=-r}^{r} w_i, \quad P = \sum_{i=-r}^{r} p_i, \quad Q = \sum_{i=-r}^{r} q_i.
\]

**Theorem 2.14.** If the functions \( G \) and \( \Psi \) are quadratic integrable on \( Q \) in \( t \) at almost all values of other arguments, satisfy the Lipschitz conditions with the above mentioned constants, and

\[
\frac{(t_1 - t_0)^2}{2} P + \frac{t_1 - t_0}{\sqrt{2}} (Q + Z) + W < 1,
\]

then there exists in \( H(Q) \) a unique solution of boundary value problem (2.32),(2.34), (2.37).

Another theorem of existence of solutions to boundary value problem (2.32),(2.34),(2.37) is based on the fixed point theorem of Krasnosel'skii. In the following theorem we shall impose the following conditions of growth on \( G \) and \( \Psi \).

(2.41) \[ |G(t, s, [u], [v])| \leq \sum_{i=-r}^{r} [\alpha_i|u(t, s + i)| + \beta_i|v(t, s + i)|] + K_1, \]

(2.42) \[ |\Psi(t, s, [u], [v])| \leq \sum_{i=-r}^{r} [\gamma_i|u(t, s + i)| + \delta_i|v(t, s + i)|] + K_2. \]

Denote

\[
\alpha = \sum_{i=-r}^{r} \alpha_i, \quad \beta = \sum_{i=-r}^{r} \beta_i, \quad \gamma = \sum_{i=-r}^{r} \gamma, \quad \delta = \sum_{i=-r}^{r} \delta.
\]

**Theorem 2.15.** Let functions \( G \) and \( \Psi \) be quadratic integrable on \( Q \) in \( t \) at any fixed values of other arguments, satisfy the Lipschitz conditions with above mentioned constants and

\[
\frac{t_1 - t_0}{2} Z + W < 1,
\]

\[
\frac{(t_1 - t_0)^2}{2} \gamma + \frac{t_1 - t_0}{\sqrt{2}} (\alpha + \delta) + \beta < 1.
\]

Then there exists at least one solution \( u \in H(Q) \) of boundary value problem (2.32),(2.34), (2.37).

**2.6. Comments and Bibliography.** The connection between variational problems for nonlocal functional and boundary value problems for
functional differential equations was described in [21]. The analysis of statements of boundary value problems for functional differential equations was made in [39]. The theorems of existence and uniqueness of solutions to boundary value problems for nonlinear equations and the Fredholm alternative for linear equations were proved in [40],[41]. To the theory of boundary value problems for linear functional differential equations is dedicated the book [48]. The question of smoothness of generalized solutions to boundary value problems was studied in [13, 45, 46]. In [46] there are proved some sufficient conditions for fulfillment of the Assumption A. It would be useful to find new sufficient conditions for validity of the Assumption A for different particular cases of equation (2.27). It is important to investigate the problem of smoothness of solutions to asymmetric boundary value problems. The simple type of mixed functional differential equation is the equation
\begin{equation}
(2.43) \quad u'(t, s) = F(t, s, u(t, s + h_1), \ldots, u(t, s + h_k)),
\end{equation}
which is a difference equation with respect to x and a differential equation with respect to t. Equations of this type and many particular cases of these equations and their applications are described in the book of E. Pinney [60].

The general type of mixed functional differential equations are equations
\begin{equation}
(2.44) \quad u'(x, t) = f(x, t, u_{xt}),
\end{equation}
and also
\begin{equation}
(2.45) \quad u^{(m)}(x, t) = f(x, t, u_{xt}, u_{xt}', \ldots, u_{xt}^{(l)}).
\end{equation}
Here \( a < x < b \) (\( a, b = \text{const}, -\infty < a < b < \infty \)), \( 0 < t < \infty \); the ranges of values of \( u \) and \( f \) lie in \( \mathbb{R}^n \); \( u_{xt}^{(k)} \) means the derivative of order \( k \) of \( u_{xt} \) with respect to \( t \); \( u_{xt}(\xi, \theta) := u(x + \xi, t + \theta) \) (\( |\xi| \leq h, -g \leq \theta \leq 0 ; h, g = \text{const} \geq 0 \)); thus \( f \) is a functional of \( u_{xt} \) in (2.43) and of \( u_{xt} \) and its derivatives in (2.44) depending on \( x \) and \( t \) as parameters. This general mixed functional differential equations are studied in [42],[43]. A survey of this theory is in [31]. Applications of equations of the type (2.43) (with discrete \( s \)) to mechanics were mentioned already in works of J. Bernoulli, L. Euler, J. L. Laplace and other mathematicians of 18-th and 19-th centuries. For description of some of these works and references see [60]. Many forms of equations of the type (2.43) and (2.45) appear now in papers dedicated to investigations of different problems of epidemiology, ecology, biology and physics. (see [31].) The symmetrical boundary value problems for mixed functional differential equations are studied in [28],[42],[43], the asymmetrical ones — in [29]. Theory of periodical solutions of mixed FDE, which is connected with the theory of BVP of mixed FDE, was investigated in [30] and [44].

3.1. Method of finite differences. The method of finite differences is a rather general method and it can be applied to different types of equations, but here this method is described for linear second order functional differential equation, which arise by solving the problem of extremum of quadratic nonlocal functional. Consider equation

\[ (3.1) \left[ \sum_{k=-N}^{N} a_k(t) y'(t + k\tau) \right]' + \sum_{k=-N}^{N} [b_k(t) y'(t + k\tau) + c_k(t + k\tau)] = f(t), \]

where \( t \in [a, b], \tau > 0, \) \( a_k, b_k, c_k \in C^1[a, b], f \in L_2(a, b), \) with the boundary conditions

\[ (3.2) \quad y(t) = 0 \quad \text{at} \quad t \in [a - N\tau, a] \cup [b, b + N\tau]. \]

Problem (3.1),(3.2) is considered here in the space \( \hat{W}_2^1(a, b) \) with the additional condition that \( y(a) = y(b) = 0. \) It is denoted \( \hat{W}_2^1(a, b). \) Solution of problem (3.1),(3.2) is the function \( y \in \hat{W}_2^1(a, b), \) which satisfies (3.2) and equation (3.1) almost everywhere on \([a, b]\) and besides the function

\[ \sum_{k=-N}^{N} a_k(t) y'(t + k\tau) \]

must belong to \( \hat{W}_2^1(a, b). \) Remark that \( y''(t) \) exists in the general case only as a distribution even at arbitrary smooth coefficients and right hand parts of (3.1). Introduce operators \( Q, R, S : L_2(a, b) \to L_2(a, b), \) defined by formulae

\[ (3.3) \quad (Qy)(t) = \sum_{k=-N}^{N} a_k(t) y(t + k\tau), \]

\[ (3.4) \quad (Ry)(t) = \sum_{k=-N}^{N} b_k(t) y(t + k\tau), \]

\[ (3.5) \quad (Sy)(t) = \sum_{k=-N}^{N} c_k(t) y(t + k\tau) \]

and boundary conditions (3.2). Denote \( L : L_2(a, b) \to L_2[a, b] \) the operator defined by the left hand part of equation (3.1) and boundary conditions (3.2) with domain \( D(L) = \{ u \in \hat{W}_2^1(a, b), L(u) \in L_2(a, b) \}. \) This operator is unbounded and has the domain dense in \( \hat{W}_2^1(a, b). \) It supposed here that for operator \( L \) the Fredholm's alternative is true. It is supposed here that \( b - a \) and \( \tau \) are commensurable. Then by variation of \( \tau \) it is possible to achieve that \( b - a = q\tau, \) where \( q \) is an integer. It is supposed that it is already done. Denote \( \hat{C}[a, b] \) the space of piecewise continuous functions with possible points of discontinuity \( a + \tau, \ldots, a + (q + 1)\tau. \) Besides it is supposed that functions \( y \in \hat{C}[a, b] \) are extended in such a way that they are semicontinuous to the right on \([a, b]\) and semicontinuous to the left at \( b. \) If the
operator $Q$ is invertible, then under our suppositions $y' \in \mathcal{C}[a,b]$. Suppose additionally that $f \in \mathcal{C}[a,b]$. Integrate (3.1) and receive that $Qy' \in \mathcal{C}[a,b]$. Construct now the difference scheme. For each integer $m > 0$ put $h = \frac{b-a}{m}$, $n = qm$, $T_n = \{a, a + h, \ldots, a + nh\}$, and let $M_n$ be a linear space of functions defined on $T_n$. Denote $M_n = \{\xi | \xi \in M_n, \xi(a) = \xi(b) = 0\}$ and define operators

$$
(\Delta^+_n \xi)(s) = \begin{cases} 
\frac{\xi(s+h) - \xi(s)}{h}, & \text{if } s \in T_n \setminus \{b\}, \\
\frac{\xi(b) - \xi(b-h)}{h}, & \text{if } s = b. 
\end{cases}
$$

(3.6)

$$
(\Delta^-_n \xi)(s) = \begin{cases} 
\frac{\xi(a+h) - \xi(a)}{h}, & \text{if } s = a, \\
\frac{\xi(s) - \xi(s-h)}{h}, & \text{if } s \in T_n \setminus \{a\}; 
\end{cases}
$$

(3.7)

$$
(Q_n \xi)(s) = \sum_{k=-N}^{N} a_k(s)(s + k\tau), \quad \text{if } s \in T_n.
$$

(3.8)

and by definition $\xi(s) = 0$ at the points $s \notin T_n$. Similarly are defined the operators $R_n$ and $S_n$. Denote the operator $[\cdot]_n : \mathcal{C}[a,b] \to M_n$ by equality $[y]_n(s) = y(s)$ at $s \in T_n$. It is evident that $[Qy]_n = Q_n[y]_n$ at $y \in \mathcal{C}[a,b]$. The approximate solution of problem (3.1),(3.2) will be found as a net function $\tilde{\xi} \in \bar{M}_n$ for which

$$
(L_n(\tilde{\xi}))(s) = ((\Delta^-_n Q_n \Delta^+_n + R_n \Delta^+_n + S_n)(\tilde{\xi}))(s) = [f]_n,
$$

(3.9)

where $s \in T_n \setminus (\{a\} \cup \{b\})$. This is a system of $n-1$ linear algebraic equations relative $n-1$ unknown values $\tilde{\xi}(s), s \in T_n \setminus (\{a\} \cup \{b\})$. Suppose that the operator $Q$ has a bounded inverse operator and define in $M_n$ the scalar product

$$
(\xi, \eta) = h \sum_{\nu=1}^{n} \xi(a + \nu h)\eta(a + \nu h)
$$

and the corresponding norm $\| \cdot \|_n$ and additionally the norm

$$
\|\xi\|_n^c = \max_{0 \leq \nu \leq n} |\xi(a + \nu h)|.
$$

It is easy to prove that

$$
\| \cdot \|_n \leq \sqrt{(n+1)h} \| \cdot \|_n^c.
$$

Define one more norm

$$
\|\xi\|_n^0 = \sqrt{h \sum_{\nu=1}^{n-1} |\xi(a + \nu h)|^2}.
$$

Formulate now the main theorems.

**THEOREM 3.16 (Approximation of the operator $L$).** If $Ly = f$, then

$$
||L_n[y]_n - [f]_n||_n^0 \to 0 \text{ as } n \to \infty
$$

(3.10)
THEOREM 3.17 (Stability of the difference scheme). The operator $L$ is injective iff there are such $C > 0$ and $n_0$ for which
\[ \|L_n \xi\|_n \geq C \{\| \Delta_n Q_n \Delta_n^+ \xi\|_n + \| Q_n \Delta_n^+ \xi\|_n + \| \Delta_n^+ \xi\|_n + \| \xi\|_n \} \]
for $\forall n \geq n_0$, $\xi \in \hat{M}_n$.

It follows from (3.11) that equation (3.9) has for $n > n_0$ exactly one solution for an arbitrary right hand member $[f]_n$.

THEOREM 3.18 (Approximation of the solution). If the operator $\hat{L}$ is injective, $L y = f$ and $\xi_n \in \hat{M}_n$ is the solution of equation (3.9), then
\[ \|[y]_n - \xi_n\|_n \to 0 \text{ as } n \to \infty. \]

3.2. Method of numerical sweep. In this section we again consider the boundary value problem
\[ \left[ \sum_{k=0}^{N} a_k(t) y'(t + k \tau) \right]' + \sum_{k=0}^{N} [b_k(t) y'(t + k \tau) + c_k(t + k \tau)] = f(t), \]
where $t \in [a, b], \tau > 0, a_k, b_k, c_k \in C^1[a, b], f \in L^2[a, b]$, with boundary conditions
\[ y(t) = 0 \text{ at } t \in [a - N \tau, a] \cup [b, b + N \tau]. \]
Construction of the difference scheme and conditions that guarantee the convergence of the approximate solutions to the generalized solutions of the boundary value problem (3.13) are described in the previous Subsection. The problem is reduced to solution of a system of linear algebraic equations. Here is described the method of solution of such a system that using the peculiarities of this system of equations permits to solve it in the most rational way. Introduce some new notations for this difference scheme.

Denote
\[ t_k = a - N \tau + (k - 1) h, \quad k = 1, \ldots, nq + 2p + 1, \]
where $q = \frac{b - a}{\tau}, \quad p = Nn; \quad h = \frac{\tau}{n}$. Then system (3.9) can be written in the form
\[ y_{k+p+1} = A_1(k) y_{k+p} + A_2(k) y_{k+p-1} + \ldots + A_{2p+2}(k) y_{k-p-1} + q_k, \quad k = p + 2, p + 3, \ldots, nq + p. \]
with boundary conditions
\[ y_k = 0 \text{ at } k = 1, 2, \ldots, p + 1. \]
\[ y_k = 0 \text{ at } k = nq + p + 1, nq + p + 2, \ldots, nq + 2p + 1. \]
We want to find the solution in the form
\[ y_k = \sum_{j=1}^{2p+2} \beta_j^k y_j + F_k \quad k = 2p + 3, \ldots, nq + 2p + 1, \]
where

\[ \beta_j^i = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3.18} \]

\[ F_1 = F_2 = \ldots = F_{2p+2} = 0. \tag{3.19} \]

In (3.17) because of (3.18) and (3.19) it will be \( y_k = 0, \ k=1, \ldots, p+1 \) and so the first of boundary conditions (3.16) will be fulfilled. Values \( y_{p+2}, \ldots, y_{2p+2} \) will be parameters and values \( y_{2p+3}, \ldots, y_{nq+p} \) that must satisfy equation (3.15) must be given by (3.17). To determine \( y_{p+2}, \ldots, y_{p+2}, \ldots, y_{2p+2} \) it must be used the second of the boundary conditions (3.16). After supposition of (3.17) in (3.15) and comparing the coefficients at \( y_1, \ldots, y_{2p+2} \) we receive following formulae for determining of \( \beta_j^k \) and \( F_k \).

\[ \beta_j^{k+p+1} = A_1(k)\beta_j^{k+p} + A_2(k)\beta_j^{k+p-1} + \ldots + A_{2p+2}(k)\beta_j^{k-p-1}, \]

\( k = p + 2, p + 3, \ldots, np + q, \ j = 1, 2, \ldots, 2p + 2. \tag{3.20} \]

\[ F_{k+p+1} = A_1(k)F_{k+p} + A_2(k)F_{k+p-1} + \ldots + A_{2p+2}(k)F_{k-p-1} + q_k \]

\( k = p + 2, p + 3, \ldots, nq + p. \tag{3.21} \]

Solve equations (3.20) and (3.21) and after that it will be possible to find \( y_{p+2}, \ldots, y_{2p+2} \) from the system of equations

\[ \sum_{j=p+2}^{2p+2} \beta_j^k y_j = -F_k - \sum_{j=1}^{p+1} \beta_j^k y_j, \]

\( k = nq + p + 1, \ldots, nq + 2p + 1. \tag{3.22} \]

By solving this system we must use the second line from conditions (3.16). When we already know \( y_{p+2}, \ldots, y_{2p+2} \) it is possible to find \( y_{2p+3}, \ldots, y_{nq+p} \) using formulae (3.17). Computations by formulae (3.20) and (3.21) is the direct move and computations by formulae (3.17) is the reverse move of the method of numerical sweep.

**Remark.** System (3.15) is a system of algebraic equation and can be solved by traditional methods without application of the more complicated numerical sweep method. But dimension of system (3.15) depends on \( n \) and, if \([a, b]\) is large and \( h \) is small, can have great dimension. Dimension of system (3.22) does not depend on the length of \([a, b]\) and depends only on \( \tau \) and \( h \) and can be substantially smaller than dimension of (3.15).

**Comments and Bibliography.** It is rather seldom possible to find the exact solutions to BVP for functional differential equations. And it is important for applications to have the approximate methods of solution of these problems. Approximate methods for solution of BVP for functional differential equations are developed: the finite difference method in [19], [48]; the collocation method in [36], the one step collocation method for functional dif-
ferential equations by A. Bellen in [4], the trigonometric collocation method for functional differential equations by A.M. Samoilenko and N.I. Ronto [62], the Bubnov-Galerkin method in [35], the finite differences method for approximate solution of mixed FDE in rectangular domains by A.F. Kopylov in [53], the local variations method in [2].

REFERENCES

[16] L.El’sgol’ts, Qualitative methods in mathematical analysis, Moscow, Gostechisdat, 1955 (in Russian.)
[18] E. Ivanova, G. Kamenskii, Initial value problems for differential-difference equations,


[57] V. Lakshmikantham, S. Leela, Differential and integral inequalities, N.Y., London,


ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS *

I. KIGURADZE † AND B. PŮŽA ‡

Dedicated to Professor A.D. Myshkis on the occasion of his jubilee

Abstract. For the functional differential equation

\[ u''(t) = f(u)(t) \]

with the continuous operator \( f : C^1_{loc}([a, b]) \to L_{loc}([a, b]) \) the unimprovable, in a certain sense, sufficient conditions for the solvability and unique solvability of the two-point boundary value problems

\[ u(a+) = 0, \quad u(b-) = 0 \]

and

\[ u(a+) = 0, \quad u'(b-) = 0 \]

are established. These conditions cover the case when for an arbitrary \( u \in C^1_{loc}([a, b]) \) the function \( f(u)(\cdot) : [a, b] \to \mathbb{R} \) is not integrable on \([a, b]\), having singularities at the points \( a \) and \( b \).

Key Words. second order singular functional differential equation, two-point boundary value problems, solvability, unique solvability, stability

AMS(MOS) subject classification. 34K10

---

* Supported by the GRDF Grant No. 3318.
† A. Razmadze Mathematical Institute of Georgian Academy of Sciences, 1, Alekidze St., 0193 Tbilisi, Georgia, E-mail: kig@rmi.acnet.ge
‡ Mathematical Institute of the Academy of Sciences of the Czech Republic, Branch in Brno, Žižkova 22, 616 62 Brno, Czech Republic; Masaryk University, Faculty of Science, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, E-mail: puza@math.muni.cz
§ 1. Statement of the Basic Results.

1.1. Formulation of the problem, main notation and definitions. On a finite interval \([a, b]\) we consider the functional differential equation

\[(1.1) \quad u''(t) = f(u)(t),\]

where \(f\) is the operator acting from the space \(C^1_{\text{loc}}([a, b])\) to the space \(L_{\text{loc}}([a, b])\). We are mainly interested in the case when \(f\) is a singular operator, i.e. when \(f(u)(\cdot) \notin L([a, b])\) for an arbitrary \(u \in C^1_{\text{loc}}([a, b])\). A simple, but important particular case of (1.1) is the differential equation with deviating arguments

\[(1.1') \quad u''(t) = f_0(t, u(\tau_1(t)), u'(\tau_2(t))),\]

where \(f_0 : [a, b] \times \mathbb{R}^2 \to \mathbb{R}\) is the function satisfying the local Carathéodory conditions, and \(\tau_i : [a, b] \to [a, b]\) \((i = 1, 2)\) are continuous functions. In the present paper, for the singular equations (1.1) and (1.1') we investigate the two-point boundary value problems

\[(1.2_1) \quad u(a+) = 0, \quad u(b-) = 0\]

and

\[(1.2_2) \quad u(a+) = 0, \quad u'(b-) = 0\]

with the additional condition

\[(1.3) \quad \int_a^b u'^2(s) \, ds < +\infty.\]

If \(f\) is the Nemytski operator, i.e. if \(f(u)(t) \equiv f_0(t, u(t), u'(t))\), then the singular problems (1.1), (1.2_1) \((i = 1, 2)\) are studied with sufficient thoroughness (see, e.g., [1]-[3], [5]-[9], [15], [16], [18], [20]). If \(f\) is the operator of general type, or \(\tau_i(t) \neq t\) \((i = 1, 2)\), then the problems (1.1), (1.2_2) \((i = 1, 2)\) and (1.1'), (1.2_2) \((i = 1, 2)\) are studied only in the so-called weakly singular cases, when

\[\int_a^b (t - a)(b - t)|f(u)(t)| \, dt < +\infty \quad \text{for} \quad u \in C^1([a, b]),\]

or

\[\int_a^b (t - a)(b - t)|f_0(t, x, y)| \, dt < +\infty \quad \text{for} \quad x \text{ and } y \in \mathbb{R} \]
ON TWO-POINT BVPs

(see [4], [11]-[14], [19], [21]-[24]) and the references therein). In strongly singular cases these problems remained in fact unstudied. The present paper is meant to fill up the existing gap to some extent.

Throughout the paper we will use the following notation.

- \( R = ] - \infty, +\infty[; R_+ = [0, +\infty[; I = ]a, b[; \text{ or } I = ]a, b] \).
- \( u(t_0 +) \) and \( u_0(t-) \) are, respectively, the right and the left limits of the function \( u \) at the point \( t_0 \).
- \( C^1_{loc}(I) \) is the topological space of continuously differentiable functions \( u : I \to R \) in which the sequence \( (u_k)_{k=1}^{\infty} \) is assumed to be converging to \( u \) if
  \[
  \lim_{k \to +\infty} u_k(t) = u(t), \quad \lim_{k \to +\infty} u_k'(t) = u'(t)
  \]
  uniformly on every compact interval contained in \( I \).
- \( D_1(a, b) = \{ u \in C^1_{loc}(a, b) : u(a+) = u(b-) = 0, \int_a^b u'^2(s) \, ds < +\infty \} \).
- \( D_2(a, b) = \{ u \in C^1_{loc}(a, b) : u(a+) = u'(b) = 0, \int_a^b u'^2(s) \, ds < +\infty \} \).
- \( \tilde{C}^1_{loc}(I) \) is the space of functions \( u : I \to R \), absolutely continuous together with their first derivative on every compact interval contained in \( I \).

- \( L^1_{loc}(I) \) is the topological space of functions \( v : I \to R \), Lebesgue integrable on every compact interval contained in \( I \), in which the sequence \( (v_k)_{k=1}^{\infty} \) is assumed to be converging to \( v \) if
  \[
  \lim_{k \to +\infty} \int_{t_i}^{t_2} |v_k(t) - v(t)| \, dt = 0 \quad \text{for } t_i \in I \quad (i = 1, 2).
  \]
- \( L^2([a, b]) \) is the space of Lebesgue square integrable functions \( v : [a, b] \to R \) with the norm
  \[
  \|v\|_{L^2} = \left( \int_a^b v^2(t) \, dt \right)^{1/2}.
  \]
- \( L^2_{loc}(I) \) is the space of functions \( v : I \to R \), Lebesgue square integrable on every compact interval contained in \( I \).
- \( L^2_{a, b}([a, b]) \) is the space of square integrable with the weight \( (t-a)^\alpha (b-t)^\beta \) functions \( v : ]a, b[ \to R \) with the norm
  \[
  \|v\|_{L^2_{a, b}} = \left( \int_a^b (t-a)^\alpha (b-t)^\beta v^2(t) \, dt \right)^{1/2}.
  \]

By a solution of Eq. (1.1) is understood the function \( u \in \tilde{C}^1_{loc}(]a, b[) \) which almost everywhere on \( ]a, b[ \) satisfies that equation. The solution of Eq. (1.1) sat-
isfying conditions (1.2) and (1.3) (conditions (1.2) and (1.3)) is called the solution of problem (1.1), (1.2), (1.3) (the solution of problem (1.1), (1.2), (1.3)).

Along with (1.1), we consider the perturbed equation

\[ u''(t) = f(u)(t) + h(t) \]

and introduce the following definition.

**Definition 1.1.** Problem (1.1), (1.2), (1.3) (problem (1.1), (1.2), (1.3)) is called stable with respect to a small perturbation of the right-hand member of Eq. (1.1) if there exists a positive number \( r \) such that for any \( h \in L^2_{2,2}([a, b]) \) (for any \( h \in L^2_{2,0}([a, b]) \) problem (1.4), (1.2), (1.3) (problem (1.4), (1.2), (1.3))) is uniquely solvable and

\[ \|u_h - u_0\|_{L^2} \leq r\|h\|_{L^2_{2,2}} \quad \left( \|u_h' - u_0'\|_{L^2} \leq r\|h\|_{L^2_{2,0}} \right), \]

where \( u_h \) and \( u_0 \) are the solutions of problems (1.4), (1.2), (1.3) and (1.1), (1.2), (1.3) (of problems (1.4), (1.2), (1.3) and (1.1), (1.2), (1.3)).

**Definition 1.2.** We say that the operator \( f : C^1_{loc}([a, b]) \to L_{loc}([a, b]) \) (the operator \( f : C^1_{loc}([a, b]) \to L_{loc}([a, b]) \)) belongs to the set \( K_1([a, b]) \) (to the set \( K_2([a, b]) \)) if it is continuous and there exists a continuous function \( \omega : [a, b] \times [a, b] \times \mathbb{R}^+ \to \mathbb{R}^+ \) (\( \omega : [a, b] \times [a, b] \times \mathbb{R}^+ \to \mathbb{R}^+ \)) such that

\[ \omega(t, t, \rho) = 0 \quad \text{for} \quad a < t < b, \quad \rho \in \mathbb{R}^+ \]

and for an arbitrary \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \)) the inequality

\[ \int_a^b |f(u)(\xi)| d\xi \leq \omega(s, t, \|u'\|_{L^2}) \quad \text{for} \quad a < s \leq t < b \]

is fulfilled.

In the case, where

\[ f \in K_1([a, b]) \quad (f \in K_2([a, b])), \]

we have proved a general theorem on the solvability of problem (1.1), (1.2), (1.3) (of problem (1.1), (1.2), (1.3)) and called it the principle of a priori boundedness. Using this principle, we have found effective and optimal in a certain sense conditions which guarantee, respectively, the solvability and unique solvability of problems (1.1), (1.2), (1.3) and (1.1'), (1.2'), (1.3) \((i = 1, 2)\) and their stability with respect to small perturbations of the right-hand member of the equation under consideration.
1.2. The principle of a priori boundedness. Let \((a_k)_{k=1}^{+\infty}\) and \((b_k)_{k=1}^{+\infty}\) be the number sequences such that

\[
1.8 \quad a < a_k < b_k < b \ (k = 1, 2, \ldots), \quad \lim_{k \to +\infty} a_k = a, \quad \lim_{k \to +\infty} b_k = b.
\]

For an arbitrary \(u \in C^1_{loc}(]a, b[)\) and natural \(k\) we put

\[
1.9 \quad f_k(u)(t) = \begin{cases} 
0 & \text{for } t \in [a, a_k] \cup [b_k, b] \\
 f(u)(t) & \text{for } a_k < t < b_k
\end{cases}
\]

and consider the auxiliary functional differential equation

\[
1.10 \quad u''(t) = \lambda f_k(u)(t)
\]

with a parameter \(\lambda \in [0, 1[.

**Theorem 1.1.** Let condition (1.7) be fulfilled, and let there exist a positive constant \(r_0\) and sequences \((a_k)_{k=1}^{+\infty}\), \((b_k)_{k=1}^{+\infty}\) satisfying conditions (1.8), such that for arbitrary \(\lambda \in [0, 1]\) and natural \(k\) every solution of problem (1.10), (1.2) (of problem (1.10), (1.2)) admits the estimate

\[
1.11 \quad \int_a^b u'^2(s) \, ds \leq r_0^2.
\]

Then problem (1.1), (1.2), (1.3) (problem (1.1), (1.2), (1.3)) has at least one solution.

Note that the analogous result for regular boundary value problems is contained in [10].

1.3. The existence and uniqueness theorems for Eq. (1.1).

**Theorem 1.2.** Let condition (1.7) be fulfilled, and let there exist constants \(\ell \in [0, 1[\), \(\ell_0 \geq 0\), \(a_0 \in ]a, b[\) and \(b_0 \in ]a_0, b[\) such that for an arbitrary \(u \in D_1(]a, b[)\) (for an arbitrary \(u \in D_2(]a, b[)\)) the inequality

\[
1.12 \quad \int_{t_0}^t f(u)(s)u(s) \, ds \geq -\ell \int_a^b u'^2(s) \, ds - \ell_0 \quad \text{for } a < t_0 \leq a_0, \ b_0 \leq t < b
\]

is fulfilled. Then problem (1.1), (1.2), (1.3) (problem (1.1), (1.2), (1.3)) has at least one solution.

Corollaries of that theorem given below deal with the functional differential equation

\[
1.13 \quad u''(t) = f_1(u)(t)u(t) + f_2(u)(t)u'(t) + f_0(u)(t),
\]
where \( f_i : C_{loc}^1([a, b]) \to L_{loc}([a, b]) \) (\( i = 0, 1, 2 \)) are continuous operators. Assume

\[
\begin{align*}
 f_{i1}^*(t; \rho) &= \sup \{|f_i(u)(t)| : u \in D_1([a, b]), \|u'\|_{L^2} \leq \rho\} \quad (i = 0, 1, 2), \\
 f_{i2}^*(t; \rho) &= \sup \{|f_i(u)(t)| : u \in D_2([a, b]), \|u'\|_{L^2} \leq \rho\} \quad (i = 0, 1, 2).
\end{align*}
\]

We are interested in the cases where the conditions

\[
(1.141)\quad f_{01}^*(\cdot; \rho) \in L^2_{x,2}([a, b]) \text{ for } \rho \in R_+, \quad \lim_{\rho \to +\infty} \rho^{-1}\|f_{01}^*(\cdot; \rho)\|_{L^2_{x,2}} = 0,
\]

\[
(1.151)\quad f_{11}^*(\cdot; \rho) \in L_{loc}([a, b]), \quad f_{21}^*(\cdot; \rho) \in L^2_{loc}([a, b]) \text{ for } \rho \in R_+,
\]

or the conditions

\[
(1.142)\quad f_{02}^*(\cdot; \rho) \in L^2_{x,0}([a, b]) \text{ for } \rho \in R_+, \quad \lim_{\rho \to +\infty} \rho^{-1}\|f_{02}^*(\cdot; \rho)\|_{L^2_{x,0}} = 0,
\]

\[
(1.152)\quad f_{12}^*(\cdot; \rho) \in L_{loc}([a, b]), \quad f_{22}^*(\cdot; \rho) \in L^2_{loc}([a, b]) \text{ for } \rho \in R_+
\]

are fulfilled.

**Corollary 1.1.** Let conditions (1.141), (1.151) (conditions (1.142), (1.152)) be fulfilled, and let there exist a constant \( \lambda \in ]0, 4[ \) such that for an arbitrary \( u \in D_1([a, b]) \) (for an arbitrary \( u \in D_2([a, b]) \)) almost everywhere on \( [a, b] \) the inequality

\[
(1.16)\quad f_2^2(u)(t) \leq \lambda f_1(u)(t)
\]

is fulfilled. Then problem (1.13), (1.21), (1.3) (problem (1.13), (1.22), (1.3)) has at least one solution.

**Example 1.1.** Consider the differential equation

\[
(1.17)\quad u''(t) = \frac{\lambda_1(t)|u(\tau(t))|^{2\mu}}{(t-a)^{2\alpha_1}(b-t)^{2\beta_1}} u(t) + \frac{\lambda_2(t)|u(\tau(t))|^{\mu}}{(t-a)^{\alpha_1}(b-t)^{\beta_1}} u'(t) + \frac{\lambda_0(t)}{(t-a)^{\alpha_0}(b-t)^{\beta_0}},
\]

where \( \alpha_i, \beta_i \ (i = 0, 1, 2) \), \( \mu \) are nonnegative constants, while \( \lambda_i : [a, b] \to R \) (\( i = 0, 1, 2 \)) and \( \tau : [a, b] \to [a, b] \) are continuous functions. According to Corollary 1.1, if

\[
\alpha_0 < \frac{3}{2}, \quad \beta_0 < \frac{3}{2} \quad \left( \alpha_0 < \frac{3}{2}, \quad \beta_0 = \beta_1 = \beta_2 = 0 \right) \quad \text{and} \quad \lambda_2^2(t) < 4\lambda_1(t),
\]

then problem (1.17), (1.21), (1.3) (problem (1.1), (1.22), (1.3)) has at least one solution. Consequently, under the conditions of Corollary 1.1 Eq. (1.13) may have singularities of arbitrary order at the points \( a \) and \( b \) (at the point \( a \)).
Example 1.2. If \( f_1(u)(t) \equiv \frac{1}{4(t-a)^2} \), \( f_2(u)(t) \equiv -\frac{1}{t-a} \), and \( f_0(u)(t) \equiv 15 \), then for Eq. (1.13) all the conditions of Corollary 1.1 hold, except (1.16), instead of which we have
\[
f_2^2(u)(t) \leq 4f_1(u)(t).
\]

On the other hand, in this case Eq. (1.13) has the form
\[
u''(t) = \frac{u(t)}{4(t-a)^2} - \frac{u'(t)}{t-a} + 15
\]
and its arbitrary solution admits the representation
\[
u(t) = c_1(t-a)^{1/2} + c_2(t-a)^{-1/2} + 4(t-a)^2,
\]
where \( c_i \in \mathbb{R} \) (\( i = 1, 2 \)). Hence it is evident that both problems (1.13), (1.21), (1.3) and (1.13), (1.22), (1.3) have no solution.

The above constructed example shows that the condition \( \ell \in [0, 1] \) (\( \lambda \in [0, 4] \)) in Theorem 1.2 (in Corollary 1.1) is unimprovable and it cannot be replaced by the condition \( \ell = 1 \) (\( \lambda = 4 \)).

Now we proceed to the consideration of the case when condition (1.16) is violated, but \( f_1 \) and \( f_2 \) satisfy either the conditions
\[
(1.18_1) \quad f_1(u)(t) \geq -\frac{\ell_1}{(t-a)^2(b-t)^2}, \quad |f_2(u)(t)| \leq \frac{\ell_2}{(t-a)(b-t)},
\]
where
\[
(1.19_1) \quad \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} < 1,
\]
or the conditions
\[
(1.18_2) \quad f_1(u)(t) \geq -\frac{\ell_1}{(t-a)^2}, \quad |f_2(u)(t)| \leq \frac{\ell_2}{(t-a)},
\]
where
\[
(1.19_2) \quad 4\ell_1 + 2\ell_2 < 1.
\]

Corollary 1.2. Let conditions (1.14_1) (conditions (1.14_2)) be fulfilled and \( f_1^*(\cdot, \rho) \in L_{loc}([a,b]) \) \( f_2^*(\cdot, \rho) \in L_{loc}([a,b]) \) for \( \rho \in \mathbb{R}^+ \). Let, moreover, there exist nonnegative constants \( \ell_1, \ell_2 \) satisfying inequality (1.19_1) (inequality (1.19_2)), such that for an arbitrary \( u \in D_1([a,b]) \) (for an arbitrary \( u \in D_2([a,b]) \)) inequalities (1.18_1) (inequalities (1.18_2)) are fulfilled almost everywhere on \([a,b] \). Then problem (1.13) (1.21), (1.3) (problem (1.13), (1.22), (1.3)) has at least one solution.

Example 1.3. Consider the differential equation
\[
u''(t) = -\frac{\lambda_1}{(t-a)^2} u(t) - \frac{\lambda_2}{t-a} u'(t) + 2 + \lambda_1 + \lambda_2,
\]
where $\lambda_1$ and $\lambda_2$ are positive constants. This equation is obtained from (1.13) in the case where

$$f_1(u)(t) = -\frac{\lambda_1}{(t-a)^2}, \quad f_2(u)(t) = -\frac{\lambda_2}{t-a}, \quad f_0(u)(t) = 2 + \lambda_1 + \lambda_2.$$ 

It is clear that in this case conditions (1.181), (conditions (1.182)), where $\ell_1 = \lambda_1(b-a)^2$, $\ell_2 = \lambda_2(b-a)$ ($\ell_1 = \lambda_1$, $\ell_2 = \lambda_2$), are fulfilled. By Corollary 1.2, it follows that the inequality

$$4\lambda_1 + 2\lambda_2 < 1$$

guarantees the solvability of problems (1.20), (1.21), (1.3) and (1.20), (1.22), (1.3). On the other hand, if

$$4\lambda_1 + 2\lambda_2 = 1,$$

then both problems (1.20), (1.21), (1.3) and (1.20), (1.22), (1.3) have no solution since an arbitrary solution of Eq. (1.20) has the form

$$u(t) = c_1(t-a)^{1/2} + c_2(t-a)^{1/2-\lambda_2} + (t-a)^2,$$

where $c_i \in R$ ($i = 1, 2$). The above constructed example shows that condition (1.191) (condition (1.192)) in Corollary 1.2 is unimprovable and it cannot be replaced by the condition

$$(1.21) \quad \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} \leq 1 \quad (4\ell_1 + 2\ell_2 \leq 1).$$

**Theorem 1.3.** Let condition (1.7) be fulfilled, and let there exist constants $\ell \in [0,1[$, $a_0 \in ]a,b]$ and $b_0 \in ]a_0, b[$ such that for arbitrary $u_i \in D_1([a,b])$ (for arbitrary $u_i \in D_2([a,b])$) ($i = 1, 2$) the inequality

$$f(u_2)(s) - f(u_1)(s)) \leq \begin{cases} \int_{t_0}^{t} (f(u_2)(s) - f(u_1)(s))(u_2(s) - u_1(s)) \, ds \\ \geq -\ell \int_{a}^{b} (u'_2(s) - u'_1(s))^2 \, ds \quad \text{for} \quad a < t_0 \leq a_0, \quad b_0 \leq t < b \end{cases}$$

is fulfilled. Let, moreover,

$$f(0)(\cdot) \in L_{2,2}^2([a,b]) \quad (f(0)(\cdot) \in L_{2,0}^2([a,b])).$$

Then problem (1.1), (1.21), (1.3) (problem (1.1), (1.22), (1.3)) is uniquely solvable and stable with respect to small perturbations of the right-hand member of Eq. (1.1).
**Remark 1.1.** By Examples 1.2 and 1.3, the condition \( \ell \in [0,1] \) in Theorem 1.3 is unimprovable and it cannot be replaced by the condition \( \ell = 1 \).

**Remark 1.2.** Under the conditions of Theorem 1.3, problem (1.1), (1.2) (problem (1.1), (1.2)) may have an infinite set of solutions. Indeed, if \( \lambda_1 > 0, \lambda_2 \geq 0, \) and \( 4\lambda_1 + 2\lambda_2 < 1 \), then for Eq. (1.20) all the conditions of Theorem 1.3 are fulfilled, and hence problems (1.20), (1.2), (1.3) and (1.20), (1.2), (1.3) are uniquely solvable. On the other hand, it is clear that both problems (1.20), (1.2) and (1.20), (1.2) have an infinite set of solutions.

**1.4. The existence and uniqueness theorems for Eq. (1.1').** Everywhere in this section we assume that the function \( f_0 : \times R^2 \to R \) is measurable in the first and continuous in the two last arguments. As for the functions \( \tau_i : \times R^2 \to \times R^2 \) (i = 1, 2), they are continuously differentiable and

\[
\tau_i'(t) \neq 0 \quad \text{for} \quad a < t < b \quad (i = 1, 2).
\]

Of special interest is the case when in \( \times R^2 \) either the inequality

\[
|f_0(t, x, y)| \leq \frac{\ell_1 |\tau_1'(t)|^{1/2}|x|}{(\tau_1(t) - a)(b - \tau_1(t))(t - a)(b - t)} + \frac{\ell_2 |\tau_2'(t)|^{1/2}|y|}{(t - a)(b - t)} + q(t, (t - a)^{-1/2}(b - t)^{-1/2}|x|),
\]

or the inequality

\[
|f_0(t, x, y)| \leq \frac{\ell_1 |\tau_1'(t)|^{1/2}|x|}{(\tau_1(t) - a)(t - a)} + \frac{\ell_2 |\tau_2'(t)| |y|}{t - a} + q(t, (t - a)^{-1/2}|x|)
\]

is fulfilled. Here \( \ell_1 \) and \( \ell_2 \) are nonnegative constants, and \( q : \times R^2 \to R^2 \) is a nondecreasing in the second argument function, satisfying the conditions

\[
q(\cdot, \rho) \in L^2_{\ell, 2}(\times R^2) \quad \text{for} \quad \rho \in R^2, \quad \lim_{\rho \to +\infty} \rho^{-1} ||q(\cdot, \rho)||_{L^2_{\ell, 2}} = 0,
\]

or

\[
q(\cdot, \rho) \in L^2_{\ell, 0}(\times R^2) \quad \text{for} \quad \rho \in R^2, \quad \lim_{\rho \to +\infty} \rho^{-1} ||q(\cdot, \rho)||_{L^2_{\ell, 0}} = 0.
\]

**Theorem 1.4.** Let there exist positive constants \( \ell_1, \ell_2 \) and a nondecreasing in the second argument function \( q : \times R^2 \to R^2 \) satisfying conditions (1.19), (1.25) (conditions (1.19), (1.25)) such that in \( \times R^2 \) condition (1.24) (condition (1.24)) is fulfilled. Then problem (1.1'), (1.2), (1.3) (problem (1.1'), (1.2), (1.3)) has at least one solution.

The theorem on the unique solvability of problems (1.1'), (1.2), (1.3) and (1.1'), (1.2), (1.3) concerns the cases where instead of (1.24) the condition

\[
|f_0(t, x_1, y_1) - f_0(t, x_2, y_2)| \leq
\]

\[
\frac{\ell_1 |\tau_1'(t)|^{1/2}|x|}{(\tau_1(t) - a)(b - \tau_1(t))(t - a)(b - t)} + \frac{\ell_2 |\tau_2'(t)|^{1/2}|y|}{(t - a)(b - t)} + q(t, (t - a)^{-1/2}(b - t)^{-1/2}|x|),
\]

or the inequality

\[
|f_0(t, x, y)| \leq \frac{\ell_1 |\tau_1'(t)|^{1/2}|x|}{(\tau_1(t) - a)(t - a)} + \frac{\ell_2 |\tau_2'(t)| |y|}{t - a} + q(t, (t - a)^{-1/2}|x|)
\]
is fulfilled, and instead of (1.242) the condition
\[ |f_0(t, x_1, y_1) - f_0(t, x_2, y_2)| \leq \frac{\ell_1 |\tau'_1(t)|^{1/2} |x_1 - x_2|}{(\tau_1(t) - a)(b - \tau_1(t))(t - a)(b - t)} + \frac{\ell_2 |\tau'_2(t)|^{1/2} |y_1 - y_2|}{(t - a)(b - t)} \]
is fulfilled.

**Theorem 1.5.** Let there exist positive constants \( \ell_1, \ell_2 \) satisfying inequality (1.191) (inequality (1.192)) such that in \([a, b] \times \mathbb{R}^2 \) condition (1.261) (condition (1.262)) is fulfilled. Moreover, let
\[ f_0(\cdot, 0, 0) \in L^2_{2,2}([a, b]) \quad (f_0(\cdot, 0, 0) \in L^2_{2,0}([a, b])). \]
Then problem (1.1'), (1.21), (1.3) (problem (1.1'), (1.22), (1.3)) is uniquely solvable, and its solution is stable with respect to small perturbations of the right-hand member of Eq. (1.1').

**Remark 1.3.** According to Example 1.3, condition (1.191) (condition (1.192)) in Theorems 1.4 and 1.5 is unimprovable and it cannot be replaced by condition (1.21).

§ 2. Auxiliary Propositions.

2.1. Lemmas on integral inequalities. Let \( \tau_i : [a, b] \to [a, b] \) (i = 1, 2) be continuously differentiable monotone functions. For an arbitrary \( u \in C_{1 \text{loc}}([a, b]) \) we put
\[ w_{10}(u)(t) = \frac{|\tau'_1(t)|^{1/2}|u(\tau_1(t))|}{(t - a)(b - \tau_1(t)(t - a)(b - \tau_1(t))}, \]
\[ w_{11}(u)(t) = \frac{|\tau'_2(t)|^{1/2}|u'(\tau_2(t))|}{(t - a)(b - t)}; \]
\[ w_{20}(u)(t) = \frac{|\tau'_1(t)|^{1/2}|u(\tau_1(t))|}{(t - a)(\tau_1(t) - a)}, \quad w_{21}(u)(t) = \frac{|\tau'_2(t)|^{1/2}|u'(\tau_2(t))|}{t - a}. \]

**Lemma 2.1.** If \( u \in D_1([a, b]) \), then
\[ \int_a^b |u(s)| w_{1i}(u)(s) \, ds \leq \left( \frac{2}{b - a} \right)^{2-i} \int_a^b u^2(s) \, ds \quad (i = 0, 1). \]
If \( u \in D_2([a, b]) \), then
\[ \int_a^b |u(s)| w_{2i}(u)(s) \, ds \leq 2^{2-i} \int_a^b u^2(s) \, ds \quad (i = 0, 1). \]
To prove this lemma, we need the following

**Lemma 2.2 (V. I. Levin [17]).** If \( u \in D_1([a, b]) \), then

\[
\int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \leq \frac{4}{(b-a)^2} \int_a^b u'^2(s) \, ds.
\]

If \( u \in D_2([a, b]) \), then

\[
\int_a^b \frac{u^2(s) \, ds}{(s-a)^2} \leq 4 \int_a^b u'^2(s) \, ds.
\]

**Proof of Lemma 2.1.** Let \( u \in D_1([a, b]) \). Then by virtue of Lemma 2.2 and the Schwartz inequality we have

\[
\int_a^b |u(s)| \omega_{10}(u)(s) \, ds \leq \left( \int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \right)^{1/2} \left( \int_a^b \frac{|\tau'_1(s)| u^2(\tau_1(s)) \, ds}{(\tau_1(s)-a)^2(b-\tau_1(s))^2} \right)^{1/2} =
\]

\[
= \left( \int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \right)^{1/2} \left| \int_{\tau_1(a)}^{\tau_1(b)} \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \right|^{1/2} \leq \int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \leq \frac{4}{(b-a)^2} \int_a^b u'^2(s) \, ds,
\]

\[
\int_a^b |u(s)| \omega_{11}(u)(s) \, ds \leq \left( \int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \right)^{1/2} \left( \int_{\tau_2(a)}^{\tau_2(b)} |u^2(\tau_2(s))| \, ds \right)^{1/2} =
\]

\[
= \left( \int_a^b \frac{u^2(s) \, ds}{(s-a)^2(b-s)^2} \right)^{1/2} \left| \int_{\tau_2(a)}^{\tau_2(b)} u'^2(s) \, ds \right|^{1/2} \leq \frac{2}{b-a} \int_a^b u'^2(s) \, ds.
\]

Consequently, inequalities (2.3) are valid.

Analogously we can show that for \( u \in D_2([a, b]) \) inequalities (2.4) are fulfilled. \( \square \)
Lemma 2.1 immediately yields the following lemma.

**Lemma 2.3.** Let \( \ell_1 \) and \( \ell_2 \) be nonnegative constants and

\[
(2.5) \quad w_i(u)(t) = \ell_1 w_{i0}(u)(t) + \ell_2 w_{i1}(u)(t) \quad (i = 1, 2).
\]

If, moreover, \( u \in D_1([a, b]) \), then

\[
\int_a^b |u(s)|w_1(u)(s) \, ds \leq \left( \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} \right) \int_a^b u'^2(s) \, ds.
\]

If \( u \in D_2([a, b]) \), then

\[
\int_a^b |u(s)|w_2(u)(s) \, ds \leq (4\ell_1 + 2\ell_2) \int_a^b u'^2(s) \, ds.
\]

**Lemma 2.4.** Let

\[
u \in D_1([a, b]), \quad q \in L^2_{1,2}([a, b]) \quad \left( u \in D_2([a, b]), \quad q \in L^2_{2,0}([a, b]) \right).
\]

Then for an arbitrary \( \varepsilon > 0 \) the inequality

\[
(2.6) \quad \int_a^b |q(s)u(s)| \, ds \leq \rho_\varepsilon(q) + \varepsilon \int_a^b u'^2(s) \, ds
\]

holds, where

\[
\rho_\varepsilon(q) = (b-a)^{-2}\varepsilon^{-1}\|q\|^2_{L^2_{1,2}} \quad \left( \rho_\varepsilon(q) = \varepsilon^{-1}\|q\|^2_{L^2_{2,0}} \right).
\]

**Proof.** Obviously,

\[
|q(s)u(s)| \leq \varepsilon^{-1}(b-a)^{-2}(s-a)^2(b-s)^2 q^2(s) + \frac{\varepsilon(b-a)^2}{4} \frac{u^2(s)}{(s-a)^2(b-s)^2}
\]

and

\[
|q(s)u(s)| \leq \varepsilon^{-1}(s-a)^2 q^2(s) + \frac{\varepsilon}{4} (s-a)^2 q^2(s).
\]

If we integrate the first (the second) of these last two inequalities from \( a \) to \( b \), and apply Lemma 2.2, then we get inequality (2.6). \( \Box \)

**Lemma 2.5.** Let \( \ell_1, \ell_2 \) be nonnegative constants satisfying inequality (1.191) (inequality (1.192)), and let \( q : [a, b[ \times R_+ \rightarrow R_+ \) be a nondecreasing in the second argument function, satisfying conditions (1.251) (conditions (1.252)). Let,
moreover, the operator $f : C^1_{loc}([a, b]) \to L_{loc}([a, b])$ be such that for an arbitrary $u \in D_1([a, b])$ ($u \in D_2([a, b])$) almost everywhere on $[a, b]$ the inequality

$$f(u)(t)\text{sgn}u(t) \geq -w_1(u)(t) - q(t, \|u\|_{L^2})$$

holds, where $w_1$ and $w_2$ are the operators, given by equality (2.5). Then there exist constants $\ell \in ]0, 1[$ and $\ell_0 \geq 0$ such that for arbitrary $a_0 \in ]a, b[, b_0 \in ]a_0, b[$ and $u \in D_1([a, b])$ ($u \in D_2([a, b])$) inequality (1.12) is satisfied.

Proof. By conditions (1.191) and (1.251) (conditions (1.192) and (1.252)), there exist constants $\varepsilon \in ]0, 1[$ and $\ell_0 \geq 0$ such that

$$\ell = \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} + \varepsilon < 1 \quad \left( \ell = 4\ell_1 + 2\ell_2 + \varepsilon < 1 \right)$$

and

$$2(b-a)^{-2} \varepsilon^{-1} \|q(\cdot, \rho)\|_{L_{2,2}}^2 \leq \frac{\varepsilon}{2} \rho^2 + \ell_0 \quad \text{for } \rho \in R_+,$$

(2.8)

Due to Lemma 2.3, from (2.7) we find

$$\int_{t_0}^{t} f(u)(s)u(s) \, ds \geq -\left( \ell - \varepsilon \right) \int_{a}^{b} u'^2(s) \, ds -$$

$$- \int_{a}^{b} q(s, \|u'\|_{L^2})|u(s)| \, ds \quad \text{for } a < t_0 < t < b.$$ On the other hand, by Lemma 2.4 and condition (2.8), we get

$$\int_{a}^{b} q(s, \|u'\|_{L^2})|u(s)| \, ds \leq$$

$$\leq \frac{\varepsilon}{2} \int_{a}^{b} u'^2(s) \, ds + 2(b-a)^{-2} \varepsilon^{-1} \|q(\cdot, \|u'\|_{L^2})\|_{L_{2,2}}^2 \leq \varepsilon \int_{a}^{b} u'^2(s) \, ds + \ell_0$$

$$\left( \int_{a}^{b} q(s, \|u'\|_{L^2})|u(s)| \, ds \leq$$

$$\leq \frac{\varepsilon}{2} \int_{a}^{b} u'^2(s) \, ds + 2\varepsilon^{-1} \|q(\cdot, \|u'\|_{L^2})\|_{L_{2,0}}^2 \leq \varepsilon \int_{a}^{b} u'^2(s) \, ds + \ell_0 \right).$$

Therefore, for arbitrary $a_0 \in ]a, b[, b_0 \in ]a_0, b[$ and $u \in D_1([a, b])$ ($u \in D_2([a, b])$) inequality (1.12) holds. □
2.2. Lemmas on a priori estimates.

**LEMMA 2.6.** Let

\[(2.9)\]
\[u \in D_1([a, b]) \cup D_2([a, b])\]

and

\[(2.10)\]
\[\liminf_{t_0 \to a, t \to b} \int_{t_0}^{t} u''(s)u(s) \, ds \geq -\ell \int_{a}^{b} u'^2(s) \, ds - \ell_0,\]

where \(\ell \in [0, 1]\) and \(\ell_0 \geq 0\). Then

\[(2.11)\]
\[\int_{a}^{b} u'^2(s) \, ds \leq \frac{\ell_0}{1 - \ell} .\]

To prove the above lemma, we need the following

**LEMMA 2.7.** If condition (2.9) is fulfilled, then

\[(2.12)\]
\[\liminf_{t \to a} |u'(t)u(t)| = 0,\]

\[(2.13)\]
\[\liminf_{t \to b} |u'(t)u(t)| = 0.\]

**Proof.** Suppose that equality (2.12) is violated. Then there exist \(a_0 \in ]a, b[\) and \(\delta > 0\) such that

\[|u'(t)u(t)| > \delta \text{ for } a < t \leq a_0,\]

whence with regard for the equality \(u(a+) = 0\), we find

\[u^2(t) > 2\delta(t - a) \text{ for } a < t \leq a_0.\]

On the other hand, by (2.9) we have

\[2\delta \leq (t - a)^{-1}u^2(t) = (t - a)^{-1}\left(\int_{a}^{t} u'(s) \, ds\right)^2 \leq \int_{a}^{t} u'^2(s) \, ds \to 0 \text{ as } t \to a.\]

The obtained contradiction shows that equality (2.12) is valid.

If \(u \in D_2([a, b])\), then equality (2.13) is obvious. If, however, \(u \in D_1([a, b])\), then it can be proved just in the same way as (2.12). \(\Box\).

**Proof of Lemma 2.6.** By Lemma 2.7, there exist sequences \((t_{0k})_{k=1}^{+\infty}\) and \((t_k)_{k=1}^{+\infty}\) such that

\[a < t_{0k} < t_k < b \text{ (} k = 1, 2, \ldots\text{), } \lim_{k \to +\infty} t_{0k} = a, \lim_{k \to +\infty} t_k = b\]
and
\[ \lim_{k \to +\infty} u'(t_{0k})u(t_{0k}) = \lim_{k \to +\infty} u'(t_k)u(t_k) = 0. \]

On the other hand,
\[ \int_{t_{0k}}^{t_k} u''(s)u(s)\,ds = u'(t_k)u(t_k) - u'(t_{0k})u(t_{0k}) - \int_{t_{0k}}^{t_k} u'^2(s)\,ds. \]

Therefore,
\[ \lim_{k \to +\infty} \int_{t_{0k}}^{t_k} u''(s)u(s)\,ds = -\int_a^b u'^2(s)\,ds. \]

This equality, according to (2.10), implies
\[ -\int_a^b u'^2(s)\,ds \geq -\ell \int_a^b u'^2(s)\,ds - \ell_0. \]

Consequently, estimate (2.11) is true. □

**Lemma 2.8.** Let
\[ u \in D_1([a, b]), \quad q \in L^2_{2,2}([a, b]) \quad \left( u \in D_2([a, b]), \quad q \in L^2_{2,0}([a, b]) \right) \]

and
\[ \liminf_{t_0 \to a, t \to b} \int_{t_0}^{t} u''(s)u(s)\,ds \geq -\ell \int_a^b u'^2(s)\,ds - \int_a^b q(s)|u(s)|\,ds, \]

where \( \ell \in [0, 1] \). Then
\[ \|u'\|_{L^2} \leq \frac{2}{(1-\ell)(b-a)}\|q\|_{L^2_{2,2}} \quad \left( \|u'\|_{L^2} \leq \frac{2}{1-\ell}\|q\|_{L^2_{2,0}} \right). \]

**Proof.** By Lemma 2.4, from (2.14) we find
\[ \liminf_{t_0 \to a, t \to b} \int_{t_0}^{t} u''(s)u(s)\,ds \geq -(\ell + \varepsilon) \int_a^b u'^2(s)\,ds - \rho_\varepsilon(q), \]

where \( \varepsilon = \frac{1-\ell}{2} \). Hence by Lemma 2.6 we have
\[ \int_a^b u'^2(s)\,ds \leq \frac{\rho_\varepsilon(q)}{1-\ell - \varepsilon} = \varepsilon^{-1}\rho_\varepsilon(q). \]
Therefore, estimate (2.15) is valid.

§ 3. Proofs of the Basic Results.

Below, under $C_0^1$ we will mean the Banach space of continuously differentiable functions $u : [a, b] \to \mathbb{R}$, satisfying the conditions

\begin{equation}
\tag{3.1}
u(a) = 0, \quad u(b) = 0,
\end{equation}

with the norm

$$\|u\|_{C_0^1} = \max \{\|u'(t)\| : a \leq t \leq b\}.$$ 

Proof of Theorem 1.1. We will consider only the case where $f \in \mathcal{K}_1([a, b])$ and prove the solvability of problem (1.1), (1.21), (1.3), since in the case where $f \in \mathcal{K}_2([a, b])$, the solvability of problem (1.1), (1.22), (1.3) is proved analogously.

By Definition 1.2, there exists a continuous function $\omega : [a, b] \times [a, b] \times R_+ \to R_+$ satisfying identity (1.5), such that for an arbitrary $u \in D_1([a, b])$ inequality (1.6) is fulfilled.

First we prove that for an arbitrary natural $k$ the functional differential equation

\begin{equation}
\tag{3.2}
u''(t) = f_k(u)(t)
\end{equation}

has at least one solution $u_k$ satisfying the conditions

\begin{equation}
\tag{3.3}u_k(a) = u_k(b) = 0, \quad \|u_k'\|_{L^2} \leq r_0.
\end{equation}

Suppose

$$\eta(\rho) = \begin{cases} 1 & \text{for } 0 \leq \rho \leq r_0 \\ 2 - \frac{\rho}{r_0} & \text{for } r_0 < \rho < 2r_0 \\ 0 & \text{for } \rho \geq 2r_0 \end{cases}, \quad \xi_k(t) = \begin{cases} t & \text{for } a_k < t < b_k \\ b_k & \text{for } t \geq b_k \end{cases}.$$ 

Then the function $\omega_k : [a, b] \times [a, b] \to R_+$ is continuous, and

$$\omega_k(t, t) = 0.$$ 

Consequently, $B_k$ is the compact set of the space $C_0^1$. On the other hand, by conditions (1.6) and (1.8), for an arbitrary $u \in C_0^1$ the inequalities

\begin{equation}
\tag{3.4}\int_s^t |\tilde{f}_k(u)(\xi)| \, d\xi \leq \omega_k(s, t) \quad \text{for } a \leq s \leq t \leq b, \quad \int_a^b |\tilde{f}_k(u)(s)| \, ds \leq r_k
\end{equation}
are fulfilled.

To prove the solvability of problem (3.2), (3.1), we need to consider the problem on the existence of a solution of the functional differential equation

$$u''(t) = \tilde{f}_k(u)(t),$$

satisfying the boundary conditions (3.1). This problem is equivalent to the following operator equation in the space $C^1_0$,

$$u(t) = g_k(u)(t),$$

where

$$g_k(u)(t) = \int_a^b g(t, s)\tilde{f}_k(u)(s)\,ds,$$

and $g$ is the Green function of the boundary value problem

$$u'' = 0, \quad u(a) = u(b) = 0.$$

The continuity of the operator $f : C^1_{loc}(]a, b[) \to L_{loc}(]a, b[)$ implies that of the operator $g_k : C^1_0 \to C^1_0$. On the other hand, by conditions (3.4), for an arbitrary $u \in B_k$ the function $v = g_k(v)$ satisfies the inequalities

$$\|v\|_{C^1_0} \leq \int_a^b |\tilde{f}_k(u)(s)|\,ds \leq r_k,$$

$$|v'(t) - v'(s)| = \left|\int_s^t \tilde{f}_k(u)(\xi)\,d\xi\right| \leq \omega_k(s, t) \quad \text{for } a \leq s \leq t \leq b.$$

Consequently, the operator $\tilde{g}_k$ transforms the convex compact $B_k$ into itself. By the Schauder principle, this implies the existence of a solution $u_k$ of Eq. (3.5), belonging to the set $B_k$. Obviously, $u_k$ is a solution of problem (3.5), (3.1) as well, i.e. a solution of problem (1.10), (1.21), where

$$\lambda = \eta_k(\|u_k\|_{L^2}) \in [0, 1]$$

and $\lambda < 1$ for $\|u_k\|_{L^2} > r_0$. However, by the condition of the theorem, for an arbitrary $\lambda \in [0, 1]$ every solution of problem (1.10), (1.21) admits estimate (1.11). Therefore,

$$\|u_k\|_{L^2} \leq r_0,$$

whence by the definition of $\tilde{f}_k$ it follows that $\tilde{f}_k(u)(t) \equiv f_k(u_k)(t)$. Thus we have proved that $u_k$ is a solution of the functional differential equation (3.2), satisfying conditions (3.3).
By virtue of conditions (1.6), (3.3), for every natural \( k \) the function \( u_k \) satisfies the inequalities

\[
|u_k(t)| \leq \frac{2r_0}{b-a} (t-a)^{1/2}(b-t)^{1/2} \quad \text{for} \quad a \leq t \leq b,
\]

\[
|u_k(t) - u_k(s)| \leq r_0 |t-s|^{1/2} \quad \text{for} \quad a \leq s \leq t \leq b,
\]

\[
\min \left\{ |u_k(t)| : \frac{3a+b}{4} \leq t \leq \frac{a+b}{2} \right\} \leq 2r_0(b-a)^{-1/2},
\]

\[
|u'_k(t) - u'_k(s)| = \left| \int_t^s f_k(u)(\xi) \, d\xi \right| \leq \int_t^s |f(u)(\xi)| \, d\xi \leq \omega(s, t, r_0) \quad \text{for} \quad a < s \leq t < b.
\]

Hence, by the Arzela-Ascoli lemma follows the existence of a subsequence \( (u_{k_j})_{j=1}^{+\infty} \) of the sequence \( (u_k)_{k=1}^{+\infty} \) and a function \( u \in D_1([a, b[) \) such that

\[
\lim_{j \to +\infty} u_{k_j}(t) = u(t), \quad \lim_{j \to +\infty} u'_{k_j}(t) = u'(t)
\]

uniformly on every compact interval contained in \( [a, b[ \). Consequently, \( (u_{k_j})_{j=1}^{+\infty} \) converges to \( u \) due to the topology of the space \( C_{loc}^1([a, b[) \).

To complete the proof of the theorem, it remains to show that \( u \) is a solution of Eq. (1.1). Indeed, let \( t_0 \) and \( t \) be arbitrarily fixed points from \( [a, b[ \). By virtue of condition (1.8), there exists a natural number \( m(t, t_0) \) such that

\[
t_0 \in ]a_{k_j}, b_{k_j}[, \quad t \in ]a_{k_j}, b_{k_j}[ \quad \text{for} \quad j \geq m(t, t_0).
\]

Owing to this fact and condition (1.9), we have

\[
u'_{k_j}(t) = u'_{k_j}(t_0) + \int_{t_0}^t f(u_{k_j})(s) \, ds \quad \text{for} \quad j \geq m(t, t_0).
\]

If in this equality we pass to the limit as \( j \to +\infty \) and take into account the continuity of the operator \( f : C_{loc}^1([a, b[) \to L_{loc}([a, b[) \), then we get

\[
u'(t) = u'(t_0) + \int_{t_0}^t f(u)(s) \, ds.
\]

Hence, due to the arbitrariness of \( t \in [a, b[ \), it follows that \( u \) is a solution of Eq. (1.1). \( \square \).

**Proof of Theorem 1.2.** Let

\[
r_0 = \left( \frac{\ell_0}{1-\ell} \right)^{1/2}, \quad a_k = a + \frac{a_0-a}{2k}, \quad b_k = b - \frac{b-b_0}{2k} \quad (k = 1, 2, \ldots),
\]

where \( \ell_0 = \ell_0(\ell) \).
and \( u \) be a solution of problem (1.10), (1.21) (of problem (1.10), (1.22)) for some \( \lambda \in [0, 1] \) and natural \( k \). Then by condition (1.12) we have

\[
\lim_{t \to b, t_0 \to a} \int_{t_0}^{t} u''(s)u(s) \, ds = \lambda \int_{a_k}^{b} f(u)(s)u(s) \, ds \geq -\ell \int_{a}^{b} u''(s) \, ds - \ell_0.
\]

However, according to Lemma 2.6, the last inequality guarantees the validity of estimate (1.11). Thus we have proved that all the conditions of Theorem 1.1 are fulfilled and, consequently, problem (1.1), (1.21), (1.3) (problem (1.1), (1.22), (1.3)) is solvable.

\[\text{Proof of Corollary 1.1.}\] Let us choose \( \epsilon \in ]0, 1[ \) such that

\[\ell \overset{\text{def}}{=} \frac{\lambda}{4} + \epsilon < 1.\]

By condition (1.141) (condition (1.142)), there exists a nonnegative constant \( \ell_0 \) such that

\[
(3.7) \quad \frac{2}{(b-a)^2 \epsilon} \| f_{01}^* (\cdot, \rho) \|_{L^2, 2}^2 \leq \frac{\epsilon}{2} \rho^2 + \ell_0 \left( \frac{1}{\epsilon} \| f_{01}^* (\cdot, \rho) \|_{L^2, 0}^2 \leq \frac{\epsilon}{2} \rho^2 + \ell_0 \right)
\]

for \( \rho \in R_+ \).

Let

\[
(3.8) \quad f(u)(t) = f_1(u)(t)u(t) + f_2(u)(t)u'(t) + f_0(u)(t).
\]

Owing to the continuity of the operators \( f_i : C^1_{loc}(]a, b[) \to L_{loc}(]a, b[) \) (\( i = 0, 1, 2 \)) and by conditions (1.141), (1.151) (by conditions (1.142), (1.152)), the operator \( f : C^1_{loc}(]a, b[) \to L_{loc}(]a, b[) \) is continuous and for an arbitrary \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \) inequality (1.6) holds, where

\[
\omega(s, t, \rho) = \int_{s}^{t} (f_{02}^* (\xi, \rho) + (b-a)^{1/2} \rho f_{11}^* (\xi, \rho)) \, d\xi + \rho \left( \int_{s}^{t} f_{21}^* (\xi, \rho) \, d\xi \right)^{1/2}
\]

\[
\left( \omega(s, t, \rho) = \int_{s}^{t} (f_{02}^* (\xi, \rho) + (b-a)^{1/2} \rho f_{12}^* (\xi, \rho)) \, d\xi + \rho \left( \int_{s}^{t} f_{22}^* (\xi, \rho) \, d\xi \right)^{1/2}. \right)
\]

Moreover, the function \( \omega : ]a, b[ \times ]a, b[ \to R_+ \) (\( \omega : ]a, b[ \times ]a, b[ \to R_+ \)) is continuous and satisfies identity (1.5). Hence the operator \( f \) satisfies condition (1.7).

It follows from (3.8) that

\[
\int_{t_0}^{t} f(u)(s)u(s) \, ds \geq \int_{t_0}^{t} \left( f_1(u)(s)u^2(s) - |f_2(u)(s)u'(s)|u(s) \right) \, ds - \int_{a}^{b} |f_0(u)(s)u(s)| \, ds \quad \text{for} \quad a < t_0 \leq t < b.
\]
However, by virtue of Lemma 2.4 and condition (3.7), for an arbitrary \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \)) the inequality

\[
\int_{a}^{b} |f_0(u)(s)u(s)| \, ds \leq \int_{a}^{b} f^*_0(s, \|u'\|_{L^2})|u(s)| \, ds + \\
\frac{2}{(b-a)^2} \|f^*_0(\cdot, \|u'\|_{L^2})\|_{L^2_{x,2}}^2 + \frac{\varepsilon}{2} \int_{a}^{b} u''(s) \, ds \leq \varepsilon \int_{a}^{b} u''(s) \, ds + \ell_0
\]

is fulfilled. On the other hand, by condition (1.16) we have

\[
f_1(u)(s)u^2(s) - |f_2(u)(s)u'(s)u(s)| \geq \left( f_1(u)(s) - \frac{1}{\lambda} f^*_2(u)(s) \right) u^2(s) - \frac{\lambda}{4} u''(s) \geq -\frac{\lambda}{4} u''(s).
\]

Therefore it is clear that for arbitrary \( a_0 \in [a, b[ , \ b_0 \in ]a_0, b[ \) and \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \)) inequality (1.12) is fulfilled.

Thus the operator \( f \), given by equality (3.8), satisfies all the conditions of Theorem 1.1, which guarantees the solvability of problem (1.1), (1.21), (1.3) (of problem (1.1), (1.22), (1.3)). \( \square \)

**Proof of Corollary 1.2.** Let \( f \) be the operator given by equality (3.8). Then, as it is proved above, condition (1.7) is fulfilled. On the other hand, by conditions (1.141) and (1.181) (by conditions (1.142) and (1.182)) for an arbitrary \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \)) almost everywhere on \([a, b[\) inequality (2.7) is fulfilled, where \( w_i \) is the operator given by equality (2.5),

\[
w_{10}(u)(t) = \frac{|u(t)|}{(t-a)^2(b-t)^2}, \quad w_{11}(t) = \frac{|u'(t)|}{(t-a)(b-t)}
\]

\[
\begin{align*}
  w_{20}(u)(t) &= \frac{|u(t)|}{(t-a)^2}, & w_{21}(u)(t) &= \frac{|u'(t)|}{t-a}, \\
  q(t, \rho) &= f^*_0(t, \rho) \left( q(t, \rho) = f^*_0(t, \rho) \right)
\end{align*}
\]

and the function \( q \) satisfies conditions (1.251) (conditions (1.252)). By virtue of Lemma 2.5, this implies that there exist constants \( \ell \in ]0,1[ \) and \( \ell_0 \geq 0 \) such that for arbitrary \( a_0 \in [a, b[ , \ b_0 \in ]a_0, b[ \) and \( u \in D_1([a, b]) \) (\( u \in D_2([a, b]) \)) inequality (1.12) is fulfilled.

Consequently, the operator \( f \) satisfies all the conditions of Theorem 1.2, which guarantees the solvability of problem (1.1), (1.21), (1.3) (of problem (1.1), (1.22), (1.3)). \( \square \)
ON TWO-POINT BVPs

Proof of Theorem 1.3. Let us choose \( \varepsilon > 0 \) such that
\[
\tilde{\ell} = \ell + \varepsilon < 1.
\]
Let \( h \in L^2_{2,2}([a,b]) \) \((h \in L^2_{2,0}([a,b]))\) and
\[
h_0(t) = f(0)(t) + h(t).
\]
Then by condition (1.23),
\[
h_0 \in L^2_{2,2}([a,b]) \quad \left( h \in L^2_{2,0}([a,b]) \right)
\]
On the other hand, if \( u \in D_1([a,b]) \) \((u \in D_2([a,b]))\), then by Lemma 2.4, we have
\[
\int_a^b |h_0(s)u(s)| \, ds \leq \ell_0 + \varepsilon \int_a^b u''(s) \, ds,
\]
where
\[
\ell_0 = \frac{1}{(b-a)^2 \varepsilon} \|h_0\|_{L^2_{2,2}}^2 \quad \left( \ell_0 = \frac{1}{\varepsilon} \|h_0\|_{L^2_{2,0}}^2 \right).
\]
This inequality and condition (1.22) imply that for an arbitrary \( u \in D_1([a,b]) \) \((u \in D_2([a,b]))\) the inequality
\[
\int_{t_0}^t [f(u)(s) + h(s)]u(s) \, ds = \int_{t_0}^t [f(u)(s) - f(0)(s)]u(s) \, ds +
\]
\[
+ \int_{t_0}^t h_0(s)u(s) \, ds \geq -\tilde{\ell} \int_a^b u''(s) \, ds - \int_a^b |h_0(s)u(s)| \, ds \geq
\]
\[
\geq -\tilde{\ell} \int_a^b u''(s) \, ds - \ell_0 \quad \text{for} \quad a < t_0 \leq a_0, \ b_0 \leq t < b
\]
is fulfilled. Using now Theorem 1.2, we can see that problem (1.4), (1.21), (1.3) \( (\text{problem (1.4), (1.22), (1.3))} \) is solvable.

Let \( u_1 \) and \( u_2 \) be two arbitrary solutions of that problem. Assume \( u(t) = u_2(t) - u_1(t) \). Then by virtue of condition (1.22), we have
\[
\liminf_{t_0 \to a, t \to b} \int_{t_0}^t u''(s)u(s) \, ds \geq -\tilde{\ell} \int_a^b u''(s) \, ds.
\]
This inequality, by Lemma 2.6 and equality \( u(a+) = 0 \), yields \( u(t) \equiv 0 \). Therefore, problem (1.4), (1.21), (1.3) \( (\text{problem (1.4), (1.22), (1.3))} \) is uniquely solvable for an arbitrary \( h \in L^2_{2,2}([a,b]) \) \((h \in L^2_{2,0}([a,b]))\).
Let \( u_0 \) and \( u_h \) be, respectively, the solutions of problems (1.1), (1.2), (1.3) and (1.4), (1.2), (1.3) (of problems (1.1), (1.2), (1.3) and (1.4), (1.2), (1.3)). Then by condition (1.22), we obtain

\[
\liminf_{t_0 \to a, t \to b} \int_{t_0}^{t} (u''_h(s) - u''_0(s))(u_h(s) - u_0(s)) \, ds \geq \frac{2}{\ell} \int_{a}^{b} (u'_h(s) - u'_0(s))^2 \, ds - \int_{a}^{b} |h(s)||u_h(s) - u_0(s)| \, ds.
\]

Hence, by Lemma 2.8, we have the estimate

\[
\|u'_h - u'_0\|_{L^2} \leq r\|h\|_{L^{\infty}} \left( \|u'_h - u'_0\|_{L^2} \leq \frac{2}{1 - \ell} \right),
\]

where

\[
r = \frac{2}{(1 - \ell)(b - a)} \quad \left( r = \frac{2}{1 - \ell} \right)
\]

is a number, independent of \( h \). Consequently, the problem under consideration is stable with respect to small perturbations of the right-hand member of Eq. (1.1).

\( \Box \)

**Proof of Theorem 1.4.** Suppose

\[
(3.9) \quad f(u)(t) = f_0(t, u(\tau_1(t)), u'(\tau_2(t))).
\]

Conditions (1.241) (conditions (1.242)) and the continuity of the function \( f_0 \) in the last two arguments imply that the operator \( f \) satisfies condition (1.7). On the other hand, according to inequality (1.241) (inequality (1.242)), for an arbitrary \( u \in D_1([a, b]) \) \((u \in D_2([a, b]))\) almost everywhere on \([a, b]\) the inequality

\[
|f(u)(t)| \leq w_1(u)(t) + q\left(t, \left(\frac{2}{b - a}\right)^{1/2}\|u'\|_{L^2}\right)
\]

\[
\left( |f(u)(t)| \leq w_2(u)(t) + q(t, \|u'\|_{L^2}) \right)
\]

holds, where \( w_1 \) and \( w_2 \) are the operators, given by equalities (2.1), (2.2) and (2.5). Hence, by conditions (1.191) and (1.251) (conditions (1.192) and (1.252)) and Lemma 2.5 follows the existence of constants \( \ell \in ]0, 1[ \) and \( \ell_0 \geq 0 \) such that for arbitrary \( a_0 \in ]a, b[ \), \( b_0 \in ]a_0, b[ \) and \( u \in D_1([a, b]) \) \((u \in D_2([a, b]))\) inequality (1.12) holds.

Therefore, the operator \( f \), given by equality (3.9), satisfies all the conditions of Theorem 1.2, which guarantees the solvability of problem (1.1'), (1.2'), (1.3) (problem (1.1'), (1.2'), (1.3)). \( \Box \)
**Proof of Theorem 1.5.** Suppose $f$ is an operator, given by equality (3.9). Then, by virtue of conditions (1.26) and (1.27) (conditions (1.26) and (1.27)), the operator $f$ satisfies condition (1.7). On the other hand, according to condition (1.26) (condition (1.26)), for arbitrary $u_i \in D_1([a,b])$ ($i=1,2$) (for arbitrary $u_i \in D_2([a,b])$ ($i=1,2$)) almost everywhere on $[a,b]$ the inequality
\[
(f(u_2)(t) - f(u_1)(t))(u_2(t) - u_1(t)) \geq -w_1(u_2 - u_1)(t)|u_2(t) - u_1(t)|
\]
is fulfilled, where $w_1$ and $w_2$ are the operators, given by equalities (2.1), (2.2) and (2.5). Hence, by inequality (1.19) (inequality (1.19)) and Lemma 2.3, follows that for arbitrary $a_0 \in [a,b]$, $b_0 \in [a_0,b]$ and $u_i \in D_1([a,b])$ ($u_i \in D_2([a,b])$) ($i=1,2$) inequality (1.22) holds, where
\[
\ell = \frac{4\ell_1}{(b-a)^2} + \frac{2\ell_2}{b-a} < 1 \quad (4\ell_1 + 2\ell_2 < 1).
\]
If now we apply Theorem 1.3, then the validity of Theorem 1.5 becomes evident.

**REFERENCES**


Abstract. Systems modeled by Volterra difference equations differ from the conventional discrete-time systems in that their present state may explicitly depend on the whole previous state history. When unknown perturbations or disturbances affect the operation of the system, boundness in average of its solutions is a desirable, stability-like property of the system. In this paper new analysis tools are developed to determine if solutions of a given system are bounded in average. Specifically, with the help of appropriate Lyapunov functionals, we formulate several conditions for boundness in average directly in terms of the coefficients of the equations.

Key Words. Volterra nonlinear difference equations, boundedness in average, Lyapunov functionals.

AMS(MOS) subject classification. 39A11, 65Q05

1. Introduction. No nominal model should be considered complete without some assessment of the effect of models uncertainties. The range of permissible uncertainties is rarely spelled out explicitly along with the model. Moreover, to model various real world phenomena, systems with aftereffect are quite often used. One of their important classes are the difference Volterra equations in which the state at the present time instant may depend explicitly on the time history of the state on the whole time interval, from the initial up to the current time instances, and on the unknown perturbations or disturbances. Let us underline the difference between the discrete Volterra systems and the conventional discrete-time systems (finite order equations with finite fixed memory). Equations with finite fixed memory can be always reduced to a system representing a one step process. Thus
it is possible to use conventional tools to develop general stability conditions in terms of existence of a Lyapunov function, and to obtain concrete stability criteria formulated in terms of characteristics of the systems under consideration. Such a reduction is not possible, in general, for the difference Volterra equations. Thus new analysis tools need to be developed that rely on the existence of appropriate functionals, defined on the solutions of the equations, and that depend, generally speaking, on the whole previous history of the solutions. When unknown perturbations or disturbances affect the operation of the system, a desirable, stability-like property of the system is boundedness in average of its solutions. The main objective of this paper is to show that the boundedness in average can be influenced by perturbations which are not known exactly. The class of perturbations that we consider in this paper enter the system additively and may represent either the effect of disturbances or model uncertainty. Therefore the boundedness in average can be used as a measure for both acceptable types of parameter uncertainties and for the ability of the system to withstand various perturbations. One of the essential roles of boundedness under unknown perturbations is to produce satisfactory plants or closed loop systems that are either not known exactly due to modeling errors, or are varying in time during the operation. When broader classes of parameter perturbations have to be considered, the frequency conditions lose much of their appeal. Then other methods based on the modified Lyapunov theory are better suited for analysis, and ultimately, for control system design in such problems. Plant perturbations can be treated by these methods, which have no precise descriptions, and uncertainty can enter in an essential way via parameters and nonlinearities. In Sections 3 and 4 of this paper are obtained concrete results that are, respectively, for classes of nonlinear difference equations of convolution and nonconvolution type.

2. Problem formulation. Let us consider a discrete-time equation:

\[ x_{n+1} = F(n, x_0, \ldots, x_n) + \xi_n, \]

where: \( n \geq 0 \) is an integer, \( x_n \in \mathbb{R}^k \), \( \{ \xi_n : \xi_n \in \mathbb{R}^k \} \) is a sequence of perturbations, \( F \) is \( \ldots \), and \( x_0 \) is a prescribed initial condition. The aim of this paper is to obtain conditions which guarantee that the solutions of equation (2.1) are bounded in average. This is accomplished through the construction of appropriate Lyapunov functionals. Here the authors generalize the direct Lyapunov methods (see [1, 2, 3, 4]) for difference equations and construct a functional \( V(n, x_0, \ldots, x_n) \) that yields the boundedness in average (see [5, 6, 7]).
DEFINITION 1. The solutions of the system (2.1) are bounded in average if:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_1(x_j) < \infty, \]

under appropriate choice of the positive scalar function \( F_1(x) \), \( F_1(0) = 0 \). Here \( \lim \) is an upper limit as \( N \to \infty \). Let us formulate simple generalization of Lyapunov theory.

LEMMA 1. Assume that there exists a nonnegative scalar functional \( V(n, x_0, \ldots, x_n) \geq 0, n \geq 0 \), such that:

\[ V(n+1, x_0, \ldots, x_{n+1}) - V(n, x_0, \ldots, x_n) \leq -F_1(x_n) + F_2(|x_n|), \]

and also:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_2(|x_j|) < \infty, \quad F_2(0) = 0, \]

where \( F_2 \) is a nonnegative scalar function; then:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_1(x_j) \leq \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_2(|x_j|). \]

Proof. Summing up both parts of (2.3) from 0 to \( N \) and dividing by \( N \), we have:

\[ \frac{1}{N} \sum_{j=0}^{N} F_1(x_j) \leq \frac{1}{N} \sum_{j=0}^{N} F_2(|x_j|) + \frac{1}{N} V(0, x_0). \]

Taking the upper limit as \( N \to \infty \) in the inequality (2.6), we obtain the proof of lemma. Note that Lemma 1 provides an explicit upper bound for the solution that may be useful in concrete applications:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_1(x_j) \leq \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} F_2(|x_j|). \]

3. Scalar convolution equation. Consider a nonlinear equation:

\[ x_{n+1} = -\sum_{j=0}^{n} a_{n-j} g(x_j) + \xi_n, \quad n \geq 0, \]

where: \( a_n \) is a given sequence and the continuous function \( g(x) \) satisfies the conditions:

\[ g(0) = 0, \quad xg(x) > 0, \quad x \neq 0, \quad |g(x)| \leq |x|, \]

and \( \xi_n \) are unknown perturbations. For investigation of boundedness in average for nonlinear equation consider the functional \( V_1 = G_1 + G_2 + G_3 \), where:

\[ G_1 = (2 - a_{-1}) x_n g(x_n), \]
\[(3.4) \quad G_2 = a_{n+1} \left[ \sum_{k=0}^{n} g(x_k) \right]^2, \]

\[(3.5) \quad G_3 = \sum_{j=0}^{n} (a_{n-j} - a_{n+1-j}) \left[ \sum_{k=0}^{n} g(x_k) \right]^2. \]

Let us calculate first difference of \(G_1\), \(G_2\) and \(G_3\):

\[\Delta G_1 = - (2 - a_{-1}) x_n g(x_n) - 2g(x_{n+1}) \sum_{j=0}^{n} a_{n-j} g(x_j) - a_{-1} x_{n+1} g(x_{n+1}) + 2\xi_n g(x_{n+1}), \]

further:

\[\Delta G_2 = (a_{n+2} - a_{n+1}) \left( \sum_{j=0}^{n+1} g(x_j) \right)^2 + a_{n+1} g(x_{n+1}) \left( g(x_{n+1}) + 2 \sum_{j=0}^{n} g(x_j) \right). \]

At last:

\[\Delta G_3 = - \sum_{j=0}^{n+1} (a_{n+2-j} - 2a_{n+1-j} + a_{n-j}) \left[ \sum_{k=j}^{n+1} g(x_k) \right]^2 + (a_{-1} - a_{n+1}) g^2(x_{n+1}) - 2g(x_{n+1}) a_{n+1} \sum_{j=0}^{n} g(x_j) + 2g(x_{n+1}) \sum_{j=0}^{n} a_{n-j} g(x_j). \]

Consequently:

\[(3.6) \quad \Delta V_0 = - (2 - a_{-1}) x_n g(x_n) - 2\xi_n g(x_{n+1}) - a_{-1} g(x_{n+1}) [x_{n+1} - g(x_{n+1})] - (a_{n+1} - a_{n+2}) \left[ \sum_{j=0}^{n+1} g(x_j) \right]^2 - \left( a_{n+2-j} - 2a_{n+1-j} + a_{n-j} \right) \left[ \sum_{k=j}^{n+1} g(x_k) \right]^2. \]

Now, let us take \(\varepsilon > 0\) such that:

\[(3.9) \quad |2\xi_n g(x_{n+1})| \leq \frac{1}{\varepsilon} \xi_n^2 + \varepsilon g^2(x_{n+1}), \]

\[(3.10) \quad (2 - a_{-1} - \varepsilon) > 0, \]

then using (3.8) and inequalities of (3.10) we obtain the following expression:

\[(3.11) \quad (2 - a_{-1}) x_n g(x_n) \leq \frac{1}{\varepsilon} \xi_n^2 + \varepsilon g^2(x_{n+1}) + V(0, x_0). \]
Summing both parts of (3.11) from 0 to N and dividing by N we have:

\[(2 - a_{-1}) \frac{1}{N} \sum_{j=0}^{N} x_j g(x_j) \leq \frac{1}{\varepsilon N} \sum_{j=0}^{N} \xi_j^2 + \]

\[(3.13) \quad \varepsilon \frac{1}{N} \sum_{j=0}^{N} g^2(x_{j+1}) + \frac{1}{N} V(0, x_0). \]

thus:

\[(3.14) \quad (2 - a_{-1} - \varepsilon) \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} x_j g(x_j) \leq \frac{1}{\varepsilon} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \xi_j^2. \]

Finally, we have proven the following theorem.

**Theorem 1.** Assume that the conditions:

\[(3.15) \quad 2 - a_{-1} - \varepsilon > 0, \quad \varepsilon > 0, \quad a_j \geq 0, \]

\[(3.16) \quad a_{j+1} \leq a_j, \quad a_{j+2} - 2a_{j+1} + a_j \geq 0, \quad j \geq -1; \]

are fulfilled. Then the solutions of nonlinear equation (3.1) are bounded in average, if:

\[(3.17) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \xi_j^2 < \infty. \]

4. **Scalar nonconvolution equation.** By the same way we can obtain boundedness in average of the solutions of some nonlinear Volterra equations of the form:

\[(4.1) \quad x_{n+1} = -\sum_{j=0}^{n} a_{n,j} g(x_j) + \xi_n, \quad n \geq 0; \]

where: \(a_{n,j}\) is a given sequence of the coefficients, \(\xi_n\) are unknown perturbations and the continuous function \(g(x)\) satisfies the conditions prescribed above in (3.2). Let us define the coefficients \(a_{n,j}\) for \(j = n+1\) and \(j \leq -1\) with \(n \geq 0\). After that consider the functional \(V_0 = G_1 + G_2 + G_3\), where:

\[(4.2) \quad G_1 = a_{n,-1} \left[ \sum_{k=0}^{n} g(x_k) \right]^2, \]

\[(4.3) \quad G_2 = -\sum_{j=0}^{n} (a_{n,j-1} - a_{n,j}) \left[ \sum_{k=j}^{n} g(x_k) \right]^2, \]

\[(4.4) \quad G_3 = (2 - a_{n-1,n}) x_n g(x_n). \]
Calculating the difference for every $G$ we have:
\[
\Delta G_1 = a_{n+1,-1} \left( \sum_{j=0}^{n} g(x_j) \right)^2 - G_1 \pm a_{n,-1} \left( \sum_{j=0}^{n+1} g(x_j) \right)^2 = \\
= (a_{n+1,-1} - a_{n,-1}) \left( \sum_{j=0}^{n+1} g(x_j) \right)^2 + \\
+ a_{n,-1}g(x_{n+1}) \left( g(x_{n+1}) + 2 \sum_{j=0}^{n} g(x_j) \right),
\]
further:
\[\Delta G_2 = G_5 + G_4,\]
where $G_4$:
\[
G_4 = -g(x_{n+1})^2 [-a_{n,n+1} + a_{n,-1}] - \\
-2g(x_{n+1}) a_{n,-1} \sum_{j=0}^{n} g(x_j) + 2g(x_{n+1}) \sum_{j=0}^{n} a_{n,j} g(x_j),
\]
and $G_5$:
\[G_5 = - \sum_{j=0}^{n+1} (a_{n+1,j-1} - a_{n+1,j} - a_{n,j-1} + a_{n,j}) \left[ \sum_{k=j}^{n+1} g(x_k) \right]^2.
\]
At last:
\[
\Delta G_3 = (2 - a_{n,n+1}x_{n+1}g(x_{n+1}) - G_3 = \\
= -G_3 - a_{n,n+1}x_{n+1}g(x_{n+1}) - 2g(x_{n+1}) \sum_{j=0}^{n} a_{n,j} g(x_j) + \\
+ 2\xi_n g(x_{n+1}).
\]
Consequently:
\[
\Delta V_0 = G_5 - G_3 - a_{n,n+1}g(x_{n+1}) [x_{n+1} - g(x_{n+1})] + \\
+ (a_{n+1,-1} - a_{n,-1}) \left[ \sum_{j=0}^{n+1} g(x_j) \right]^2 + 2\xi_n g(x_{n+1}).
\]
Summing up from 0 to $N$ and dividing by $N$ we obtain that:
\[\Delta V_0 \leq \frac{1}{\varepsilon N} \sum_{j=0}^{N} \xi_j^2 + \frac{\varepsilon}{N} \sum_{j=0}^{N} g^2(x_{j+1}) + \frac{1}{N} V(0, x_0),\]
thus taking upper limit as $N \rightarrow \infty$:

$$
(4.8) \quad (2 - a_{n-1,n} - \varepsilon) \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N} x_{j} g(x_{j}) \leq \frac{1}{\varepsilon} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N} \xi_{j}^{2}.
$$

Then we have proven the following theorem.

**THEOREM 2.** Assume that the conditions:

$$
(4.9) \quad 2 - \varepsilon - \sup_{n \geq 0} \{a_{n-1,n}\} > 0, \quad \varepsilon > 0;
$$

are fulfilled. Then the solutions of nonlinear equation (4.1) are bounded in average, if:

$$
(4.10) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N} \xi_{j}^{2} < \infty.
$$

**5. Conclusion.** In the paper, first is proved the general stability result for systems described by Volterra difference equations using Lyapunov functionals. Subsequently, are constructed the concrete Lyapunov functionals for convolution and nonconvolution nonlinear equations. The paper demonstrated that boundedness in average can be verified directly in terms of equation coefficients and properties of perturbations acting on the system.

**REFERENCES**


ON SOME DEVELOPMENTS OF THE METHOD OF INTEGRAL GUIDING FUNCTIONS*

S. KORNEV† AND V. OBUKHOVSKII†

This paper is dedicated to the jubilee of Professor Anatolii Dmitrievich Myshkis

Abstract. We introduce the notion of a non-smooth integral guiding function and apply it to the existence of a periodic solution for a functional differential inclusion.

Key Words. Non-smooth integral guiding function, functional differential inclusion, periodic solution.

AMS(MOS) subject classification. 34K13; 34A60

1. Introduction. The study of functional differential inclusions was initiated in the pioneering work of A.D. Myshkis [14]. In the present paper we apply topological methods of the theory of multivalued maps (see, e.g., [1] - [3], [9], [10]) to the search of periodic solutions for some classes of functional differential inclusions. Extending the notion introduced by A. Fonda [8] we define a non-smooth integral guiding function for a functional differential inclusion and prove the existence of periodic solutions using the topological degree methods. It should be mentioned that the method of guiding functions whose base was laid by M.A. Krasnosel’skii and A.I. Perov (see, e.g., [13]) was expanded to differential inclusions and demonstrated its effectiveness to the study of periodic problems (see, e.g., [3], [9], [11]) but its direct application to functional differential inclusions is embarrassed.

* The work is supported by the RFBR grants 04-01-00081, 05-01-00100 and by U.S. CRDF - R.F. Ministry of Education Award VZ-010-0.
† Voronezh State Pedagogical University, 394043 Voronezh, Russia
e-mail: kornev.vrn@rambler.ru
† Voronezh State University, 394006 Voronezh, Russia
e-mail: valerio@org.vru.ru
2. Preliminaries. For $\tau > 0$, denote by the symbol $C$ the space of continuous functions $C([-\tau,0]; R^n)$. For a function $x(\cdot) \in C([-\tau,T]; R^n)$, $T > 0$ by the symbol $x_t \in C$ will be denoted the function given as $x_t(\theta) = x(t + \theta), \theta \in [-\tau,0]$. By the symbol $Kv(R^n)$ we will denote the collection of all nonempty compact convex subsets of $R^n$. At first we will consider the periodic problem for a functional differential inclusion of the following form:

$$\begin{cases}
  x'(t) \in F(t, x_t) \quad \text{a.e.} \quad t \in [0,T], \\
  x(0) = x(T),
\end{cases}$$

assuming that a multivalued map (multimap) $F : R \times C \rightarrow Kv(R^n)$ satisfies the next conditions (we refer to [1] - [3], [10] concerning the terminology):

- $F_T$) the multifunction $F$ is $T$-periodic in the first argument:

$$F(t, \varphi) = F(t + T, \varphi) \quad \text{for all } t \in R, \ varphi \in C;$$

(it is clear that this condition allows to consider the multimap $F$ only on $[0,T] \times C$);

- $F_1$) for each $\varphi \in C$ the multifunction $F(\cdot, \varphi) : [0,T] \rightarrow Kv(R^n)$ admits a measurable selection;

- $F_2$) for almost every $t \in [0,T]$ the multimap $F(t, \cdot) : C \rightarrow Kv(R^n)$ is upper semicontinuous;

- $F_3$) for each bounded subset $\Omega \subset C$ there exists a function $\alpha_\Omega(\cdot) \in L^1_+ [0,T]$ such that for each $\varphi \in \Omega$

$$\|F(t, \varphi)\| := \max_{y \in F(t, \varphi)} \|y\| \leq \alpha_\Omega(t)$$

for almost every $t \in [0,T]$.

**Remark 1.** To satisfy condition $(F_1)$ it is sufficient to assume that the multifunction $F(\cdot, \varphi)$ is measurable for each $\varphi \in C$ (see, e.g. [3], [10]).

**Remark 2.** Under conditions $(F_1) - (F_3)$ the superposition multiooperator $P_F : C([-\tau,T]; R^n) \rightarrow P(L^1([0,T]; R^n))$, assigning to each function $x(\cdot)$ the set of all summable selections of the multifunction $F(t, x_t)$ is well defined. It is known that this multioperator is closed (see, e.g. [3]). By a solution of problem (1), (2) we mean an absolutely continuous function $x(\cdot)$, satisfying the periodicity condition (2) and inclusion (1) a.e. on $[0,T]$. To study problem (1), (2) we will need a coincidence point theorem based on the evaluation of the topological degree of a multivalued vector field (see, e.g., [1] - [3], [9], [10]). Let $E_1, E_2$ be Banach spaces; $U \subset E_1$ a bounded open subset; $l : dom l \subseteq E_1 \rightarrow E_2$ a linear Fredholm operator of zero
index such that \( Iml \subset E_2 \) is a closed subspace. Consider continuous linear projection operators \( p : E_1 \to E_1 \) and \( q : E_2 \to E_2 \) such that \( Iml = \ker l, Iml = \ker q \). By the symbol \( l_p \) we denote the restriction of the operator \( l \) on \( \text{dom} l \cap \ker p \). Let the operator \( k_{p,q} : E_2 \to \text{dom} l \cap \ker p \) be given by the relation \( k_{p,q}(y) = l_p^{-1}(y - q(y)), y \in E_2 \). The canonical projection operator \( \pi : E_2 \to E_2/Iml \) is of the form \( \pi(y) = y + Iml \); and let \( \phi : \text{Coker} l \to \ker l \) be a continuous linear isomorphism.

**Theorem 3.** ([5], Lemma 13.1). Let \( G : \overline{U} \to \text{Ker}(E_2) \) be a multimap such that compositions \( \pi G \) and \( k_{p,q} G \) are compact and upper semicontinuous. Suppose that the following conditions hold:

1) \( l(x) \notin \lambda \Gamma(x) \) for all \( \lambda \in (0,1), x \in \text{dom} l \cap \partial U \);
2) \( 0 \notin \pi G(x) \) for all \( x \in \ker l \cap \partial U \);
3) \( \text{deg}_{\ker l}(\phi \pi G|_{\overline{U} \cap \ker l}) \neq 0 \), where the symbol \( \text{deg}_{\ker l} \) denotes the topological degree of a multivalued vector field, evaluating in the space \( \ker l, \overline{U} \cap \ker l \).

Then there exists a coincidence \( l(x) \in G(x) \) for some \( x \in \overline{U} \). Now let us recall some notions of non-smooth analysis (see, e.g. [4], [6], [7]). Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a locally Lipschitzian function. For \( x_0 \in \mathbb{R}^n \) and \( \nu \in \mathbb{R}^n \) the generalized derivative \( V^0(x_0; \nu) \) of \( V \) at a point \( x_0 \) in the direction \( \nu \) is defined by the formula

\[
V^0(x_0; \nu) = \lim_{t \to 0^+} \frac{V(x + tu) - V(x)}{t}.
\]

Then the generalized gradient \( \partial V(x_0) \) of a function \( V \) at a point \( x_0 \) is defined by:

\[
\partial V(x_0) = \left\{ x \in \mathbb{R}^n : (x, \nu) \leq V^0(x_0; \nu) \text{ for every } \nu \in \mathbb{R}^n \right\}.
\]

It is known (see, e.g., [4]) that the multimap \( \partial V \) is upper semicontinuous with nonempty compact convex values. In particular, it means that for each continuous function \( x : [0,T] \to \mathbb{R}^n \) the set of summable selections of the multifunction \( \partial V(x(t)) \) is nonempty. A locally Lipschitzian function \( V : \mathbb{R}^n \to \mathbb{R} \) is said to be regular if, for each \( x \in \mathbb{R}^n \) and \( \nu \in \mathbb{R}^n \), there exists the directional derivative \( V'(x, \nu) \) and \( V'(x, \nu) = V^0(x, \nu) \). It is known that convex functions are regular. Denote by \( C_T \) the space of continuous \( T \)-periodic functions \( x : \mathbb{R} \to \mathbb{R}^n \) with the norm \( \|x\|_C = \sup_{t \in [0,T]} \|x(t)\| \). By \( \|x\|_2 \) we will denote the norm of the function \( x \) given by \( \|x\|_2 = \left( \int_0^T \|x(s)\|^2 ds \right)^{\frac{1}{2}} \). Now we may introduce the following notion.
DEFINITION 4. A regular function \( V : \mathbb{R}^n \to \mathbb{R} \) is said to be a non-smooth integral guiding function for problem (1), (2) if there exists \( N > 0 \) such that for each absolutely continuous function \( x \in C_T \) with \( \| x \|_2 \geq N \) and \( \| x'(t) \| \leq \| F(t, x_t) \| \) for a.e. \( t \in [0, T] \) we have that

\[
\int_0^T (v(s), f(s)) \, ds > 0
\]

for every summable selections \( v(s) \in \partial V(x(s)) \) and \( f(s) \in F(s, x_s) \).

3. Existence of Periodic Solutions. The following statement holds true.

**THEOREM 5.** Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a non-smooth integral guiding function of problem (1), (2) such that there exists \( K > 0 \) with \( 0 \not\in \partial V(u) \) for all \( \| u \| \geq K \) and

\[
\text{deg}(\partial V; B_K) \neq 0,
\]

where \( B_K \subset \mathbb{R}^n \) is the ball of the radius \( K \) centered at the origin.

Then problem (1), (2) has a solution.

**REMARK 6.** Let us mention that condition of the theorem is fulfilled if, for example, a guiding function \( V \) is even or \( \lim_{\| x \| \to +\infty} V(x) = \pm \infty \) (see [7]).

For the proof of the theorem we will need the next statement which may be proved following the same reasonings as in [6], Section 1.5.

**LEMMA 7.** Let a function \( V : \mathbb{R}^n \to \mathbb{R} \) be regular, \( x : [0, T] \to \mathbb{R}^n \) an absolutely continuous function. Then the function \( V(x(t)) \) is absolutely continuous and

\[
V(x(t)) - V(x(0)) = \int_0^t V^0(x(s), x'(s)) \, ds, \quad t \in [0, T].
\]

**Proof.** [Proof of Theorem 5] For the proof we will use Theorem 3. Consider the following operators:

\[
l : \{ \text{dom}, l := \{ x \in C_T : x \text{ is absolutely continuous} \} \subset C_T \to L^1_T,
\]

\[
l(x) = x',
\]

a superposition multioperator \( G = P_F : C_T \to P(L^1_T) \) and the projection \( \pi : L^1_T \to \mathbb{R}^n \), given as

\[
\pi f = \frac{1}{T} \int_0^T f(s) \, ds.
\]
Using Theorem 1.5.19 of [3] it is easy to verify that multioperators $\pi G$ and $k_{p,q} G$ are compact and upper semicontinuous.

Let us mention that for $\lambda \in (0,1)$ an arbitrary solution $x \in \text{dom} l$ of the inclusion $l(x) \in \lambda G(x)$ satisfies the relations

$$
\begin{align*}
    x'(t) &\in \lambda F(t, x_t), \\
    x(0) &= x(T).
\end{align*}
$$

It means that $x(\cdot)$ is an absolutely continuous function such that $x'(t) = \lambda f(t)$ for a.e. $t \in [0, T]$, where $f \in P_P(x)$.

Then, applying Lemma 7, we obtain

$$
\int_0^T (v(s), f(s)) \, ds = \frac{1}{\lambda} \int_0^T (v(s), x'(s)) \, ds \leq 
$$

$$
\leq \frac{1}{\lambda} \int_0^T V^0(x(s), x'(s)) \, ds = \frac{1}{\lambda}(V(x(T)) - V(x(0))) = 0,
$$

for each summable selection $v(s) \in \partial V(x(s))$, and hence

$$
\|x\|_2 < N.
$$

Then from condition $(F_3)$ it follows that there exists a constant $M' > 0$ such that $\|x'\|_2 < M'$. But then in turn there exists also $M > 0$ such that

$$
\|x\|_C < M
$$

for all such functions.

Now let us take as $U$ the ball $B_r \subset C_T$ centered at the origin with the radius $r = \max\{K, M, NT^{-1/2}\}$. Then we have

$$
l(x) \notin \lambda G(x)
$$

for all $x \in \partial U, \lambda \in (0,1)$.

Further, take an arbitrary point $u \in \partial U \cap \ker l$. Since $\|u\| \geq NT^{-1/2}$, from the definition of the integral guiding function we obtain that

$$
\int_0^T (v(s), f(s)) \, ds > 0
$$

for all measurable selections $v(s) \in \partial V(u), f(s) \in F(s,u)$. But taking a constant selection $v(s) \equiv v$ we obtain

$$
\int_0^T (v, f(s)) \, ds = \left(v, \int_0^T f(s) \, ds \right) = T(v, \pi f) > 0
$$
for all \( v \in \partial V(u) \), and hence
\[
(v, y) > 0
\]
for all \( v \in \partial V(u), y \in \pi G(u) \).
But it means that multifields \( \partial V(u) \) and \( \pi G(u) \) are homotopic on \( \partial U \cap \ker l \) (see, e.g., [1] - [3], [10]) and hence
\[
\deg_{\ker l}(\pi G|_{U_{\ker l}}, \overline{U}_{\ker l}) = \deg(\partial V, \overline{U}_{\ker l}) \neq 0.
\]
So, all conditions of Theorem 3 are fulfilled and we come to the conclusion that problem (1), (2) has a solution in the ball \( \overline{U} \). □

As an example, we consider the periodic problem for a gradient functional differential inclusion of the following form:
\[
\begin{align*}
&x'(t) \in \partial G(x(t)) + F(t, x_t) \\
&x(0) = x(T),
\end{align*}
\]
where the multimap \( F \) satisfies conditions \( (F_T), (F_1) - (F_3) \), and \( \partial G \) is the generalized gradient of a locally Lipschitz and differentiable in all directions function \( G : \mathbb{R}^n \rightarrow \mathbb{R} \).

**Theorem 8.** Suppose that the following conditions are satisfied:
1) there exist constants \( \varepsilon > 0, K > 0 \) and \( \beta \geq 1 \) such that
\[
\|g\| \geq \varepsilon \|u\|^\beta - K
\]
for all \( g \in \partial G(u), u \in \mathbb{R}^n \);
2) \[
\lim_{\|x\|_2 \rightarrow \infty} \frac{\|P_F(x)\|_2}{\|x\|_2^\beta} < \varepsilon T^{(1-\beta)/2}
\]
for all absolutely continuous \( x \in C_T \);
3) the generalized gradient \( \partial G \) has a non-zero topological degree:
\[
\deg(\partial G, \overline{B}_N) \neq 0
\]
for sufficiently large \( N > 0 \).

Then problem (6), (7) has a solution.

**Proof.** Let us demonstrate that \( G \) is a non-smooth integral guiding function for problem (6), (7). Notice that the embedding \( L^{2\beta} \subset L^2 \) gives the following estimation for each absolutely continuous function \( x(\cdot) \in C_T \) and every summable selection \( g(t) \in \partial G(x(t)) \):
\[
\|g\|_2 \geq \varepsilon \|x\|_{2\beta}^\beta - K \sqrt[2\beta]{T} \geq \varepsilon T^{(1-\beta)/2} \|x\|_2^\beta - K \sqrt{T}.
\]
But then for each summable selections \( f \in P_F(x) \) and \( g(t) \in \partial G(x(t)) \) we have

\[
\int_0^T (g(s), g(s) + f(s)) \, ds \geq
\]

\[
\geq \|g\|_2 (\|g\|_2 - \|f\|_2) \geq
\]

\[
\geq \|g\|_2 (\|g\|_2 - \|P_F(x)\|_2) \geq
\]

\[
\geq \|g\|_2 \left( \varepsilon T^{(1-\beta)/2} - \frac{K \sqrt{T}}{\|x\|_2^\beta} - \frac{\|P_F(x)\|_2}{\|x\|_2^\beta} \right) \|x\|_2^\beta > 0
\]

for \( \|x\|_2 \) sufficiently large. \( \Box \)

The method of non-smooth integral guiding functions can be extended to the case when the right-hand side of functional differential inclusion has non-convex values. In this situation we must substitute the Caratheodory type conditions have been imposed on \( F \) by some lower semicontinuity assumptions. Denote by \( K(R^n) \) the collection of all nonempty compact subsets of \( R^n \). Let a multimap \( F : R \times C \to K(R^n) \) satisfies conditions \( (F_7) \), \( (F_3) \), and the following assumption:

\( F_L \) there exists a sequence of disjoint compact subsets \( \{J_n\}, J_n \subseteq [0,T], \) such that

(i) \( \mu([0,T] \setminus I) = 0, \) where \( \mu \) is Lebesgue measure, \( I = \bigcup_n J_n; \) (ii) the restriction of \( F \) on each set \( J_n = I_n \times C \) is lower semicontinuous. Under above conditions the superposition multioperator

\[
P_F : C([-\tau,T]; R^n) \to P(L^1([0,T]; R^n)),
\]

is well defined. It is lower semicontinuous with closed decomposable values and hence admits a continuous selection (see, e.g. [5], [10]). For this multioperator the topological coincidence degree with respect to a linear Fredholm map can be defined (see [12]). Using this fact we may obtain the analogue of Theorem 5 for this class of functional differential inclusions.

**REFERENCES**


RYABOV'S THEOREM ON SMALL LAG:
THE DICHOTOMY OF SOLUTIONS POINT OF VIEW

V.G. KURBATOV* AND V.I. KUZNETSOVA†

Abstract. The article discusses the Ryabov phenomenon: any solution of the equation

$$\dot{x}(t) + \sum_{m=1}^{\infty} b_m(t)x(t - \varepsilon h_m) = 0$$

with a small $\varepsilon > 0$ can be represented as the sum of an exponentially decreasing solution of the same equation and a solution of a special ordinary differential equation. A new proof based on the theorem on exponential dichotomy of solutions is given.

1991 Mathematics Subject Classification 34K15

Key words: functional differential equation, small lag, exponential dichotomy

Dedicated to A.D. Myshkis on the occasion of his 85th birthday

1. Introduction. The phenomenon discussed in this article was first noticed by Yu. A. Ryabov [37]. He observed that a functional differential equation with a small lag has a unique special solution which can be interpreted as a solution of an ordinary differential equation; thus an equation with a small lag can be approximated by an ordinary differential equation. Further this fact was discussed by many mathematicians from different points of view; see, e.g., [1-3], [5, 6], [8-19], [22-39], [41-44].

The goal of this article is to discuss the connection of Ryabov's phenomenon with exponential dichotomy of solutions. Namely, we give a proof of a result of Ryabov's type (theorem 23) based on the theorem on exponential dichotomy of solutions [4, 20, 21]. The idea of the proof is as follows.

---

* All-Russian Distance Institute of Finance and Economics, Lipetsk Branch, Internatsionalnaya St., 12b, Lipetsk 398050 Russia; e-mail: kv51@inbox.ru
† Lipetsk State Technical University, Dept. of Math., Moskovskaya St., 30, Lipetsk 398055 Russia; e-mail: kvi@stu.lipetsk.su
One can discuss the dependence of the equation

$$\dot{x}(t) + \sum_{m=1}^{\infty} b_m(t)x(t - \varepsilon h_m) = 0$$

from the parameter $\varepsilon \geq 0$. The smallness of the lag allows one to include into the consideration the case $\varepsilon = 0$ when the equation becomes ordinary. The operator corresponding to the equation depends on $\varepsilon$ in a continuous way (proposition 21). For $\varepsilon = 0$ it is invertible in spaces with exponential weight (proposition 7). Therefore it is invertible for $\varepsilon > 0$ small enough as well, which implies (theorem 24) the exponential dichotomy of solutions. It turns out that the solutions from the subspace of increasing functions are determined by their values at the initial point and thus can be interpreted as solutions of an ordinary differential equation.

In Section 6 we formulate the main result (theorem 23) and in Section 8 give its proof. Other Sections contain auxiliary definitions, constructions, and assertions.

The authors are grateful to A.D. Myshkis for helpful discussion.

2. Functional spaces. Let $X$ and $Y$ be Banach spaces. We denote by $B = B(X,Y)$ the Banach space of all bounded linear operators $B$ acting from $X$ to $Y$. If $X = Y$ we employ the brief notation $B = B(X)$. The symbols $1$ and $1_X$ denote the identity operator acting in $X$.

**Proposition 1.** Let $X$ and $Y$ be Banach spaces, and $A, B \in B(X,Y)$. If the operator $A$ is invertible and

$$\|B\| \cdot \|A^{-1}\| < 1,$$

then the operator $A - B$ is also invertible. Moreover,

$$\|(A - B)^{-1}\| \leq \frac{\|A\|^{-1}}{1 - \|B\| \cdot \|A^{-1}\|}, \quad (1)$$

$$\|(A - B)^{-1} - A^{-1}\| \leq \frac{\|B\| \cdot \|A^{-1}\|^2}{1 - \|B\| \cdot \|A^{-1}\|}. \quad (2)$$

**Proof.** We consider the series

$$A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \ldots$$

By assumption this series converges absolutely. A straightforward calculation shows that the result of the multiplication of the sum of this series by $A - B$ is the identity operator.
Estimates (1) and (2) are evident estimates of the sums of the series

\[(A - B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \ldots\]

\[(A - B)^{-1} - A^{-1} = A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \ldots\]

\[\Box\]

**Corollary 2.** Let the operator \(T_x\) depend continuously on the parameter \(x\) (\(x\) runs over some topological space). Let \(T_x\) be invertible for all \(x\). Then the operator \((T_x)^{-1}\) depends continuously on the parameter \(x\) too.

**Proof.** Proof follows from estimate (2). \(\Box\)

**Proposition 3.** Let the operator \(T_x\) depend continuously on the parameter \(x \in K\), where \(K\) is a compact topological space. Then the family \(T_x\), \(x \in K\), is uniformly bounded, i.e., there exists a constant \(L\) such that \(|T_x| \leq L\) for all \(x \in K\). If additionally \(T_x\) is invertible for all \(x \in K\) then the family of the inverses \((T_x)^{-1}\), \(x \in K\), is uniformly bounded too.

**Proof.** The first assertion is a corollary of the compactness principle. The second one follows from the first and corollary 2. \(\Box\)

Throughout this article we denote by \(\mathbb{E}\) a fixed Banach space with the norm \(|\cdot|\).

We denote by \(C = C(\mathbb{R}) = C(\mathbb{R}, \mathbb{E})\) the Banach space of all bounded continuous functions \(x: \mathbb{R} \to \mathbb{E}\) with the supremum norm

\[\|x\| = \|x\|_C = \sup_{t \in \mathbb{R}} |x(t)|.\]

Similarly we define the spaces \(C(-\infty, b] = C((-\infty, b], \mathbb{E}), C[a, b] = C([a, b], \mathbb{E})\), and so on.

We denote by \(C^{1}_{loc} = C^{1}_{loc}(\mathbb{R}, \mathbb{E})\) the linear space of all continuous functions \(x: \mathbb{R} \to \mathbb{E}\). The (classical) derivative of a function \(x \in C^{1}_{loc}\) is the function

\[\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.\]

In a similar way we define derivatives if the domain of \(x\) is \((-\infty, b], [a, b]\) etc. We denote by \(C^{1}_{loc} = C^{1}_{loc}(\mathbb{R}) = C^{1}_{loc}(\mathbb{R}, \mathbb{E})\) the linear space of all functions \(x \in C^{1}_{loc}\) such that the derivative \(\dot{x}\) of \(x\) exists and is continuous, i.e., belongs to \(C^{1}_{loc}\).

We denote by \(C^1 = C^1(\mathbb{R}) = C^1(\mathbb{R}, \mathbb{E})\) the linear space of all functions \(x \in C\) such that the derivative \(\dot{x}\) of \(x\) exists and belongs to \(C\). We endow \(C^1\) with the norm

\[\|x\| = \|x\|_{C^1} = \|x\|_C + \|\dot{x}\|_C.\]
It is easy to show (see, e.g., [21, 2.3.2]) that the space $C^1$ is complete, i.e., Banach. The spaces of the kind $C^1[a,b]$ are defined similarly.

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a function. The closure of the set $\{ t: \psi(t) \neq 0 \}$ is called the support of $\psi$ and is denoted by the symbol $\text{supp} \psi$. We denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}, \mathbb{R})$ the linear space of all infinitely many times differentiable functions $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with a compact support.\footnote{If $E$ is complex, one should consider functions $\psi: \mathbb{R} \rightarrow \mathbb{C}$; see the next paragraph.} We say that a sequence $\psi_k \in \mathcal{D}$ converges (in the sense of $\mathcal{D}$) to a function $\psi \in \mathcal{D}$ if:

(a) the supports of $\psi_k$ are uniformly bounded, i.e., there exists a segment $[a, b]$ such that $\text{supp} \psi_k \subseteq [a, b]$ for all $k$;

(b) the sequence $\psi_k^{(n)}$ converges uniformly to $\psi^{(n)}$ for all $n = 0, 1, 2, \ldots$; here $\psi^{(n)}$ is the $n$-th derivative of $\psi$, in particular, $\psi^{(0)}$ is $\psi$ itself.

We note that the operator of differentiation $\psi \mapsto \dot{\psi}$ acts continuously in $\mathcal{D}$, i.e., if $\psi_k \in \mathcal{D}$ converges to $\psi \in \mathcal{D}$ then $\dot{\psi}_k$ converges to $\dot{\psi}$.

Let $f: \mathcal{D} \rightarrow \mathbb{E}$ be a linear (vector-valued) functional. We denote the value of the functional $f$ at $\psi$ by $\langle \psi, f \rangle$. We say that the functional $f$ is continuous if $\langle \psi_k, f \rangle$ converges to $\langle \psi, f \rangle$ whenever $\psi_k$ converges to $\psi$. Clearly, if $f$ is continuous at the point $0 \in \mathcal{D}$, it is continuous at all points $\psi \in \mathcal{D}$.

Any continuous linear functional $f: \mathcal{D} \rightarrow \mathbb{E}$ is called [40] a distribution (with values in $\mathbb{E}$). We denote by $\mathcal{D}' = \mathcal{D}'(\mathbb{R}, \mathbb{E})$ the space of all distributions $f$. By misuse of language, elements of $\mathcal{D}'$ are usually called functions. We say that a sequence $f_k \in \mathcal{D}'$ converges to $f \in \mathcal{D}'$ if $\langle \psi, f_k \rangle \rightarrow \langle \psi, f \rangle$ for all $\psi \in \mathcal{D}$.

Let $f \in \mathcal{D}'$. The functional $\dot{f}$ defined by the rule

$$\langle \psi, \dot{f} \rangle = -\langle \psi, f \rangle$$

is called the derivative of the functional $f$. It is straightforward to verify that $\dot{f}$ is actually linear and continuous; thus it belongs to $\mathcal{D}'$.

Let $x \in C_\text{loc}(\mathbb{R}, \mathbb{E})$. It is easy to see that the functional $f_x$ defined by the rule

$$\langle \psi, f_x \rangle = \int_{-\infty}^{+\infty} \psi(t) x(t) \, dt$$

belongs to $\mathcal{D}'$. We call the distribution $f_x$ induced by a function $x \in C_\text{loc}$ regular. It is easy to see that $f_x = f_y$ implies $x = y$. Thus the mapping $x \mapsto f_x$ is an injective embedding of $C_\text{loc}$ into $\mathcal{D}'$. We call the derivative $\dot{f}_x \in \mathcal{D}'$ of the functional $f_x \in \mathcal{D}'$ the distribution derivative of $x \in C_\text{loc}$.\footnote{If $E$ is complex, one should consider functions $\psi: \mathbb{R} \rightarrow \mathbb{C}$; see the next paragraph.}
We say that $F \in \mathcal{D}'$ is a primitive of $f \in \mathcal{D}'$ if

$$\hat{F} = f.$$ 

Let $f, g \in \mathcal{D}'$. We say that $f$ is equal to $g$ on an open subset $U \subseteq \mathbb{R}$ or, briefly, $f = g$ on $U$ if $\langle \psi, f \rangle = \langle \psi, g \rangle$ for all $\psi \in \mathcal{D}$ such that $\text{supp} \, \psi \subseteq U$. (We stress that the equality of distributions makes sense only on open sets $U$.) Clearly, if $f$ and $g$ are regular, the equality of $f$ and $g$ on $U$ is equivalent to the equality $f(t) = g(t)$ for all $t \in U$.

Let $\eta: \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function and $f \in \mathcal{D}'$. We define the product $\eta f$ by the rule

$$\langle \psi, \eta f \rangle = \langle \eta \psi, f \rangle.$$  

It is straightforward to verify that $\eta f$ is really a linear continuous functional, i.e., an element of $\mathcal{D}'$. Clearly, if $f$ is regular, this definition agrees with the pointwise one.

**Proposition 4** ([21, proposition 2.3.5]). Let $-\infty \leq a < b \leq +\infty$.

(a) Any distribution $f \in \mathcal{D}'$ has a primitive.

(b) Let $F \in \mathcal{D}'$, and let $\hat{F}$ be equal to zero on $(a, b)$. Then $F$ coincides with a constant function on $(a, b)$, i.e., there exists $c \in \mathbb{E}$ such that $F = f_x$ on $(a, b)$, where $x(t) = c$ for all $t \in \mathbb{R}$.

Consequently, two primitives $F_1$ and $F_2$ of the same $f \in \mathcal{D}'$ differ by a constant, i.e., $F_1 - F_2$ is a regular distribution induced by a constant function.

(c) Assume that $F \in \mathcal{D}'$ has a regular derivative on $(a, b)$, i.e., there exists $y \in C^1_{\text{loc}}$ such that $F = f_y$ on $(a, b)$. Then $F$ is regular on $(a, b)$ itself, namely, $F = f_x$ on $(a, b)$ for a continuous $x$. Moreover, the classical derivative of $x$ exists and coincides with $y$ on $(a, b)$.

In particular, if $x \in C^1_{\text{loc}}$ then the distribution derivative of $x$ is regular and coincides with $f_x$, where $\dot{x}$ is the classical derivative of $x$.

(d) Let $\eta$ be an infinitely many times differentiable function and $f \in \mathcal{D}'$. Then

$$(\eta f)' = \eta f + \eta \dot{f}.$$  

We denote by $\mathcal{C}^{-1} = \mathcal{C}^{-1}(\mathbb{R}, \mathbb{E})$ the space of all $f \in \mathcal{D}'$ representable in the form $f = \dot{u} + v$, where $u, v \in C$ and $\dot{u}$ is the distribution derivative of $u$. We stress that the representation $f = \dot{u} + v$ is not unique. We define the norm on $\mathcal{C}^{-1}$ by the formula

$$\|f\|_{\mathcal{C}^{-1}} = \inf \{ \|u\|_{C} + \|v\|_{C} : f = \dot{u} + v, u, v \in C \}.$$
It can be shown (see, e.g., [21, 2.3.6]) that the space $C^{-1}$ is complete, i.e., Banach. The space $C^{-1}(-\infty, a]$ and so on are defined in a similar way.

Clearly, the operator $\dot{x} \mapsto x$ of differentiation acts continuously from $C$ to $C^{-1}$.

3. Operators with exponential memory. For any $x: \mathbb{R} \to \mathbb{E}$ and $\nu \in \mathbb{R}$ we define the function $\Psi_\nu x$ by the rule

$$(\Psi_\nu x)(t) = e^{\nu t} x(t).$$

Let $X$ be one of the spaces

$$C = C(\mathbb{R}, \mathbb{E}), \quad C^1 = C^1(\mathbb{R}, \mathbb{E}), \quad C^{-1} = C^{-1}(\mathbb{R}, \mathbb{E}).$$

We denote by $X_\nu$ the space of all distributions of the form $\Psi_\nu x \in \mathcal{D}'$, where $x$ runs over $X$. We endow $X_\nu$ with the norm

$$\|y\| = \|y\|_{X_\nu} = \|\Psi_{-\nu} y\|_{X}.$$  

Thus by definition, $\Psi_\nu: X \to X_\nu$ is an isometric isomorphism. The detailed notation for $X_\nu$ will appear as $C_\nu, C^{-1}_\nu(\mathbb{R}, \mathbb{E})$ and so on. The spaces $C_\nu(-\infty, a] = C_\nu((-\infty, a], \mathbb{E})$ and $C^1_\nu(-\infty, a] = C^1_\nu((-\infty, a], \mathbb{E})$ are defined in a similar way. We mention that the space $C^{-1}_\nu(-\infty, a] = C^{-1}_\nu((-\infty, a], \mathbb{E})$ can be identified with the space $(C^{-1}_\nu)^{\alpha}$, see proposition 10 below, but we shall not use it.

**Proposition 5 ([21, proposition 3.5.1]).**

(a) For all $\nu \in \mathbb{R}$ the space $C_\nu$ consists of all $y \in C_{\text{loc}}$ bounded in the norm $\|\Psi_{-\nu} y\|_C$.

(b) For all $\nu \in \mathbb{R}$ the space $C^1_\nu$ consists of all $y \in C_\nu$ such that its derivative $\dot{y}$ also belongs to $C_\nu$. The norm

$$\|y\| = \|\dot{y}\|_{C_\nu} + \|y\|_{C_\nu}$$

on $C^1_\nu$ is equivalent to the initial norm (4).

(c) For all $\nu \in \mathbb{R}$ the space $C^{-1}_\nu$ consists of all distributions $g$ of the form $g = \dot{u} + v$, where $u, v \in C_\nu$. The norm

$$\|g\| = \inf \{ \|u\|_{C_\nu} + \|v\|_{C_\nu} : g = \dot{u} + v; u, v \in C_\nu \}$$

on $C^{-1}_\nu$ is equivalent to the initial norm (4).

Let $\nu \in \mathbb{R}$ and $(X, Y)$ be one of the pairs

$$(C_\nu, C_\nu), \quad (C^1_\nu, C_\nu), \quad (C_\nu, C^1_\nu), \quad (C_\nu, C^{-1}_\nu), \quad (C^{-1}_\nu, C_\nu).$$  

(5)
A bounded linear operator $T: X \to Y$ is called [45, 21] **causal** if for all $x \in X$ and $t \in \mathbb{R}$

$$x(s) = 0, \quad s < t, \quad \Rightarrow \quad (Tx)(s) = 0, \quad s < t. \quad (6)$$

We denote by $B^+ = B^+(X, Y)$ the set of all causal operators $T \in B(X, Y)$. If $X = Y$ we employ the brief notation $B^+ = B^+(X)$. **Clearly, the sum and the composition of operators of the class $B^+$ is also an operator of the class $B^+$.**

A bounded linear operator $T: X \to Y$ is called **anticausal** if for all $x \in X$ and $t \in \mathbb{R}$

$$x(s) = 0, \quad s > t, \quad \Rightarrow \quad (Tx)(s) = 0, \quad s > t.$$

We say that an operator $T$ is **memoryless** if it is both causal and anticausal.

**Proposition 6.** Let $\nu > \lambda$.

(a) **The memoryless operator**

$$U_\lambda x = \dot{x} - \lambda x \quad (7)$$

considered as acting from $C^1_\nu$ to $C_\nu$ is invertible. The inverse operator is defined by the formula

$$(U_\lambda^{-1} f)(t) = \int_0^{+\infty} e^{\lambda s} f(t-s) \, ds = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s) \, ds.$$  

**Clearly, $U_\lambda^{-1}$ is causal.**

(b) **The memoryless operator** (7) **considered as acting from** $C_\nu$ **to** $C^{-1}_\nu$ **is invertible. The inverse operator is defined by the formula**

$$(U_\lambda^{-1} f)(t) = u(t) + \int_0^{+\infty} e^{\lambda s} (\nu(t-s) + \lambda u(t-s)) \, ds = u(t) + \int_{-\infty}^{t} e^{\lambda(t-s)} (\nu(s) + \lambda u(s)) \, ds,$$

where $f = u + \nu$. **Clearly, $U_\lambda^{-1}$ is causal.**

Let $\nu < \lambda$.

(c) **The memoryless operator** (7) **considered as acting from** $C^1_\nu$ **to** $C_\nu$ **is invertible. The inverse operator is defined by the formula**

$$(U_\lambda^{-1} f)(t) = -\int_{-\infty}^{0} e^{\lambda s} f(t-s) \, ds = -\int_{t}^{+\infty} e^{\lambda(t-s)} f(s) \, ds.$$  

**Clearly, $U_\lambda^{-1}$ is anticausal.**
(d) The memoryless operator (7) considered as acting from $C_{\nu}$ to $C_{\nu}^{-1}$ is invertible. The inverse operator is defined by the formula

$$
(U_{\lambda}^{-1} f)(t) = u(t) - \int_{-\infty}^{0} e^{\lambda s} (v(t-s) + \lambda u(t-s)) \, ds
$$

$$
= u(t) - \int_{t}^{+\infty} e^{\lambda(t-s)} (v(s) + \lambda u(s)) \, ds,
$$

where $f = \dot{u} + v$. Clearly, $U_{\lambda}^{-1}$ is anticausal.

Proof. The case of $\lambda = -1$ and $\nu = 0$ is considered, e.g., in [21, 2.3.4, 2.3.9, 3.5.1]. The general case can be handled in a similar way. □

**Proposition 7.** The ordinary differential operator

$$(Lx)(t) = \dot{x}(t) + a(t)x(t)$$

with a bounded continuous coefficient $a: \mathbb{R} \rightarrow \mathcal{B}(E)$ is causally invertible in all pairs $(C_{\nu}^{1}, C_{\nu})$ and $(C_{\nu}, C_{\nu}^{-1})$ with $\nu$ large enough, and anticausally invertible in all pairs $(C_{\nu}^{1}, C_{\nu})$ and $(C_{\nu}, C_{\nu}^{-1})$ with $\nu$ small enough.

Proof. We consider $L: C_{\nu}^{1} \rightarrow C_{\nu}$ or $L: C_{\nu} \rightarrow C_{\nu}^{-1}$. Clearly $\Psi_{-\nu}L\Psi_{\nu}$ acts from $C^{1}$ to $C$ or from $C$ to $C^{-1}$ respectively. Easy calculations show that $(\Psi_{-\nu}L\Psi_{\nu}x)(t) = \dot{x}(t) + \nu x(t) + a(t)x(t)$.

We distinguish the two cases. In the case of the pair $(C_{\nu}^{1}, C_{\nu})$ we consider the operator $(\Psi_{-\nu}L\Psi_{\nu})(U_{-\nu})^{-1} = 1 + A(U_{-\nu})^{-1} : C \rightarrow C$. In the case of the pair $(C_{\nu}, C_{\nu}^{-1})$ we consider $(U_{-\nu})^{-1}(\Psi_{-\nu}L\Psi_{\nu}) = 1 + (U_{-\nu})^{-1}A : C \rightarrow C$. In the both cases $U_{-\nu}$ is defined as in proposition 6 and $A$ means the memoryless operator $(Ax)(t) = a(t)x(t)$. Clearly $\|A: C \rightarrow C\| \leq \sup_{t \in \mathbb{R}} \|a(t)\|$ and (see, e.g., [21, 4.4.1])

$$
\|(U_{-\nu})^{-1} : C \rightarrow C\| \leq \int_{0}^{+\infty} e^{-\nu s} \, ds = 1/\nu \quad \text{if } \nu > 0,
$$

$$
\|(U_{-\nu})^{-1} : C \rightarrow C\| \leq \int_{-\infty}^{0} e^{-\nu s} \, ds = 1/|\nu| \quad \text{if } \nu < 0.
$$

Therefore if $|\nu|$ is large enough, then $\|A(U_{-\nu})^{-1}\| < 1$ and $\|(U_{-\nu})^{-1}A\| < 1$. Hence (see, e.g., [21, 1.4.2, 2.2.6]) the operators $1 + A(U_{-\nu})^{-1}$ and $1 + (U_{-\nu})^{-1}A$ are causally invertible if $\nu > 0$ (in this case $(U_{-\nu})^{-1}$ is causal, see proposition 6) or anticausally invertible if $\nu < 0$ (in this case $(U_{-\nu})^{-1}$ is anticausal). Consequently $L$ is, respectively, causally or anticausally invertible. □

Let $(X, Y)$ be one of the pairs

$$(C, C), \ (C^{1}, C), \ (C, C^{1}), \ (C, C^{-1}), \ (C^{-1}, C).$$

(8)
Let \( \mu, \nu \in \mathbb{R} \). We say that an operator \( T_\mu \in \mathcal{B}(X_\mu, Y_\mu) \) is a continuation of an operator \( T_\nu \in \mathcal{B}(X_\nu, Y_\nu) \) if \( T_\nu x = T_\mu x \) for all \( x \in X_\nu \cap X_\mu \). (Here, by the equality of \( x \in X_\nu \) and \( z \in X_\mu \) we mean the equality of \( x \) and \( z \) as elements of \( \mathcal{D}' \). By proposition 5(a), if \( X = C \) or \( X = C^1 \) this is equivalent to the pointwise equality.) We say that \( T_\nu \in \mathcal{B}(X_\nu, Y_\nu) \) admits a continuation to an operator \( T_\mu \in \mathcal{B}(X_\mu, Y_\mu) \) if a continuation \( T_\mu \in \mathcal{B}(X_\mu, Y_\mu) \) exists.

We say that an operator \( T_\nu \in \mathcal{B}(X_\nu, Y_\nu) \) has exponential memory if it admits a continuation to an operator \( T_\mu \in \mathcal{B}(X_\mu, Y_\mu) \) for all \( \mu \in \mathbb{R} \) close enough to \( \nu \). We denote by \( e = e(X_\nu, Y_\nu) \) the class of all operators \( T_\nu \in \mathcal{B}(X_\nu, Y_\nu) \) with exponential memory. If \( X = Y \) we employ the brief notation \( e(X_\nu) \). Clearly, the sum and the composition of operators of the class \( e \) is also an operator of the class \( e \). We use the notation \( e^+ \) for the intersection \( e \cap B^+ \).

**Proposition 8.** Let \((X, Y)\) be one of the pairs (8). Then any operator \( T_\nu \in \mathcal{B}^+(X_\nu, Y_\nu) \), \( \nu \in \mathbb{R} \), admits a continuation to an operator \( T_\mu \in \mathcal{B}^+(X_\mu, Y_\mu) \) for all \( \mu > \nu \). This continuation is unique.

**Proof.** Let \((X, Y)\) be \((C, C)\). We take an arbitrary \( x \in C_\mu \), \( \mu > \nu \). By the definition of \( C_\mu \), we can represent \( x \) in the form

\[
x = z = \Psi_\mu z = \Psi_{\mu-\nu} \Psi_\nu z,
\]

where \( z \in C \). Clearly, in this representation the function \( \tilde{z} = \Psi_\mu z \) belongs to \( C_\nu \). We recall that, by definition, \( \|z\|_C = \|x\|_{C_\mu} = \|\tilde{z}\|_{C_\nu} \).

Let \( a \in \mathbb{R} \) be an arbitrary point. Since \( \mu - \nu > 0 \), we have \( e^{(\mu-\nu)t} \leq e^{(\mu-\nu)a} \) for \( t \in (-\infty, a] \). Therefore

\[
|x(t)| = e^{(\mu-\nu)t} |\tilde{z}(t)| \leq e^{(\mu-\nu)a} |\tilde{z}(t)| \quad \text{for } t \in (-\infty, a].
\]

Consequently, the restriction of \( x \) to \((-\infty, a]\) belongs to \( C_\nu(-\infty, a] \) with

\[
\|x\|_{C_\nu(-\infty, a]} \leq e^{(\mu-\nu)a} \|\tilde{z}\|_{C_\nu(-\infty, a]} \leq e^{(\mu-\nu)a} \|\tilde{z}\|_{C_\nu} = e^{(\mu-\nu)a} \|x\|_{C_\mu}.
\]

We replace \( x \) with a function \( \tilde{x} \) such that \( \tilde{x}(t) = x(t) \) for \( t \leq a \) and \( \|\tilde{x}\|_{C_\nu} = \|x\|_{C_\nu(-\infty, a]} \leq e^{(\mu-\nu)a} \|x\|_{C_\mu} \). Obviously \( \tilde{x} \in C_\nu \cap C_\mu \); consequently the operators \( T_\nu \) and \( T_\mu \) must coincide on \( \tilde{x} \). Next, since the operator \( T_\mu \) must be causal, we have

\[
(T_\mu x)(t) = (T_\nu \tilde{x})(t) = (T_\nu \tilde{x})(t) \quad \text{for } t \leq a.
\]

Therefore, \( T_\mu x \) can be defined only by the rule

\[
(T_\mu x)(t) = (T_\nu \tilde{x})(t) \quad \text{for } t \leq a.
\]
We prove that this definition is correct. Indeed, since $T_\nu$ is causal, this definition does not depend on $\alpha$ and the continuation $\tilde{x}$. Further it is easy to see that

$$\|T_\mu x\|_{C_\nu(-\infty, a]} \leq \|T_\nu : C_\nu \to C_\nu\| \|\tilde{x}\|_{C_\nu} \leq e^{(\mu-\nu)a} \|T_\nu : C_\nu \to C_\nu\| \|x\|_{C_\mu}.$$  

Since $Y = C$, this inequality implies that

$$e^{-\nu a} |T_\mu x(a)| \leq e^{(\mu - \nu) a} \|T_\nu : C_\nu \to C_\nu\| \|x\|_{C_\mu} \quad \text{for all } a \in \mathbb{R},$$  

or

$$e^{-\mu a} |T_\mu x(a)| \leq \|T_\nu : C_\nu \to C_\nu\| \|x\|_{C_\mu} \quad \text{for all } a \in \mathbb{R},$$  

i.e., $\|T_\mu x\|_{C_\mu} \leq \|T_\nu : C_\nu \to C_\nu\| \|x\|_{C_\mu}$ or

$$\|T_\mu : C_\mu \to C_\mu\| \leq \|T_\nu : C_\nu \to C_\nu\|,$$

which shows that the operator $T_\mu$ is bounded.

The proof for other pairs (8) is based on the common idea. For example, let $(X, Y) = (C, C^1)$. So, let $T_\nu \in \mathcal{B}^+(C_\nu, C^1_\nu)$. We consider the auxiliary operator $R_\nu = T_\nu U_\lambda^{-1}$ for some $\lambda < \nu$; see proposition 6 for the definition of $U_\lambda$. Clearly, $R_\nu$ is causal and acts from $C$ to $C$. By the proved it admits a continuation to an operator $R_\mu : C_\mu \to C_\mu$. Clearly, $T_\mu = R_\mu U_\lambda : C_\mu \to C^1_\mu$ is a continuation of $T_\nu$.

**Corollary 9.** Let $(X, Y)$ be one of the pairs (8), and $\nu \in \mathbb{R}$. An operator $T \in \mathcal{B}^+(X_\nu, Y_\nu)$ has exponential memory if and only if $T$ is a continuation of an operator $T_\mu \in \mathcal{B}^+(X_\mu, Y_\mu)$ for some $\mu < \nu$. In other words $\mathcal{E}^+(X_\nu, Y_\nu)$ can be interpreted as the union of $\mathcal{B}^+(X_\mu, Y_\mu)$ over all $\mu < \nu$.

**Proof.** The proof is evident. 

According to corollary 9 we give the following definition: we say that a causal operator $T_\nu \in \mathcal{B}^+(X_\nu, Y_\nu)$ has exponential memory of the type $\mu$ if $T$ admits a continuation to an operator $T_\mu \in \mathcal{B}^+(X_\mu, Y_\mu)$ for $\mu < \nu$. In particular, an operator $T$ of the class $\mathcal{E}^+(X, Y)$ is an operator of the class $\mathcal{B}^+(X, Y)$ with an exponential memory of a negative type.

Usually we shall omit the index $\mu$ in the notation of the kind $T_\mu$ and denote the continuations $T_\mu$ of $T$ by the initial symbol $T$.

**4. Causal operators and quotient spaces.** Definition (6) of a causal operator can be rewritten in the following way. Let $X$ and $Y$ be linear spaces of functions defined on $\mathbb{R}$. Let $X_t$ and $Y_t$ denote the subspaces of $X$ and $Y$, respectively, consisting of all functions that are equal to zero on $(-\infty, t)$.  

Clearly, with this notation a linear operator $T: X \to Y$ is causal if and only if

$$TX_t \subseteq Y_t \quad \text{for all } t.$$  \hfill (9)

Clearly the collection of subspaces $\{X_t: t \in \mathbb{R}\}$ possesses the property

$$X_a \supseteq X_b \quad \text{for all } a \leq b.$$

We call such a collection of subspaces a direction on $X$. For completeness we set $X_{-\infty} = X$ and $X_{+\infty} = \{0\}$. Clearly, for $t = +\infty$ and $t = -\infty$ condition (9) is fulfilled for any linear operator $T$.

Let $X$ be a linear space with a direction and $-\infty \leq a < b \leq +\infty$. We denote by $X_{a/b}$ the quotient space $X_a/X_b$. Clearly, $X_{a/b}$ is a Banach space provided $X$ is Banach and $X_t$ is closed. We identify the space $X_{a/+\infty} = X_a/X_{+\infty} = X_a/\{0\}$ with $X_a$; and we denote briefly the space $X_{-\infty/b} = X_{-\infty}/X_b = X/X_b$ by $X/b$. We denote by

$$Q_b = Q_{a/b}: X_a \to X_{a/b}$$

the natural restriction that assigns to a function $x \in X_a$ its quotient class in $X_{a/b}$. Evidently, one can identify $(C_v)_{-\infty/b}$ with $C_v(-\infty, b]$, and $(C_v^1)_{-\infty/b}$ with $C_v^1(-\infty, b]$, and $(C_v)_{a/b}$ with the subspace $(C_v)_a(-\infty, b]$ of $C_v(-\infty, b]$ consisting of all functions that are equal to zero on $(-\infty, a]$, and $(C_v^1)_{a/b}$ with the subspace $(C_v^1)_a(-\infty, b]$ of $C_v^1(-\infty, b]$ consisting of all functions that are equal to zero on $(-\infty, a]$. Moreover the norms on these spaces (respectively) are equivalent uniformly with respect to $a$, and $Q_a$ can be identified with the restrictions $Q_{a/a}: C_v \to C_v(-\infty, a]$ and $Q_a: C_v^1 \to C_v^1(-\infty, a]$. We need not the space $C_v^{-1}(-\infty, b]$, but nevertheless we mention that according to proposition 10, see below, the subspace $(C_v^{-1})_b$ has a natural complement $(C_v^{-1})^b$; so the quotient space $(C_v^{-1})_{-\infty/b}$ can be identified with $(C_v^{-1})^b$.

We also note that $(C_v)_{a/b}$, $(C_v^1)_{a/b}$, and $(C_v^{-1})_{a/b}$ do not depend on $\nu$ if $a$ and $b$ are finite.

Let $X$ be a linear space with a direction. Clearly, for any fixed $a \in \mathbb{R}$ the family $\{X_t \cap X_a: -\infty \leq t \leq +\infty\}$ forms a direction on the space $X_a$. In a similar way, the natural projections of the subspaces $X_t$ into $X_{t/b}$ form a direction on $X_{t/b}$. Finally, the natural projections of the subspaces $X_t \cap X_a$ into $X_{a/b} = X_a/X_b$ form a direction on $X_{a/b}$. Of course, these directions are trivial for $t < a$ and $t > b$. Nevertheless, the consideration of directions with a superfluous set of indices allows one to treat all examples as special cases of the common definition.
Below in formula (36) we use the causal projector $P_a : X \to X$ onto $X_a$ and the causal continuation $R_a : X_{-\infty/a} \to X$. We define them only for special cases we actually need. Assume $a \in \mathbb{R}$ and $\nu > -1$. If $X = C_\nu$, we set

$$(R_a\varphi)(t) = \begin{cases} \varphi(t) & \text{for } t \leq a, \\ \varphi(a)e^{-(t-a)} & \text{for } t \geq a. \end{cases}$$

(10)

If $X = C_\nu^1$, we set

$$(R_a\varphi)(t) = \begin{cases} \varphi(t) & \text{for } t \leq a, \\ (\varphi(a)(t-a) + \varphi(a)(t-a+1))e^{-(t-a)} & \text{for } t \geq a. \end{cases}$$

(11)

If $X = C_\nu$, we set

$$(P_a f)(t) = \begin{cases} 0 & \text{for } t \leq a, \\ f(t) - f(a)e^{-(t-a)} & \text{for } t \geq a. \end{cases}$$

(12)

Clearly, $P_a$ is a projector onto $(C_\nu)_a$, and $P_aP_b = P_bP_a = P_b$ for $a < b$.

We define the projector $P_a : C^{-1}_\nu \to C^{-1}_\nu$ as follows. For $f \in C^{-1}_\nu$ we take a representation $f = \hat{u} + v$, $u, v \in C_\nu$, such that $u(a) = 0$ and $v(a) = 0$ (it is easy to show [21, 2.3.8] that such a representation always exists). Then we set

$$P_a f = \hat{u}_1 + v_1,$$

where

$$u_1(t) = \begin{cases} 0 & \text{for } t \leq a, \\ u(t) & \text{for } t \geq a; \end{cases} \quad v_1(t) = \begin{cases} 0 & \text{for } t \leq a, \\ v(t) & \text{for } t \geq a. \end{cases}$$

It can be shown [21, 2.3.8] that this definition does not depend on the choice of the representation $f = \hat{u} + v$ with $u(a) = 0$ and $v(a) = 0$.

**Proposition 10** ([21, proposition 2.3.8]). The operator $P_a : C^{-1}_\nu \to C^{-1}_\nu$ is a projector possessing the following properties.

(a) The image of $P_a$ is the subspace $(C^{-1}_\nu)_a$ of all distributions that are equal to zero on $(-\infty,a)$.

(b) The kernel of $P_a$ is the subspace $(C^{-1}_\nu)_a$ of all distributions that are equal to zero on $(a, +\infty)$.

(c) $P_aP_b = P_bP_a = P_b$ for $a < b$.

(d) The projector $P_a$ is memoryless.
The projector $P_a$ is determined uniquely by (a) and (b). Namely, we have
\[ C_{\nu}^{-1} = (C_{\nu}^{-1})^a \oplus (C_{\nu}^{-1})^b. \]

In terms of $P_a$ the definition of a causal operator can be rewritten as
\[ \text{for all } t. \]

Let $X$ and $Y$ be linear spaces with directions, let $T: X \to Y$ be a linear causal operator, and let $-\infty \leq a < b \leq +\infty$. Let $T_a: X_a \to Y_a$ denote the natural restriction of $T$ (see formula (9)), $T_{/b}: X_{/b} \to Y_{/b}$ denote the quotient operator induced by $T$, and $T_{a/b}: X_{a/b} \to Y_{a/b}$ denote the quotient operator induced by $T_a$. We call the operator $T_{a/b}$ the restriction of the operator $T$ to the segment $[a, b]$. Clearly,
\[ ||T_{a/b}|| \leq ||T|| \quad (13) \]
if $X$ and $Y$ are Banach spaces and $T: X \to Y$ is a bounded operator.

Let $X$ and $Y$ be linear spaces with directions, and let $T: X \to Y$ be a linear causal operator, and let $-\infty \leq a < b \leq +\infty$. Evidently,
\[ T_{a/b} X_{t/b} \subseteq Y_{t/b}. \]
Thus $T_{a/b}$ is causal, too.

**Proposition 11 ([21, proposition 2.1.1]).** Let $X$, $Y$, and $Z$ be linear spaces with directions, and $T: X \to Y$ and $S: Y \to Z$ be linear causal operators. Then the operator $ST$ is also causal, and
\[ (ST)_{a/b} = S_{a/b} T_{a/b} \quad \text{for all } a \text{ and } b. \]

Let $X$ and $Y$ be linear spaces with directions, and let $T: X \to Y$ be a linear causal operator. We say that the operator $T$ is causally invertible if $T$ is invertible and $T^{-1}$ is causal. Proposition 6 shows that the causal invertibility does not always coincide with the ordinary one: the inverse of the causal operator $U_{\lambda}$ can be both causal and anticausal. Let $-\infty \leq a < b \leq +\infty$. We say that the operator $T$ is causally invertible on $[a, b]$ if $T_{a/b}$ is invertible and $(T_{a/b})^{-1}$ is causal.

**Corollary 12 ([21, proposition 2.1.3]).** If a causal operator $T$ is causally invertible (on the whole $\mathbb{R}$), it is causally invertible on $[a, b]$ for all $-\infty \leq a < b \leq +\infty$. Moreover
\[ ||(T^{-1})_{a/b}|| \leq ||T||. \]
Proof. Proof follows from the identities
\[(T^{-1})_{a/b} T_{a/b} = (T^{-1}T)_{a/b} = 1_{a/b}, \quad T_{a/b} (T^{-1})_{a/b} = (TT^{-1})_{a/b} = 1_{a/b}.\]
The estimate follows from (13). \(\square\)

5. Initial value problem. Let \(\nu \in \mathbb{R}\) and \((X, Y)\) be one of the pairs
\[(C_{\nu}^{1}, C_{\nu}), \quad (C_{\nu}, C_{\nu}^{-1}),\]and let \(\mathcal{L} \in \mathcal{B}^{+}(X, Y)\). In this Section we discuss the equation
\[\mathcal{L}x = f.\]

Let \(-\infty < a < b < +\infty\). We consider the initial value problem
\[
\begin{align*}
(Lx)(t) &= f(t), & a < t < b, \\
x(t) &= \varphi(t), & t \leq a.
\end{align*}
\]

We assume that \(f \in C_{\nu}\) and \(\varphi \in C_{\nu}^{1}(-\infty, a]\) (respectively \(f \in C_{\nu}^{-1}\) and \(\varphi \in C_{\nu}(-\infty, a]\)) and look for a solution \(x \in C_{\nu}^{1}(-\infty, b]\) (respectively \(x \in C_{\nu}(-\infty, b]\)). We recall that equivalently we can think that \(\varphi \in X_{-\infty/a}\) and \(x \in X_{-\infty/b}\).

It is convenient to reduce the solubility of the initial value problem (16), (17) to the homogeneous case \(\varphi = 0\). We do it in the following way.

Assume \(\varphi \in C_{\nu}^{1}(-\infty, a]\) (respectively \(\varphi \in C_{\nu}(-\infty, a]\)). We denote by \(\psi \in C_{\nu}^{1}(-\infty, b]\) (respectively \(\psi \in C_{\nu}(-\infty, b]\)) an arbitrary continuously differentiable (continuous) continuation of the function \(\varphi\) to \((-\infty, b]\). We change \(x\) in (16), (17) to
\[x = \psi + z,
\]
where \(z \in C_{\nu}^{1}(-\infty, b]\) (respectively \(z \in C_{\nu}(-\infty, b]\)) is a new unknown function. Substituting \(x = \psi + z\) in (16) and (17) we obtain the new initial value problem
\[
\begin{align*}
(Lz)(t) &= g(t), & a < t < b, \\
z(t) &= 0, & t \leq a,
\end{align*}
\]
where
\[g = f - \mathcal{L}\psi.\]
We observe that condition (19) means that $z$ belongs to $(C^{1}_{\nu}(-\infty, b])_{a}$ (respectively $(C^{1}_{\nu}(-\infty, b])_{a}$) which can be interpreted as $z \in X_{a/b}$. Thus instead of the pair of the equalities (18) and (19) we may use only (18), but look for $z$ in $X_{a/b}$. So we arrive at the following statement.

**Lemma 13.** Initial value problem (16), (17) has a unique solution $x \in X_{-\infty/b}$ if and only if initial value problem (18), (19) with the forcing function (20) has a unique solution $z \in X_{a/b}$.

Let us look at the features of the pair $(C^{1}_{\nu}, C_{\nu})$. So, let us assume that $(X, Y) = (C^{1}_{\nu}, C_{\nu})$ and consider initial value problem (18), (19). From (19) and the causality of $\mathcal{L}$ we have $(\mathcal{L}z)(t) = 0$ for $t < a$. Since $\mathcal{L}z$ is a continuous function, from here and (18) we obtain

$$g(a) = 0.$$  

This equality is called the compatibility condition. Initial value problem (18), (19) can have a solution $z \in C^{1}_{\nu}(-\infty, b]$ only if $g$ satisfies this condition. From (20) it follows that in terms of the initial value problem (16), (17) the compatibility condition can be rewritten as

$$(\mathcal{L}\varphi)(a) = f(a).$$

We say that equation (15) is uniquely soluble on $[a, b]$ if the initial value problem (16), (17) has a unique solution $x \in C^{1}_{\nu}(-\infty, b]$ for all $f \in C_{\nu}$ and $\varphi \in C^{1}_{\nu}(-\infty, a]$ satisfying the compatibility condition.

Let us return to the equation (18). We assume that the compatibility condition $g(a) = 0$ be fulfilled. We denote by $\tilde{g}$ the function that coincides with $g$ on $[a, b]$ and is equal to zero on $(-\infty, a]$; thus $\tilde{g}$ can be considered as an element of $Y_{a/b} = (C_{\nu})_{a/b}$. We note that in these terms equation (18) can be rewritten as

$$\mathcal{L}z = \tilde{g},$$

where $\tilde{g} \in (C_{\nu})_{a/b}$ and $z \in (C^{1}_{\nu})_{a/b}$. Thus from lemma 13 we obtain the following proposition.

**Proposition 14.** Let $(X, Y)$ be $(C^{1}_{\nu}, C_{\nu})$. Equation (15) is uniquely soluble on $[a, b]$ if and only if the operator

$$\mathcal{L}_{a/b}: X_{a/b} \to Y_{a/b}$$

is invertible.

Now we assume that $(X, Y) = (C_{\nu}, C_{\nu}^{-1})$, i.e., we discuss the features of equation (15) with $\mathcal{L}: C_{\nu} \to C_{\nu}^{-1}$. We say that equation (15) is uniquely
soluble on \([a, b]\) if the initial value problem (16), (17) has a unique solution \(x \in C_v(-\infty, b)\) for all \(f \in C_v^{-1}\) and \(\varphi \in C_v(-\infty, a]\).

We denote by \(\widetilde{g}\) the function \(P_\alpha g\), where \(P_\alpha\) is defined in proposition 10. By virtue of proposition 10 \(g\) coincides with \(g\) on \((a, b)\) and is equal to zero on \((-\infty, a]\); thus \(\widetilde{g}\) can be considered as an element of \((C_v^{-1})_{a/b}\). We note that equation (18) can be rewritten as

\[
\mathcal{L}z = \widetilde{g},
\]

where \(\widetilde{g} \in (C_v^{-1})_{a/b}\) and \(z \in (C_v)_{a/b}\). From lemma 13 we obtain the following proposition.

**Proposition 15.** Let \((X, Y)\) be \((C_v, C_v^{-1})\). Equation (15) is uniquely soluble on \([a, b]\) if and only if the operator

\[
\mathcal{L}_{a/b} : X_{a/b} \to Y_{a/b}
\]

is invertible.

Let \((X, Y)\) be one of the pairs (14). We say that equation (15) is **locally soluble** if it is uniquely soluble on \([a, b]\) for all \(-\infty < a < b < +\infty\).

Assume \((X, Y) = (C_v, C_v^{-1})\). We say that equation (15) is **uniformly soluble** if it is locally soluble and there exist \(\delta > 0\) and \(K < \infty\) such that for any segment \([a, b], b - a < \delta\), and \(f \in C_v^{-1}\) and \(\varphi \in C_v(-\infty, a]\) the solution \(x \in C_v(-\infty, b]\) of the initial value problem

\[
\begin{align*}
(Lx)(t) &= f(t), \quad a < t < b, \\
x(t) &= \varphi(t), \quad t \leq a,
\end{align*}
\]

satisfies the estimate

\[
\|x\|_{C_v(-\infty, b]} \leq K(\|\varphi\|_{C_v(-\infty, a]} + \|f\|_{C_v^{-1}}).
\]

Assume \((X, Y) = (C_v^1, C_v)\). We say that equation (15) is **uniformly soluble** if it is locally soluble and there exist \(\delta > 0\) and \(K < \infty\) such that for any segment \([a, b], b - a < \delta\), and \(f \in C_v\) and \(\varphi \in C_v^1(-\infty, a]\) satisfying the compatibility condition

\[
(L\varphi)(a) = f(a)
\]

the solution \(x \in C_v^1(-\infty, b]\) of (21) satisfies the estimate

\[
\|x\|_{C_v^1(-\infty, b]} \leq K(\|\varphi\|_{C_v^1(-\infty, a]} + \|f\|_{C_v}).
\]
Obviously we can rewrite (22) and (23) equivalently as the common estimate

\[ \|x\|_{X_{-\infty/b}} \leq K(\|\varphi\|_{X_{-\infty/a}} + \|f\|_Y), \]

where \( \|\varphi\|_{X_{-\infty/a}} = \|Q_\alpha \varphi\| \).

Sometimes we shall specify that we consider solubility in the pair \((X, Y)\).

**Proposition 16 ([21, Proposition 3.5.5]).**

(a) The uniform solubility of (15) is equivalent to the uniform invertibility of all operators \( L_{a/b} \) for \( b - a \leq \delta \), where \( \delta > 0 \) is fixed.

(b) The definition of uniform solubility does not depend on \( \delta \).

**Corollary 17.** The definition of uniform solubility does not depend on \( \nu \).

**Proof.** It is easy to see that the spaces \( (X_\nu)_{a/b}, -\infty \leq a < b \leq +\infty \), with the same \( X \) and different \( \nu \) are naturally isomorphic. Consequently, the operators \( (L_\nu)_{a/b} \) with different \( \nu \) are naturally similar and one may not distinguish them. Moreover, it is easy to verify that for any \( \nu, \mu \in \mathbb{R} \) and \( \delta > 0 \) there exist \( M \) and \( N \) such that

\[ M\|T_\nu\| \leq \|T_\mu\| \leq N\|T_\nu\| \]

for \( b - a \leq \delta \) for any pair of similar operators \( T_\nu: (X_\nu)_{a/b} \rightarrow (Y_\nu)_{a/b} \) and \( T_\mu: (X_\mu)_{a/b} \rightarrow (Y_\mu)_{a/b} \), particularly for the pair \( ((L_\nu)_{a/b})^{-1} \) and \( ((L_\mu)_{a/b})^{-1} \).

**Proposition 18.** Let the operator \( L: X_\nu \rightarrow Y_\nu \) be causally invertible for at least one \( \nu \). Then equation (15) is uniformly soluble.

**Proof.** Proof follows from proposition 16 and corollaries 12 and 17.

Often it is convenient to consider the initial value problem

\[ (Lx)(t) = f(t), \quad t > a, \]
\[ x(t) = \varphi(t), \quad t \leq a, \]

on the semi-axis \([a, +\infty)\). By a solution of such a problem we mean a function \( x \in X_{\text{loc}} \) whose restriction to \((-\infty, b]\) belongs to \( X_{-\infty/b} \) and satisfies the equation on \((a, b)\) for any \( b > a \).

6. The formulation of the main result. In this Section we formulate theorem 23, which is the main result of the article. We begin with some preliminary discussion.

Let \( b_m: \mathbb{R} \rightarrow \mathcal{B}(\mathcal{E}), \ m = 1, 2, \ldots, \) be bounded continuous functions and \( h_m \geq 0, \ m = 1, 2, \ldots, \) be arbitrary numbers. We assume that the following two assumptions are fulfilled.
There exist numbers $\beta_m \geq 0$, $m = 1, 2, \ldots$, and a number $\mu > 0$ such that
\[
\forall t \in \mathbb{R} \quad \|b_m(t)\| \leq \beta_m
\]
and
\[
\sum_{m=1}^{\infty} e^{\mu h_m} \beta_m < \infty. \tag{25}
\]

There exists a continuous function $\theta: [0, +\infty) \to [0, +\infty)$ such that
\[
\theta(0) = 0, \quad \forall t, h \in \mathbb{R} \quad \|b_m(t + h) - b_m(t)\| \leq \theta(h) \beta_m.
\]
Clearly, we may additionally assume that $\theta(t) \leq 2$ for all $t$.

We consider the family of operators
\[
(B[\varepsilon]\{x\})(t) = \sum_{m=1}^{\infty} b_m(t)x(t - \varepsilon h_m) \tag{26}
\]
depending on the parameter $\varepsilon \geq 0$. We stress that for $\varepsilon = 0$ the operator $B[\varepsilon]$ is the operator of multiplication
\[
(B[0]\{x\})(t) = b_0(t)x(t)
\]
by the function
\[
b_0(t) = \sum_{m=1}^{\infty} b_m(t).
\]

**Proposition 19.** Formula (26) defines the causal operators $B[\varepsilon] \in B^+(C_{-\mu/\varepsilon}, C_{-\mu/\varepsilon})$ depending on the parameter $\varepsilon \geq 0$. In particular $B[\varepsilon] \in e^+(C_{\nu}, C_{\nu})$ for $\nu > -\mu/\varepsilon$.

**Proof.** Let $x \in C_{-\mu/\varepsilon}$, i.e., $x(t) = e^{-\mu t/\varepsilon}z(t)$ for some $z \in C$. For such an $x$ we have
\[
(B[\varepsilon]\{x\})(t) = \sum_{m=1}^{\infty} b_m(t)x(t - \varepsilon h_m)
\]
\[
= \sum_{m=1}^{\infty} b_m(t)e^{-\mu(t-\varepsilon h_m)/\varepsilon}z(t - \varepsilon h_m)
\]
\[
= \sum_{m=1}^{\infty} b_m(t)e^{-\mu t/\varepsilon}e^{\mu h_m}z(t - \varepsilon h_m)
\]
\[
= e^{-\mu t/\varepsilon} \sum_{m=1}^{\infty} b_m(t)e^{\mu h_m}z(t - \varepsilon h_m).
\]
From estimate (25) it follows that the function

$$y(t) = \sum_{m=1}^{\infty} b_m(t) e^{\mu h_m} z(t - \varepsilon h_m)$$

belongs to $C$; indeed,

$$\|y\|_C \leq \sum_{m=1}^{\infty} e^{\mu h_m} \beta_m \|z\|_C.$$ 

Now it is easy to see that $B[\varepsilon]x \in C_{-\mu/\varepsilon}$.

The last assertion follows from corollary 9. $\Box$

For each $\varepsilon \geq 0$ we consider the equation

$$\dot{x}(t) + (B[\varepsilon]x)(t) = 0.$$ 

(27)

If $\varepsilon = 0$ this equation becomes the ordinary one

$$\dot{x}(t) + b_0(t)x(t) = 0.$$ 

For each $\varepsilon \geq 0$ we also consider the operator $\mathcal{L} = \mathcal{L}[\varepsilon]$ defined by the formula

$$(\mathcal{L}[\varepsilon]x) = \dot{x}(t) + (B[\varepsilon]x)(t).$$ 

(28)

In particular

$$(\mathcal{L}[0]x) = \dot{x}(t) + b_0(t)x(t).$$ 

(29)

**Corollary 20.** Formula (28) defines the causal operators $\mathcal{L}[\varepsilon] \in B^+(C_{-\mu/\varepsilon}, C_{-\mu/\varepsilon})$ and $\mathcal{L}[\varepsilon] \in B^+(C_{-\mu/\varepsilon}, C_{-\mu/\varepsilon}^{-1})$ depending on the parameter $\varepsilon \geq 0$. In particular $\mathcal{L}[\varepsilon] \in e^+(C^1_{\nu}, C_{\nu})$ and $\mathcal{L}[\varepsilon] \in e^+(C_{\nu}, C_{\nu}^{-1})$ for $\nu > -\mu/\varepsilon$.

**Proof.** Proof follows from proposition 19. $\Box$

Below we use the continuous dependence of the operators $\mathcal{L}[\varepsilon]: C^1_{\nu} \rightarrow C_{\nu}$ and $\mathcal{L}[\varepsilon]: C_{\nu} \rightarrow C_{\nu}^{-1}$ from the parameter $\nu$. One cannot formulate such a statement directly, because the operators $\mathcal{L}[\varepsilon]: C^1_{\nu} \rightarrow C_{\nu}$ and $\mathcal{L}[\varepsilon]: C_{\nu} \rightarrow C_{\nu}^{-1}$ act in functional spaces depending on the parameter $\nu$. We overcome this difficulty in the following proposition by means of a change of variables.

**Proposition 21.** The operators

$$\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_{\nu} : C^1 \rightarrow C,$$ 

where $\mathcal{L}[\varepsilon]: C^1_{\nu} \rightarrow C_{\nu},$

$$\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_{\nu} : C \rightarrow C^{-1},$$ 

where $\mathcal{L}[\varepsilon]: C_{\nu} \rightarrow C_{\nu}^{-1},$
depend on $\epsilon \geq 0$ and $\nu \in \mathbb{R}$ in a continuous way.

Proof. The both cases are analysed in a similar way. So, we restrict ourselves to the second one.

First we calculate the operator $\Psi_{-\nu} \mathcal{L}[\epsilon] \Psi_\nu$:

$$
(\Psi_{-\nu} \mathcal{L}[\epsilon] \Psi_\nu x)(t) = e^{-\nu t} \frac{d}{dt}(e^{\nu t} x(t)) + \sum_{m=1}^{\infty} e^{-\nu t} b_m(t) (e^{\nu(t-\epsilon h_m)} x(t - \epsilon h_m))
$$

$$
= \dot{x}(t) + \nu x(t) + \sum_{m=1}^{\infty} e^{-\nu h_m} b_m(t) x(t - \epsilon h_m).
$$

Then we consider the auxiliary operator

$$
T[\epsilon, \nu] = U^{-1} \Psi_{-\nu} \mathcal{L}[\epsilon] \Psi_\nu : C \to C,
$$

where $U$ is the operator $U_\lambda$ with $\lambda = -1$ defined in proposition 6 (in the case of operator $\mathcal{L}[\epsilon]: C_2 \to C_\nu$ we define $T[\epsilon, \nu]$ by the formula $T[\epsilon, \nu] = \Psi_{-\nu} \mathcal{L}[\epsilon] \Psi_\nu U^{-1}: C \to C$). Clearly it is enough to prove that the operator $T[\epsilon, \nu]$ depends on $\epsilon \geq 0$ and $\nu \in \mathbb{R}$ continuously.

It is easy to see that (here for simplicity we assume that $\nu > -1$; only this case will be used below)

$$
(T[\epsilon, \nu] x)(t) = x(t) + \int_{-\infty}^{t} e^{-(t-s)} \left( (\nu - 1) x(s) + \sum_{m=1}^{\infty} e^{-\nu \epsilon h_m} b_m(s) x(s - \epsilon h_m) \right) ds
$$

$$
= x(t) + (\nu - 1) \int_{-\infty}^{t} e^{-(t-s)} x(s) ds
$$

$$
+ \sum_{m=1}^{\infty} e^{-\nu \epsilon h_m} \int_{-\infty}^{t} e^{-(t-s)} b_m(s) x(s - \epsilon h_m) ds
$$

$$
= x(t) + (\nu - 1) \int_{-\infty}^{t} e^{-(t-s)} x(s) ds
$$

$$
+ \sum_{m=1}^{\infty} e^{-(\nu - 1) \epsilon h_m} \int_{-\infty}^{t-\epsilon h_m} e^{-(t-s)} b_m(s + \epsilon h_m) x(s) ds.
$$

So we consider the operator $T[\epsilon_1, \nu_1] - T[\epsilon_2, \nu_2]$ (for the sake of definiteness
we assume that $\varepsilon_1 \leq \varepsilon_2$:

\[
(T[\varepsilon_1, \nu_1] - T[\varepsilon_2, \nu_2])x(t) = (\nu_1 - 1) \int_{-\infty}^{t} e^{-(t-s)} x(s) \, ds \\
+ \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_1 h_m} \int_{-\infty}^{t-\varepsilon_1 h_m} e^{-(t-s)} b_m(s + \varepsilon_1 h_m) x(s) \, ds \\
- (\nu_2 - 1) \int_{-\infty}^{t} e^{-(t-s)} x(s) \, ds \\
- \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_2 h_m} \int_{-\infty}^{t-\varepsilon_2 h_m} e^{-(t-s)} b_m(s + \varepsilon_2 h_m) x(s) \, ds
\]

\[
= (\nu_1 - \nu_2) \int_{-\infty}^{t} e^{-(t-s)} x(s) \, ds \\
+ \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_1 h_m} \int_{-\infty}^{t-\varepsilon_1 h_m} e^{-(t-s)} b_m(s + \varepsilon_1 h_m) x(s) \, ds \\
- \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_2 h_m} \int_{-\infty}^{t-\varepsilon_2 h_m} e^{-(t-s)} b_m(s + \varepsilon_2 h_m) x(s) \, ds
\]

From this representation we obtain the estimate

\[
||T[\varepsilon_1, \nu_1] - T[\varepsilon_2, \nu_2]|| \leq |\nu_1 - \nu_2| \int_{-\infty}^{t} e^{-(t-s)} \, ds \\
+ \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_1 h_m} \beta_m \int_{-\infty}^{t-\varepsilon_1 h_m} e^{-(t-s)} \, ds \\
+ \sum_{m=1}^{\infty} e^{-(\nu_1-1) \varepsilon_1 h_m} \beta_m \theta((\varepsilon_1 - \varepsilon_2) h_m) \int_{-\infty}^{t-\varepsilon_2 h_m} e^{-(t-s)} \, ds
\]
which after some transformation gives

\[ \sum_{m=1}^{\infty} (e^{-(\nu-1)e_{1}h_m} - e^{-(\nu-1)e_{2}h_m}) \beta_m \int_{-\infty}^{t-\epsilon_2 h_m} e^{-(t-s)} ds, \]

where \( \theta \) is the function from assumption (ii). We observe that the members of the series tend to zero not slower that \( \beta_m \). On the other hand the members of the series corresponding to small \( m \) can be made arbitrary small if one takes \( \epsilon_1 \) and \( \epsilon_2 \), and \( \nu_1 \) and \( \nu_2 \) close enough to each other. Therefore the right hand side of the last estimate is small provided \( \epsilon_1 \) and \( \epsilon_2 \), and \( \nu_1 \) and \( \nu_2 \) are close enough to each other.

**Proposition 22.** Equation (15) with operator (28) is uniformly solvable.

**Proof.** We state that the operators \( \mathcal{L} : C_{\nu} \rightarrow C_{\nu}^{-1} \) and \( \mathcal{L} : C_{\nu}^1 \rightarrow C_{\nu} \) are causally invertible for \( \nu \) large enough. The proof of this fact is similar to that of proposition 7. The rest of the proof follows from proposition 18.

**Theorem 23.** There exists \( \alpha > -\infty \) (\( \alpha \) is independent from \( \epsilon, \alpha, \) and \( \varphi \)) such that for any \( \omega < \alpha \) and for all \( \epsilon \geq 0 \) small enough there exists an ordinary differential equation

\[ \dot{x}(t) + b_\epsilon(t)x(t) = 0 \]

with a bounded continuous coefficient \( b_\epsilon : \mathbb{R} \rightarrow B(\mathbb{R}) \) such that for any \( \alpha \in \mathbb{R} \) and \( \varphi \in C_{\nu}(-\infty, a] \) with \( \nu \in [\omega, \alpha] \) the solution of the initial value problem

\[ \dot{x}(t) + (B[\epsilon]x)(t) = 0, \quad t > a, \]
\[ x(t) = \varphi(t), \quad t \leq a, \]

can be uniquely represented as the sum

\[ x = x^- + x^+, \]
where $x^{-}$ is a solution of the equation

$$\dot{x}^{-}(t) + b_{c}(t)x^{-}(t) = 0, \quad t \in \mathbb{R},$$

and $x^{+}$ is a solution of the equation

$$\dot{x}^{+}(t) + (B[e]x^{+})(t) = 0, \quad t > a,$$

and $x^{-}$ and $x^{+}$ satisfy the estimates

$$|x^{-}(t)| \geq Me^{\alpha(t-a)}|x^{-}(a)|, \quad t \geq a,$$

$$|x^{+}(t)| \leq Ne^{\nu(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \in \mathbb{R},$$

with $M$ and $N$, and some $\delta > 0$ independent from $a$ and $\varphi$.

If $\varphi$ belongs to $C^{1}_{\nu}(-\infty, a]$ and satisfies the compatibility condition $\varphi(a) + (B[e]\varphi)(a) = 0$, then $x^{+}$ also belongs to $C^{1}_{\nu}(-\infty, a]$ and satisfies the compatibility condition $\dot{x}^{+}(a) + (B[e]x^{+})(a) = 0$, and

$$|x^{+}(t)| + |x^{-}(t)| \leq Ne^{\nu(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}(|\varphi(s)| + |\varphi(s)|), \quad t \in \mathbb{R}.$$

We postpone the proof of theorem 23 until Section 8.

7. The theorem on exponential dichotomy. In this Section we formulate the theorem on exponential dichotomy. This theorem is the basic tool of our proof of theorem 23.

Let $(X, Y)$ be one of the pairs (14), $\nu \in \mathbb{R}$, and let $\mathcal{L} \in e^{+}(X, Y)$. We consider equation (15). We assume that equation (15) is uniformly soluble in $(X, Y)$. The main example of equation (15) is equation (27), see proposition 22.

First we assume that $(X, Y) = (C_{\nu}, C_{\nu}^{-1})$. We say that equation (15) possesses exponential dichotomy in $(C_{\nu}, C_{\nu}^{-1})$ if the following four conditions are fulfilled.

(i) For any $a \in \mathbb{R}$ the phase space $C_{\nu}(-\infty, a]$ is decomposed by the direct sum

$$C_{\nu}(-\infty, a] = C_{\nu}^{+}(-\infty, a] \oplus C_{\nu}^{-}(-\infty, a].$$

We denote briefly the corresponding projections of $\varphi \in C_{\nu}(-\infty, a]$ into the subspaces $C_{\nu}^{+}(-\infty, a]$ and $C_{\nu}^{-}(-\infty, a]$ by $\varphi^{+}$ and $\varphi^{-}$, respectively. And we denote by $x^{+}$ and $x^{-}$ the solutions of the initial value problems

$$\mathcal{L}x^{\pm}(t) = 0, \quad t > a, \quad (31)$$

$$x^{\pm}(t) = \varphi^{\pm}(t), \quad t \leq a. \quad (32)$$
For all $b > a$ the natural restriction $Q_b x^+$ of $x^+$ to $(-\infty, b]$ belongs to $C^+_\nu(-\infty, b]$.

The function $x^-$ satisfies the equation

$$(Lx^-)(t) = 0, \quad t \in \mathbb{R},$$

and for all $b \in \mathbb{R}$ the natural restriction $Q_b x^-$ of $x^-$ to $(-\infty, b]$ belongs to $C^-\nu(-\infty, b]$.

There exist $N < \infty$ and $\delta > 0$ such that for all $\varphi \in C^-\nu(-\infty, a]$ one has $x^+ \in C^-\nu-\delta$ and $\varphi^- \in C^-\nu+\delta(-\infty, a]$ (note $+\delta$ and $-\delta$), and the following estimates hold:

$$|x^+(t)| \leq Ne^{(\nu-\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \in \mathbb{R}, \quad (34)$$

$$|\varphi^-(t)| \leq Ne^{(\nu+\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \leq a. \quad (35)$$

**Remark 1.**

(a) The intensive part of estimate (34) is

$$|x^+(t)| \leq Ne^{(\nu-\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \geq a.$$

(b) By (32) $\varphi^-$ and $x^-$ coincide on $(-\infty, a]$; thus (35) can be rewritten as

$$|x^-(t)| \leq Ne^{(\nu+\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \leq a.$$

(c) Since $e^{\delta(t-a)} < 1$ for $t \leq a$, from estimate (35) it follows that

$$|\varphi^-(t)| \leq Ne^{\nu(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}|\varphi(s)|, \quad t \leq a,$$

or equivalently

$$\|\varphi^-\|_{C^-\nu(-\infty, a]} \leq N\|\varphi\|_{C^-\nu(-\infty, a]}.$$

Thus the projector $\Pi^+\nu : C^+\nu(-\infty, a] \to C^+\nu(-\infty, a]$ onto $C^-\nu(-\infty, a]$ parallel to $C^+_\nu(-\infty, a]$ is bounded uniformly with respect to $a \in \mathbb{R}$. Consequently the complementary projector $\Pi^-\nu = 1 - \Pi^+\nu$ in $C^+\nu(-\infty, a]$ onto $C^-\nu(-\infty, a]$ parallel to $C^-\nu(-\infty, a]$ is bounded uniformly with respect to $a \in \mathbb{R}$ as well. Particularly, since the image of a bounded projector is closed, the subspaces $C^\pm\nu(-\infty, a]$ are closed in $C^\nu(-\infty, a]$. 
Now we assume that \((X,Y) = (C^1_\nu, C_\nu)\). We denote by \(\hat{C}^1_\nu(-\infty,a]\) the subspace of all \(\varphi \in C^1_\nu(-\infty,a]\) satisfying the compatibility condition \((\mathcal{L}\varphi)(a) = 0\). We say that equation (15) possesses exponential dichotomy in \((C^1_\nu, C_\nu)\) if the following four conditions are fulfilled.

(i) For any \(a \in \mathbb{R}\) the phase space \(\hat{C}^1_\nu(-\infty,a]\) is decomposed by the direct sum

\[
\hat{C}^1_\nu(-\infty,a]\ = C^1_\nu^+(a] \oplus C^1_\nu^-(a].
\]

We shall denote briefly the projections of \(\varphi \in \hat{C}^1_\nu(-\infty,a]\) into the subspaces \(C^1_\nu^+(a]\) and \(C^1_\nu^-(a]\) by \(\varphi^+\) and \(\varphi^-\), respectively. And we shall denote by \(x^+\) and \(x^-\) the solutions of the initial value problems

\[
(\mathcal{L}x^\pm)(t) = 0, \quad t \geq a,
\]

\[
x^\pm(t) = \varphi^\pm(t), \quad t \leq a.
\]

(ii) For all \(b > a\) the natural restriction \(Q_b x^+\) of \(x^+\) to \((-\infty,b]\) belongs to \(C^1_\nu^+(\infty,b]\).

(iii) The function \(x^-\) satisfies the equation

\[
(\mathcal{L}x^-)(t) = 0, \quad t \in \mathbb{R}.
\]

For all \(b \in \mathbb{R}\) the natural projection of \(x^-\) into \(C^1_\nu(-\infty,b]\) lies in \(C^1_\nu^-(\infty,b]\).

(iv) There exist \(N < \infty\) and \(\delta > 0\) such that for all \(\varphi \in \hat{C}^1_\nu(-\infty,a]\) one has \(x^+ \in C^1_\nu_{-\delta}\) and \(\varphi^- \in C^1_\nu_{+\delta}(-\infty,a]\), and the following estimates hold:

\[
|\dot{x}^+(t)| + |x^+(t)| \leq Ne^{(\nu-\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}(|\varphi(s)| + |\varphi(s)|), \quad t \in \mathbb{R},
\]

\[
|\dot{\varphi}^-(s)| + |\varphi^-(s)| \leq Ne^{(\nu+\delta)(t-a)} \sup_{s \leq a} e^{-\nu(s-a)}(|\varphi(s)| + |\varphi(s)|), \quad t \leq a.
\]

An analogue of remark 1 holds.

**Theorem 24 ([4, 20], [21, theorem 3.6.4]).** Let \(\nu \in \mathbb{R}\) and \((X,Y)\) be one of the pairs (14), let \(\mathcal{L} \in e^+ (X,Y)\), and let equation (15) be uniformly soluble. Then the following assumptions are equivalent:

(a) The operator \(\mathcal{L} : X \to Y\) is invertible.

(b) Equation (15) possesses exponential dichotomy in \((X,Y)\). Under these assumptions the projector \(\Pi^+_a : C^1_\nu(-\infty,a] \to C^1_\nu(-\infty,a]\) parallel to \(C^1_\nu^-(\infty,a]\) (respectively, \(\Pi^-_a : \hat{C}^1_\nu(-\infty,a]\ \to C^1_\nu(-\infty,a]\) onto \(C^1_\nu^+(-\infty,a]\) parallel to \(C^1_\nu^-(\infty,a]\) and the complementary projector \(\Pi^-_a = 1 - \Pi^+_a\) can be calculated by the formulae

\[
\Pi^+_a = Q_a(1 - P_a)LR_a, \quad \Pi^-_a = Q_a P_aLR_a,
\]

where \(R_a\), \(P_a\), and \(Q_a\) are defined as in Section 4.
8. The proof of theorem 23.

Proof. We break the proof of theorem 23 into nine steps.

(i) For $\varepsilon \geq 0$ we consider the operator $\mathcal{L} = \mathcal{L}[\varepsilon]$ defined by formula (28) and its special case (29) as acting from $C_\nu$ to $C_{\nu}^{-1}$ and from $C_\nu^1$ to $C_\nu$ with $\nu > -\mu/\varepsilon$, see corollary 20. For definiteness we shall consider the case of the pair $(C_\nu, C_{\nu}^{-1})$ as the main. As for the case of the pair $(C_\nu^1, C_\nu)$ we restrict ourselves to pointing to distinctions from the case of the pair $(C_\nu, C_{\nu}^{-1})$.

We choose $\alpha > -\infty$ in such a way that the operator $\Psi_{-\nu} \mathcal{L}[0] \Psi_\nu : C \to C^{-1}$ is also invertible for all $\nu \leq \alpha$. Clearly, the operator $\Psi_{-\nu} \mathcal{L}[0] \Psi_\nu : C \to C^{-1}$ is also invertible for all $\nu \leq \alpha$.

We fix some $\omega \in (-\infty, \alpha)$ and consider the operators $\Psi_{-\nu} \mathcal{L}[0] \Psi_\nu : C \to C^{-1}$ for $\nu \in [\omega, \alpha]$. All of them are invertible. By proposition 21 and corollary 2 the norms of the inverses depend continuously on $\nu$. Hence by proposition 3 the norms of the inverses are uniformly bounded.

Next we consider the operators $\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_\nu : C \to C^{-1}$ for $\nu \in [\omega, \alpha]$ and $\varepsilon \in [0, \varepsilon_0]$ with some $\varepsilon_0 > 0$. By proposition 21 the operator

$$\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_\nu - \Psi_{-\nu} \mathcal{L}[0] \Psi_\nu$$

depends continuously on $(\varepsilon, \nu) \in [0, \varepsilon_0] \times [\omega, \alpha]$. At the same time it is zero when $\varepsilon = 0$. Hence taking $\varepsilon_0 > 0$ small enough we can make the supremum of the norms of the operators $\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_\nu - \Psi_{-\nu} \mathcal{L}[0] \Psi_\nu$ over all $(\varepsilon, \nu) \in [0, \varepsilon_0] \times [\omega, \alpha]$ arbitrarily small. In this case, by estimate (1), we can state that the operators $\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_\nu : C \to C^{-1}$, $(\varepsilon, \nu) \in [0, \varepsilon_0] \times [\omega, \alpha]$, are invertible and, by proposition 3, the norms of the inverses are uniformly bounded. The same assertion is true for the case of the pair $(C_\nu^1, C_\nu)$, namely, for the operators $\Psi_{-\nu} \mathcal{L}[\varepsilon] \Psi_\nu : C^1 \to C$, $(\varepsilon, \nu) \in [0, \varepsilon_0] \times [\omega, \alpha]$.

Since the norms of the operators $\Psi_{\nu}$ and $\Psi_{-\nu}$, $\nu \in [\omega, \alpha]$, are uniformly bounded, we have that the operators $\mathcal{L}[\varepsilon] : C_\nu \to C_{\nu}^{-1}$, $(\varepsilon, \nu) \in [0, \varepsilon_0] \times [\omega, \alpha]$, are invertible and the norms of the inverses are uniformly bounded as well. A similar assertion is true for the case of the pair $(C_\nu^1, C_\nu)$.

(ii) Our subsequent reasoning uses references to [21]. In [21], for simplicity of notation the operator $U_\lambda$ (see proposition 6) is considered only with $\lambda = -1$. Therefore in [21] some results for weighted spaces are proved only for $\nu > -1$ (though they are true for all $\nu \in \mathbb{R}$). To make references to [21] formally correct we shall assume that $\alpha = 0$ and $\omega > -1$. This assumption can be made without loss of generality. Indeed, one can perform the change of variables $y = \Psi_{-\alpha} x$; as a result $\alpha$ becomes 0. Next one can perform the change of variables $z(t) = y(t/(2|\omega|))$; as a result $\omega$ becomes $-1/2$. 
We take any $\varepsilon \in [0, \varepsilon_0]$. By (i) and theorem 24 we have an exponential dichotomy of solutions of the equation $L[\varepsilon]x = 0$ in each of the pairs $(C_\varphi, C_\varphi^{-1})$, $\nu \in [\omega, \alpha] = [-1/2, 0]$. A similar assertion is true for the case of the pair $(C_\varphi^1, C_\varphi)$.

(iii) We consider the operator $L[0]$. We stress that the operator $L[0]$ is an ordinary differential one. We recall from (ii) that we have an exponential dichotomy in each of the pairs $(C_\varphi, C_\varphi^{-1})$ and $(C_\varphi^1, C_\varphi)$, $\nu \in [\omega, \alpha] = [-1/2, 0]$. We prove that $C_\varphi^+(-\infty, a]$ coincides with the subspace $C_\varphi(-\infty, a] = \{ \varphi \in C_\varphi(-\infty, a): \varphi(a) = 0 \}$ of all functions vanishing at $a$; and $C_\varphi^-(-\infty, a]$ consists of all solutions $\varphi \in C_\varphi(-\infty, a]$ of the homogeneous equation $\varphi(t) + b_0(t)\varphi(t) = 0$ (remember that $\varepsilon = 0$). And a similar assertion holds for the case of the pair $(C_\varphi^1, C_\varphi)$; namely, $C_\varphi^1(-\infty, a]$ coincides with the subspace $C_\varphi^1(-\infty, a] = \{ \varphi \in C_\varphi^1(-\infty, a): \varphi(a) = 0 \}$ of all functions vanishing at $a$; and $C_\varphi^1(-\infty, a]$ consists of all solutions $\varphi \in C_\varphi^1(-\infty, a]$ of the homogeneous equation $\varphi(t) + b_0(t)\varphi(t) = 0$. (We note that since $C_\varphi^1(-\infty, a]$ consists of functions $\varphi \in C_\varphi^1(-\infty, a]$ satisfying the compatibility condition $\varphi(a) + b_0(a)\varphi(a) = 0$, the assumption $\varphi(a) = 0$ implies that $\varphi(a) = 0$.)

We make use of (36). We take an arbitrary function $\varphi \in C_\varphi(-\infty, a]$ (respectively $\varphi \in C_\varphi^1(-\infty, a]$). From (29), and (10), and (11) we have

\[
(LR_a \varphi)(t) = \begin{cases} 
\varphi(t) + b_0(t)\varphi(t) & \text{for } t < a, \\
\int C(t) & \text{for } t > a 
\end{cases} \quad \text{if } X = C,
\]

\[
(LR_a \varphi)(t) = \begin{cases} 
\varphi(t) + b_0(t)\varphi(t) & \text{for } t \leq a, \\
\int C^1(t) & \text{for } t \geq a 
\end{cases} \quad \text{if } X = C^1,
\]

where

\[
f_C(t) = \varphi(a)(b_0(t) - 1)e^{-(t-a)},
\]

\[
f_{C^1}(t) = \left(\varphi(a)(1 + (t-a)(b_0(t) - 1)) + \varphi(a)(1 + (t-a+1)(b_0(t) - 1))\right)e^{-(t-a)}.
\]

(We stress that in the first formula $LR_a \varphi$ must lie in $C_\varphi^{-1}$; and we recall from proposition 10 that a function from $C_\varphi^{-1}$ is determined completely by its values on $(-\infty, a)$ and $(a, +\infty)$; hence the first formula is correct. It is straightforward to verify that the second formula defines a continuously differentiable function, thus the second formula is correct too.) Further from
proposition 10 and formula (12) we have

$$
((1 - P_a)\mathcal{L}R_a \varphi)(t) = \begin{cases} 
\varphi(t) + b_0(t)\varphi(t) & \text{for } t < a, \\
0 & \text{for } t > a
\end{cases}
$$

if \( X = C \),

$$
(P_a \mathcal{L}R_a \varphi)(t) = \begin{cases} 
0 & \text{for } t < a, \\
f_C(t) & \text{for } t > a
\end{cases}
$$

if \( X = C \),

$$
((1 - P_a)\mathcal{L}R_a \varphi)(t) = \begin{cases} 
\varphi(t) + b_0(t)\varphi(t) & \text{for } t \leq a, \\
0 & \text{for } t \geq a
\end{cases}
$$

if \( X = C^1 \),

$$
(P_a \mathcal{L}R_a \varphi)(t) = \begin{cases} 
0 & \text{for } t \leq a, \\
f_C(t) & \text{for } t \geq a
\end{cases}
$$

if \( X = C^1 \).

(We recall that if \( X = C^1 \), we assume that \( \varphi \in \check{C}_\nu^1(-\infty, a] \). This means that \( f_C(a) = \varphi(a) + b_0(a)\varphi(a) = 0 \), which implies the evident simplification of formula (12).)

Since \( \mathcal{L}^{-1} \) is anticausal, from above we obtain

$$
(\mathcal{L}^{-1}(1 - P_a)\mathcal{L}R_a \varphi)(t) = \begin{cases} 
\varphi^+(t) & \text{for } t \leq a, \\
0 & \text{for } t > a
\end{cases}
$$

(37)

where \( \varphi^+ \) is some function. Recalling that \( Q_a \) is the natural restriction acting from \( C_\nu \) to \( C_\nu(-\infty, a] \) (respectively, from \( C_\nu^1 \) to \( C_\nu(-\infty, a] \)), we obtain that

$$
\Pi^+_a \varphi = Q_a \mathcal{L}^{-1}(1 - P_a)\mathcal{L}R_a \varphi = \varphi^+.
$$

Particularly, from this formula it is seen that \( \varphi^+ \) in (37) coincides with the function \( \varphi^+ \) in the definition of exponential dichotomy.

From the continuity of the function \( \mathcal{L}^{-1}(1 - P_a)\mathcal{L}R_a \varphi \) and (37) we obtain that \( \varphi^+(a) = 0 \), thus \( \varphi^+ \in \check{C}_\nu(-\infty, a] \). Since \( \varphi^- + \varphi^+ = \varphi \), we have \( \varphi^-(a) = \varphi(a) \). Similar assertions are true for the case of the pair \((C_\nu^1, C_\nu)\).

From (iii) it follows that \( \varphi^- \) is a solution of the homogeneous equation \( \dot{\varphi}^- + b_0\varphi^- = 0 \) on \((-\infty, a) \). We verify that \( C_\nu^-(-\infty, a] \) and \( C_\nu^1(-\infty, a] \) consist exactly of all solutions of the homogeneous equation \( \dot{\varphi}^- + b_0\varphi^- = 0 \) on \((-\infty, a) \). Indeed, it is evident that for any vector \( e \in \mathbb{E} \) there exists a function \( \varphi \in C_\nu(-\infty, a] \) (respectively \( \varphi \in \check{C}_\nu^1(-\infty, a] \)) such that \( \varphi(a) = e \). Taking \( \varphi^- \) that corresponds to this \( \varphi \) we obtain a function \( \varphi^- \) from \( C_\nu^-(-\infty, a] \) or \( C_\nu^1(-\infty, a] \) that satisfies the condition \( \varphi^-(a) = \varphi(a) = e \). By the proved, \( \varphi^- \) is a solution of the homogeneous equation \( \dot{\varphi}^- + b_0\varphi^- = 0 \). It remains to recall that a solution of an ordinary differential equation is determined completely by the initial condition \( \varphi^-(a) = 0 \).
Finally we show that $C^+_\nu(-\infty, a]$ exhausts the whole $C^\circ_\nu(-\infty, a]$, and a similar assertion holds for the case of $X = C^1$. Indeed, we take an arbitrary \( \varphi \in C^\circ_\nu(-\infty, a] \) or \( \varphi \in C^{1+}_\nu(-\infty, a] \). Then

\[
(R_a \varphi)(t) = \begin{cases} 
\varphi(t) & \text{for } t \leq a, \\
0 & \text{for } t > a.
\end{cases}
\]

(Here we use the definitions of $R_a$ and the facts that $C^\circ_\nu(-\infty, a]$ consists of functions \( \varphi \) such that \( \varphi(a) = 0 \), and $C^{1+}_\nu(-\infty, a]$ consists of functions \( \varphi \) such that \( \varphi(a) = 0 \) and \( \varphi'(a) = 0 \)). Further (in the both cases) we have

\[
(\mathcal{L} R_a \varphi)(t) = \begin{cases} 
\varphi(t) + b_0(t)\varphi(t) & \text{for } t < a, \\
0 & \text{for } t > a,
\end{cases}
\]

\[
(1 - P_a) \mathcal{L} R_a \varphi = \mathcal{L} R_a \varphi,
\]

\[
\mathcal{L}^{-1}(1 - P_a) \mathcal{L} R_a \varphi = \mathcal{L}^{-1} \mathcal{L} R_a \varphi = R_a \varphi,
\]

\[
\Pi^+_a \varphi = Q_a \mathcal{L}^{-1}(1 - P_a) \mathcal{L} R_a \varphi = Q_a R_a \varphi = \varphi
\]

(here we use the evident identity $Q_a R_a = 1$). Thus \( \varphi \) under consideration belongs to $\text{Im} \, \Pi^+_a$, which is $C^+_\nu(-\infty, a]$ or $C^{1+}_\nu(-\infty, a]$ respectively.

(iv) We show that for $\varepsilon = 0$, for all $\varphi \in C^\circ_\nu(-\infty, a]$ and $\varphi \in C^{1-}_\nu(-\infty, a]$

\[
|\varphi^-(t)| + |\varphi^-(t)| \leq Ke^{\alpha(t-a)}|\varphi^-(a)|, \quad t \leq a,
\]

where $K$ is independent from $a$ and $\varphi^-$. Indeed, we make use of the explicit formulae derived in (iii). We see that the functions $f_{C^1}$ and $f_C$ are determined completely by the values $\varphi(a)$ and $\varphi'(a)$. Moreover, from the definitions of $f_C$ and $f_{C^1}$ it follows the evident estimates (recall that $\nu \in [\omega, \alpha] = [-1/2, 0]$)

\[
\|f_C\|_{C^\nu(-\infty, a]} \leq Ke^{-\nu a}|\varphi(a)|, \quad \text{if } X = C,
\]

\[
\|f_{C^1}\|_{C^\nu(-\infty, a]} \leq Ke^{-\nu a}(|\varphi(a)| + |\varphi(a)|), \quad \text{if } X = C^1,
\]

where $K$ is independent from $a$ and $\varphi$. Further from formulae derived in (iii) we see that $P_a \mathcal{L} R_a \varphi$ and, consequently, $\varphi^- = \Pi^-_a \varphi = Q_a \mathcal{L}^{-1} P_a \mathcal{L} R_a \varphi$ are determined by $f_C$ or $f_{C^1}$, and (consequently) are determined by the values $\varphi(a)$ and $\varphi'(a)$. Hence the norm of $\varphi^-$ possesses the similar estimate:

\[
|\varphi^-(t)| \leq Ke^{-\nu(t-a)}|\varphi(a)| \quad \text{for } t \leq a, \quad \text{if } X = C,
\]

\[
|\varphi^-(t)| + |\varphi^-(t)| \leq Ke^{-\nu(t-a)}(|\varphi(a)| + |\varphi(a)|) \quad \text{for } t \leq a, \quad \text{if } X = C^1,
\]
where $K$ is independent from $a$ and $\varphi$. Since $\Pi_a^-$ is a projector, in the last formulae we can take $\varphi^-$ for $\varphi$. Recalling that $\varphi^-$ satisfies the equation $\varphi^-(t) + b_0(t)\varphi^-(t) = 0$ and hence $|\varphi^-(t)| \leq ||b_0(t)|| \cdot |\varphi^-(t)|$ we arrive at the common estimate (38).

We recall that $C_{\nu}^-(-\infty, a]$ and $C_{\nu}^1(-\infty, a]$ consist of all solutions of the equation $\varphi + b_0\varphi = 0$. Hence these spaces coincide and does not depend on $\nu$. So we can take $a$ for $\nu$.

(v) We take some $\nu \in [\omega, \alpha]$. We prove that for $\varepsilon \geq 0$ small enough (not only for $\varepsilon = 0$) functions $\varphi$ from $C_{\nu}^1(-\infty, a]$ are determined completely by their values $\varphi(a)$ at the point $a$. And a similar assertion holds for the case of $X = C_{\nu}^1$.

Let us denote by $\pi_a^- : C_{\nu}(-\infty, a] \to E$ the projection

$$\pi_a^- \varphi = e^{-\nu a} \varphi(a)$$

(the factor $e^{-\nu a}$ makes the norm of $\pi_a^-$ bounded uniformly with respect to $a$).

As above we denote by $\hat{\cdot} \in C_{\nu}^1(-\infty, a]$ the subspace of all $\varphi \in C_{\nu}(-\infty, a]$ satisfying the condition $\varphi(a) = 0$. And we denote by $\pi_a^+ : C_{\nu}(-\infty, a] \to \hat{C}_{\nu}(-\infty, a]$ the projection

$$\pi_a^+ \varphi(t) = \varphi(t) - e^{\alpha(t-a)} \varphi(a),$$

for $t \leq a$. For any $\varepsilon \geq 0$ we consider the mapping

$$H_{\varepsilon} = \pi_a^+ \Pi_a^+ \oplus \pi_a^- \Pi_a^- : C_{\nu}(-\infty, a] \to \hat{C}_{\nu}(-\infty, a] \oplus E$$

that assigns to a function $\varphi \in C_{\nu}(-\infty, a]$ the pair $(\pi_a^+ \Pi_a^+ \varphi, \pi_a^- \Pi_a^- \varphi)$. (If $X = C^1$, we define $\pi_a^- : \hat{C}_{\nu}^1(-\infty, a] \to E$ and $\pi_a^+ : \hat{C}_{\nu}^1(-\infty, a] \to \hat{C}_{\nu}^1(-\infty, a]$, and

$$H_{\varepsilon} = \pi_a^+ \Pi_a^+ \oplus \pi_a^- \Pi_a^- : \hat{C}_{\nu}^1(-\infty, a] \to \hat{C}_{\nu}^1(-\infty, a] \oplus E$$

by the same formulae.)

We notice that $H_{\varepsilon}$ can be represented as the composition of the isomorphism

$$\Pi_a^+ \oplus \Pi_a^- : C_{\nu}(-\infty, a] \to C_{\nu}^1(-\infty, a] \oplus C_{\nu}^1(-\infty, a]$$

(39)

that assigns to a function $\varphi \in C_{\nu}(-\infty, a]$ the pair $(\varphi^+, \varphi^-)$, and the mapping

$$\pi_a^+ \oplus \pi_a^- : C_{\nu}^1(-\infty, a] \oplus C_{\nu}^1(-\infty, a] \to \hat{C}_{\nu}(-\infty, a] \oplus E$$

(40)
that assigns to a pair \((\varphi^+, \varphi^-)\) the pair \((\pi_a^+ \varphi^+, \pi_a^- \varphi^-)\).

The result of (iii) states that if \(\epsilon = 0\) then \(\pi_a^+: C_\nu^+(-\infty, a] \to C_\nu(-\infty, a]\)

is the identical operator and \(\pi_a^-: C_\nu^-(-\infty, a] \to \mathbb{E}\) is an isomorphism. Consequently, \(H_a\) is an isomorphism too, provided \(\epsilon = 0\). Further we show that the inverse \(H_a^{-1}\) (with \(\epsilon = 0\)) is bounded uniformly with respect to \(a \in \mathbb{R}\). Indeed, simple direct calculations show that (for \(\epsilon = 0\))

\[
H_a \varphi = (\varphi - \varphi^-, e^{-\nu a} \varphi(a)),
\]

where \(\varphi^-\) is the solution of the equation \(\dot{\varphi}^- + b_0 \varphi^- = 0\) satisfying the condition \(\varphi^-(a) = \varphi(a)\). Therefore

\[
H_a^{-1}(z, c) = z + \varphi^-,
\]

where \(\varphi^-\) is the solution of the equation \(\dot{\varphi}^- + b_0 \varphi^- = 0\) satisfying the condition \(\varphi^-(a) = e^{\nu a} \cdot c\). From estimate (38) it is clear that the norm of \(H_a^{-1}\) (with \(\epsilon = 0\)) is bounded uniformly with respect to \(a \in \mathbb{R}\).

From formula (36) it is seen that \(\Pi_a^\pm\) depend on \(\epsilon\) continuously in the norm (uniformly with respect to \(a\), i.e., \(\|\Pi_a^\pm(\epsilon_1) - \Pi_a^\pm(\epsilon_2)\|\) tends to 0 as \(\epsilon_1 \to \epsilon_2\) uniformly with respect to \(a\)). At the same time \(\pi_a^\pm\) do not depend on \(\epsilon\). Consequently, by proposition 1 the operator \(H_a\) remains to be invertible for \(\epsilon > 0\) small enough; moreover we can state that the norm of the inverse is bounded uniformly with respect to \(a\) and \(\epsilon\). If necessary, we change \(\epsilon_0 > 0\) so that \(H_a\) be invertible for all \(\epsilon \in [0, \epsilon_0]\).

Since \(\Pi_a^\pm \oplus \Pi_a^-\) is invertible for \(\epsilon \in [0, \epsilon_0]\), the factor (40) of \(H_a\) is invertible as well. Particularly, \(\pi_a^-: C_\nu^-(-\infty, a] \to \mathbb{E}\) is invertible. Thus for \(\epsilon > 0\) small enough the projection \(\pi_a^-\) establishes an isomorphism from \(C_\nu^-(-\infty, a]\) to \(\mathbb{E}\).

By its implication, \((\pi_a^-)^{-1}: \mathbb{E} \to C_\nu^-(-\infty, a]\) restores the initial function \(\varphi^-\) by the value \(e^{-\nu a} \varphi^-\). We notice that \(H_o^{-1}\) takes the pair \((0, e^{-\nu a} \varphi^-\)) to \(\varphi^-\), i.e., does essentially the same as \((\pi_a^-)^{-1}\). Therefore \(\|\pi^-\| \leq \|H_o^{-1}\|\).

Hence the norm of the inverse \((\pi_a^-)^{-1}: \mathbb{E} \to C_\nu^-(-\infty, a]\) is bounded uniformly with respect to \((\epsilon, a) \in [0, \epsilon_0] \times \mathbb{R}\). And a similar assertion is true for the case of \(X = C^1\).

(vi) Below we shall need a corollary of the last result. The uniform boundedness of the norms of the inverses to \(\pi_a^-: C_\nu^-(-\infty, a]\) \(\to \mathbb{E}\), i.e., the uniform continuity of the dependence \(e^{-\nu a} \varphi^-\) \(\to\) \(\varphi^-\) or equivalently \(\varphi^-\) \(\to\) \(e^{\nu a} \varphi^-\) in the norm of \(C_\nu^-(-\infty, a]\) means the estimate

\[
e^{-\nu(t-a)}|\varphi^-(t)| \leq K |\varphi^-(a)|, \quad t \leq a, \quad \text{if } X = C,
\]

\[
e^{-\nu(t-a)}(|\dot{\varphi}^-(t)| + |\varphi^-(t)|) \leq K |\varphi^-(a)|, \quad t \leq a, \quad \text{if } X = C^1,
\]
with $K$ independent from $a$ and $\varepsilon$, and $\varphi$. Changing $t$ by $a$ and vise versa, and $\varphi^-$ by $x^-$ we can rewrite this estimate as

$$
|x^-(t)| \geq M e^{\nu(t-a)|x^-(a)|}, \quad t \geq a, \quad \text{if } X = C,
$$

$$
|x^-(t)| \geq M e^{\nu(t-a)(|\dot{x}^-(a)| + |x^-(a)|)}, \quad t \geq a, \quad \text{if } X = C^1, \quad (41)
$$

which shows that $x^-$ grows as $t \to +\infty$ no slower then $t \mapsto e^{\nu t}$. (Since we can take any $\nu \in [\omega, \alpha]$, we can state that $|x^-(t)| \geq M e^{a(t-a)}|x^-(a)|$ for $t \geq a$; see the formulation of theorem 23.)

At the same time form assumption (iv_e) in the definition of exponential dichotomy we have

$$
|x^+(t)| \leq N e^{(\nu-\delta)(t-a)} \sup_{s \leq a} e^{-(s-a)}|\varphi(s)| \quad \text{if } X = C,
$$

$$
|\dot{x}^+(t)| + |x^+(t)| \leq N e^{(\nu-\delta)(t-a)} \sup_{s \leq a} e^{-(s-a)}(|\dot{\varphi}(s)| + |\varphi(s)|) \quad \text{if } X = C^1.
$$

Since $\delta > 0$, this estimate implies that $x^+$ decreases faster then $t \mapsto e^{\nu t}$ as $t \to +\infty$.

(vii) Now we show that the solution $x^-$ in the pairs $(C^1_\nu, C_\nu)$ and $(C_\nu, C^{-1}_\nu)$ (with the same value $x^-(a)$) is the same.

Let $c \in \mathbb{E}$ be given, and $x^-_1 \in C^{-1}_\nu(-\infty, a]$ be the function that satisfies the initial condition $x^-_1(a) = c$. Since $C^1_\nu \subset C_\nu$, and we have exponential dichotomy in the pair $(C_\nu, C^{-1}_\nu)$ too, we can represent $x^-_1$ in the form

$$
x^-_1 = x^+ + x^-\n$$

with $x^\pm$ corresponding to the direct sum $C^1_\nu(-\infty, a]\oplus C^{-1}_\nu(-\infty, a]$. According to (vi) $|x^+(t)|$ decreases as $t \to +\infty$ faster then $t \mapsto e^{\nu t}$. At the same time $|x^-_1(t)|$ and $|x^-(t)|$ grow as $t \to +\infty$ no slower then $t \mapsto e^{\nu t}$. Therefore $x^+ = 0$. This means that $x^-_1 = x^-.$

(viii) From (v) and (vii) we know that $x^-$ is a function of the class $C^1$, which does not vanish at any point $a$. Therefore we can define the coefficient $b_\varepsilon$ in equation (30) by the formula $b_\varepsilon(a)x^-(a) = -\dot{x}^-(a)$, where $a$ runs over $\mathbb{R}$ and $x^-(a)$ runs over $\mathbb{E}$. From (41) with $t = a$ it is seen that the dependence $x^-(a) \mapsto \dot{x}^-(a)$ is continuous uniformly with respect to $a$. Hence the linear operator $b_\varepsilon(a)$ is continuous uniformly with respect to $a$, i.e., $a \mapsto b_\varepsilon(a)$ is a bounded function.

(ix) We show that the function $a \mapsto b_\varepsilon(a)$ is continuous.

We fix $\varepsilon \geq 0$ and $\nu \in [\omega, \alpha]$, and consider the dependence of the operator

$$
L[\tau] = S_\tau L S_{-\tau},
$$
where
\[(S_\tau x)(t) = x(t - \tau),\]
from \(\tau \in \mathbb{R}\). It is straightforward to verify that
\[\mathcal{L}[\tau]x = \dot{x} + B[\tau]x,\]
where
\[(B[\tau]x)(t) = \sum_{m=1}^{\infty} b_m(t - \tau)x(t - \varepsilon h_m).\]

From assumption (ii), see the beginning of Section 6, it is evident that the operator \(\mathcal{L}[\tau]\) depends continuously on the parameter \(\tau\). By corollary 2 the operator \((\mathcal{L}[\tau])^{-1}\) depends on \(\tau\) in a continuous way too.

We notice the rule: when one passes from \(\mathcal{L}\) to \(\mathcal{L}[\tau]\) the argument \(t\) should be replaced by \(t - \tau\). Particularly, the coefficient \(b_\varepsilon[\tau]\) that corresponds to \(\mathcal{L}[\tau]\) is \(b_\varepsilon[\tau](a) = b_\varepsilon(a - \tau)\), where \(b_\varepsilon\) corresponds to \(\mathcal{L}\), see (viii).

From formula (36) it is seen that the projectors \(\Pi^\pm_a\) depend on \(\tau\) in a continuous way. Clearly, the operators \(\pi^\pm_a\) considered on the whole \(C^1_{\nu}(-\infty, a]\) do not depend on \(\tau\). We also recall that the operator \(H_a\) can be represented as the composition of (39) and (40). Therefore \(H_a\) depends on \(\tau\) in a continuous way as well.

We recall from the last paragraph of (v) that \((\pi_a^-)^{-1}\) is essentially the restriction of \((H_a)^{-1}\) to \(\{0\} \oplus \mathbb{E}\). Hence \((\pi_a^-)^{-1}\) depends on \(\tau\) continuously as well. But \((\pi_a^-)^{-1}\) is the dependence \(\varphi^-(a) \mapsto e^{\nu a} \varphi^-\) from \(\mathbb{E}\) to \(C^1_{\nu}(-\infty, a]\). Particularly, this means that the operator \(\varphi^-(a) \mapsto \varphi^-(a)\) depend continuously from \(\tau\). But this operator is just the dependence \(\tau \mapsto b_\varepsilon(a - \tau)\).

\[\square\]

REFERENCES


TWO-POINT BOUNDARY VALUE PROBLEMS WITH MONOTONICALLY BOUNDARY CONDITIONS FOR ONE-DIMENSIONAL \( \varphi \)-LAPLACIAN EQUATIONS

A.J.A. LEPIN, L.A. LEPIN AND F. zh. SADYRBAEV

Abstract. The results of laborious work on detecting monotonicity types (classes) of the boundary conditions in the form

\[
H_1(x(a), x(b), x'(a), x'(b)) = h_1,
\]

\[
H_2(x(a), x(b), x'(a), x'(b)) = h_2,
\]

for which the respective boundary value problem for one-dimensional \( \varphi \)-laplacian equation \((\varphi(t, x, x'))' = f(t, x, x')\) is solvable in the region between given the upper and lower functions \( \beta \) and \( \alpha \), are presented. The carried out analysis is complete in the sense that for any possible variant either the respective theorem is proved or the contradictory example is constructed.

AMS(MOS) subject classification. 34 B 15

Key Words. One-dimensional \( \varphi \)-Laplacian, nonlinear boundary conditions, upper and lower functions, monotonicity classes.

1. Introduction. We study boundary value problems for equation

\[
(\varphi(t, x, x'))' = f(t, x, x'), \quad t \in I := [a, b],
\]

where a function \( \varphi \in C(I \times R^2, R) \) strictly increases in \( x' \) and \( f \) satisfies the Carathéodory conditions, that is, (i) \( f(\cdot, x, y) \) is measurable in \( I \) for \((x, y) \in R^2 \) fixed; (ii) \( f(t, \cdot, \cdot) \) is continuous in \( R^2 \) for a.e. \( t \in I \); (iii) for any compact set \( P \subset R^2 \) there exists a function \( g \in L_1(I, R) \) such that \( |f(t, x, y)| \leq g(t) \) holds in \( I \times P \).
DEFINITION 1. A function \( x \in C^1(I, \mathbb{R}) \) is a solution of the equation (1), if \( \varphi(t, x(t), x'(t)) : I \to \mathbb{R} \) is an absolutely continuous function and equation (1) is satisfied a.e. in \( I \).

A set of solutions \( x : I \to \mathbb{R} \) of (1) is denoted by \( S(I, \mathbb{R}) \).

We will focus on the boundary value problem

(2) \[(\varphi(t, x, x'))' = f(t, x, x'),\]

(3) \[H_1(x(a), x(b), x'(a), x'(b)) =: H_1 x = h_1,\]

(4) \[H_2(x(a), x(b), x'(a), x'(b)) =: H_2 x = h_2,\]

(5) \[\alpha \leq x \leq \beta, \quad U,\]

where \( H_1, H_2 \in C(R^4, \mathbb{R}), \ h_1, h_2 \in \mathbb{R}, \) \( \alpha, \beta : I \to \mathbb{R} \) are properly ordered (\( \alpha \leq \beta \)) the lower and upper functions. By \( U \) is meant a subset of the set of the following conditions:

1. \( \alpha(a) = \beta(a) ; \)
2. \( \alpha'(a) < \beta'(a) ; \)
3. \( \alpha'(a) = \beta'(a) ; \)
4. \( \alpha'(a) > \beta'(a) ; \)
5. \( \alpha(b) = \beta(b) ; \)
6. \( \alpha'(b) < \beta'(b) ; \)
7. \( \alpha'(b) = \beta'(b) ; \)
8. \( \alpha'(b) > \beta'(b) ; \)
9. \( (\forall x, y \in S(I, \mathbb{R}))((x \leq y \land x'(a) \leq y'(a) \Rightarrow x'(b) \leq y'(b)) \land (x \leq y \land x'(b) \geq y'(b) \Rightarrow x'(a) \geq y'(a)))) ; \)

A. \( \alpha \in S(I, \mathbb{R}) ; \)
B. \( \beta \in S(I, \mathbb{R}) ; \)
C. \( H_1 \alpha = H_1 \beta ; \)
D. \( H_2 \alpha = H_2 \beta . \)

Functions \( H_1 \) and \( H_2 \) are supposed to belong to certain monotonicity classes, which are defined below.

DEFINITION 2. We denote the type of monotonicity of a function \( H \in C(R^4, \mathbb{R}) \) by a quadruple \( (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \), where \( \sigma_i \in \{0, -, +, 1\}, \quad i = 1, 2, 3, 4. \)

Meaning of the symbols follows:

\( \sigma_i = 0 \) \( -H \) is independent of the \( i \)-th argument;
\( \sigma_i = "-" \) \( -H \) does not increase in the \( i \)-th argument;
\( \sigma_i = "+" \) \( -H \) does not decrease in the \( i \)-th argument;
\( \sigma_i = 1 \) \( -H \) does not restrict on the behavior of \( H \) with respect to the \( i \)-th argument.
By $M(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is denoted a class of functions $H$ with the type of monotonicity $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.

Our approach is based on considering the Dirichlet problems

$$
(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) = h_1, \quad x(b) = h_2, \quad \alpha_1 \leq x \leq \beta_1,
$$

where $\alpha_1$ and $\beta_1$ are the lower and upper functions respectively.

We will say that the condition $E$ is satisfied for given $\alpha$ and $\beta$ if the Dirichlet problems (6) are solvable for any $\alpha_1$, $\beta_1$, $h_1$, $h_2$ such that $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$ and $h_1 \in [\alpha_1(a), \beta_1(a)]$, $h_2 \in [\alpha_1(b), \beta_1(b)]$, and the set of solutions is compact.

The condition $E$ holds, for example, if: 1) $\alpha, \beta \in \text{Lip}(I, R)$ and for any $t_1 \in (a, b)$, $t_2 \in (t_1, b)$ the existence of derivatives involved implies the inequalities

$$
\varphi(t_2, \alpha(t_2), \alpha'(t_2)) - \varphi(t_1, \alpha(t_1), \alpha'(t_1)) \geq \int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))ds,
$$

and 2) the so called Schrader condition ([7]) fulfills in the region between $\alpha$ and $\beta$.

The analogue of the Schrader condition for one-dimensional $\varphi$-Laplacian equation (6) holds if some Nagumo type condition for (6) fulfills. For various versions of the Nagumo type condition for one-dimensional $\varphi$-Laplacian equations one may consult the works [1] – [3].

It can be proved that for $\alpha$ and $\beta$ defined above there exist $\alpha'(a)$, $\beta'(a)$, $\alpha'(b)$ and $\beta'(b)$. The definition of $\alpha$ and $\beta$ above is quite general and contains the classical definition of upper and lower functions in the case of $\varphi \equiv x'$ (the second order equation $x'' = f(t, x, x')$) and various generalizations. For alternative definitions of upper and lower functions one may consult the works [1] – [4]. It should be mentioned that definition of $\alpha$ and $\beta$ as given in [3] is equivalent to our definition above as shown in [6].

2. **Statement of the problem.** The Dirichlet problem

$$
(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) = h_1, \quad x(b) = h_2, \quad \alpha \leq x \leq \beta
$$

is a particular case of the problem (2)-(5) for $H_1 x = x(a)$, $H_2 x = x(b)$ and $U = \emptyset$. The conditions $h_1 \in [\alpha(a), \beta(a)]$ and $h_2 \in [\alpha(b), \beta(b)]$ are necessary
for solvability of the Dirichlet problem. They write in terms of the functions $H_1$ and $H_2$ as $h_1 \in [H_1, H_1, \alpha]$ and $h_2 \in [H_2, H_2, \beta]$. These conditions are natural also for the problem (2) - (5). It is reasonable to require that $H_1, \alpha \leq H_1, \beta$ and $H_2, \alpha \leq H_2, \beta$. If this is not the case, the respective boundary condition can be multiplied by $-1$.

Let $M_1$ and $M_2$ be the monotonicity classes. Our goal in the sequel is to clarify which $M_1, M_2$ and $U$ make the assertion below true.

**Proposition I.** For any $H_1 \in M_1$, $H_2 \in M_2$ the conditions $H_1 \alpha \leq H_1, \beta$, $H_2 \alpha \leq H_2, \beta$, $U$ and $E$ together imply the existence of a solution to the problem (2) - (5) for any $h_1 \in [H_1, \alpha, H_1, \beta]$ and $h_2 \in [H_2, \alpha, H_2, \beta]$.

Proposition I is valid, for instance, for $M_1 = M(1, 0, 0, 0)$, $M_2 = M(0, 1, 0, 0)$ and $U = 0$, as follows from solvability of the Dirichlet problem. On the other hand, the problem (2) - (5) does not have a solution for $H_1 x = x(a) = h_1$ and $H_2 x = x(a) = h_2$ if $\alpha(a) < \beta(a)$, where $h_1 = \alpha(a)$ and $h_2 = \beta(a)$. Therefore Proposition I is false for $M_1 = M(1, 0, 0, 0)$, $M_2 = M(1, 0, 0, 0)$ and $U = 0$.

We will call the collection $(M_1, M_2, U)$ by a **theorem**, if its substitution in Proposition I makes it true. If the collection is not a theorem in the sense above, a contradicting example can be constructed. We will call such a collection by an **example**. Let $TE$ stand for the set of all collections $(M_1, M_2, U)$, $\forall T$ be a set of all theorems, and $\forall E$ be a set of all examples. Let us make $TE$ partially ordered by introducing order as follows: $(M_1, M_2, U_1) \leq (M_3, M_4, U_2)$, if $M_1 \subset M_3$, $M_2 \subset M_4$ and $U_2 \subset U_1$. Let $T_{\max}$ be a set of maximal elements of $\forall T$, and $E_{\min}$ be a set of minimal elements of $\forall E$. It follows from $t_1 \in \forall T$, $t_2 \in TE$ and $t_2 \leq t_1$ that $t_2$ is a particular case of theorem $t_1$. Hence $t_2 \in \forall T$. Similarly $e_1 \in \forall E$, $e_2 \in TE$ and $e_1 \leq e_2$ imply $e_2 \in \forall E$. Therefore $T_{\max}$ and $E_{\min}$ completely define the sets $\forall T$ and $\forall E$ respectively.

The number of elements in $TE$ is approximately $5 \cdot 10^8$. It was shown previously in the book [5] for the case of $\varphi(t, x, y) \equiv y$ that $T_{\max}$ and $E_{\min}$ consist of 931 and 2554 elements respectively. The problem (2) - (5) and the conditions 1 - D have symmetries, which can be used in order to diminish the number of theorems to be proved. In this way a set $T_g$ of generated theorems was described, which consists of 106 elements. Similarly a set $E_g$ of generated examples was extracted consisting of 315 examples. The number of theorems needed proofs still can be reduced. Generated theorems, containing the conditions 2 or 4 instead of 3 (as well as theorems containing the condition 7 instead of 6), can be proved similarly. Taking this into account one obtains a set $T_b$ of base theorems. A set $T_b$ consists of 95 elements. Thus in order
to prove all theorems in $\forall T$ it suffices to prove theorems in $Tb$. To be sure that no theorem is missed it suffices to construct all examples in $E_g$. In the case of equation $x'' = f(t, x, x')$ (where $\varphi(t, x, y) \equiv y$) this work was carried out in the book [5].

3. Base theorems. The collection $(M_1, M_2, U)$, where $M_1 = M(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and $M_2 = M(\sigma_5, \sigma_6, \sigma_7, \sigma_8)$, completely determines a theorem or an example. We will use the following notation for base theorems:

$$Tbn. \sigma_1\sigma_2\sigma_3\sigma_4.\sigma_5\sigma_6\sigma_7\sigma_8. u_1u_2u_3u_4u_5u_6u_7u_8u_9u_Au_Bu_Cu_D.\text{comm}$$

where Tbn means the number n of base theorem, $u_i = i$, if the $i$-th condition is included in $U$ and $u_i$ is void otherwise. The identifier "comm" stands for a group of possible comments, which contain additional information on the theorems. The notation used in the comments is described after the list of base theorems, which follows.

List of base theorems

- Tb01. 1 1 - +. - - 0 0. - -. NS
- Tb02. 1 - - 0. - 1 0 +. - -. NN. m
- Tb03. 1 1 - +. 1 - - 0. 1. + -. NS. m
- Tb04. - 1 - +. - - - 0. 2. + -. NS
- Tb05. - 1 1 +. - - - 1 0. 3. + -. NS
- Tb06. - 1 1 +. - - - + 0. 4. + -. NS
- Tb07. - 1 + +. - - - 1 0. 4. + -. NN
- Tb08. 1 1 1 +. 1 - - 0 1. 1 3. + -. NS. m
- Tb09. 1 1 - +. 1 1 - +. 1 5. + +. SS. m
- Tb10. 1 - - 1. 1 - - -. 1 6. + +. NS
- Tb11. 1 - - 1. 1 - - 1. 1 7. + +. SS
- Tb12. 1 - - +. 1 - - +. 1 8. + +. SS. m
- Tb13. - - - 1. - - - -. 2 6. + +. NS
- Tb14. - - - 1. - - - 1. 2 7. + +. SS
- Tb15. - - - +. - - - +. 2 8. + +. SS
- Tb16. - - 1 1. - - - 1. 3 6. + +. NS
- Tb17. - - 1 1. - - - 1. 3 7. + +. SS
- Tb18. - - 1 1. - - - +. 4 6. + +. NS
- Tb19. - - 1 -. - - + 1. 4 6. + +. NN
- Tb20. 1 1 1 +. 1 1 1 +. 1 3 5. + +. SS. m
- Tb21. 1 - 1 1 1 - 1 -. 1 3 6. + +. NS
- Tb22. 1 - 1 1. 1 - 1 1. 1 3 7. + +. SS
- Tb23. 1 - 1 +. 1 - 1 +. 1 3 8. + +. SS. m
Tb24. 1 1 1 1. 1 1 1 1. 1 3 5 7. ++. SS. m
Tb25. 1 1 1 1. -- 0 0. 9 A B. ++. NS
Tb26. 1 -- -. -- 1. 9 A B. --. NN. m
Tb27. 1 1 1 1. 1 -- -. 1 9 A B. ++. NS. m
Tb28. 1 1 1 1. -- -. 2 9 A B. ++. NS
Tb29. 1 -- -. -- 1 --. 2 9 A B. ++. NN
Tb30. 1 -- -. -- + 1. 2 v 3 9 A B. --. NN
Tb31. 1 1 1 1. -- 1 1. 3 9 A B. ++. NS
Tb32. 1 1 1 1. -- + 0. 4 9 A B. ++. NS
Tb33. -- 1 ++. -- 1 --. 4 9 A B. ++. NS
Tb34. 1 1 1 1. 1 -- 1. 1 3 9 A B. ++. NS. m
Tb35. 1 1 1 1. 1 1 1 1. 1 5 9 A B. ++. SS. m
Tb36. 1 1 1 1. 1 -- 1. 1 7 9 A B. ++. NS. m
Tb37. 1 1 1 1. -- 1 1. 3 7 9 A B. ++. NS
Tb38. 1 1 1 1. -- --. 4 6 9 A B. ++. NS
Tb39. 1 + 0 ++. -- 0 ++. C. --. N
Tb40. + -- 0. -- 1 ++. C. --. N
Tb41. ++ 0 0. 1 1 ++. C. --. N
Tb42. 1 + 0 ++. 1 -- 0 1. C. ++. N
Tb43. 0 + 1 ++. -- 1 1. 3 C. ++. N
Tb44. ++ --. -- 0 ++. 3 4 3 C. --. N
Tb45. + -- -. -- 0 ++. 3 4 3 C. --. N
Tb46. 0 + 1 ++. -- 0 ++. 4 C. ++. N
Tb47. 0 ++ ++. -- 1 ++. 4 C. ++. N
Tb48. 0 ++ ++. -- 1 1 ++. C. ++. N
Tb49. 1 + 1 ++. 1 -- 1 1. 1 3 C. ++. N
Tb50. + -- -. -- 0 ++. 3 4 6 V 7 C. --. S
Tb51. 0 0 ++. -- 1 1. 4 6 C. ++. N
Tb52. 1 1 1 1. -- 0 0. A C. ++. S.*
Tb53. 1 ++ ++. -- 0 ++. A C. --. N
Tb54. + -- -. -- 0 ++. A C. --. N
Tb55. 1 1 1 1. 1 -- 0 1. A C. ++. S.*
Tb56. 1 ++ ++. 1 -- 0 1. A C. ++. N
Tb57. 1 1 1 1. -- 0 0 2 A C. ++. S.*
Tb58. + ++ ++. -- 0 2 2 A C. ++. N
Tb59. 1 1 1 1. -- 1 0 3 A C. ++. S.*
Tb60. ++ 1 ++. -- 1 ++. 3 A C. ++. N
Tb61. 1 1 1 1. -- 0 0 4 A C. ++. S.*
Tb62. ++ 1 ++. -- 0 0 4 A C. ++. N
BVP FOR $\varphi$-LAPLACIAN EQUATIONS

Tb63. 1 1 1 1. 1 - 1 0. 1 3 A C. ++. S.*
Tb64. 1 1 1 1. 1 1 - + . 1 5 A C. ++. S.*
Tb65. 1 1 1 1. 1 -- . 1 6 A C. ++. S.*
Tb66. 1 1 1 1. 1 - 1. 1 7 A C. ++. S.*
Tb67. 1 1 1 1. 1 -- +. 1 8 A C. ++. S.*
Tb68. 1 1 1 1. --- -. 2 6 A C. ++. S.*
Tb69. 1 1 1 1. --- 1. 2 7 A C. ++. S.*
Tb70. 1 1 1 1. --- + . 2 8 A C. ++. S.*
Tb71. 1 1 1 1. -- 1 -. 3 6 A C. ++. S.*
Tb72. 1 1 1 1. -- 1 1. 3 7 A C. ++. S.*
Tb73. 1 1 1 1. -- + 1. 4 6 A C. ++. S.*
Tb74. 1 1 1 1. 1 1 1 + . 1 3 5 A C. ++. S.*
Tb75. 1 1 1 1. 1 - 1 -. 1 3 6 A C. ++. S.*
Tb76. 1 1 1 1. 1 - 1 1. 1 3 7 A C. ++. S.*
Tb77. 1 1 1 1. 1 - 1 +. 1 3 8 A C. ++. S.*
Tb78. 1 0 0 0. - 1 + +. 3 V 4 9 A C. --. N
Tb79. 1 0 0 0. -- + 1. 3 V 4 6 9 A C. --. N
Tb80. + -- -- . 1 1 1 1. 9 A B C. --. N
Tb81. 1 1 0 0. 1 1 0 0. C D. --. m
Tb82. + 1 - 0. 0 1 0 0. 3 V 4 C D. --
Tb83. 0 1 1 0. 0 1 1 0. 3 V 4 C D. + -
Tb84. 1 1 1 0. 1 1 1 0. 1 3 C D. + -. m
Tb85. + 0 - 1. 0 0 0 1. 3 V 4 6 C D. --
Tb86. 0 0 1 1. 0 0 1 1. 3 V 4 6 C D. --
Tb87. 1 0 1 1. 1 0 1 1. 1 3 6 C D. --
Tb88. 1 1 1 1. 1 1 1 1. A C D. + +.*
Tb89. 1 1 1 1. 1 1 1 1. 1 4
Tb90. 1 1 1 1. 1 1 1 1. 2 3
Tb91. 1 1 1 1. 1 1 1 1. 2 4
Tb92. 1 1 1 1. 1 1 1 1. 3 4
Tb93. 1 1 1 1. 1 1 1 1. 1 8 9 A B
Tb94. 1 1 1 1. 1 1 1 1. 2 7 9 A B
Tb95. 1 1 1 1. 1 1 1 1. 2 8 9 A B

Conventions and comments on the notation

In the theorem Tb30 the condition $2V3$ corresponds to $\alpha'(a) \leq \beta'(a)$, and in the theorems Tb44, Tb45, Tb50, Tb78, Tb79, Tb82, Tb83, Tb85 and Tb86 the condition $3V4$ corresponds to $\alpha'(a) \geq \beta'(a)$. In the theorem Tb50
the condition $6V7$ corresponds to $\alpha'(b) \leq \beta'(b)$. It appears that a solution of the problem (2) - (5) can satisfy the inequalities

$$
\min\{\alpha'(a), \beta'(a)\} \leq x'(a) \leq \max\{\alpha'(a), \beta'(a)\},
$$

$$
\min\{\alpha'(b), \beta'(b)\} \leq x'(b) \leq \max\{\alpha'(b), \beta'(b)\}.
$$

If both inequalities hold, then the double plus sign "++" follows the identifier for $U$; the double signs "+-" or "-+" indicate that the inequality (7) or the inequality (8) hold respectively; the double sign "--" shows that both conditions (7) - (8) can be violated. The boundary conditions $H_1x = h_1$ and $H_2x = h_2$ in theorems Tb01 - Tb38 can degenerate. We will say that the boundary condition $H_1x = h_1$ degenerates if it follows from the conditions of a theorem that $H_1\alpha = H_1\beta$. Similarly the boundary condition $H_2x = h_2$ is said to be degenerate if the conditions of a theorem imply that $H_2\alpha = H_2\beta$. For example both boundary conditions of the theorem Tb09 are degenerated. This is denoted by identifier SS. In the theorem Tb01 the second boundary condition is degenerated. This is denoted by identifier NS. There is not degeneration in the theorem Tb02. This is denoted by NN. The more interesting theorems are those non-degenerated. It makes not sense to speak on non-degeneracy of the first boundary condition in the theorems Tb39-Tb80. Thus the degeneracy identifier refers only to the second boundary condition. For instance the second boundary condition is degenerated in the theorem Tb50. The non-degeneracy makes not sense for the theorems Tb81 - Tb95. In the theorems Tb52, Tb55, Tb57, Tb59, Tb61, Tb63-Tb77 and Tb88 the lower function $\alpha$ is a solution to the boundary value problem. This is denoted by asterisk *. There exist maximal and minimal solutions for some theorems. This is denoted by the symbol "m".

For base theorem Tbn by TbnC will be denoted a theorem which is obtained from Tbn by adding the condition C and replacing $H_1$ with $-H_1$. If theorem Tbn is proved then the respective TbnC is true also. Similarly by TbnD will be denoted a theorem which is obtained from Tbn by adding the condition D and replacing $H_2$ with $-H_2$, by TbnH will be denoted a theorem, which is obtained from Tbn, if $H_1$ and $H_2$ are interchanged, by TbnÎ is denoted a theorem, obtained from Tbn by the variable change $t$ to $-t$. Analogously Tbnx stands for a theorem, obtained from Tbn by the dependent variable $x$ change to $-x$. These theorems show what kind symmetries were used to get $T_9$.

Let the following theorem be true. The symbol Id stands for the theorem identifier, for example, Tb02C.
Theorem 1. For any $H_1 \in M_1$, $H_2 \in M_2$, $h_1 \in [H_1 \alpha, H_1 \beta]$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ the conditions $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$ and $U$, $E$ imply the existence of a solution to the boundary value problem (2)-(5).

Then the following theorems are true also.

Theorem 1.I. For any $H_1 \in M_1$, $H_2 \in M_2$ and $h_1 \in [H_1 \alpha, H_1 \beta]$ there exists $h_2 \in [H_2 \alpha, H_2 \beta]$ such that $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$, and the conditions $U \setminus \{H_2 \alpha = H_2 \beta\}$ and $E$ imply the existence of a solution to the boundary value problem (2)-(5).

Theorem 1.II. For any $H_1 \in M_1$, $H_2 \in M_2$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ there exists $h_1 \in [H_1 \alpha, H_1 \beta]$ such that $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$, and the conditions $U \setminus \{H_1 \alpha = H_1 \beta\}$ and $E$ imply the existence of a solution to the boundary value problem (2)-(5).

Theorem 1.III. For any $H_1 \in M_1$ and $H_2 \in M_2$ there exist $h_1 \in [H_1 \alpha, H_1 \beta]$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ such that $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$, and the conditions $U \setminus \{H_1 \alpha = H_1 \beta\}$ and $E$ imply the existence of a solution to the boundary value problem (2)-(5).

Indeed, let $\varphi_i(t) = t - H_i \alpha$ for $t < H_i \alpha$, $\varphi_i(t) = 0$ for $H_i \alpha \leq t \leq H_i \beta$ and $\varphi_i(t) = t - H_i \beta$ for $H_i \beta < t$, $i = 1, 2$. Replacing $H_2$ with $\varphi_2(H_2)$, one obtains that Theorem 1.I is true. Replacing $H_1$ with $\varphi_1(H_1)$ one obtains Theorem 1.II. Replacing $H_1$ with $\varphi_1(H_1)$ and $H_2$ with $\varphi_2(H_2)$, one obtains Theorem 1.III.

Let us mention that for certain functions $\alpha$ and $\beta$ at least two of the conditions 1-8 hold. This is the evidence that they are natural. The conditions A and B occur often. They will be used in the proofs of base theorems. It is interesting to mention, that adding the conditions A and B to the conditions 1-8 do not yield new theorems. In contrast, adding the condition 9 to the conditions A and B yield new theorems. The condition C is employed when studying the periodic problem.

Let us show how well-known results for the most intensively studied boundary value problems follow from the base theorems above. Solvability of the Dirichlet problem

$$(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) = h_1, \quad x(b) = h_2, \quad \alpha \leq x \leq \beta,$$

where $\alpha(a) \leq h_1 \leq \beta(a)$, $\alpha(b) \leq h_2 \leq \beta(b)$, follows from the theorem Tb02.

Solvability of the Neumann problem

$$(\varphi(t, x, x'))' = f(t, x, x'), \quad -x'(a) = -h_1, \quad x'(b) = h_2, \quad \alpha \leq x \leq \beta,$$
where \( \beta'(a) \leq h_1 \leq \alpha'(a), \alpha'(b) \leq h_2 \leq \beta'(b) \), follows from the theorem Tb02. \( 1 - 0 - 1 0 + \) and the conditions \( \alpha'(a) \geq \beta'(a) \) and \( \alpha'(b) \leq \beta'(b) \), which follow from the conditions \( H_1 \alpha \leq H_1 \beta \) and \( H_2 \alpha \leq H_2 \beta \).

Solvability of the periodic problem

\[
(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) - x(b) = 0, \quad x'(a) - x'(b) = 0, \quad \alpha \leq x \leq \beta,
\]

where \( \alpha(a) - \alpha(b) \leq 0 \leq \beta(a) - \beta(b) \) and \( \alpha'(b) - \alpha'(a) \leq 0 \leq \beta'(b) - \beta'(a) \), follows from the theorem Tb40. \( + - 0 - . - 1 + . C \) or from the theorem Tb41. \( + - 0 0 \). \( 1 \ 1 - + . C \) and the conditions \( \alpha(a) - \alpha(b) = \beta(a) - \beta(b) = 0 \), \( \alpha'(b) - \alpha'(a) \leq 0 \leq \beta'(b) - \beta'(a) \). The equality \( \alpha(a) - \alpha(b) = \beta(a) - \beta(b) \) follows from the condition C.

4. Proofs of base theorems. Any base theorem for the problem (2)-(5) can be proved using the approach elaborated in [5] for the case of \( \varphi(t, x, x') \equiv x' \). The only exception is the theorem Tb50. We demonstrate this approach by proving the theorems Tb01-Tb09. For better understanding we formulate the first theorem in extended form and then in the compact one. All further formulations are compact using the above notation and conventions.

**Theorem Tb01** (extended formulation). The boundary value problem

\[
(\varphi(t, x, x'))' = f(t, x, x'),
\]

\[
H_1(x(a), x(b), x'(a), x'(b)) = h_1, \quad H_2(x(a), x(b)) = h_2, \quad \alpha \leq x \leq \beta,
\]

where \( \alpha \) and \( \beta \) are the lower and upper functions, is solvable for any \( h_1 \in [H_1 \alpha, H_1 \beta], h_2 \in [H_2 \alpha, H_2 \beta] \) if:
1) \( H_1 \alpha \leq H_1 \beta, \quad H_2 \alpha \leq H_2 \beta \);  
2) the condition E fulfills;  
3) the function \( H_1 \) does not increase in the third argument and does not decrease in the fourth one;  
4) the function \( H_2 \) does not increase in both arguments.

**Theorem Tb01** (compact formulation). \( 1 \ 1 - +. \ - - 0 0 . \ - - \).

For any \( H_1 \in M(1, 1, - , +), H_2 \in M(-, - , 0, 0, ), h_1 \in [H_1 \alpha, H_1 \beta] \) and \( h_2 \in [H_2 \alpha, H_2 \beta] \) the conditions \( H_1 \alpha \leq H_1 \beta, H_2 \alpha \leq H_2 \beta \) and \( E \) imply the existence of a solution to the boundary value problem (2) - (5).

**Proof.** Define sequences \( \alpha_k, \beta_k, k = 1, 2, ... \) by induction. Let \( \alpha_1 = \alpha, \beta_1 = \beta \). By \( y \) denote a solution (it exists due the condition E) of the Dirichlet problem

\[
(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) = (\alpha_k(a) + \beta_k(a))/2,
\]
$$x(b) = (\alpha_k(b) + \beta_k(b))/2, \quad \alpha_k \leq x \leq \beta_k.$$  

If $H_1 y \leq h_1$, then set $\alpha_{k+1} = y$ and $\beta_{k+1} = \beta_k$. If $H_1 y > h_1$, then set $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = y$. Evidently, $\alpha_k$ and $\beta_k$ are the lower and upper functions respectively, $\alpha = \alpha_1 \leq \alpha_2 \leq \ldots \leq \beta_2 \leq \beta_1 = \beta$ and $H_1 \alpha_k \leq h_1 \leq H_1 \beta_k$. The inequalities $H_2 \alpha \geq H_2 \alpha_k \geq H_2 \beta_k \geq H_2 \beta \geq H_2 \alpha$ follow from $H_2 \beta \geq H_2 \alpha$ and the monotonicity conditions for $H_2$. Hence $H_2 \alpha_k = h_2 = H_2 \beta_k$. The limits $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$ satisfy the conditions $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta$, $\alpha_0(a) = \beta_0(a)$, $\alpha_0(b) = \beta_0(b)$, $H_1 \alpha_0 \leq h_1 \leq H_1 \beta_0$, $H_2 \alpha_0 = h_2 = H_2 \beta_0$, $\alpha_0 \in S(I, R)$ or $\beta_0 \in S(I, R)$. The monotonicity conditions for $H_1$ imply $H_1 \alpha_0 \geq H_1 \beta_0$. Therefore, $H_1 \alpha_0 = h_1 = H_1 \beta_0$. It follows that either $\alpha_0$ or $\beta_0$ solve the problem (2)-(5). □

**Theorem Tb02.** 1) -- 0. -- 0 +. --. For any $H_1 \in M(1,-,-,0)$, $H_2 \in M(-,1,0,+)$, $h_1 \in [H_1 \alpha, H_1 \beta]$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ the conditions $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$ and $E$ imply the existence of a solution to the boundary value problem (2) - (5).

**Proof.** First prove that for any $B \in [\alpha(b), \beta(b)]$ there exists a solution to the boundary value problem

$$\begin{align*}
(\varphi(t, z, z'))' &= f(t, z, z'), \\
H_1 z &= h_1, \\
z(b) &= B, \\
\alpha \leq z \leq \beta.
\end{align*}$$

Let $\alpha_*$ and $\beta_*$ be the respective solutions of the Dirichlet problems

$$\begin{align*}
(\varphi(t, \alpha_*, \alpha_*'))' &= f(t, \alpha_*, \alpha_*'), \\
\alpha_*(a) &= \alpha(a), \\
\alpha_*(b) &= B, \\
\alpha \leq \alpha_* \leq \beta,
\end{align*}$$

$$\begin{align*}
(\varphi(t, \beta_*, \beta_*'))' &= f(t, \beta_*, \beta_*'), \\
\beta_*(a) &= \beta(a), \\
\beta_*(b) &= B, \\
\alpha_* \leq \beta_* \leq \beta.
\end{align*}$$

Then $\alpha \leq \alpha_* \leq \beta_* \leq \beta$ and the monotonicity conditions for $H_1$ imply the inequalities $H_1 \alpha_* \leq H_1 \alpha \leq h_1 \leq H_1 \beta \leq H_1 \beta_*$. It follows from Theorem Tb01 that there exists a solution $z$ of the boundary value problem

$$\begin{align*}
(\varphi(t, z, z'))' &= f(t, z, z'), \\
H_1 z &= h_1, \\
\alpha_* \leq z \leq \beta_*.
\end{align*}$$

It is clear that $z(b) = B$ and $\alpha \leq z \leq \beta$. Therefore $z$ solves the BVP (9).

Define sequences $\alpha_k, \beta_k, k = 1, 2, \ldots$ by induction. Let $\alpha_1 = \alpha$, $\beta_1 = \beta$. It follows from the above arguments that there exists a solution $y$ of the problem

$$\begin{align*}
(\varphi(t, y, y'))' &= f(t, y, y'), \\
H_1 y &= h_1, \\
y(b) &= (\alpha_k(b) + \beta_k(b))/2, \\
\alpha_k \leq y \leq \beta_k.
\end{align*}$$

If $H_2 y \leq h_2$, then set $\alpha_{k+1} = y$ and $\beta_{k+1} = \beta_k$. If $H_2 y > h_2$, then set $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = y$. It is clear that $\alpha_k$ and $\beta_k$ are the lower and
upper functions respectively, \( \alpha = \alpha_1 \leq \alpha_2 \leq \ldots \leq \beta_2 \leq \beta_1 = \beta \) and 
\( H_2 \alpha_k \leq h_2 \leq H_2 \beta_k \). The limits \( \alpha_k \to \alpha_0 \) and \( \beta_k \to \beta_0 \) satisfy the conditions 
\( \alpha' \leq \alpha_0 \leq \beta_0 \leq \beta, \alpha_0(b) = \beta_0(b), H_2 \alpha_0 \leq h_2 \leq H_2 \beta_0, \alpha_0 \in S(I, R) \wedge H_1 \alpha_0 = h_1 \) 
or \( \beta_0 \in S(I, R) \wedge H_1 \beta_0 = h_1 \). We derive from the monotonicity conditions 
for \( H_2 \) that \( H_2 \alpha_0 \geq H_2 \beta_0 \). Hence \( H_2 \alpha_0 = h_2 = H_2 \beta_0 \). Therefore either \( \alpha_0 \) or \( \beta_0 \) solves the BVP (2) - (5).

**Theorem Tb03.** \( 1 \ 1 - +. \ 1 - 0. \ 1 + - \). For any \( H_1 \in M(1, 1, -,+), \)
\( H_2 \in M(1, -,-,0), h_1 \in [H_1 \alpha, H_1 \beta] \) and \( h_2 \in [H_2 \alpha, H_2 \beta] \) the conditions 
\( H_1 \alpha \leq H_1 \beta, H_2 \alpha \leq H_2 \beta \), the conditions 1 and \( E \) imply the existence of a 
solution to the BVP (2) - (5), which satisfies the condition (7).

**Proof.** It follows from Theorem Tb01 that there exists a solution \( x \) to the problem

\[(\varphi(t, x, x'))' = f(t, x, x'), \quad H_1 x = h_1, \quad \alpha \leq x \leq \beta.\]

It is clear that \( \alpha'(a) \leq x'(a) \leq \beta'(a) \), and the monotonicity conditions for 
\( H_2 \) imply the inequalities \( H_2 \alpha \geq H_2 x \geq H_2 \beta \). Thus \( H_2 x = h_2 \). Therefore \( x \)
solves the problem (2) - (5) and satisfies the condition (7). \( \Box \)

**Theorem Tb04.** \( -1 - +. \ - - 0. \ 2 + - \). For any \( H_1 \in M(-,1,-,+), \)
\( H_2 \in M(-,-,-,0), h_1 \in [H_1 \alpha, H_1 \beta] \) and \( h_2 \in [H_2 \alpha, H_2 \beta] \) the conditions 
\( H_1 \alpha \leq H_1 \beta, H_2 \alpha \leq H_2 \beta \) and the conditions 2 and \( E \) imply the existence of 
a solution to the problem (2) - (5), satisfying the condition (7).

**Proof.** Define sequences \( \alpha_k, \beta_k, k = 1, 2, \ldots \) by induction. Let \( \alpha_1 = \alpha \)
and \( \beta_1 = \beta \). It follows from Theorem Tb02 C.II. \( 1 + + 0. \ - 1 0 +. \ C \) that there 
extists a solution \( y \) of the BVP

\[(\varphi(t, y, y'))' = f(t, y, y'), \quad \alpha'_k(a) \leq y'(a) \leq \beta'_k(a), \]

\[y(b) = (\alpha_k(b) + \beta_k(b))/2, \quad \alpha_k \leq y \leq \beta_k.\]

If \( H_1 y \leq h_1 \), then set \( \alpha_{k+1} = \alpha_k \) and \( \beta_{k+1} = \beta_k \). If \( H_1 y > h_1 \) then set 
\( \alpha_{k+1} = \alpha_k \) and \( \beta_{k+1} = \beta_k \). Obviously \( \alpha_k \) and \( \beta_k \) are respectively the lower 
and upper functions, \( \alpha = \alpha_1 \leq \alpha_2 \leq \ldots \leq \beta_2 \leq \beta_1 = \beta, \alpha'(a) = \alpha'_1(a) \leq \alpha'_2(a) \leq \ldots \leq \beta'_2(a) \leq \beta'_1(a) = \beta'(a) \) and 
\( H_1 \alpha_k \leq h_1 \leq H_1 \beta_k \). The inequality 
\( H_2 \beta \geq H_2 \alpha \) and the monotonicity conditions for \( H_2 \) imply the inequalities 
\( H_2 \alpha \geq H_2 \alpha_k \geq H_2 \beta_k \geq H_2 \beta \geq H_2 \alpha \). Hence \( H_2 \alpha_k = h_2 = H_2 \beta_k \). The limits 
\( \alpha_k \to \alpha_0, \beta_k \to \beta_0 \) satisfy the conditions \( \alpha \leq \alpha_0 \leq \beta_0 \leq \beta, \alpha'(a) \leq \alpha'_0(a) \leq \beta'_0(a) \leq \beta'(a), \alpha_0(b) = \beta_0(b), H_1 \alpha_0 \leq h_1 \leq H_1 \beta_0, \)
\( H_2 \alpha_0 = h_2 = H_2 \beta_0, \alpha_0 \in S(I, R) \) or \( \beta_0 \in S(I, R) \). It follows from the monotonicity of \( H_1 \) that 
\( H_1 \alpha_0 \geq H_1 \beta_0 \). Thus \( H_1 \alpha_0 = h_1 = H_1 \beta_0 \). Therefore either \( \alpha_0 \) or \( \beta_0 \) solves the 
BVP (2) - (5) and satisfies the condition (7). \( \Box \)
Theorem Tb05. For any $H_1 \in M(-1,1,+,1)$, $H_2 \in M(-1,-1,0)$, $h_1 \in [H_1 \alpha, H_1 \beta]$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ the conditions $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$ and the conditions 3 and 4 imply the existence of a solution to the BVP (2) - (5), satisfying the condition (7).

Proof. Define sequences $\alpha_k, \beta_k$, $k = 1, 2, ...$ by induction. Let $\alpha_1 = \alpha$ and $\beta_1 = \beta$. It follows from Theorem Tb02 that there exists a solution $y$ to the problem

$$(\varphi(t, y, y'))' = f(t, y, y'),$$

$$-y'(a) = -\alpha'(a), \ y(b) = (\alpha_k(b) + \beta_k(b))/2, \ \alpha_k \leq y \leq \beta_k.$$ 

If $H_1 y \leq h_1$ set $\alpha_{k+1} = y$ and $\beta_{k+1} = \beta_k$. If $H_1 y > h_1$ set $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = y$. Obviously $\alpha_k$ and $\beta_k$ are respectively the lower and upper functions, $\alpha = \alpha_1 \leq \alpha_2 \leq ... \leq \beta_2 \leq \beta_1 = \beta$, $\alpha_k(a) = \beta_k(a) = \alpha'(a)$ and $H_1 \alpha_k \leq h_1 \leq H_1 \beta_k$. The inequality $H_2 \beta \geq H_2 \alpha$ and the monotonicity of $H_2$ imply the inequalities $H_2 \alpha \geq H_2 \alpha_k \geq H_2 \beta_k \geq H_2 \beta \geq H_2 \alpha$. Hence $H_2 \alpha_k = h_2 = H_2 \beta_k$. The limits $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$ satisfy the conditions $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta$, $\alpha_0(a) = \beta_0(a) = \alpha'(a)$, $\alpha_0(b) = \beta_0(b)$, $H_1 \alpha_0 \leq h_1 \leq H_1 \beta_0$, $H_2 \alpha_0 = h_2 = H_2 \beta_0$, $\alpha_0 \in S(I, R)$ or $\beta_0 \in S(I, R)$. The monotonicity of $H_1$ implies $H_1 \alpha_0 \geq H_1 \beta_0$. Thus $H_1 \alpha_0 = h_1 = H_1 \beta_0$. Therefore either $\alpha_0$ or $\beta_0$ solves the BVP (2) - (5) and satisfies the condition (7).

Theorem Tb06. For any $H_1 \in M(-1,1,+,1)$, $H_2 \in M(-1,-1,+,0)$, $h_1 \in [H_1 \alpha, H_1 \beta]$ and $h_2 \in [H_2 \alpha, H_2 \beta]$ the conditions $H_1 \alpha \leq H_1 \beta$, $H_2 \alpha \leq H_2 \beta$, the conditions 4 and 5 imply the existence of a solution to the BVP (2) - (5), satisfying the condition (7).

Proof. Define sequences $\alpha_k, \beta_k$, $k = 1, 2, ...$ by induction. Let $\alpha_1 = \alpha$ and $\beta_1 = \beta$. It follows from Theorem Tb02 that there exists a solution $y$ of the BVP

$$(\varphi(t, y, y'))' = f(t, y, y'),$$

$$-y'(a) = -\alpha'(a), \ y(b) = (\alpha_k(b) + \beta_k(b))/2, \ \alpha_k \leq y \leq \beta_k.$$ 

If $H_1 y \leq h_1$ set $\alpha_{k+1} = y$ and $\beta_{k+1} = \beta_k$. If $H_1 y > h_1$ set $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = y$. It follows that $\alpha_k$ and $\beta_k$ are respectively the lower and upper functions, $\alpha = \alpha_1 \leq \alpha_2 \leq ... \leq \beta_2 \leq \beta_1 = \beta$, $\alpha'(a) = \alpha'(a)$ and $H_1 \alpha_k \leq h_1 \leq H_1 \beta_k$. The inequality $H_2 \beta \geq H_2 \alpha$ and the monotonicity of $H_2$ imply the inequalities $H_2 \alpha \geq H_2 \alpha_k \geq H_2 \beta_k \geq H_2 \beta \geq H_2 \alpha$. Hence $H_2 \alpha_k = h_2 = H_2 \beta_k$. The limits $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$
satisfy the conditions $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta$, $\beta'(a) \leq \beta'(a) = \alpha'(a)$, $\alpha_0(b) = \beta_0(b)$, $H_1\alpha_0 = h_1 \leq H_1\beta_0$, $H_2\alpha_0 = h_2 = H_2\beta_0$, $\alpha_0 \in S(I, R)$ or $\beta_0 \in S(I, R)$. The monotonicity of $H_1$ implies $H_1\alpha_0 \geq H_1\beta_0$. Hence $H_1\alpha_0 = h_1 = H_1\beta_0$. Therefore either $\alpha_0$ or $\beta_0$ solves the BVP (2) - (5) and satisfies the condition (7).

Theorem Tb07. For any $H_1 \in M(-, 1, +, +)$, $H_2 \in M(-, - 1, 0)$, $h_1 \in [H_1\alpha, H_1\beta]$ and $h_2 \in [H_2\alpha, H_2\beta]$ the conditions $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$, the conditions 4 and $E$ imply the existence of a solution to the BVP (2) - (5), satisfying the condition (7).

Proof. Let us prove first that for any $A' \in [\beta'(a), \alpha'(a)]$ there exists a solution to the BVP

$$(10) (\varphi(t, z, z'))' = f(t, z, z'), \quad H_1z = h_1, \quad -z'(a) = -A', \quad \alpha \leq z \leq \beta.$$  

Let $\alpha_*$ and $\beta_*$ be solutions of the boundary value problems

$$(\varphi(t, \alpha_*, \alpha'_*))' = f(t, \alpha_*, \alpha'_*), \quad -\alpha'_*(a) = -A', \quad \alpha_*(b) = \alpha(b), \quad \alpha \leq \alpha_* \leq \beta,$$

$$(\varphi(t, \beta_*, \beta'_*))' = f(t, \beta_*, \beta'_*), \quad -\beta'_*(a) = -A', \quad \beta_*(b) = \beta(b), \quad \alpha \leq \beta_* \leq \beta,$$

which are solvable by virtue of Theorem Tb02. Then $\alpha \leq \alpha_* \leq \beta_* \leq \beta$ and the monotonicity conditions for $H_1$ imply the inequalities $H_1\alpha_* \leq H_1\alpha \leq h_1 \leq H_1\beta \leq H_1\beta_*$. It follows from Theorem Tb05 that there exists a solution $z$ of the BVP

$$(\varphi(t, z, z'))' = f(t, z, z'), \quad H_1z = h_1, \quad -z'(a) = -A', \quad \alpha_* \leq z \leq \beta_*.$$  

It follows that $\alpha \leq z \leq \beta$. Therefore $z$ solves the BVP (10).

Define sequences $\alpha_k, \beta_k$, $k = 1, 2, \ldots$ by induction. Let $\alpha_1 = \alpha$ and $\beta_1 = \beta$. It follows from the above considerations that there exists a solution $y$ of the BVP

$$(\varphi(t, y, y'))' = f(t, y, y'),$$

$$H_1y = h_1, \quad -y'(a) = -(\alpha'_k(a) + \beta'_k(a))/2, \quad \alpha_k \leq y \leq \beta_k.$$  

If $H_2y \leq h_2$ then set $\alpha_{k+1} = y$ and $\beta_{k+1} = \beta_k$. If $H_2y > h_2$, set $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = y$. It follows that $\alpha_k$ and $\beta_k$ are respectively the lower and upper functions, $\alpha = \alpha_1 \leq \alpha_2 \leq \ldots \leq \beta_2 \leq \beta_1 = \beta$, $\beta'(a) \leq \beta'(a)$, $\alpha_k \leq \alpha'_k(a) \leq \alpha'_j(a)$ and $H_2\alpha_k \leq h_2 \leq H_2\beta_k$. The limits $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$ satisfy the conditions $\alpha \leq \alpha_0 \leq \beta_0 \leq \beta$, $\beta'(a) \leq \beta'(a)$, $\alpha_0(b) \leq \alpha'(a)$, $H_2\alpha_0 \leq h_2 \leq H_2\beta_0$, $\alpha_0 \in S(I, R) \land H_1\alpha_0 = h_1$ or $\beta_0 \in S(I, R) \land H_1\beta_0 = h_1$. The monotonicity conditions for $H_2$ imply the inequality $H_2\alpha_0 \geq H_2\beta_0$. Hence $H_2\alpha_0 = h_2 = H_2\beta_0$. Therefore either $\alpha_0$ or $\beta_0$ solves the BVP (2) - (5) and satisfies the condition (7).
Theorem Tb08. For any $H_1 \in M(1,1,1,+) \cup M(1,1,1,-)$, $H_2 \in M(1,1,1,+) \cup M(1,1,1,-)$, $h_1 \in [H_1\alpha, H_1\beta]$ and $h_2 \in [H_2\alpha, H_2\beta]$, the conditions $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$, the conditions 1, 3 and $E$ imply the existence of a solution to the BVP (2) - (5), satisfying the condition (7).

Proof. By Theorem Tb03 there exists a solution $x$ to the BVP

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad H_1(x(a),x(b),\alpha'(a),x'(b)) = h_1,$$

$$H_2(x(a),x(b),\alpha'(a),x'(b)) = h_2, \quad \alpha \leq x \leq \beta.$$  

It follows that $\alpha'(a) = x'(a) = \beta'(a)$, $H_1x = h_1$ and $H_2x = h_2$. Therefore $x$ solves the BVP (2) - (5) and satisfies the condition (7). \qed

Theorem Tb09. For any $H_1, H_2 \in M(1,1,1,+) \cup M(1,1,1,-)$, $h_1 \in [H_1\alpha, H_1\beta]$ and $h_2 \in [H_2\alpha, H_2\beta]$, the conditions $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$, the conditions 1, 5 and $E$ imply the existence of a solution to the BVP (2) - (5), satisfying the condition (7) - (8).

Proof. Let $x$ be a solution of the BVP

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = \alpha(a), \quad x(b) = \alpha(b), \quad \alpha \leq x \leq \beta,$$

which is solvable by Theorem Tb02. It follows that $\alpha'(a) \leq x'(a) \leq \beta'(a)$ and $\beta'(b) \leq x'(b) \leq \alpha'(b)$. The monotonicity conditions for $H_1$ and $H_2$ imply that $H_1\alpha \geq H_1x \geq H_1\beta$ and $H_2\alpha \geq H_2x \geq H_2\beta$. Hence $H_1x = h_1$ and $H_2x = h_2$. Therefore $x$ solves the BVP (2) - (5) and satisfies the conditions (7) - (8). \qed

For the reader convenience we give here the extended formulations of some base theorems which refer to conditions A, B and C of U.

Theorem Tb26 (extended formulation). The boundary value problem

$$(\varphi(t,x,x'))' = f(t,x,x'),$$

$H_1(x(a),x(b),x'(a),x'(b)) = h_1, \quad H_2(x(a),x(b),x'(a),x'(b)) = h_2, \quad \alpha \leq x \leq \beta,$

where $\alpha$ and $\beta$ are the lower and upper functions, is solvable for any $h_1 \in [H_1\alpha, H_1\beta], h_2 \in [H_2\alpha, H_2\beta]$ if:

1) $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$;
2) the condition $E$ fulfils;
3) the function $H_1$ arbitrarily depends on the first argument and does not increase in the remaining arguments;
4) the function $H_2$ does not increase in the first argument, depends arbitrarily on the second argument and does not decrease in the remaining two arguments;

5) the condition 9 of $U$ fulfills and both $\alpha$ and $\beta$ are solutions of the one-dimensional $\varphi$-Laplacian equation.

**Theorem Tb41** (extended formulation). The boundary value problem

$$(\varphi(t,x,x'))' = f(t,x,x'),$$

$$H_1(x(a),x(b)) = h_1, \quad H_2(x(a),x(b),x'(a),x'(b)) = h_2, \quad \alpha \leq x \leq \beta,$$

where $\alpha$ and $\beta$ are the lower and upper functions, is solvable for any $h_1 \in [H_1\alpha, H_1\beta]$, $h_2 \in [H_2\alpha, H_2\beta]$ if:

1) $H_1\alpha \leq H_1\beta$, $H_2\alpha \leq H_2\beta$;

2) the condition $E$ fulfills;

3) the function $H_1$ does not decrease in the first argument and does not increase in the second one;

4) the function $H_2$ depends arbitrarily on the first and the second arguments, does not increase in the third argument and does not decrease in the fourth argument;

5) the condition $H_1\alpha = H_1\beta$ holds.

5. **Existence of maximal solution.** Let us state the result.

**Theorem 1.** For the theorems Tb02 - Tb03, Tb08 - Tb09, Tb12, Tb20, Tb23 - Tb24, Tb26 - Tb27, Tb34 - Tb36, Tb42, Tb49, Tb81 and Tb84 there exist maximal and minimal solutions.

**Proof.** Let us consider the case of Theorem Tb02. Let $S$ stand for a set of solutions of the BVP (2)-(5), $x,y \in S$ and $s = \max\{x,y\}$. Then $s$ is a lower function in accordance to the definition above. Let us show that $H_1s \leq h_1$ and $H_2s \leq h_2$. Let $x(a) > y(a)$ and $x(b) < y(b)$. Then $h_1 = H_1x \geq H_1(x(a), y(b), x'(a), x'(b)) = H_1s$, $h_2 = H_2y \geq H_2(x(a), y(b), y'(a), y'(b)) = H_2s$. Other cases can be treated similarly. Hence there exists a solution $z \in S$, satisfying the estimate $z \geq s$. If $S$ consists of a finite number of solutions, the existence of a maximal solution is evident. Let $S$ be infinite. Let $r_i \in I$, $i = 1, 2, \ldots$ and $x_i \in S$, $i = 1, 2, \ldots$ be such that $r_i$ are rational points and $x_i(r_i) = \max\{x(r_i) : x \in S\}$. Define a sequence $z_i$, $i = 1, 2, \ldots$ as follows: $z_1 = x_1$ and $z_i \in S$, $i = 2, 3, \ldots$ satisfy the inequalities $z_i \geq \max\{z_{i-1}, x_i\}$. It follows that $\lim_{i \to \infty} z_i = z \in S$ and $z = \max\{x : x \in S\}$.

The existence of a minimal solution can be proved in analogous way. Similarly can be proved the existence of maximal and minimal solutions for other theorems. □
Remark 1. It is possible to show (by constructing appropriate examples) that for the remaining theorems maximal solutions do not exist. For instance, the example showing the absence of a maximal solution in case of Theorem Tb01 follows: \(a = -1, b = 1, x'' = 0, \alpha = -\beta = -1,\)

\[ H_1 x = (| x(b) - \alpha(b) | + x(b) - \alpha(b))( | x'(a) - 1 | - x'(a) + 1) = 0 \]

and \(H_2 = 0.\) This example is suitable also for the theorems Tb05, Tb25, Tb31 Tb37.

Remark 2. Simple examples for the theorems Tb12, Tb23, Tb42 and Tb49 show that the condition (8) can be violated for a maximal solution.

REFERENCES


Felix Sadyrbaev - contact person
Institute of Mathematics
and Computer Science,
University of Latvia
Rainis blvd. 29
LV-1459 Riga, Latvia
E-mail: felix@cclu.lv
THE STRUCTURE OF THE RESOLVENT OF THE DISCRETE RENEWAL EQUATION WITH NONSUMMABLE KERNEL

I.L. OYNAS * AND Z.B. TSALYUK *

Abstract. The equation \( x_n = \sum_{k=0}^{n} a_{n-k} x_k + f_n \) \((k = 0, 1, \ldots)\) is considered under the assumptions that the kernel \( a = \{a_n\} = a^{(0)} + a^{(1)} \), where \( a^{(0)} \in l_1 \), \( a^{(1)} \) is a sum of quasipolynomials, and the symbol has a finite set of zeros with finite multiplicities in the closed unit disk.

The conditions are obtained under which the resolvent \( r \) has a representation \( r = u + b + b^* u \), where \( b \) is constructed properly on the basis of the zeros of the symbol, and \( u \in l_1 \).

In the case \( a \in l_1 \), these conditions are also necessary.

Key Words. Discrete renewal equation, resolvent, generating function, convolution.

AMS(MOS) subject classification. 39A11

We consider the difference equation of renewal type

\[ x_n = \sum_{k=0}^{n} a_{n-k} x_k + f_n, \quad (n = 0, 1, \ldots). \] (1)

For the sequences \( a = \{a_n\} \), we define the generating function \( \hat{a}(z) = \sum_{k=0}^{\infty} a_k z^k \) and the convolution product \( a \ast b = \left\{ \sum_{k=0}^{n} a_{n-k} b_k \right\} \) (see [4]). The sequence \( \delta = \{1, 0, \ldots, 0, \ldots\} \) is the identity with respect to this operation, that is, \( a \ast \delta = a \). The functional sequence \( \beta^{(r)}(\lambda) = \{\beta^{(r)}_n(\lambda)\} \) (where

* Kuban state University, Department of Differential Equations, Stavropol’skaya, 149, Krasnodar, RUSSIA–350040. E-mail: du@math.kubsu.ru
λ ∈ ℂ, r ∈ ℝ) is defined as β_n^{(r)}(λ) = \frac{\lambda^n}{n!} \prod_{i=1}^{n} (r + i). By E(x) we denote the greatest integer not exceeding the number x ∈ ℝ, and by P_r(n) the polynomials of degree ≤ r. The space l_1 consists of sequences a with the norm ||a|| = \sum_{n=0}^{∞} |a_n| < ∞. Note that we assume \prod_{i=1}^{0} = 1 and \sum_{i=1}^{0} = 0.

In the sequel, we suppose that a_0 ≠ 1. So the solution of the equation

(1) \quad (δ - a) * x = f

may be obtained as x = (δ + r) * f, where r is the resolvent for the kernel a. That's why the behavior of solutions of the equation (1) with given f depends on the structure of the resolvent.

The resolvent r is a solution of the equation

r = (δ + r) * a.

Since \hat{r}(z) = \hat{a}(z)(1 - \hat{a}(z))^{-1}, the behavior of the sequence r depends on zeros of function 1 - \hat{a}(z). In the case of the sequence a ∈ l_1 such that the function 1 - \hat{a}(z) has a finite set of zeros with finite multiplicity, the structure of the resolvent was studied in [1, 2]. Here we consider a certain class of the kernels a ∉ l_1.

Let the function 1 - \hat{a}(z) have the finite set of zeros λ_1, ..., λ_k and the multiplicities m_1, ..., m_k respectively in the disk |z| ≤ 1. Let, further, m_1, ..., m_s be integers and m_j (j > s) be not integer numbers. We say that the sequence represented in the form

\begin{align*}
b &= \sum_{j=1}^{s} \sum_{r=0}^{m_j-1} c_{jr} β^{(r)}(λ_j^{-1}) \\
  &+ \sum_{j=s+1}^{k} \left[ \sum_{r=0}^{E(m_j)-1} c_{jr} β^{(r)}(λ_j^{-1}) + \sum_{r=0}^{E(m_j)} \tilde{c}_{jr} β^{(m_j-r-1)}(λ_j^{-1}) \right]
\end{align*}

is a-regular if \psi(z) = \prod_{j=1}^{k} (z - λ_j)^{m_j} (1 - \hat{b}(z)) ≠ 0 for |z| ≤ 1. (We recall that λ_j and m_j are defined by a.)

To construct an a-regular sequence, we can multiply δ - a consequently by the multipliers δ - β^{(r)}(λ_j^{-1}), j = 1, ..., k, with suitable values of r (see also [2]). Note that \min_{|z| ≤ 1} |\psi(z)| continuously depends on the coefficients c_{jr} and \tilde{c}_{jr}; hence the small variation of them does not violate the a-regularity.
In this paper we consider kernels of the following structure:

\[ a = a^{(0)} + a^{(1)}, \]

where \( a^{(0)} \in l_1 \) and

\[ a_n^{(1)}(z) = \sum_{j=1}^{\infty} P_n(z)\mu_j^{-n}, \quad |\mu| \leq 1. \]

**Theorem 1.** Let the function \( 1 - \hat{a}(z) \) have the finite set of zeros \( \lambda_1, \ldots, \lambda_k \) of the multiplicities \( m_1, \ldots, m_k \), respectively, in the disk \( |z| \leq 1 \). Let, further, \( m_1, \ldots, m_s \) be integer numbers and \( m_j (j > s) \) be fractional ones. Besides, suppose that

\[ (\hat{a}^{(0)}(\lambda_j)\delta - a^{(0)}) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) \in l_1 \]

for all \( j \) such that \( |\lambda_j| = 1 \). Then for any \( a \)-regular sequence \( b \) there exists a sequence \( u \in l_1 \) such that the resolvent \( r \) of the kernel \( a \) has the representation

\[ r = u - b - b \ast u. \]

The proof is based on the methods of the paper [3] and on the following theorem, which develops the results of the paper [2].

**Theorem 2.** Let \( a \in l_1 \) and the function \( 1 - \hat{a}(z) \) have the finite set of zeros \( \lambda_1, \ldots, \lambda_k \) of the multiplicities \( m_1, \ldots, m_k \), respectively, in the disk \( |z| \leq 1 \). Let, further, \( m_1, \ldots, m_s \) be integer numbers and \( m_j (j > s) \) be fractional ones. Besides, suppose that

\[ (\delta - a) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) \in l_1 \]

for all \( j \) such that \( |\lambda_j| = 1 \). Then for any \( a \)-regular sequence \( b \) there exists a sequence \( u \in l_1 \) such that the resolvent \( r \) of the kernel \( a \) has the representation (4).

Conversely, if for any \( a \)-regular sequence \( b \) the equality (4) holds with some \( u \in l_1 \) then the inclusion (5) is true.

**Proof.** Denote

\[ c^{(r,j)} = (\delta - a) \ast \beta^{(r)}(\lambda_j^{-1}). \]

Let \( |\lambda_j| = 1 \). We have

\[ c^{(r,j)} - c^{(r-1,j)} = \lambda_j^{-1} \cdot \{ c_n^{(r,j)} \} \quad (r \in \mathbb{R}). \]
So

\[ \|c^{(r-1,j)}\| \leq 2\|c^{(r,j)}\|. \]

Thus, due to (5), \( c^{(r,j)} \in l_1 \) for \( r = m_j - E(m_j) - 1, m_j - E(m_j), \ldots, m_j - 1 \).

By the book [4], it follows that for arbitrary \( p, q \) and \( \lambda \in \mathbb{C} \)

\[ \beta^{(p-1)}(\lambda) * \beta^{(q-1)}(\lambda) = \beta^{(p+q-1)}(\lambda), \]

hence

\[ c^{(E(m_j)-1,j)} = c^{(m_j-1,j)} * \beta^{(E(m_j)-m_j-1)}(\lambda_j^{-1}). \]

The inequality \(-m_j + E(m_j) - 1 \leq -1\) yields the inclusion

\[ \beta^{(E(m_j)-m_j-1)}(\lambda_j^{-1}) \in l_1 \]

(see [2]). So the sequence \( c^{(E(m_j)-1,j)} \), being the convolution of two sequences from \( l_1 \), belongs also to \( l_1 \). From (6) we get \( c^{(r,j)} \in l_1 \) for all \( r = 0, 1, \ldots, E(m_j) - 2 \).

Now let \( |\lambda_j| < 1 \). Then, in virtue of the equalities

\[ \hat{a}(\lambda_j) = \sum_{l=0}^{\infty} a_l \lambda_j^l = 1, \]

\[ \frac{d^n}{d z^n} \hat{a}(\lambda_j) = (-1)^i i! \sum_{l=i}^{\infty} \beta^{(l-1)}(1) a_l \lambda_j^{-i} = 0, \quad i = 1, \ldots, m_j - 1, \]

and (7), we obtain that

\[ \sum_{l=0}^{\infty} \beta^{(n-1)}_r(1) a_l \lambda_j^l = \sum_{l=0}^{r} \beta^{(n)}_r(1) \sum_{l=i}^{\infty} \beta^{(l-1)}(1) a_l \lambda_j^l = \beta^{(n)}_r(1) \]

for \( r = 0, 1, \ldots, m_j - 1 \).

Using the identity \( \beta^{(r)}_m(1) = \beta^{(m)}_r(1) \) (for \( m = 0, 1, \ldots \)) we get

\[ c^{(r,j)} = \sum_{l=n+1}^{\infty} \beta^{(n-l)}_r(1) a_l \lambda_j^{l-n}. \]

So the estimate

\[ \sum_{n=0}^{\infty} \left| \sum_{l=n+r+1}^{\infty} \beta^{(n-r)}_r(1) a_l \lambda_j^{l-n} \right| \leq \sum_{l=r+1}^{\infty} |a_l| \sum_{n=0}^{l-r-1} \beta^{(r)}_l(\lambda_j^{l}) \]

\[ \leq ||a|| \|\beta^{(r)}(\lambda_j^{l})\| \]
implies that \( c^{(r,j)} \in l_1 \).

Thus we have \( c^{(r,j)} \in l_1 \) for all \( |\lambda_j| \leq 1 \). So
\[
c = (\delta - a) \ast b \in l_1
\]
for every \( a \)-regular sequence \( b \).

Let \( a^{(1)} = a + c \). Then \( a^{(1)} \in l_1 \) and
\[
1 - \hat{a}^{(1)}(z) = (1 - \hat{a}(z))(1 - \hat{b}(z)) \neq 0 \quad \text{if } |z| \leq 1.
\]

According to the Wiener theorem [1], the resolvent \( u \) of the kernel \( a^{(1)} \) belongs to the space \( l_1 \). To conclude the proof of the first part of the theorem, we refer to the equality
\[
\delta + \tau = (\delta - a) \ast (\delta + u).
\]

For the converse part of the theorem, suppose that for any \( a \)-regular sequence \( b \) the equality (4) takes place, where \( u \in l_1 \). Let \( -a^{(1)} \) be a resolvent of the kernel \( -u \). It follows from (4) that
\[
1 + \hat{r}(z) = (1 - \hat{b}(z))(1 + \hat{u}(z)) \quad \text{for } |z| < 1, \ z \neq \lambda_j.
\]

Hence
\[
(1 - \hat{a}(z))(1 - \hat{b}(z))(1 + \hat{u}(z)) = 1,
\]
which implies (8) for \( z \neq \lambda_j \). Therefore, the function \( 1 - \hat{a}^{(1)}(z) \) is defined for all \( z \) such that \( |z| \leq 1, \ z \neq \lambda_j \), and it has a nonzero finite limit as \( z \to \lambda_j \). Due to (9), it means that \( 1 + \hat{u}(z) \neq 0 \) for \( |z| \leq 1 \). Referring to the Wiener theorem, we get \( -a^{(1)} \in l_1 \). So the equality
\[
\delta - a^{(1)} = (\delta - a) \ast (\delta - b)
\]
implies
\[
(\delta - a) \ast b \in l_1.
\]

It is noted above that if \( c \) is sufficiently small then the sequence
\[
b^* = b + c\beta^{(m_j-1)}(\lambda_j^{-1}), \quad |\lambda_j| = 1,
\]
is still \( a \)-regular one. Repeating the reasonings for \( b^* \) instead of \( b \), we get
\[
(\delta - a) \ast b^* \in l_1.
\]
Thus, due to (10), we get (5). □

\textit{Proof of Theorem 1.} Let the kernel \( c \) be defined by the formula

\[ \delta - c = (\delta - d) * (\delta - a), \]

where the sequence \( d = \{d_n\} \) is chosen in such a way that

i) \( c \in l_1; \)

ii) the sets of zeros of the functions \( 1 - \hat{c}(z) \) and \( 1 - \hat{a}(z) \) in the disk \( |z| \leq 1 \) coincide;

iii) if \( |\lambda_j| = 1 \) then

\begin{equation}
(\delta - c) * \beta^{(m_j - 1)}(\lambda_j^{-1}) \in l_1.
\end{equation}

Namely, we put \( d_n = \sum_{i=0}^{m} \alpha_i \beta^{(i)} \left( \frac{1}{2} \right) \) with \( m = \sum_{j=1}^{p} n_j + p - 1 \) (see (2)),

\[ \alpha_i = \frac{1}{2} (-2)^{-i} \eta_{m-i}, \quad \eta_i \text{ being the coefficients of the expansion of } \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} \text{ in powers of } z - 2. \]

So we have

\begin{equation}
\hat{d}(z) = \sum_{i=0}^{m} \alpha_i \frac{(-2)^{i+1}}{(z - 2)^{i+1}}
\end{equation}

and

\begin{equation}
(1 - \hat{d}(z)) = \frac{(z - 2)^{m+1} - \sum_{i=0}^{m} \alpha_i (-2)^{i+1} (z - 2)^{m-i}}{(z - 2)^{m+1}} = \frac{\prod_{j=1}^{p} (z - \mu_j)^{n_j+1}}{(z - 2)^{m+1}}.
\end{equation}

First, we prove the property ii) of the sequence \( c \). Obviously,

\[ \hat{\alpha}^{(1)}(z) = \frac{P_m(z)}{\prod_{j=1}^{p} (z - \mu_j)^{n_j+1}}, \]

where \( P_m(z) = \sum_{j=1}^{p} \sum_{l=0}^{n_j} c_{lj} (z - \mu_j)^{n_j-l} \prod_{i=1}^{p} (z - \mu_i)^{n_i+1}. \) Hence

\[ 1 - \hat{a}(z) = \left( (1 - \hat{\alpha})^{(0)}(z) \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} - P_m(z) \right) \prod_{j=1}^{p} (z - \mu_j)^{-n_j-1}. \]
The polynomials in (2) have their leading coefficients being distinct from zero; so \( P_m(\mu_j) \neq 0 \). Therefore, \( \mu_j \) do not satisfy the equation \( 1 - \tilde{a}(z) = 0 \), i.e. \( \mu_j \neq \lambda_l \). So

\[
1 - \tilde{c}(z) = (1 - \tilde{a}(z)) (1 - \tilde{d}(z)) = \prod_{i=1}^{k} (z - \lambda_i)^{m_i} q(z) \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} \prod_{i=1}^{k} (z - \lambda_i)^{m_i} q(z) = \prod_{i=1}^{k} (z - \lambda_i)^{m_i} q(z)
\]

\[
= \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} (z - 2)^{m+1},
\]

where \( q(z) \neq 0 \) for \( |z| \leq 1 \). Thus the function \( 1 - \tilde{c}(z) \) has the same zeros that \( 1 - \tilde{a}(z) \) has.

Now take the property i). We have from (2) and the construction of \( c \) that

\[
\delta - c = (\delta - d) \ast (\delta - a^{(0)} - a^{(1)}).
\]

Since \( d \in l_1 \), also \( \delta - (\delta - d) \ast (\delta - a^{(0)}) \in l_1 \). Therefore, to prove that \( d \in l_1 \) we need the inclusion \( g = \{g_n\} = (\delta - d) \ast a^{(1)} \in l_1 \).

Due to (13),

\[
(14) \quad \tilde{g}(z) = \tilde{a}^{(1)}(z)(1 - \tilde{d}(z)) = \frac{P_m(z)}{\prod_{j=1}^{p} (z - \mu_j)^{n_j+1}} \prod_{j=1}^{p} (z - \mu_j)^{n_j+1} \prod_{i=1}^{k} (z - \lambda_i)^{m_i} \frac{q(z)}{(z - 2)^{m+1}}.
\]

Consequently, \( g = \sum_{i=0}^{m} c_i \beta^{(i)}(\frac{1}{2}) \in l_1 \). Thus \( c \in l_1 \).

Next, we prove iii). Take the sequences

\[
A_j^{(i)} = (\tilde{a}^{(i)}(\lambda_j) \delta - a^{(i)}) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) \quad (i = 0, 1).
\]

Since \( 1 = \tilde{a}^{(0)}(\lambda_j) + \tilde{a}^{(1)}(\lambda_j) \), then we have

\[
(\delta - c) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) = (\delta - d) \ast (A_j^{(0)} + A_j^{(1)}).
\]
According to (3), \((\delta - d) \ast A_j^{(0)} \in l_1\). Therefore, to establish (11), we have to show that

\[(\delta - d) \ast A_j^{(1)} = (\delta - d) \ast (\hat{a}^{(1)}(\lambda_j) \delta - a^{(1)}) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) \in l_1.\]

The sequence \(\eta^{(j)}\) is defined as

\[
\eta^{(j)} = (\delta - d) \ast (\hat{a}^{(1)}(\lambda_j) \delta - a^{(1)}) \ast \beta^{(c)}(\lambda_j^{-1}),
\]

where

\[
\epsilon = \begin{cases} 
    m_j - 1 & \text{if } m_j \text{ is integer}, \\
    E(m_j) & \text{otherwise}.
\end{cases}
\]

Considering (12) and (14) we have

\[
\eta^{(j)}(z) = \left(\hat{a}^{(1)}(\lambda_j) - \frac{P_m(z)}{(z - 2)^{m+1}}\right) \frac{(-\lambda_j)^{\epsilon+1}}{(z - \lambda_j)^{\epsilon+1}}.
\]

Expand \(\eta^{(j)}(z)\) to simple fractions:

\[
\eta^{(j)}(z) = \sum_{l=1}^{s+1} \frac{A_l}{(z - \lambda_j)^l} + \sum_{l=1}^{m+1} \frac{B_l}{(z - 2)^l}.
\]

On the other hand, due to (15),

\[
\eta^{(j)}(z) = (1 - \hat{d}(z)) (\hat{a}^{(1)}(\lambda_j) - \hat{a}^{(0)}(z)) (-\lambda_j)^{\epsilon+1}(z - \lambda_j)^{-\epsilon-1}.
\]

According to (3) we have \(A_j^{(0)} \in l_1\); so the generating function

\[
\hat{A}_j^{(0)}(z) = (\hat{a}^{(0)}(\lambda_j) - \hat{a}^{(0)}(z)) (-\lambda_j)^{m_j}(z - \lambda_j)^{-m_j}
\]

is analytic in the disk \(|z| < 1\) and is continuous on its closure.

Therefore, \(z = \lambda_j\) is a root of the order \(\geq m_j\) for the equation

\[\hat{a}^{(0)}(\lambda_j) - \hat{a}^{(0)}(z) = 0\]

And \(z = \lambda_j\) is a root of equation

\[1 - \hat{a}^{(1)}(z) - \hat{a}^{(0)}(z) = 0\]

with the order \(m_j\). So the equation

\[\hat{a}^{(1)}(\lambda_j) - \hat{a}^{(1)}(z) = 0\]
has also the root $\lambda_j$ with order not less than $m_j$. We have a factorization
\[ \hat{\varphi}^{(1)}(\lambda_j) - \hat{\varphi}^{(1)}(z) = (z - \lambda_j)^t \varphi(z), \]
where integer $t \geq m_j$ and $\varphi(z)$ has no singularity on $\lambda_j$. Using this, we get the representation
\[ \eta^{(j)}(z) = (1 - \hat{\varphi}(z)) \frac{(-\lambda_j)^{\varepsilon+1} \varphi(z)}{(z - \lambda_j)^{\varepsilon+1-t}}. \]

The inequality $\varepsilon \leq t - 1$ implies that the function $\eta^{(j)}(z)$ has no singularity at the point $\lambda_j$ and, clearly, we have all $A_t = 0$ in the representation (16).

So $\eta^{(j)} = \{P_m(n)2^{-n}\} \in l_1$.

Due to (7),
\[ \beta^{(m_j-1)}(\lambda_j^{-1}) = \beta^{(\varepsilon)}(\lambda_j^{-1}) \ast \beta^{(m_j-\varepsilon-2)}(\lambda_j^{-1}). \]

Then $\beta^{(m_j-\varepsilon-2)}(\lambda_j^{-1}) \in l_1$ for $|\lambda_j| = 1$ because $m_j - \varepsilon - 2 \leq -1$. Thus
\[ (\delta - c) \ast \beta^{(m_j-1)}(\lambda_j^{-1}) = (\delta - d) \ast A^{(0)}_j + \eta^{(j)} \ast \beta^{(m_j-\varepsilon-2)}(\lambda_j^{-1}) \in l_1. \]

The property iii) is shown.

We see that $c$ satisfies the hypotheses of Theorem 2. By this theorem, for arbitrary $c$-regular sequence $b$ there exists $h \in l_1$ such that the resolvent $r^{(0)}$ of the kernel $c$ is represented in the following way:
\[ r^{(0)} = h - b - b \ast h. \]

By the construction of $c$ the sequence $b$ is also $a$-regular (see property ii)) and we have
\[ r = r^{(0)} - d - d \ast r^{(0)} = u - b - b \ast u, \]
where $u \overset{\text{def}}{=} h - d - d \ast h \in l_1$, which completes the proof of the theorem. \(\square\)

REFERENCES

ANALYTIC CONTINUATION OF CAUCHY-TYPE INTEGRALS *
N. ROYTVARF AND Y. YOMDIN†

Abstract. We consider Cauchy type integrals $I(t) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)dz}{z-t}$ with $g(z)$ an analytic function. The main goal is to give conditions on $g(z)$ for $I(t)$ to satisfy linear differential equations with rational coefficients. In particular, we show that this is the case for $g(z)$ satisfying a Fuchsian differential equation.

To prove this result we study the global analytic continuation of $I(t)$, and compute its monodromy group, relating the monodromy group of the analytic function $g$, the geometry of the integration curve $\gamma$, and the analytic properties of the Cauchy type integrals.

1. Introduction. In this paper, which is a continuation of [17], we study the Cauchy integrals of the form

\[ I(t) = I(\gamma, g, t) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)dz}{z-t}, \]

where $\gamma$ is a curve in the complex plane $\mathbb{C}$, while the integrand $g(z)$ is an analytic function with a "controlled" analytic continuation, in particular, an algebraic function. In many cases we prove that $I(t)$ satisfies a linear differential equation with rational coefficients. In particular, this is the case for $g(z)$ - solutions of Fuchsian linear differential equations.

The study of this property of the Cauchy-type integrals is motivated by the role these integrals play in the investigation of the behavior of periodic solutions of the Abel differential equation (see [24]). In analogy with the classical case of the plane polynomial vector-fields (see [3, 14]) the Cauchy-type integrals play for the Abel equation the role of the Abelian integrals. The importance of the fact that Abelian integrals satisfy certain Fuchsian

---

* This research was supported by the ISF, Grant No. 264/02, by the BSF, grant No. 2002243, and by the Minerva Foundation
† Dept. of Mathematics, The Weizmann Institute of Science, Rehovot, Israel
linear differential equations is well known - see [3, 14], [5]-[11],[18]. Following [10, 11] we hope to extend the results of the present paper also to the “Iterated Cauchy integrals”.

We believe that our approach will be useful also in the study of the generalized moments ([1, 17, 24]) and double and multiple moments ([2, 4, 13, 22, 23]).

In many cases the Cauchy-type integrals $I(t)$ turn out to be very similar to algebraic functions. This important property is related both to the behavior of their Taylor coefficients (compare [20, 21]) and to their apparent similarity to automorphic functions, in the most classical setting (compare [19]). We hope that the study of the Fuchsian linear differential equations satisfied by the functions $I(t)$ will be the first step in the investigation of this “almost-algebraicity” properties.

Let us now describe more accurately the setting of the problem. In (1.1) we assume that after removing from $\gamma$ a finite set of points $\Sigma$ (which includes all the double points of $\gamma$ and its end points) on each segment of $\gamma \setminus \Sigma$ the function $g(z)$ is given by an analytic continuation of a germ of a certain possibly multivalued analytic function. We also assume that the corresponding branches are regular on $\gamma \setminus \Sigma$. Accordingly, it is always assumed below that the branches of $g(z)$ on each segment of $\gamma \setminus \Sigma$ are chosen in advance and in this sense $g(z)$ is univalued on $\gamma$. Of course, analytic continuation of $g(z)$ outside $\gamma \setminus \Sigma$ may ramify.

The main problem considered in this paper is to describe the analytic continuation of the function $I(t)$. On this base we give conditions on $g(z)$ sufficient for $I(t)$ to satisfy a linear differential equation with rational coefficients.

Let $C \setminus \gamma$ be the union of the domains $D_i$ (with $D_0$ being the infinite domain). The expression (1.1) defines $I(t)$ as a collection of regular analytic functions $I_i$ on the domains $D_i$. A simple classical description exists for the behavior of $I(t)$ in the process of crossing the curve $\gamma$ (see, for example, [16]): for the adjacent domains $D_i$ and $D_j$ the function $I_j$ is obtained from $I_i$ by the analytic continuation into $D_j$ combined with the addition of the local branch of $g$ at the crossing point (also analytically continued into $D_j$). This last operation (as extended to several crossings of $\gamma$) is “combinatorial” in its nature. It is captured by the notion of the “combinatorial monodromy of $I$” introduced in [17] and restated in Section 4 below. The combinatorial monodromy depends only on the monodromy of $g$ and on the geometry of $\gamma$ and in principle it can be explicitly computed.

We analyze the analytic continuation of $I_i(t)$ from each of the domains
and show that it is essentially described by the combinatorial monodromy. On this base we get a sufficient condition for \( I(t) \) to satisfy a linear differential equation with rational coefficients.

The authors would like to thank L. Gavrilov for inspiring discussions.

2. Partition of \( \mathbb{C} \) by \( \gamma \). Below we always assume the curve \( \gamma \) to be oriented, piecewise-smooth, and to have only transversal self-intersections. In this section we also assume that \( \gamma \) is closed. A classical description of the geometry of \( \gamma \) given below closely follows [2, 22, 23].

The curve \( \gamma \) subdivides \( \mathbb{C} \) into a finite number of open domains \( D_i \). One of these domains which we denote by \( D_0 \) is unbounded and the rest are bounded and simply connected. For a point \( z \in \mathbb{C} \setminus \gamma \) define \( \mu(\gamma, z) \) as the rotation number of \( \gamma \) around \( z \). Clearly, \( \mu(\gamma, z) \) is constant on each \( D_i \) and we will denote this constant by \( \mu_i, \mu_0 = 0 \). Alternatively, \( \mu(\gamma, z) \) can be defined as the (signed) number of the intersection points of \( \gamma \) with any path joining \( \infty \) to \( z \) or as the linking number of the curve \( \gamma \) and the point \( z \).

According to this last definition for any complex one-dimensional chain \( Z \) in \( \mathbb{C} \) with \( \gamma = \partial Z \) the number \( \mu(\gamma, z) \) is the (signed) “intersection number of \( Z \) with \( z \)” or, in other words, simply the number of times the chain \( Z \) covers the point \( z \).

It is natural to call the union of \( D_j \) with \( \mu_j = 0 \) “an outside part” of the curve \( \gamma \) and the union of \( D_i \) with \( \mu_j \neq 0 \) “an inside part”.

As a simple example, consider the following situation: let \( U \) be a simply-connected open domain in \( \mathbb{C} \) and let \( g(z) \) be a meromorphic function in \( U \) with the finite number of poles at the points \( z_1, \ldots, z_l \) with the orders \( k_1, \ldots, k_l, \ k_i \geq 1 \), respectively.

**Proposition 2.1.** The Cauchy integral \( I(\gamma, g, t) \) is given by

\[
I(\gamma, g, t) = \mu(\gamma, t)g(t) - \sum_{i=1}^{l} \mu(\gamma, z_i)R_i(t),
\]

where for \( i = 1, \ldots, l \), \( R_i(z) \) denotes the essential part of \( g(z) \) at the pole \( z_i \). In particular, for such \( g(z) \) the Cauchy integral \( I(\gamma, g, t) \) is a rational function on the exterior domain \( D_0 \).

3. Local structure of \( I(t) \). Let \( \gamma \) be a curve (closed or non-closed) and let \( z \in \gamma \). We say that the integrand function \( g \) has a “jump” at \( z \) if the branches \( g_0 \) and \( g_1 \) of \( g \) on the two sides of \( z \) on \( \gamma \) cannot be obtained from one another by a local analytic continuation (i.e. a continuation along a curve inside any given neighborhood of \( z \)). Equivalently, the full local germs of \( g_0 \) and \( g_1 \) at a jump point \( z \) do not coincide.
Let us remind that we have denoted by $\Sigma$ the set containing the endpoints of $\gamma$, all its multiple points, and all the points $z$ on $\gamma$ where the integrand function $g$ has either a jump or a ramification point. (In this paper we exclude the possibility for $g$ to have poles on $\gamma$).

Lemmas 3.1-3.3 below provide an elementary and well-known description of the behavior of the Cauchy type integral near the integration curve $\gamma$ (see, for example, [16]).

Consider first $z_0 \in \gamma$ and $z_0 \notin \Sigma$. In particular, $g$ is regular at $z_0$.

**Lemma 3.1.** $I(\gamma, g, t)$ near $z_0$ is represented by two regular analytic functions: $I_-(t)$ on the left side of $\gamma$ and $I_+(t)$ on the right side. Both $I_-$ and $I_+$ are analytically extendible into an entire neighborhood $U$ of $z_0$ and $I_+ = I_- + g$ in $U$.

Let now $z_0$ be a double point of $\gamma$ and let $\gamma_0$ and $\gamma_1$ denote the two local segments of $\gamma$ crossing at $z_0$. We assume that the restrictions of $g$ to $\gamma_0$ and $\gamma_1$ are both regular at $z_0$.

**Lemma 3.2.** $I(\gamma, g, t)$ in a neighborhood $U$ of $z_0$ is represented as $I(t) = I_0^0 + I_1^1$, where the combination of the signs is chosen according to the part of $U \setminus \gamma$ considered, and $I_0^0$ and $I_1^1$ have with respect to $\gamma_0$ and $\gamma_1$ all the properties stated in Lemma 3.1.

Assume now that $z_0$ is the end-point of $\gamma$ with the positive integration direction from $z_0$ along $\gamma$. Let $g$ be regular near $z_0$.

**Lemma 3.3.** In a neighborhood of $z_0$,

$$I(t) = \bar{I}(t) - \frac{1}{2\pi i} g(t) \log(t - z_0)$$

with $\bar{I}(t)$ regular near $z_0$. The proof of Lemma 3.3, stressing the analytic regularity of $\bar{I}(t)$, as well as a much more involved description of the structure of $I(t)$ near the end point of $\gamma$ in the case of a finite ramification of $g$ can be found in [17].

4. **Global structure of $I(t)$**. Integral representation (1.1) defines $I(t)$ as a collection of univalent regular functions $I_i(t)$ in each domain $D_i$ of the complement of $\gamma$ in $\mathbb{C}$. In this section we study the relation between $I_i(t)$ in the neighboring domains $D_i$ and on this base analyze their global analytic continuation.

Denote by $\gamma_s$ the segments of $\gamma \setminus \Sigma$. So $g$ is regular in a neighborhood of each interior point of $\gamma_s$. According to Lemma 3.1 for two adjoint domains $D_i$ and $D_j$ separated by their common segment $\gamma_s$ of the curve $\gamma$, $I_j$ is obtained from $I_i$ as follows:
a. $I_i$ is analytically continued through $\gamma_s$ into a certain neighborhood $G$ of $\gamma_s$ in $D_j$.

b. An algebraic function $g_s$ in $G$ (obtained by the analytic continuation to $G$ of the branch $g_s$ of $g$ on $\gamma_s$) is added to $I_i$ (multiplied by $-1$ if the crossing orientation of $\gamma$ is negative).

c. $I_i + g_s$ is analytically extended from $G$ to the entire domain $D_j$. $I_j$ is equal to this continued function $I_i + g_s$.

To proceed with computing the analytic continuation of $I(t)$ we have to restrict the class of the integrands $g$ considered. We shall do this in several steps.

**Assumption A.** There exists a finite set of points $\Sigma_1 = \{z_1, \ldots, z_m\}$ such that each branch of $g$ appearing in the integral (1.1) can be analytically continued as a regular function along any path in $\mathbb{C} \setminus \Sigma_1$. Shortly we shall say that $g$ is a regular multivalued function over $\mathbb{C} \setminus \Sigma_1$.

In this paper we assume in addition that the points of $\Sigma_1$ do not belong to $\gamma$. Denote by $U$ the set $U = \mathbb{C} \setminus (\Sigma \cup \Sigma_1)$.

For $g$ satisfying the Assumption A the operation of forming the sums of the branches of $g$ and of their analytic continuations (as extended to several crossings of $\gamma$) is "combinatorial" in its nature. It depends only on the monodromy of $g$ and on the geometry of $\gamma$ and in principle it can be explicitly computed.

To define this operation accurately let us consider curves $S$ (or $S_{c,d}$) starting at $c \in \mathbb{C} \setminus \gamma$ and ending at $d \in \mathbb{C} \setminus \gamma$. Say that $S$ is admissible if it avoids singularities of $g$ and crosses $\gamma$ transversally and only at the interior points of the segments $\gamma_s$. For any admissible curve $S_{c,d}$ denote by $S_{c,d}^*$ the operator of the analytic continuation along $S_{c,d}$ of the analytic germs at $c$ to the analytic germs at $d$.

The presentation below follows closely [17], Section 4. Let $S_{c,d}$ be an admissible curve with $c \in D_i$ and $d \in D_j$. Suppose that $S \cap \gamma = \{a_1, a_2, \ldots, a_r\}$ and let $\{g_1, g_2, \ldots, g_r\}$ be the germs of $g$ at $a_i$, $1 \leq i \leq r$.

Define a sum of branches $g(S_{c,d}, \gamma)$ of $g$ along $S_{c,d}$ across $\gamma$ as follows: it is a regular germ at $d$ defined by

$$g(S_{c,d}, \gamma) = \sum_{i=1}^{r} \text{sgn}(a_i)S_{a_i,d}^*(g_i),$$

where $S_{a_i,d}$ denotes the part of $S$ connecting $a_i$ and $d$ and $\text{sgn}(a_i)$ is equal to plus or minus one according to the orientation of the crossing of $S$ and $\gamma$ at $a_i$.

The following property of the sum of branches along $S$ is immediate:
PROPOSITION 4.1. Let the admissible curve $S_{c,d}$ be the union of the admissible curves $S_{c,e}$ and $S_{e,d}$. Then

$$g(S_{c,d}, \gamma) = S_{e,d}^*(g(S_{c,e}, \gamma)) + g(S_{e,d}, \gamma).$$

Remind that $\Sigma_1$ denotes the set of the singular points of (all the branches) of $g$ and that we assume that these points do not lie on $\gamma$ (and hence do not belong to $\Sigma$). As above, $U$ denotes the complex plane $\mathbb{C}$ with $\Sigma$ and $\Sigma_1$ deleted. By definition, admissible curves $S$ lie entirely in $U$.

PROPOSITION 4.2. The sum of branches along $S_{c,d}$ depends only on the homotopy class (with the fixed end-points) of this curve in $U$.

Proof. As we deform $S$ in $U$ preserving the transversality of the intersection of $S$ and $\gamma$, each term in (4.1) remains the same, being the analytic continuation of the same function along a continuously deformed path. Now for a generic deformation of $S$ at each moment of a non-transversal intersection of $S$ and $\gamma$ a couple of transversal intersections appears (or disappears). The corresponding terms in (4.1) cancel one another. □

Remark. The result follows also from Lemma 4.1 below since the analytic continuation of $I_i$ depends only on the homotopy class of $S$ in $U$.

The following lemma shows that the sum of branches along $S$ measures the difference between the analytic continuation $S^*(I_i)$ of $I_i$ into the domain $D_j$ and the function $I_j$. Let $S_{c,d}$ be any admissible curve with $c \in D_i$ and $d \in D_j$.

LEMMA 4.1. The germ of $I_i$ at $c$ can be analytically continued along $S$. The resulting germ $S^*(I_i)$ at $d$ satisfies

$$S^*(I_i)(t) = I_j(t) - g(S, \gamma)(t).$$

Proof. It follows by induction on the number $r$ of the intersection points of $S$ and $\gamma$. If we write (4.2) in the form

$$I_j(t) = S^*(I_i)(t) + g(S, \gamma)(t)$$

then for the first crossing of $S$ and $\gamma$ the equality (4.3) follows directly from the description of the behavior of $I$ on $\gamma$ given in the steps a, b, c above. Assuming that (4.3) holds after $l$ crossings of $S$ and $\gamma$ and combining the above description with the definition of the sum of branches along $S$, we get that (4.3) is valid also after $l + 1$ crossings. □

Let $S_{c,c}$ be a closed admissible curve with $c \in D_i$. According to Proposition 4.2, the sum of branches along $S$ across $\gamma$ depends only on the element $\hat{S}$ of the fundamental group $\pi_1(U, c)$ defined by $S$. We define a combinatorial...
monodromy of \( I_i \) at \( c \) as the mapping \( A_i \) of \( \pi_1(U, c) \) to the analytic germs at \( c \) which associates to each \( \hat{S} \in \pi_1(U, c) \) the germ \( A_i(\hat{S}) \) at \( c \) equal to the sum of branches along \( S \) across \( \gamma \).

The following definition specifies the main properties of the combinatorial monodromy we work with in the present paper.

**Definition 4.1.** We say that the combinatorial monodromy of \( I_i \) at \( c \) is finite-dimensional if the image of \( \pi_1(U, c) \) under \( A_i \) spans a finite-dimensional linear subspace in the space of all the analytic germs at \( c \).

We say that the combinatorial monodromy of \( I_i \) at \( c \) is finite if the image of \( \pi_1(U, c) \) under \( A_i \) is finite, and we say that the combinatorial monodromy of \( I_i \) at \( c \) is trivial if \( A_i(\hat{S}) = 0 \) for any \( \hat{S} \in \pi_1(U, c) \).

The combinatorial monodromy depends only on the monodromy of \( g \) and on the geometry of \( \gamma \) and in principle it can be explicitly computed.

Now we are ready to prove the main results of this section. The following Theorem 4.1 provides a description of the complete analytic continuation of the Cauchy integral \( I_i(t) \) from the domain \( D_i \) where it was initially defined by the expression (1.1). So fix a point \( c \in D_i \) and let \( I_i(t) \) be the function in \( D_i \) defined by (1.1). Remind that the usual monodromy mapping \( MA_i \) of the fundamental group \( \pi_1(U, c) \) to the analytic germs at \( c \) is given by

\[
MA_i(\hat{S}) = S^*(I_i(t)),
\]

for any \( \hat{S} \in \pi_1(U, c) \).

**Theorem 4.1.** Let the integrand \( g(z) \) in (1.1) satisfy the Assumption A. Then the function \( I_i(t) \) allows for a complete analytic continuation as a regular multivalued function \( \tilde{I}_i(t) \) in \( U \). For any admissible curve \( S_{c,d}^* \) with \( c \in D_i \) and \( d \in D_j \) the analytic continuation \( S_{c,d}^*(I_i) \) of \( I_i \) along \( S_{c,d} \) is given by

\[
S_{c,d}^*(I_i) = I_j - g(S, \gamma).
\]

In particular, the monodromy mapping \( MA_i \) of \( I_i(t) \) is given by \( MA_i(\hat{S}) = I_i(t) - A_i(\hat{S}) \) for any \( \hat{S} \in \pi_1(U, c) \), where \( A_i \) is the combinatorial monodromy of \( I_i \).

The singularities of \( \tilde{I}_i(t) \) at the points of \( \Sigma_1 \) are those of certain sums of the branches of \( g \), up to a regular addition.

At each point of \( \Sigma \) all the leaves of the function \( \tilde{I}_i(t) \) are simultaneously either regular or have an infinite logarithmic ramification. In the last case the analytic representation of each of the leaves of the function \( \tilde{I}_i(t) \) at these points (up to an addition of a regular germ) is given by Lemma 3.3.

**Proof.** The analytic continuation of \( I_i(t) \) along any admissible curve \( S \) and its expression via the sum of branches of \( g \) across \( \gamma \) is provided by Lemma 4.1. Applying this expression to the closed curve \( \hat{S} = S_{c,c} \) we get the required description of the monodromy action \( MA_i \).
Now let us take $S$ with the end-point $d$ near a singular point $w_0 \in \Sigma_1$ of $g$. By the representation above we get the leave of the function $\hat{I}_i(t)$ obtained by the analytic continuation along $S$ as the difference of the regular function $I_j(t)$ and a certain sum of the branches of $g$. This implies the required description of the singularities of the leaves of $\hat{I}_i(t)$ at the points of $\Sigma_1$.

Taking $S$ with the end-point $d$ near a point $w_1 \in \Sigma$ we get the corresponding leave of $\hat{I}_i(t)$ as the difference of the possibly singular at $w_1$ function $I_j(t)$ and a certain finite sum of the branches of $g$. By our assumption, all the branches of $g$ are regular at $w_1$. Therefore the singularity of the considered leave (and hence of each leave) of $\hat{I}_i(t)$ at $w_1$ is the same, up to addition of a regular germ, as of $I_j(t)$. Now the analytic representation of $I_j(t)$ at $w_1$ is given by Lemmas 3.1-3.3. Accordingly, $I_j(t)$ is regular at the interior points $w_1$ of $\gamma$, and it has a logarithmic ramification at the end points $w_1$ of $\gamma$. In the last case, Lemma 3.3 provides an analytic representation of the singularity of $I_j(t)$ at $w_1$, and hence of each leave of $I(t)$. This completes the proof of Theorem 4.1.

The combinatorial monodromy has been explicitly computed in some examples given in [17] (for $g$ an algebraic function). In particular, these examples show that the combinatorial monodromy of $I_i$ may be infinite even in the case where all the local ramifications of $I$ are finite. Below we propose some additional natural questions concerning the structure of the combinatorial monodromy.

5. Fuchsian differential equations. First we remind shortly the main facts related to the structure of linear differential equations in complex domain (see, for example, [15, 12]).

Consider the equation

\[ y^{(k)} + a_{k-1}(x)y^{(k-1)} + \ldots + a_1(x)y' + a_0(x)y = 0 \]

with the coefficients $a_{k-1}(x), \ldots, a_0(x)$ regular and univalued in the domain $\Omega = \mathbb{C} \setminus \{x_0, \ldots, x_m\}$. We do not specify at this stage the character of possible singularities of $a_j(x)$ at the points $x_0, \ldots, x_m$.

**Proposition 5.1.** Any solution $y(x)$ of (5.1) is a regular multivalued function in $\Omega$, satisfying the following additional property (F): For any point $w \in \Omega$ the linear subspace $L_w$ spanned by all the branches of $y(x)$ at $w$ in the space $O(w)$ of all the analytic germs at $w$, has dimension at most $k$.

Any regular multivalued function $v(x)$ in $\Omega$ with the property (F) satisfies a certain equation of the form (5.1) of order at most $k$ with all the coefficients regular and univalued in the domain $\Omega$. 
ANALYTIC CONTINUATION OF CAUCHY-TYPE INTEGRALS 383

Proof. This is a classical fact (see, for example, [15, 12]). To simplify our presentation we give a sketch of the proof. First of all, for a given equation (5.1) the germs of its solutions at any point \( w \in \Omega \) form a linear subspace \( V_w \) of dimension \( k \) in the space \( \mathcal{O}(w) \) of all the analytic germs at \( w \). This subspace \( V_w \) can be identified with the complex space \( \mathbb{C}^k \) via the isomorphism associating to each solution \( y(x) \) its initial data \( y(w), y'(w), \ldots, y^{k-1}(w) \). Now let \( y(x) \) be a given solution of (5.1). This solution can be analytically continued as a regular function along any path \( \sigma \) in \( \Omega \) (see, for example, [15]), and this analytic continuation remains a solution to (5.1). Hence the full analytic continuation of \( y(x) \) is a regular multivalued function in \( \Omega \). The germs of all the branches of \( y(x) \) at any point \( w \in \Omega \) belong to \( V_w \) and therefore they span a linear subspace in \( \mathcal{O}(w) \) of the dimension at most \( k \).

In the opposite direction, let \( v(x) \) be a regular multivalued function \( v(x) \) in \( \Omega \) with the property (F). Let us denote, as above, by \( L_w \) the linear subspace of \( \mathcal{O}(w) \) generated by the germs of all the branches of \( v(x) \) at \( w \). Notice first of all that the dimension \( k \) of \( L_w \) does not depend on \( w \). Indeed, the analytic continuation along any path \( \sigma \) in \( \Omega \) joining \( w_1 \) and \( w_2 \) (whenever defined) is a linear mapping of \( \mathcal{O}(w_1) \) to \( \mathcal{O}(w_2) \), and the backward continuation provides an inverse to it. Therefore, this analytic continuation induces a linear isomorphism \( \Lambda(\sigma) \) between \( L_{w_1} \) and \( L_{w_2} \).

Now let us fix a point \( w_0 \in \Omega \) and a basis \( v_1, \ldots, v_k \) of \( L_{w_0} \). The germs \( v_1, \ldots, v_k \) are linear combinations with the constant coefficients of certain branches of \( v(x) \) at \( w_0 \). Hence they allow an analytic continuation to the regular multivalued function \( v_j(x), j = 1, \ldots, k \) in \( \Omega \).

Now we can write the required differential equation for as follows:

\[
W = \det \begin{pmatrix} v & v_1 & \cdots & v_k \\ v' & v'_1 & \cdots & v'_k \\ \vdots & \vdots & \ddots & \vdots \\ v^{(k-1)} & v^{(k-1)}_1 & \cdots & v^{(k-1)}_k \\ v^{(k)} & v^{(k)}_1 & \cdots & v^{(k)}_k \end{pmatrix} = 0.
\]

It is well known that a function \( v(x) \) satisfies the differential equation (5.2) if and only if it is a linear combination of the functions \( v_j(x), j = 1, \ldots, k \). To bring this equation to the form (5.1), we open the determinant (5.2) with respect to the first column, and divide all the coefficients by the coefficient
of $v^{(k)}$, which is the Wronskian

$$W = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_k \\ v'_1 & v'_2 & \cdots & v'_k \\ \vdots & \vdots & \ddots & \vdots \\ v_{(k-1)} & v_{(k-1)} & \cdots & v_{(k-1)} \end{pmatrix}$$

of the functions $v_j(x), j = 1, \ldots, k$. Since these functions are linearly independent everywhere in $\Omega$, this Wronskian does not vanish in $\Omega$. Therefore, we get a $k$-th order differential equation of the form (5.1). The coefficients $a_j$ are the ratios of the appropriate minors of the matrix (5.2) and of $W$, so $a_j, j = 1, \ldots, k$ are regular in $\Omega$.

To see that these coefficients are univalued in $\Omega$ we notice that as we follow a closed path $\sigma$ starting and ending at the point $w_0$, the analytic continuations $\tilde{v}_j(x), j = 1, \ldots, k$ of the functions $v_j(x), j = 1, \ldots, k$ are obtained from $v_j(x), j = 1, \ldots, k$ by a non-degenerate linear transformation $M(\sigma)$. After this transformation all the minors of the matrix (5.2) obtained by omitting the first column and a certain row, are multiplied by $\det M(\sigma)$. Therefore $a_j$, being the ratios of such minors, do not change. This completes the proof of Proposition 5.1.

Let us give a couple of examples, illustrating the last theorem.

**Example 1.** $v(x) = \log G(x)$, with $G$ a meromorphic function. Let us take $\Omega$ as $\mathbb{C}$ with all the poles and zeroes of $G$ deleted. Fix a point $w_0 \in \Omega$ and let $v_0$ be a germ of a certain branch of $v$ at $w$. Then any other branch $v_j$ of $v$ satisfies $v_j = v_0 + 2n\pi i$. Therefore, the subspace $L_w$ of $\mathcal{O}(w)$ generated by the germs of all the branches of $v(x)$ at $w$ is two-dimensional, and it is spanned by $v_0(x)$ and 1. According to Theorem 5.1, $v$ satisfies an equation (5.1) of the second order. Of course, one can guess the form of this equation:

$$y'' - \left( \frac{G''}{G'} - \frac{G'}{G} \right) y' = 0.$$

**Example 2.** $v(x) = \log(x-x_1)\log(x-x_2)\log(x-x_3)$, with $x_1, x_2, x_3$ three different points in $\mathbb{C}$. We take $\Omega = \mathbb{C} \setminus \{x_1, x_2, x_3\}$. Fix a point $w \in \Omega$ and wix certain branches $h_1, h_2, h_3$ of $\log(x-x_1), \log(x-x_2), \log(x-x_3)$, respectively, at $w$. Denote by $v_0$ the branch of $v$ at $w$ which is the product $h_1h_2h_3$. Then any other branch $\tilde{v}$ of $v$ has the form $v_0$ plus a linear combination with constant coefficients of $h_1, h_2, h_3$ and $h_1h_2, h_1h_3, h_2h_3$. Consequently, $v$ satisfies an equation (5.1) of the order at most 7.
Example 3. The algebraic function

\[ g = c_1 \prod_{i=1}^{i=n_1} (x - a_1^i)^{r_1^i} + c_2 \prod_{i=1}^{i=n_2} (x - a_2^i)^{r_2^i} + \ldots + c_l \prod_{i=1}^{i=n_l} (x - a_l^i)^{r_l^i}, \]

where \( r_i^j \) are any rational numbers, and \( c_m \) and \( a_q^k \) are any complex numbers, satisfies the linear algebraic differential equation of order \( l \).

Indeed, the branches of each product in this sum differ from one another by a constant coefficient. Hence all the branches of the sum spend the space of the dimension at most \( l \).

Notice that in all the three examples the coefficients of (5.1) are rational, since the functions \( v \) considered satisfy the growth conditions at singularities, as stated below.

Example 4. Let us give now an example of a multivalued function which does not satisfy any linear equation with univalued coefficients. Let \( f(x) = \exp(\frac{1}{2\pi} \log^2 x) = x^{\frac{1}{2\pi}} \log^2 x \). This function is regular multivalued in \( \Omega = \mathbb{C} \setminus \{0\} \). From the definition we see that for any branch \( f_i(x) \) the result \( f_{i+1}(x) \) of its analytic continuation around zero satisfies \( f_{i+1}(x) = x f_i(x) \). Therefore the germs of all these branches are linearly independent at any point \( w \) of \( \Omega \). By Proposition 5.1 we conclude that \( f \) does not satisfy any linear equation with univalued coefficients in \( \Omega \).

Now we are ready to prove our main results, concerning the conditions for the Cauchy-type integral \( I(t) \) given by (1.1) to satisfy a linear differential equations with regular and univalued coefficients. Let us remind that we assume that the integrand \( g(z) \) in (1.1) satisfies the following:

Assumption A. There exists a finite set of points \( \Sigma_1 = \{z_1, \ldots, z_m\} \) such that each branch of \( g \) appearing in the integral (1.1) can be analytically continued as a regular function along any path in \( \Omega = \mathbb{C} \setminus \Sigma_1 \). Shortly we shall say that \( g \) is a regular multivalued function on \( \Omega \).

In addition we’ve assumed that the points of \( \Sigma_1 \) do not belong to \( \gamma \), and we’ve denoted by \( U \) the set \( U = \mathbb{C} \setminus (\Sigma \cup \Sigma_1) \).

Let us remind also that the integral representation (1.1) defines \( I(t) \) as a collection of univalent regular functions \( I_i(t) \) in each domain \( D_i \) of the complement of \( \gamma \) in \( \mathbb{C} \). We’ve denoted by \( \hat{I}_i(t) \) the complete analytic continuations of the functions \( I_i(t) \), which by Theorem 4.1 are regular multivalued functions over \( U \).

Theorem 5.1. If the integrand \( g \) satisfies in \( \Omega \) a linear differential equation of order \( k \) with regular and univalued in \( \Omega \) coefficients, then each \( \hat{I}_i(t) \) satisfies in \( U \) a linear differential equation of order at most \( k + 1 \) with
regular and univalued in $U$ coefficients. If one of the functions $I_j(t)$ in the domain $D_j$ vanishes identically, then each $\hat{I}_i(t)$ satisfies an equation of order at most $k$.

Proof. By Theorem 4.1 all the branches of $\hat{I}_i(t)$ over the domain $D_j$ are obtained by adding to $I_j(t)$ certain sums of the branches of $g$. Now if the branches of $g$ span at each point of $U \subset \Omega$ a linear subspace of dimension $k$, we conclude that all the branches of $\hat{I}_i(t)$ span at each point of $D_j$ a linear subspace of dimension at most $k + 1$. In the case where $I_j(t) \equiv 0$ in $D_j$, this dimension is at most $k$. Application of Proposition 5.1 now completes the proof of the theorem. □

Corollary 5.1. If one of the functions $I_j(t)$ in the domain $D_j$ vanishes identically, then each $\hat{I}_i(t)$ satisfies the same differential equation as the integrand $g$ satisfies.

Proof. This follows from the fact that in the case under consideration the subspaces spanned by the branches of $\hat{I}_i(t)$ are contained in the subspaces spanned by the branches of $g$, as shown in the proof of Theorem 5.1 above. This follows also directly from the fact that in our case $\hat{I}_i(t)$ are equal to certain sums of the branches of $g$ (see Theorem 4.1 above and [17]). □

Question. Is it possible to relate the equations for $g$ and for $\hat{I}_i(t)$ in general?

In some cases the inverse conclusion to Theorem 5.1 is also true.

Theorem 5.2. Let all the singular points of $g$ belong to one of the domains $D_j$. In this case if one of the functions $\hat{I}_i(t)$ satisfies in $U$ a linear differential equation with regular and univalued in $U$ coefficients, the same is true for $g$.

Proof. By Proposition 5.1 and Theorem 4.1 we see that the combinatorial monodromy of $I_i(t)$ is finite-dimensional. But since all the singular points of $g$ belong to one of the domains $D_j$, we see that the image of the combinatorial monodromy contains, in particular, the sums of a certain analytic germ with all the branches of $g$. Hence these branches span a finite-dimensional space. □

Question. Is the result of Theorem 5.2 true without restrictions on the position of singularities of $g$?

Let us now turn to the case where the Cauchy-type integrals $I(t)$ satisfy linear differential equations with rational coefficients. It is well known that if we assume in Proposition 5.1 that the function $v(x)$ (in addition to the property (F)) has at most a polynomial growth at its singularities (including infinity) then the coefficients of the resulting linear differential equation for $v$ are rational. Such functions naturally appear as solutions of Fuchsian equations (see [15, 12]). We obtain the following result:
THEOREM 5.3. If the integrand $g$ in (1.1) satisfies a Fuchsian differential equation, then $\tilde{I}_i(t)$ satisfy linear differential equations with rational coefficients. In particular, this is true for $g$ an algebraic function.

Question. Is it possible to give a “sharp” bound for the order of the Fuchsian equations to which satisfies a given algebraic function? In what cases the resulting equation has order 1? Order 2? (Compare Example 3 above). Having an answer to this question, we can bound the order of the equations for the Cauchy integrals of algebraic functions.

Question. Is it possible to give an accurate description of the ”group-theoretic” properties of the combinatorial monodromy (and hence, by Theorem 4.1, of the monodromy group of the Cauchy integral)? In particular, can we give in the case of Cauchy integrals an answer to the questions studied in [19] for general Fuchsian equations?

REFERENCES


Abstract. The paper deals with approximate null-controllability problem for a linear distributed control system with unbounded input operator, connected in a series to another distributed system without control. Initial state of the second distributed system is considered as a control parameter. Approximate null-controllability conditions by initial states of the second distributed system are obtained. Several applications to the approximate null-controllability for control partial equations by hyperbolic controller are considered.


AMS(MOS) subject classification. 93B05, 93C25, 93C20.

1. Introduction. Controllability problem is a classical problem in Control Theory. There is a literature on this topic. Investigations in this field were very intensive in the last years, so it is impossible to describe even the main progress has been made within a single paper.

Research in this field has been made in most cases for a single control system.

In this paper we will deal with linear structured distributed system consisting of two linear distributed systems interconnected in a series in such a way that the control function of the first system is the output of the second system without control, so the control system is controlled by the initial state of the second distributed system.

---

* Holon Academic Institute of Technology, Holon, Israel
† Bialystok Technical University, Bialystok, Poland
Byelorussian State Technological University, Minsk, Republic Belarus
The case when the input operator of the control distributed system is bounded has been investigated in our previous paper [10].

The paper continues research has been made [10]. The approximatenull-controllability problem for the case of inbounded input operator is considered.

2. Problem statement. Let $X, U, Z$ be Hilbert spaces, and let $A, C$ be infinitesimal generators of strongly continuous $C_0$-semigroups $S_A(t)$ in $X$ and $S_C(t)$ in $Z$ correspondingly in the class $C_0$ [5],[8]. Consider the abstract evolution control equation [5],[8]

\begin{equation}
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad u(t) = Kz(t), \quad 0 \leq t < +\infty,
\end{equation}

where $z(t)$ is a mild solution of the another evolution equation of the form

\begin{equation}
\dot{z}(t) = Cz(t), \quad z(0) = z_0, \quad 0 \leq t < +\infty.
\end{equation}

Here $x(t), x_0 \in X$, $u(t) \in U$, $z(t), z_0 \in Z$, $B : U \to X$ is a linear unbounded operator, $K : Z \to U$ is a linear possibly unbounded onto operator.

Equation (2.2) is said to be a controller equation. The control $u(t)$ is defined by $u(t) = Kz(t)$ as an output of controller equation (2.2).

Let $x(t, 0, x_0, u(\cdot))$ be a mild solution of equation (2.1) with initial condition $x(0) = x_0$, and let $u(t, 0, z_0) = Kz(t, 0, z_0)$, where $z(t, 0, z_0)$ is a mild solution of equation (2.2) with initial condition $z(0) = z_0$.

If $u = Kz, (u \in U$ and $z \in Z)$, then we say that $u$ is the image of $z$, generated by the operator $K$.

We will use the functional analytic approach developed by Salamon [15] and Weiss [20].

Let $\mu \notin \sigma_A$. We will consider the spaces $W$ and $V$ defined as follows (see, for example, [3],[11],[20]):

$W$ is the domain of the operator $A$ with the norm $\|x\|_{\mu} = \|(\mu I - A)x\|$;

$V$ is the closure of $X$ with respect to the norm $\|x\|_{-\mu} = \|(R_A(\mu)x)\|$,

where $R_A(\mu) = (\mu I - A)^{-1}$.

Obviously $W \subset X \subset V$ with continuous dense imbeddings.

It is known that for each $\mu_1, \mu_2, \mu_1 \neq \mu_2$ the norm $\|\cdot\|_{\mu_1}$ is equivalent to the norm $\|\cdot\|_{\mu_2}$, the norm $\|\cdot\|_{-\mu_1}$ is equivalent to the norm $\|\cdot\|_{-\mu_2}$, and all $\|\cdot\|_{\mu}$ are equivalent to the appropriate graph norm on $D(A)$, so the Banach spaces $W$ and $V$ do not depend on $\mu$ [20].

It is well-known [5],[9],[8],[15],[20], etc. :

- for each $t \geq 0$ the operator $S(t)$ has an unique continuous extension $S(t)$ on the space $V$ and the family of operators $S(t) : V \to V$ is the semigroup in the class $C_0$ with respect to the norm of $V$ and the corresponding infinitesimal generator $A$ of the semigroup $S(t)$ is the closed dense extension of the operator $A$ on the space $V$ with domain $D(A) = X$.
• the sets of the generalized eigenvectors of operators $A, A^*$ and $A, A^*$ are the same;
• for each $\mu \notin \sigma (A)$ the operator $R_A (\mu)$ has a unique continuous extension to the operator $R_A (\mu) : V \to X$;
• a mild solution $x (t, 0, x_0, u (\cdot))$ of equation (2.1) with initial condition (2.2) is defined by the following representation formula

$$x(t, 0, x_0, u(\cdot)) = S(t)x_0 + \int_0^t S(t - \tau)Bu(\tau)d\tau,$$

where the integral in (2.3) is understood in the Bochner's sense [5] with respect to the topology of $V$.

We say that the control $u (\cdot) \in L_2^{\text{loc}} ([0, +\infty), U)$ is vanishing after time moment $t_2$, if $u (t) = 0$ a.e. on $[t_2, +\infty)$ [10].

Denote

$$u_{t_2} (t, 0, z_0) = \begin{cases} u (t, 0, z_0) & \text{if } 0 \leq t \leq t_2, \\ 0 & \text{if } t > t_2. \end{cases}$$

The initial data $z_0 \in Z$ for equation (2.2) is considering as a control.

**Definition 2.1.**

*Equation (2.1) is said to be approximately null-controllable on $[0, t_1]$ in the class of controls vanishing after time moment $t_2, 0 < t_2 < t_1$, if for each $x_0 \in X$ and $\varepsilon > 0$ there exists a control $u (t), t \geq 0$, such that $u (\cdot) \in L_2 ([0, t_2], U), u (t) = 0$ a.e. on $[t_2, +\infty)$, and

$$\| x (t_1, 0, x_0, u (\cdot)) \| < \varepsilon.$$*

**Definition 2.2.**

*Equation (2.1) is said to be approximately null controllable on $[0, t_1]$ by controller (2.2), if for each $x_0 \in X$ and $\varepsilon > 0$ there exists $z_0 \in Z$, such that

$$\| x (t_1, 0, x_0, u_{t_2} (\cdot, 0, z_0)) \| < \varepsilon.$$*

**2.1. The assumptions.** The assumptions on the operators $A, C$ and $B$ are listed below.

1. The operators $A$ and $C$ have purely point spectrums $\sigma_A$ and $\sigma_C$ with no finite limit points. Eigenvalues of both $A$ and $C$ have finite multiplicities.

2. There exists $T \geq 0$ such that all mild solutions of the equation $\dot{x} (t) = Ax (t)$ are expanded in a series of generalized eigenvectors of the operator $A$ converging$^1$ for any $t > T$ uniformly in each segment $[T_1, T_2], T < T_1 < T_2$.

3. The unbounded operator $B$ is bounded as an operator from $U$ to $V$.

---

$^1$ in the topology of $X$. 
4. \( \int_0^t S_A(t-\tau)Bu(\tau)\,d\tau \in X \) for any \( u(\cdot) \in L_2([0,t],U) \) and the operator \( \Phi(t) : L_2([0,t],U) \to X \) defined by the formula
\[
(2.5) \quad \Phi(t)u(\cdot) = \int_0^t S_A(t-\tau)Bu(\tau)\,d\tau
\]
is bounded for each \( t \geq 0 \).

5. Let the spectrum \( \sigma_A \) of the operator \( A \) is infinite without finite limit points and consists of numbers \( \lambda_j, \ j = 1,2,\ldots \), with multiplicities \( \alpha_j \), enumerated in such a way that their absolute values are non-decreasing with respect to \( j \) (i.e. \(|\lambda_j| \geq |\lambda_{j+1}|\)), and let the sequence
\[
(2.6) \quad t^k \exp \lambda_j t, j=1,2,\ldots, k=0,\ldots,\alpha_j - 1
\]
be minimal on \([0,\delta]\) for some \( \delta > 0 \). It follows from this assumption, that there exists \( T \geq 0 \) such that the closure of the attainable set \( K_t \) defined by
\[
(2.7) \quad K_t = \left\{ x \in X : \exists u(\cdot) \in L_2([0,t],U) : \right\}
\]
do not depends on \( t_1 \) for any \( t_1 > T + \delta \).

We consider the operator \( K \) with domain \( D(K) \) such that \([15], [19]\)
1. (a) \( z(t) \in D(K) \) for a.e. \( t > 0 \);
2. (b) \( z(\cdot) \in L_2([0,t],Z), \forall t > 0 \);
3. (c) the operator \( Q : Z \to L_2([0,t],Z), Qx = z(t), t \in [0,t_1] \) is bounded for each \( t_1 > 0 \).

3. Main results. Denote:
\[
\text{Range}\{\lambda I - A, \mathcal{R}_A(\mu)B\} = \{ z \in X : \exists x \in X, \exists u \in U, z = (\lambda I - A)x + \mathcal{R}_A(\mu)Bu \}.
\]

**Theorem 3.1.**

1. For equation (2.1) to be approximately null-controllable on \([0,t_1]\), \( t_1 > T \), by distributed controller (2.2), it is necessary that the condition
\[
(3.1) \quad \text{Range}\{\lambda I - A, \mathcal{R}_A(\mu)B\} = X, \forall \lambda \in \sigma_A, \forall \mu \notin \sigma_A,
\]
or, equivalently,
\[
\text{Ker}(\lambda I - A^*) \cap \text{Ker}B^* = \{0\}, \forall \lambda \in \sigma_A.
\]
holds.

2. If the subspace \( K_{S_C}(\cdot) Z \) of \( L_2([0,t_2],U) \) is dense in \( L_2([0,t_2],U) \) for some \( t_2 > 0 \), then condition (3.1) is sufficient for approximate null controllability of equation (2.1) on \([0,t_1]\), \( t_1 > T + \delta \), by distributed controller

---

2 The integral in (2.5) is considered in the topology of \( V \).

3 i.e. there exists the sequence biorthogonal to the the above sequence with respect to the scalar product in \( L_2[0,\delta] \).
(2.2).

Proof. Sufficiency. Assume that condition (3.1) holds true and \( t_1 > T + \delta \).

Using (2.4) in (2.7) define the attainable set \( K^0_{t_1} \) generated by controls \( u(t) \), vanishing after \( t_1 - T \), by the formula

\[
K^0_{t_1} = \left\{ x \in X : \exists u(\cdot) \in L_2([0,t_1-T],U) : x = \int_{t_1-T}^{t_1} S_A(t_1-\tau) Bu(\cdot) d\tau \right\}
\]

Let \( g \in (K^0_{t_1})^\perp = \{ g \in X^* : (x,g) = 0, \forall x \in K^0_{t_1} \} \). By (3.2), we obtain

\[
\int_0^{t_1-T} S_A(t-\tau) Bu(\tau) d\tau, g) = 0, \forall u(\cdot) \in L_2([0,t_1-T],U), \forall t \geq 0.
\]

If the linear operator \( B \) is bounded, then

\[
\int_0^{t_1-T} S_A(t-\tau) Bu(\tau) d\tau, g) = \int_0^{t_1-T} (S_A(t-\tau) Bu(\tau), g) d\tau,
\]

\( \forall u(\cdot) \in L^2_{\text{loc}}([0,+\infty),U), \forall t < T \), so one can continue the proof as well as in the [10]. If the linear operator \( B \) is unbounded, then we cannot use (3.4), because there exists \( u(\cdot) \in L_2([0,t_1],U) \) and \( \tau \in [0,t_1] \), such that

\[
S_A(t-\tau) Bu(\tau) \in V, \text{but } S_A(t-\tau) Bu(\tau) \notin X.
\]

We have \( g \in X^* \), but we cannot assure \( g \in V^* \).

Let \( P_j \) be a projector to a generalized eigenspace of \( A \) at \( \lambda_j \in \sigma_A \), \( j = 1, 2, \ldots \), and let

\[
\Lambda_j = \text{diag}\{\Lambda_{j1}, \ldots, \Lambda_{jm_j}\},
\]

where

\[
\Lambda_{jk} = \begin{pmatrix}
\lambda_j & 1 & 0 & \cdots & 0 \\
0 & \lambda_j & 1 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \lambda_j
\end{pmatrix}
\]

be a Jordan \((\beta_{jk} \times \beta_{jk})\)-matrix. We have

\[
(S(t)P_jx, g) = (\Phi_j, g) \exp(\Lambda_j t) (x, \Psi_j)^T, \forall x \in X,
\]

where

\[
\Phi_j = \left\{ \left\{ \varphi_{jk1}, \varphi_{jk2}, \ldots, \varphi_{jk\beta_{jk}} \right\} : k = 1, \ldots, m_j \right\},
\]

\[
\Psi_j = \left\{ \left\{ \psi_{jk1}, \psi_{jk2}, \ldots, \psi_{jk\beta_{jk}} \right\} : k = 1, \ldots, m_j \right\},
\]

\[
(\Phi_j, g) = \left\{ \left\{ (\varphi_{jk1},g), \ldots, (\varphi_{jk\beta_{jk}},g) \right\} : k = 1, \ldots, m_j \right\},
\]

\[
(x, \Psi_j) = \left\{ \left\{ (x, \psi_{jk1}), \ldots, (x, \psi_{jk\beta_{jk}}) \right\} : k = 1, \ldots, m_j \right\}.
\]

Since the sets of generalized eigenvectors of operators \( AA^* \) and \( A, A^* \) are the same, so \( (v, \Psi_j), j = 1, 2, \ldots \) is well defined for any \( v \in V \), so one can
show that

\begin{equation}
(\Phi_{jk}, g) \exp (\Lambda_j t) (v, \Psi_j^T) = \sum_{k=0}^{\alpha_j-1} \exp (\lambda_j t) \frac{t^k}{k!} (\Phi_j, g) E_j^k (v, \Psi_j^T),
\end{equation}

where $\alpha_j \times \alpha_j$-matrix $E_j$ is defined by

\[ E_j = \text{diag} \{E_{j1}, \ldots, E_{jm_j}\}, \]

where

\[
E_{jk} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

is $\beta_{jk} \times \beta_{jk}$ matrix.

In accordance with assumption 3 (see the list of assumptions above) we obtain by (3.5), that

\begin{equation}
S(t)v = \sum_{j=1}^{\infty} \Phi_j \exp (\Lambda_j t) (v, \Psi_j)^T = \sum_{j=1}^{\infty} \sum_{k=0}^{\alpha_j-1} \exp (\lambda_j t) \frac{t^k}{k!} (\Phi_j, g) E_j^k (v, \Psi_j^T), \forall t > T, \forall v \in V,
\end{equation}

where $\sum_{j=1}^{\infty}$ is considered in the topology of $V$.

Let $\gamma_{kl}(t), k = 1, 2, \ldots, l = 0, 1, \ldots, \alpha_k - 1$ be the sequence of functions biorthogonal to the sequence (2.6) on $[0, t_1 - T]$ ($t_1 - T > \delta$). Using (3.6) in (3.3) with $u(t) = u\gamma_{kl}(t_1 - t), k = 1, 2, \ldots, l = 0, 1, \ldots, \alpha_k - 1$, where $u \in U$, we obtain that

\begin{equation}
(\Phi_j, g) E_j^k B^* \Psi_j^T = 0, j = 1, 2, \ldots, k = 0, 1, \ldots, \alpha_j - 1.
\end{equation}

Solving linear algebraic system (3.7) with respect to $(\Phi_j, g)$ provided that (3.1) holds, we obtain

\begin{equation}(\Phi_j, g) = 0, j \in \mathbb{N}.\end{equation}

(3.7) and (3.8) imply $S^*(t_1)g = 0$, therefore, $g \in (S(t_1)X)^\perp$. We have $K(t_1)^\perp \subseteq (\text{Range} S(t_1))^\perp$, so

\begin{equation}
S(t_1)X \subseteq K(t_1)^\perp, \forall t_1 > T + \delta.
\end{equation}

This shows that equation (2.1) is approximately null-controllable on $[0, t_1]$ in the class of controls vanishing after $t_1 - T, \forall t_1 > T + \delta$.

Let $\delta < t_1 - T < t_2$. In accordance with assertion proven above and provided that (3.1) holds, we have:

for each $x_0 \in X$ and $\varepsilon > 0$, there exists control $\hat{u}(\cdot) \in L_2([0, t_1 - T], U)$ such that $\|x(t_1, 0, x_0, \hat{u}(\cdot))\| < \frac{\varepsilon}{2}$.

In virtue of the density of the subspace $KS_C(\cdot)B$ in $L_2([0, t_2], U)$ one
can approximate the control \( \hat{u}(\cdot) \) by outputs of equation (2.2) with any arbitrary small accuracy in the topology of \( L_2([0,t_2],U) \), i.e. for given 
\( \hat{u}(\cdot) \in L_2([0,t_2],U) \) and any \( \varepsilon > 0 \) there exists \( z_0 \in Z \) such that \( u(t,0,z_0) \) can be taken arbitrary closely to \( \hat{u}(\tau) \) and, taking into account linearity of the system, we obtain 
\[ ||x(t_1,0,x_0,u(\cdot,0,z_0)) - x(t_1,0,x_0,\hat{u}(\cdot))|| < \frac{\varepsilon}{2}. \]
Hence
\[ ||x(t_1,0,x_0,\hat{u}(\cdot,0,u_0))|| < ||x(t_1,0,x_0,\hat{u}(\cdot))|| + 
+ ||x(t_1,0,x_0,u(\cdot,0,u_0)) - x(t_1,0,x_0,\hat{u}(\cdot))|| < \varepsilon. \]
This proves the sufficiency.

**Necessity.** The necessity of condition (3.1) immediately follows from approximate null-controllability conditions proven in [17].

This completes the proof of the theorem.

**Remark 1.**

Theorem 3.1 shows that if the subspace \( KSC(\cdot)Z \) of \( L_2([0,t_2],U) \) is dense in \( L_2([0,t_2],U) \) for some \( t_2 > 0 \), then equation (2.1) is approximately null-controllable on \([0,t_1], \forall t_1 > T + \delta, \) by distributed controller (2.2), if and only if it is approximately null-controllable on \([0,t_1], \forall t_1 > T + \delta, \) in the class of control vanishing after \( t_2 = t_1 - T. \)

4. Approximate null-controllability of abstract boundary control problem by abstract boundary controller. Let \( X, Y, U, Z, Y_1, Y_2 \) be Hilbert spaces. Consider the abstract boundary control problem

\[ \begin{align*}
(4.1) \quad \dot{x}(t) &= Lx(t), \\
\Gamma x(t) &= Bu(t), \\
x(0) &= x_0, \\
u(t) &= Kz(t),
\end{align*} \]

where \( z(t) \) is a mild solution of the another evolution equation of the form

\[ \begin{align*}
(4.2) \quad z(t) &= Mz(t), \\
Hz(t) &= 0, \\
(4.3) \quad z(0) &= z_0.
\end{align*} \]

Equation (4.2)-(4.3) is called boundary controller,

Here \( L : X \to X \) and \( M : Z \to Z \) are linear unbounded operators with dense domains \( D(L) \) and \( D(M); B : U \to Y_1 \) is a linear bounded one-to-one operator, \( K : Z \to U \) is a linear (possibly unbounded) onto operator, \( \Gamma : X \to Y_1 \) and \( H : Z \to Y_2 \) are linear operators satisfying the following conditions:

- \( \Gamma \) and \( H \) are onto, \( \text{Ker}\Gamma \) is dense in \( X, \text{Ker}\, H \) is dense in \( Z; \)
there exists a \( p \in \mathbb{R} \) such that \( \mu I - L \) is onto and \( \ker(\mu I - L) \cap \ker \Gamma = \{0\} \);

there exists a \( p \in \mathbb{R} \) such that \( \mu I - M \) is onto and \( \ker(\mu I - M) \cap \ker \mathcal{H} = \{0\} \).

Problems (4.1) and (4.3) are assumed to be well-posed. The problem (4.1) is an abstract model for classical control problems described by linear partial differential equations of both parabolic and hyperbolic type when a control acts through the boundary and it is a mild solution of linear partial differential equation without control. The control process is released by initial condition (4.3) which is considered as a control.

Now we transform abstract boundary control problems (4.1)-(4.3) to problem (2.1)-(2.2) by the process described in [15],[17]. Consider the space \( W_1 = \ker \mathcal{G} \). We have \( W \subset D(L) \subset X \) with continuous dense injection. Define the operator \( A : W_1 \rightarrow X \) by

\[
Ax = Lx \quad \text{for} \quad x \in W_1.
\]

For any \( y \in Y_1 \) define

\[
By = Lx - Ax, \quad x \in \Gamma^{-1}(y) = \{ z \in D(L) : \Gamma x = y \}.
\]

Given \( u \in U \) denote \( \tilde{B}u = \hat{B}Bu \). The operator \( B : U \rightarrow V \) is bounded, but the operator \( \hat{B} : Y_1 \rightarrow X \) defined by (4.5) is unbounded, so the operator \( \hat{B} : U \rightarrow X \) is unbounded. It follows from (4.5) that

\[
Lx = Ax + \hat{B}u,
\]

\[
\Gamma x = Bu.
\]

Since the abstract boundary control problem under consideration is uniformly well-posed, the operator \( A \) generates a strongly continuous semigroup of bounded operators in the class \( C_0 \). Hence given abstract boundary control problem is equivalent to the control problem

\[
\dot{x}(t) = Ax(t) + \hat{B}u(t),
\]

\[
x(0) = x_0.
\]

By the same way consider the space \( W_2 = \ker \mathcal{H} \). Again, we have \( W_2 \subset D(M) \subset Z \) with continuous dense injection. Define the operator \( C : W_2 \rightarrow Z \) by

\[
Cz = Mz \quad \text{for} \quad z \in W_2.
\]

Hence

\[
\dot{z}(t) = Cz(t),
\]

\[
z(0) = z_0.
\]

We assumed all the assumptions of subsection "The assumptions" for above operators \( A, C \) and \( K \) to be hold.

So the following result can is obtained as an application of Theorem 3.1:
Theorem 4.1.
1. For equation (4.1) to be approximately null-controllable on \([0, t_1]\) by boundary controller (4.2), it is necessary, that the condition

\[
\text{Range}\{\lambda I - A, R_A(\mu) \hat{BB}\} = X, \forall \mu \notin \sigma_A, \forall \lambda \in \sigma_A, \tag{4.10}
\]

or, equivalently,

\[
\text{Ker}(\lambda I - A^*) \cap \text{Ker}B^*\hat{B}^* = \{0\}, \forall \lambda \in \sigma_A. \tag{4.11}
\]

2. If the set of functions \(u(\cdot), u(t) = Kz(t)\), where \(z(t)\) is a solution of boundary problem (4.2), are dense in \(L_2([0, t_2], U)\) for some \(t_2 > 0\), then condition (4.10) is sufficient for approximate null-controllability of equation (4.1) on \([0, t_1], t_1 > T + \delta\), by boundary controller (4.2).

Together with equation (4.1)-(4.3) consider the abstract elliptic equation

\[
Lx = \mu x \tag{4.12}
\]
\[
\Gamma x = y \tag{4.13}
\]

Since problem (4.1)-(4.3) is uniformly well-posed then for any \(y \in Y_1\) there exists the solution \(x_\mu = D_A(\mu)y\) of equation (4.12)-(4.13), where \(D_A(\mu) : Y_1 \to X\) is a linear bounded operator\(^4\).

It follows from (4.12)-(4.13) and (4.6) that

\[
Ax_\mu + \hat{B}y = \mu x_\mu, \quad Gx_\mu = y.
\]

Hence

\[
x_\mu = R_A(\mu) \hat{B}y
\]

so

\[
D(\mu) = R_A(\mu) \hat{B}. \tag{4.14}
\]

Using (4.14) in (4.10) we obtain that the following theorem is valid:

**Theorem 4.2.** For equation (4.1) to be approximately null-controllable on \([0, t_1]\) by boundary controller (4.2), it is necessary, that the condition

\[
\text{Range}\{\lambda I - A, D_A(\mu) B\} = X, \forall \mu \notin \sigma_A, \forall \lambda \in \sigma_A, \tag{4.15}
\]

or, equivalently,

\[
\text{Ker}(\lambda I - A^*) \cap \text{Ker}B^*\hat{B}^* = \{0\}, \forall \lambda \in \sigma_A. \tag{4.16}
\]

If the set of functions \(u(\cdot), u(t) = Kz(t)\), where \(z(t)\) is a solution of boundary problem (4.2), are dense in \(L_2([0, t_2], U)\) for some \(t_2 > 0\), then condition (4.15)(or (4.16)) is sufficient for approximate null-controllability of equation (4.1) on \([0, t_1], t_1 > T + \delta\), by boundary controller (4.2).

\(^4\) The operator \(D_\mu\) is defined by well-known Green formula for given boundary problem.
Remark. Theorem 4.2 shows that if the second condition of the theorem holds then

5. Approximate null-controllability of partial differential equations by hyperbolic controller. In this section we consider a linear partial control equation with boundary control governed by distributed controller described by partial differential equations. The results of the previous section can be applied for the investigation of approximate null-controllability by distributed controller for the boundary control problem obtained below.

We will restrict our research by the case when boundary problem (4.1) is self-adjoint.

Consider partial parabolic differential equation [12]
\[
\frac{\partial y}{\partial t}(t, x) = \frac{\partial}{\partial x} \left( p_1(x) \frac{\partial y}{\partial x}(t, x) \right) + p_2(x) y(t, x), \quad t \geq 0, \ 0 \leq x \leq l.
\]
with non-homogeneous regular boundary conditions [12]
\[
\begin{align*}
    a_0 y(t, 0) + a_1 \frac{\partial y}{\partial x}(t, 0) &= a_2 u(t) \\
    b_0 y(t, l) + b_1 \frac{\partial y}{\partial x}(t, l) &= b_2 u(t)
\end{align*}
\]
subject to the initial conditions
\[
y(0, x) = \varphi_0(x), \quad 0 \leq x \leq l,
\]
where \( p_1(x) \) and \( p_2(x) \) are real functions, continuous in the segment \([0, l] \); \( p_1(x) > 0, p_2(x) \leq 0, x \in [0, l] ; \varphi_0(\cdot), \varphi_1(\cdot) \in L^2[0, l] ; a_j, b_j \in \mathbb{R}, |a_j| + |b_j| \neq 0, j = 0, 1, 2; a_0 a_1 \leq 0, b_0 b_1 \geq 0.

Here \( u(t) = z(t, \alpha), t \geq 0, \alpha \in [0, m] \), where \( z(t, x), t \geq 0, 0 \leq x \leq m \), is a mild solution of the partial hyperbolic equation
\[
\frac{\partial^2 z}{\partial t^2}(t, x) = \frac{\partial}{\partial x} \left( q_1(x) \frac{\partial y}{\partial x}(t, x) \right) + q_2(x) y(t, x), \quad t \geq 0, \ 0 \leq x \leq m.
\]
with homogeneous regular boundary conditions [12]
\[
\begin{align*}
    \alpha_0 z(t, 0) + \alpha_1 \frac{\partial z}{\partial x}(t, 0) &= 0 \\
    \beta_0 z(t, m) + \beta_1 \frac{\partial z}{\partial x}(t, m) &= 0
\end{align*}
\]
subject to the initial conditions
\[
z(0, x) = \psi_0(x), \frac{\partial z}{\partial x}(0, x) = \psi_1(x), \quad 0 \leq x \leq m,
\]
where \( q_1(x) \) and \( q_2(x) \) are real functions, continuous in the segment \([0, m] \); \( q_1(x) > 0, q_2(x) \leq 0, x \in [0, m] ; \psi_0(\cdot), \psi_1(\cdot) \in L^2[0, m] ; \alpha_j, \beta_j \in \mathbb{R}, |\alpha_j| + |\beta_j| \neq 0, j = 0, 1; \alpha_0 \alpha_1 \leq 0, \beta_0 \beta_1 \geq 0.

Partial equation (5.5) with boundary condition (5.6)-(5.7) will be called a hyperbolic controller. The pair \((\psi_0(x), \psi_1(x))\), \(0 \leq x \leq m\), inserted by
(5.8), is considered as a control of equation (5.1)-(5.3) governed by hyperbolic controller (5.5)-(5.8).

Let \( L : C^2([0, l]) \rightarrow L_2[0, l] \) be a differential operator with domain \( D(L) = C^2([0, l]) \), defined by
\[
L\varphi = (p_1(x)\varphi')' + p_2(x)\varphi, \varphi(\cdot) \in C^2([0, l]).
\]
Denote by \( C^2_0([0, l]) \) the set of functions \( \varphi(\cdot) \in C^2([0, l]) \) satisfying the homogeneous conditions
\[
(5.10) \quad a_0\varphi(0) + a_1\varphi'(t, 0) = 0, \\
(5.11) \quad b_0\varphi(l) + b_1\varphi'(l) = 0.
\]
Consider the linear unbounded operator \( A\varphi = L\varphi \), defined in the domain \( D(A) = C^2_0([0, l]). \) The operator \( A \) defined above is a linear symmetric operator having self-adjoint extension [12],[7].

Denote:
\[
(5.12) \quad \Gamma y = \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix}, y(\cdot) \in C(1)[0, l],
\]
where
\[
(5.13) \quad g_1(y) = a_0 y(0) + a_1 y'(0), \\
(5.14) \quad g_2(y) = b_0 y(l) + b_1 y'(l), \\
(5.15) \quad Bu = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} u, u \in \mathbb{R}.
\]

By (5.12)-(5.15) we will rewrite boundary conditions (5.2)-(5.3) in the form \( \Gamma y = Bu \).

Using (5.9)-(5.15) one can rewrite equation (5.1)-(5.3) in the form of (4.1) with the state space \( X = L_2[0, l] \) and with unbounded input operator; the corresponding operator \( A \) generates a self-adjoint semigroup (see, for example, [1],[6],[14],[15] and corresponding references for details).

Equation (5.5) can be written in new variables
\[
z_1(t, x) = z(t, x), z_2(t, x) = \frac{\partial z}{\partial t}(t, x)
\]
by the \( 2 \times 2 \) partial system of the form
\[
(5.16) \quad \begin{pmatrix} \frac{\partial z_1}{\partial t}(t, x) \\ \frac{\partial z_2}{\partial t}(t, x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\partial}{\partial x} (q_1(x)\frac{\partial}{\partial x}) + q_2(x) & 0 \end{pmatrix} \begin{pmatrix} z_1(t, x) \\ z_2(t, x) \end{pmatrix},
\]
subject the boundary conditions (5.6)-(5.7) with \( z_1(t, x) \) instead of \( z(t, x) \).

Denote:
\[
(5.17) \quad z(t) = \begin{pmatrix} z_1(t, \cdot) \\ z_2(t, \cdot) \end{pmatrix}, z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}, x \in [0, m],
\]
\[
(5.18) \quad Hz = \begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix}, z_1(\cdot) \in C(1)[0, m],
\]
where
(5.19) \( g_1 (z) = \alpha_0 z_1 (0) + \alpha_1 z_1' (0) \),
(5.20) \( g_2 (z) = \beta_0 z_1 (m) + \beta_1 z_1' (m) \).

By (5.17)-(5.20) one can rewrite boundary conditions (5.6)-(5.7) in the form \( H z (t) = 0. \)

Denote by \( C_0 (2) ([0, m]) \) the set of functions \( \psi (\cdot) \in C (2) ([0, m]) \) satisfying the conditions (5.10)-(5.11) and afterwards consider the linear unbounded operator \( C z = M z, z (\cdot) = \left( \begin{array}{c} \psi_1 (x) \\ \psi_2 (x) \end{array} \right), \) defined in the domain \( D (C) = C_0 (2) [0, m] \times L_2 [0, m]. \) The operator \( C \) defined above is a linear unbounded operator satisfying all the required assumptions (see the list of assumptions in subsection "The assumptions").

Using (5.16)-(5.20) one can rewrite equation (5.5)-(5.7) in the form of (4.2)-(4.3) with the state space \( X = L_2 [0, m] \times L_2 [0, m]; \) the corresponding operator \( C \) generates a \( C_0 \)-semigroup.

Here all the assumptions 1-4 of subsection "The assumptions" for above operators \( A \) and \( C \) are valid with \( T = 0. \)

We have
(5.21) \( D_B u = \int_0^l G (x, \xi, \mu) (\omega_0 (\xi) a_2 + \omega_l (\xi) b_2) u d\theta, \)
where \( G (x, \xi, \mu) \) is the Green function of the boundary problem [12]
(5.22) \( (p_1 (x) y' (x))' + p_2 (x) y (x) = \mu y (x), 0 \leq x \leq l, \)
\( \alpha_0 y (0) + \alpha_1 y' (0) = a_2 u, \)
\( b_0 y (l) + b_1 y' (l) = b_2 u, \)
and \( \omega_0 (\xi), \omega_l (\xi) \) are defined by [2]
(5.23) \( \omega_0 (\xi) = \begin{cases} -\frac{p_1 (0)}{\alpha_0} \delta (\xi), & \text{if } a_1 \neq 0, \\ \frac{p_1 (\xi)}{\alpha_0} \delta' (\xi), & \text{if } a_0 \neq 0, \end{cases} \)
(5.24) \( \omega_l (\xi) = \begin{cases} -\frac{p_1 (l)}{b_1} \delta (\xi - l), & \text{if } b_1 \neq 0, \\ \frac{p_1 (\xi)}{b_0} \delta' (\xi - l), & \text{if } b_0 \neq 0. \end{cases} \)

We have here \( U = \mathbb{R}^2; \) the operator \( K : Z \mapsto U \) is defined for given \( \alpha \in [0, m] \) by
(5.25) \( K z (\cdot) = z_1 (\alpha), \forall z (\cdot) = \left( \begin{array}{c} z_1 (\cdot) \\ z_2 (\cdot) \end{array} \right) \in L^2 [0, m] \times L^2 [0, m]. \)
The operator \( K, \) defined by (5.25) satisfies the assumptions listed in subsection "Assumptions", so approximate controllability conditions of equation (5.1)-(5.3) by controller (5.5)-(5.7) can be obtained by Theorem 3.1 or by Theorems 4.1-4.2. Hence one can obtain the following
Theorem 5.1. Condition (4.11) holds true if and only if for each \( \lambda \) the boundary value problem

\[(p_1(x) \varphi)' + p_2(x) \varphi - \lambda \varphi = 0, \quad x \in [0, l], \quad \lambda \in \sigma_A\]

subject the boundary conditions

\[
\begin{align*}
\alpha_0 \varphi(0) + \alpha_1 \varphi'(0) &= 0, \\
\beta_0 \varphi(l) + \beta_1 \varphi'(l) &= 0
\end{align*}
\]

and to the conditions

\[
\int_0^l \varphi(\xi) (\omega_0(\xi) a_2 + \omega_1(\xi) b_2) d\xi = 0, \quad \forall \mu_1, \mu_2 \in \mathbb{R}, \forall \mu \notin \sigma_A
\]

has only trivial solution.

Proof. Let \( G(x, \xi, \mu), \mu \notin \sigma_A \) be the Green function for the boundary problem

\[(p_1(x) \varphi)' + p_2(x) \varphi - \mu \varphi = 0, \quad x \in [0, l], \]

subject the boundary conditions (5.27)-(5.28). It follows from (5.9) and from the definition of the operator \( A \) that \( A = A^* \) and

\[
(D_\mu Bu, \varphi) = \int_0^l \left( \int_0^l G(x, \xi, \mu) (\omega_0(\xi) a_2 + \omega_1(\xi) b_2) \right) \varphi(x) dx =
\]

\[
= \left( \int_0^l \left( \int_0^l G(x, \xi, \mu) \varphi(x) dx \right) (\omega_0(\xi) a_2 + \omega_1(\xi) b_2) d\xi \right) u.
\]

Since \( \left( \int_0^l G(x, \xi, \mu) \varphi(x) dx \right) = \frac{1}{\mu} \varphi(\xi)[12] \), we obtain by (5.32) that

\[
(D_\mu Bu, \varphi) = \left( \int_0^l \varphi(\xi) (\omega_0(\xi) a_2 + \omega_1(\xi) b_2) d\xi \right) u, \quad \forall u \in \mathbb{R}.
\]

Hence the condition \( (D_\mu Bu, \varphi) = 0, \quad \forall u \in \mathbb{R}^2 \) is equivalent to the conditions (5.29).

It proves the theorem.

Theorem 5.2. Condition (4.11) holds true if and only if for each \( \lambda \in \sigma_A \) the boundary value problem (5.26) subject the boundary conditions (5.27)-(5.28) and the boundary conditions:

\[
\begin{align*}
\frac{p(0) \varphi(0)}{a_1} &\frac{a_2}{b_1} + \frac{p(l) \varphi(l)}{b_1} \frac{b_2}{b_1} = 0, \quad \text{if} \quad a_1 \neq 0 \text{ and } b_1 \neq 0, \\
\frac{p(0) \varphi'(0)}{a_0} &\frac{a_2}{b_1} - \frac{p(l) \varphi'(l)}{b_1} \frac{b_2}{b_1} = 0, \quad \text{if} \quad a_0 \neq 0 \text{ and } b_1 \neq 0, \\
\frac{p(0) \varphi(0)}{a_1} &\frac{a_2}{b_0} - \frac{p(l) \varphi(l)}{b_0} \frac{b_2}{b_0} = 0, \quad \text{if} \quad a_1 \neq 0 \text{ and } b_0 \neq 0, \\
\frac{p(0) \varphi'(0)}{a_0} &\frac{a_2}{b_0} + \frac{p(l) \varphi'(l)}{b_0} \frac{b_2}{b_0} = 0, \quad \text{if} \quad a_0 \neq 0 \text{ and } b_0 \neq 0,
\end{align*}
\]
has only trivial solution.

Proof. One should prove solely that (5.34)-(5.37) are equivalent to (5.29).

Let (5.34)-(5.37) hold and \(a_1 \neq 0\) and \(b_1 \neq 0\). According to (5.23)-(5.24) we have

\[
\int_0^l \varphi (\xi) \omega_0 (\xi) d\xi = -\frac{p(0)}{a_1} \int_0^l \varphi (\xi) \delta (\xi) d\xi = -\frac{p(0)}{a_1} \varphi (0),
\]

\[
\int_0^l \varphi (\xi) \omega_l (\xi) d\xi = -\frac{p(l)}{b_1} \int_0^l \varphi (\xi) \delta (\xi - l) d\xi = -\frac{p(l)}{b_1} \varphi (l) = 0;
\]

\[
\int_0^l \varphi (\xi) (\omega_0 (\xi) a_2 + \omega_l (\xi) b_2) d\xi = -\left( p(0) \varphi (0) \frac{a_2}{a_1} + p(l) \varphi (l) \frac{b_2}{b_1} \right)
\]

Hence the equivalence (5.34)-(5.37) \(\Rightarrow\) (5.29) holds.

Another cases (\(a_1 \neq 0\) and \(b_0 \neq 0\), \(a_0 \neq 0\) and \(b_1 \neq 0\), \(a_0 \neq 0\) and \(b_0 \neq 0\)) can be investigated by the same way.

It proves the theorem.

Theorem 5.2 is in essence the first condition of Theorem 4.2 adopted for the partial equation (5.1). Now we adopt the second condition of Theorem 4.2 for equation (5.1).

It is well-known, that any mild solution \(z (t, x)\) of equation (5.5) subject boundary conditions (5.6)-(5.7) and initial conditions (5.8) is expanded in the Fourier series

\[
z (t, x) = \sum_{k=1}^{\infty} (\alpha_k \cos \mu_k t + \beta_k \sin \mu_k t) \phi_k (x),
\]

where \(\phi_k (x)\) are eigenfunctions of boundary value problem (5.26)-(5.28) corresponding to its eigenvalues \(\mu_k, k = 1, 2, \ldots\). Here

\[
\alpha_k = \int_0^m \psi_0 (x) \phi_k (x) dx, \beta_k = \frac{1}{\mu_k} \int_0^m \psi_1 (x) \phi_k (x) dx k = 1, 2, \ldots.
\]

According to the definition of the control \(u (t)\) we have \(u (t) = Kz (t, \cdot) = z (t, \alpha), \alpha \in \{0, m\}\), i.e.

\[
u (t) = \sum_{k=1}^{\infty} (\alpha_k \cos \mu_k t + \beta_k \sin \mu_k t) \phi_k (\alpha), \alpha \in \{0, m\}.
\]

One can easily prove that if the sequence of functions \(\cos \mu_k t, \sin \mu_k t, k = 1, 2, \ldots\) is dense in \(L_2 [0, l]\) and \(\inf_{k=1,2,\ldots} |\phi_k (\alpha)| > 0\), \(k = 1, 2, \ldots\), then the second conditions of the theorem will be valid.

By asymptotic formulas for eigenvalues of boundary value problem (5.26)-(5.28), we have [7]

\[
\mu_k = \begin{cases}
\frac{\kappa \pi}{l} + \frac{\pi}{k}, & \text{if } a_1 \neq 0, b_1 \neq 0, \\
\frac{\pi}{2l} + \frac{B_1}{2k+1} + O \left(\frac{1}{k^2}\right), & \text{if } a_1 \neq 0, b_1 = 0, b_0 = 1, \\
\frac{\pi}{2l} + \frac{B_1}{2k+1} + O \left(\frac{1}{k^2}\right), & \text{if } a_1 = 0, b_1 \neq 0, a_0 = 1, \\
-\frac{C}{k+1} + O \left(\frac{1}{k^2}\right), & \text{if } a_1 = b_1 = 0, a_0 = b_0 = 1,
\end{cases}
\]

\[k \in \mathbb{N}, l, \kappa, A, B_1, B_2, C \in \mathbb{R} \setminus \{0\}.\]
where \( k = 1, 2, \ldots; A, B_1, B_2, C \) are some constant [7].

It is well-known, that the functions \( \cos \frac{k\pi}{l} t, \sin \frac{k\pi}{l} t, k = 1, 2, \ldots \) are dense in \( L_2 [0, l] \). It follows from (5.41) that \( |\mu_k - \frac{k\pi}{l}| < D = \text{const}, k = 1, 2, \ldots \). So the Wiener’s Theorem [13] and (5.41) implies the density of the functions \( \cos \mu_k t, \sin \mu_k t, k = 1, 2, \ldots \) in \( L_2 [0, l] \).

Using asymptotic formulas [7]

\[
(5.42) \quad \phi_k (x) = \begin{cases} 
\cos \frac{k\pi}{l} x + \frac{F_1(x)}{k} \sin \frac{k\pi}{l} x + O \left( \frac{1}{k^2} \right), & \text{if } a_1 \neq 0, b_1 \neq 0, \\
\cos \frac{(2k+1)\pi}{2l} x + \frac{F_2(x)}{2k+1} \sin \frac{(2k+1)\pi}{2l} x + O \left( \frac{1}{k^2} \right), & \text{if } a_1 \neq 0, b_1 = 0, b_0 = 1, \\
\sin \frac{(2k+1)\pi}{2l} x + \frac{F_3(x)}{2k+1} \cos \frac{(2k+1)\pi}{2l} x + O \left( \frac{1}{k^2} \right), & \text{if } a_1 = 0, b_1 \neq 0, a_0 = 1, \\
\sin \frac{(k+1)\pi}{l} x + \frac{F_4(x)}{2k+1} \cos \frac{(2k+1)\pi}{2l} x + O \left( \frac{1}{k^2} \right), & \text{if } a_1 = b_1 = 0, a_0 = b_0 = 1
\end{cases}
\]

where \( F_j (x), 0 \leq x \leq l, j = 1, 2, 3, 4 \) are some functions, continuous on \([0, l] \), for eigenfunctions of boundary value problem (5.26)-(5.28), we obtain that \( \inf_{k=1,2,\ldots} |\phi_k (\alpha)| > 0, k = 1, 2, \ldots \), if and only if \( \frac{\alpha}{l} \) is irrational number.

Hence above considerations imply the validity of the following

**Theorem 5.3.** Let \( \frac{\alpha}{l} \) be irrational number. For equation (5.1) to be approximately null-controllable on \([0, t_1]\), \( \forall t_1 > l \), by boundary controller (5.2)-(5.3), it is necessary and sufficient, that for each \( \lambda \) the boundary value problem (5.26)-(5.28) subject boundary conditions (5.27)-(5.28) and boundary conditions (5.34)-(5.37) has only trivial solution.

5.1. Remark on approximate null-controllability of partial differential equations by parabolic controller. If the boundary controller under consideration is described by partial boundary value problem of the form

\[
(5.43) \quad \frac{\partial z}{\partial t} (t, x) = \frac{\partial}{\partial x} \left( q_1 (x) \frac{\partial y}{\partial x} (t, x) \right) + q_2 (x) y(t, x), t \geq 0, 0 \leq x \leq m.
\]

with homogeneous regular boundary conditions (5.7)-(5.8) subject to the initial conditions

\[
(5.44) \quad z(0, x) = \psi_0 (x), 0 \leq x \leq m,
\]

where function \( \psi_0 (x), 0 \leq x \leq m \), defined by (5.44), is considered as a control of equation (5.1)-(5.3) by distributed controller (5.43)-(5.5)-(5.7)-(5.44) called a parabolic controller.

It is well-known, that any mild solution \( z(t, x) \) of equation (5.43) subject boundary conditions (5.6)-(5.7) and initial conditions (5.44) is expanded in the Fourier series

\[
(5.45) \quad z(t, x) = \sum_{k=1}^{\infty} \alpha_k e^{-\nu_k^2 t} \phi_k (x),
\]
where $\phi_k(x)$ are eigenfunctions of boundary value problem (5.26)-(5.28) corresponding to its eigenvalues $\mu_k$, $k = 1, 2, \ldots$, and

$$\alpha_k = \int_0^m \psi_0(x) \phi_k(x) \, dx, \ k = 1, 2, \ldots.$$  

Using asymptotical formulas (5.41) one can that $\sum_{k=1}^{\infty} \frac{1}{|\mu_k|^p} < +\infty$. Hence by Muntz's Theorem [4] the functions $e^{-\mu_k^2 t}$, $t \in [0, m]$, $k = 1, 2, \ldots$, are not dense in $L_2 [0, m]$, so it is impossible to finish the investigation and to obtain result like Theorem 5.3. The question is still open.

6. Concluding remarks. Controllability conditions of Theorems 3.1, 4.1, 4.2, obtained above is sufficient conditions. The problem of approximate null-controllability in the case when the space $KSC (\cdot) Z$ of $L_2 ([0, t_2], U)$ is not dense in $L_2 ([0, t_2], U)$ for any $t_2 > 0$, is still open.

The results of Sections 5 show that if controller (4.2) is hyperbolic controller (5.2), than approximate null-controllability of parabolic control equation on $[0, t_1]$, $\forall t_1 > T + \delta$ is necessary and sufficient condition for approximate null-controllability of parabolic control equation on $[0, t_1]$, $\forall t_1 > T + \delta$, by any hyperbolic controller (5.2).

Results of section 5 can be extended to the case of partial hyperbolic equation

$$\frac{\partial^2 y(t, x)}{\partial t^2} = \frac{\partial}{\partial x} \left( p_1(x) \frac{\partial y(t, x)}{\partial x} \right) + p_2(x) y(t, x), \ t \geq 0, 0 \leq x \leq l,$$

subject boundary conditions (5.2)-(5.3) governed by hyperbolic controller (5.6)-(5.8)

If controller (4.2) is a parabolic controller, than the problem of approximate null-controllability of equation (4.1) by parabolic controller (4.2) is still open.

It means that if there exists a possibility to choice a distributed controller for construction then it is worthwhile to construct hyperbolic controller.

The necessary and sufficient conditions for approximate controllability of equation (2.1) on $[0, t_1]$ by distributed controller (2.2) requires a much deeper research that one presented here.

REFERENCES


ON STRUCTURE OF SOME $\sigma$-ALGEBRAS RELATED TO MEASURABILITY OF SUPERPOSITIONS

I. V. SHRAGIN *

Abstract. Elements of some $\sigma$-algebras which are defined by means of maps preimages are described. These $\sigma$-algebras figure in criteria of measurability of superpositions.

AMS(MOS) subject classification. 28A05, 28A20

Key Words. $\sigma$-algebra, measurable function, superposition

1. Introduction. Let $(T, \mathcal{T})$ be a measurable space ($T \neq \emptyset$), $X$ be a nonempty set. Then every function $\varphi : T \to X$ generates on $X$ the $\sigma$-algebra

$$\mathcal{M}_\varphi := \{ B \subset X : \varphi^{-1}(B) \in \mathcal{T} \},$$

and every nonempty set $\Phi$ of functions $\varphi : T \to X$ (i.e. $\Phi \subset X^T$) generates on $X$ the $\sigma$-algebra

$$\mathcal{M}_\Phi := \bigcap_{\varphi \in \Phi} \mathcal{M}_\varphi = \{ B \subset X : (\forall \varphi \in \Phi) \varphi^{-1}(B) \in \mathcal{T} \}$$

(if we write "$B \subset X$", then it is possible that $B = X$).

Moreover, if a measurable space $(Y, \mathcal{B})$ and a function $g : X \to Y$ are given, then we can consider superpositions (otherwise, compositions) of kind $g \circ \varphi : T \to Y$. Naturally, the question arises: under which condition (for the function $g$) the superposition $g \circ \varphi$ is $(\mathcal{T}, \mathcal{B})$-measurable (i.e. $(\forall C \in \mathcal{B}) (g \circ \varphi)^{-1}(C) \in \mathcal{T}$) for fixed function $\varphi$ or for any function $\varphi \in \Phi$. The following proposition gives an answer to this question.

* Florenzer Str., 2-6, 20/2, 50765, Köln, Germany

407
Proposition 1. The superposition $g \circ \varphi$ is $(\mathcal{T}, \mathcal{B})$-measurable for the given function $\varphi : T \to X$ (for any function $\varphi \in \Phi$ with $\emptyset \neq \Phi \subset X^T$) if and only if the function $g$ is $(\mathcal{M}_\varphi, \mathcal{B})$-measurable ($(\mathcal{M}_\varphi, \mathcal{B})$-measurable, respectively).

This statement follows directly from the equality

$$(g \circ \varphi)^{-1}(C) = \varphi^{-1}[g^{-1}(C)], \quad C \in \mathcal{B},$$

and the definition of the $\sigma$-algebras $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$.

The Proposition 1 stimulates the interest in studying the $\sigma$-algebras $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$. In particular, in this paper the problem of the explicit description (i.e. without use preimage operation) of elements of $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$ is solved.

Note that Darst [1] proved (in the situation, when $T = [0, 1]$, $\mathcal{T}$ is the $\sigma$-algebra of Lebesgue measurable subsets of $T$, $X = \mathbb{R}$, $\Phi$ is the collection of all Lebesgue measurable functions $\varphi : [0, 1] \to \mathbb{R}$) that $\sigma$-algebra $\mathcal{M}_\Phi$ coincides with the family of all universally measurable sets in $\mathbb{R}$ (a subset of $\mathbb{R}$ is called universally measurable if it belongs to the completion of the $\sigma$-algebra $\mathcal{B}_\mathbb{R}$ of Borel sets with respect to any finite measure on $\mathcal{B}_\mathbb{R}$).

In various branches of mathematics we find superpositions that are generated by functions $f : T \times X \to Y$, i.e. superpositions of the kind $f(t, \varphi(t))$. In order to answer the question, under which condition (for the function $f$) such superposition is $(\mathcal{T}, \mathcal{B})$-measurable, we associate to every function $\varphi : T \to X$ its “graph”-function $G_\varphi : T \to T \times X$, where $G_\varphi(t) = (t, \varphi(t))$, $t \in T$ (so that $G_\varphi(T) = \text{gr} \varphi$ is the graph of $\varphi$). Now our superposition has the form $f \circ G_\varphi : T \to Y$, and therefore it can be considered as a special case of $g \circ \varphi$, where the roles $X, g, \varphi$ are played by $T \times X, f, G_\varphi$, respectively. In this situation we have in the capacity of $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$ the $\sigma$-algebras (on $T \times X$)

$$\mathcal{M}_\varphi := \{D \subset T \times X : G_\varphi^{-1}(D) \in \mathcal{T}\},$$

and

$$\mathcal{M}_\Phi := \{D \subset T \times X : (\forall \varphi \in \Phi) G_\varphi^{-1}(D) \in \mathcal{T}\}.$$
2. The classes $L(\varphi, A)$. Let us fix a function $\varphi : T \to X$ and consider the set classes

$$L(\varphi, A) := \{ B \subset X : \varphi^{-1}(B) = A \}, \quad A \subset T.$$ 

Evidently,

$$\bigcup_{A \subset T} L(\varphi, A) = \mathcal{P}(X)$$

($\mathcal{P}(X)$ is the collection of all subsets of $X$) and if $A_1 \neq A_2$, then $L(\varphi, A_1) \cap L(\varphi, A_2) = \emptyset$. Also evidently that for fixed $A \subset T$ the class $L(\varphi, A)$ is absolutely additive and multiplicative, i.e. closed relative to the union and the intersection of any collection of its sets. From this fact it follows that if the class $L(\varphi, A)$ is nonempty, then it has the greatest and smallest elements, i.e. the sets $B_{\text{max}}$ and $B_{\text{min}}$ such that $(\forall B \in L(\varphi, A)) B_{\text{min}} \subset B \subset B_{\text{max}}$. Moreover, the class $L(\varphi, A)$ has the evident "convexity" property: if $B_1 \subset B \subset B_2$ with $B_i \in L(\varphi, A), i = 1, 2$, then $B \in L(\varphi, A)$ too. Therefore, if $L(\varphi, A) \neq \emptyset$, then

$$L(\varphi, A) = \{ B \subset X : B_{\text{min}} \subset B \subset B_{\text{max}} \}.$$ 

We make more precise this result as follows.

**Theorem 1.** For every set $A \subset T$

(1) $$L(\varphi, A) = \{ B \subset X : \varphi(A) \subset B \subset X \setminus \varphi(T \setminus A) \}.$$ 

**Proof.** Let $B \in L(\varphi, A)$, i.e. $\varphi^{-1}(B) = A$. Hence $\varphi(A) \subset B$. But since $\varphi^{-1}(X \setminus B) = T \setminus A$, then $\varphi(T \setminus A) \subset X \setminus B$, i.e. $B \subset X \setminus \varphi(T \setminus A)$. Thus

(2) $$\varphi(A) \subset B \subset X \setminus \varphi(T \setminus A).$$

Let, conversely, the condition (2) be realized for a set $B$. Then

(3) $$\varphi(A) \subset X \setminus \varphi(T \setminus A),$$

hence $\varphi^{-1}(\varphi(A)) \subset T \setminus \varphi^{-1}(\varphi(T \setminus A))$. But since $T \setminus A \subset \varphi^{-1}(\varphi(T \setminus A))$, then $\varphi^{-1}(\varphi(A)) \subset T \setminus (T \setminus A) = A$. From this fact and the inclusion $A \subset \varphi^{-1}(\varphi(A))$ it follows that

(4) $$\varphi^{-1}(\varphi(A)) = A,$$

i.e. $\varphi(A) \in L(\varphi, A)$. 

Note that (4) was deduced from inclusion (3), which is equivalent to inclusion \( \varphi(T \setminus A) \subset X \setminus \varphi(A) \). Therefore, we can replace \( A \) in (4) by \( T \setminus A \), so that \( \varphi^{-1}(\varphi(T \setminus A)) = T \setminus A \), and hence

\[
\varphi^{-1}(X \setminus \varphi(T \setminus A)) = T \setminus (T \setminus A) = A,
\]
i.e. \( X \setminus \varphi(T \setminus A) \in \mathcal{L}(\varphi, A) \).

So the extreme sets in (2) belong to \( \mathcal{L}(\varphi, A) \). Hence, by “convexity” property \( B \in \mathcal{L}(\varphi, A) \), and consequently the equality (1) has been proved. \( \square \)

The following criterion of nonemptiness of class \( \mathcal{L}(\varphi, A) \) holds.

**Proposition 3.** The class \( \mathcal{L}(\varphi, A) \) is not empty if and only if

\[
(5) \quad \varphi(A) \cap \varphi(T \setminus A) = \emptyset.
\]

**Proof.** Assume that \( \mathcal{L}(\varphi, A) \neq \emptyset \) and take \( B \in \mathcal{L}(\varphi, A) \). Then, by Theorem 1, we have (2), from where (3) follows, i.e. (5) holds.

Now assume that equality (5) takes place, i.e. (3) holds. From this fact it follows (see the proof of Theorem 1) that \( \varphi(A) \in \mathcal{L}(\varphi, A) \), i.e. \( \mathcal{L}(\varphi, A) \neq \emptyset \). \( \square \)

Set by fixed function \( \varphi : T \to X \)

\[
T_\varphi := \{ A \subset T : \mathcal{L}(\varphi, A) \neq \emptyset \}.
\]

Evidently, \( T_\varphi = \{ \varphi^{-1}(B) : B \subset X \} \). From this fact it follows that the family \( T_\varphi \) is absolutely additive and multiplicative, and moreover \( T_\varphi \) contains complements (with respect to \( T \)) of sets, belonging to it.

3. The structure of \( \sigma \)-algebras \( \mathfrak{M}_\varphi \) and \( \mathfrak{M}_\varphi^\varphi \). At first let us fix a function \( \varphi : T \to X \) and consider the corresponding \( \sigma \)-algebra \( \mathfrak{M}_\varphi \). Clearly,

\[
\mathfrak{M}_\varphi = \bigcup_{A \in T} \mathcal{L}(\varphi, A),
\]

so that \( \mathfrak{M}_\varphi \) is decomposed to disjoint classes of sets. In order to exclude from this representation the empty classes, put

\[
T_\varphi = T \cap T_\varphi = \{ A \in T : \mathcal{L}(\varphi, A) \neq \emptyset \}.
\]

Evidently, \( T_\varphi \) is a \( \sigma \)-algebra on \( T \), and \( \mathfrak{M}_\varphi = \bigcup_{A \in T_\varphi} \mathcal{L}(\varphi, A) \), or, by Theorem 1,

\[
\mathfrak{M}_\varphi = \bigcup_{A \in T_\varphi} \{ B \subset X : \varphi(A) \subset B \subset X \setminus \varphi(T \setminus A) \}.
\]
For example, if the function \( \varphi \) is a constant, i.e. \( (\forall t \in T) \varphi(t) = x_0 \),
then, by Proposition 3, \( \mathcal{T}_\varphi = \{\emptyset, T\} \), and
\[
\mathcal{M}_\varphi = \mathcal{L}(\varphi, T) \cup \mathcal{L}(\varphi, \emptyset) = \{B : x_0 \in B\} \cup \{B : x_0 \notin B\} = \mathcal{P}(X).
\]
On the other hand, if \( \varphi \) is injective, then \( \mathcal{T}_\varphi = \mathcal{T} \).

Now we turn to clarification of the structure of the \( \sigma \)-algebra \( \mathcal{M}_\Phi \) by some \( \Phi \subset X^T \). It follows from definition of \( \mathcal{M}_\Phi \) and representation of \( \mathcal{M}_\varphi \) above that
\[
\mathcal{M}_\Phi = \bigcap_{\varphi \in \Phi} \bigcup_{A \in \mathcal{T}} \mathcal{L}(\varphi, A).
\]
We shall find more suitable representation for \( \mathcal{M}_\Phi \). To this end put for arbitrary map \( F : \Phi \to \mathcal{T} \) (i.e. \( F \in \mathcal{T}^\Phi \))
\[
N(\Phi, F) = \bigcap_{\varphi \in \Phi} \mathcal{L}(\varphi, F(\varphi)).
\]
Since, by Theorem 1,
\[
\mathcal{L}(\varphi, F(\varphi)) = \{B \subset X : \varphi(F(\varphi)) \subset B \subset X \setminus \varphi(T \setminus F(\varphi))\},
\]
then
\[
(6) \quad N(\Phi, F) = \left\{ B : \bigcup_{\varphi \in \Phi} \varphi(F(\varphi)) \subset B \subset X \setminus \bigcup_{\varphi \in \Phi} \varphi(T \setminus F(\varphi)) \right\}.
\]

**Proposition 4.** The following equality holds
\[
(7) \quad \mathcal{M}_\Phi = \bigcup_{F \in \mathcal{T}^\Phi} N(\Phi, F).
\]

**Proof.** Let \( B \in \mathcal{M}_\Phi \), i.e. \( (\forall \varphi \in \Phi) \varphi^{-1}(B) \in \mathcal{T} \). Put \( F_0(\varphi) = \varphi^{-1}(B) \), \( \varphi \in \Phi \). Then \( F_0 \in \mathcal{T}^\Phi \), and \( (\forall \varphi \in \Phi) B \in \mathcal{L}(\varphi, F_0(\varphi)) \), whence \( B \in N(\Phi, F_0) \).

Conversely, if \( (\exists F \in \mathcal{T}^\Phi) B \in N(\Phi, F) \), then \( (\forall \varphi \in \Phi) \varphi^{-1}(B) = F(\varphi) \in \mathcal{T} \), i.e. \( B \in \mathcal{M}_\Phi \). \( \square \)

From disjointness of the classes \( \mathcal{L}(\varphi, A) \) for fixed \( \varphi \) and different \( A \) it follows easily the disjointness of \( N(\Phi, F) \) for different \( F \).

Thus the equality (7) is a decomposition of the \( \sigma \)-algebra \( \mathcal{M}_\Phi \) to disjoint classes of subsets of \( X \).

From (6) it follows that \( N(\Phi, F) \neq \emptyset \) if and only if
\[
(8) \quad \forall (\varphi, \psi \in \Phi) \varphi(F(\varphi)) \cap \psi(T \setminus F(\psi)) = \emptyset.
\]
Hence in (7) only those \( F \) are present actually that satisfy the condition (8).

For example, the “constants” \( F_0 \) and \( F_T \) relate to such maps, where \( (\forall \varphi \in \Phi) F_0(\varphi) = \emptyset, F_T(\varphi) = T \). In these cases
\[
N(\Phi, F_0) = \left\{ B : B \subset X \setminus \bigcup_{\varphi \in \Phi} \varphi(T) \right\}, \quad N(\Phi, F_T) = \left\{ B : \bigcup_{\varphi \in \Phi} \varphi(T) \subset B \right\}.
\]
4. The structure of $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$. As we know (cf. Introduction), it is possible to consider the $\sigma$-algebras $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$ as special cases of $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$, respectively. Therefore by investigation of the structure of $\mathcal{M}_\varphi$ and $\mathcal{M}_\Phi$ we can use the results from Sections 2 and 3.

At first consider analogs of classes $\mathcal{L}(\varphi, A)$. Namely, fix a function $\varphi : T \rightarrow X$ and put
\[
\mathcal{K}(\varphi, A) = \{ D \subset T \times X : G_\varphi^{-1}(D) = A \}, \quad A \subset T.
\]
Then
\[
\bigcup_{A \in T} \mathcal{K}(\varphi, A) = \mathcal{P}(T \times X),
\]
where the classes $\mathcal{K}(\varphi, A)$ are disjoint by different sets $A$. It follows from Theorem 1 that for every set $A \subset T$
\[
(9) \quad \mathcal{K}(\varphi, A) = \{ D \subset T \times X : G_\varphi(A) \subset D \subset (T \times X) \setminus G_\varphi(T \setminus A) \}.
\]
Moreover, since $G_\varphi(A) \cap G_\varphi(T \setminus A) = \emptyset$, by Proposition 3 $\mathcal{K}(\varphi, A) \neq \emptyset$ for every $A \subset T$.

Definition of the $\sigma$-algebra $\mathcal{M}_\varphi$ implies that
\[
(10) \quad \mathcal{M}_\varphi = \bigcup_{A \in T} \mathcal{K}(\varphi, A).
\]
Thus we obtained decomposition of $\mathcal{M}_\varphi$ to nonempty disjoint classes of subsets of $T \times X$, where these classes have a simple structure, as one can see in (9).

Now we turn to the $\sigma$-algebra $\mathcal{M}_\Phi$ for arbitrary nonempty set $\Phi \subset X^T$. By Proposition 4
\[
(11) \quad \mathcal{M}_\Phi = \bigcup_{F \in T^\Phi} M(\Phi, F),
\]
where
\[
M(\Phi, F) = \bigcap_{\varphi \in \Phi} \mathcal{K}(\varphi, F(\varphi)), \quad F \in T^\Phi.
\]
It follows from (6) that
\[
(12) \quad M(\Phi, F) = \left\{ D : \bigcup_{\varphi \in \Phi} G_\varphi(F(\varphi)) \subset D \subset (T \times X) \setminus \bigcup_{\varphi \in \Phi} G_\varphi(T \setminus F(\varphi)) \right\},
\]
and it is clear that $M(\Phi, F) \neq \emptyset$ if and only if
\[
(13) \quad \forall (\varphi, \psi \in \Phi) \quad G_\varphi(F(\varphi)) \cap G_\psi(T \setminus F(\psi)) = \emptyset.
\]
ON STRUCTURE OF SOME $\sigma$-ALGEBRAS

413

(In particular, if $F_0$ is a "constant", i.e. $(\forall \varphi \in \Phi) F_0(\varphi) = A_0 \in \mathcal{T}$, then $M(\Phi, F_0) \neq \emptyset$, since

$$\forall (\varphi, \psi \in \Phi) \quad G_\varphi(A_0) \cap G_\psi(T \setminus A_0) = \emptyset.$$ 

Thus decomposition (11) of $\mathcal{M}_\Phi$ to disjoint classes $M(\Phi, F)$ of subsets $T \times X$ is obtained, where the structure of $M(\Phi, F)$ is described in (12). Remind that in (11) only those $F$ are present actually that satisfy the condition (13).

For example, let us consider the case of trivial $\sigma$-algebra $\mathcal{T} = \{\emptyset, T\}$. Then by (9) and (10) $\mathcal{M}_\varphi = \mathcal{K}(\varphi, \emptyset) \cup \mathcal{K}(\varphi, T) = \{D : D \subset (T \times X) \setminus G_\varphi(T)\} \cup \{D : G_\varphi(T) \subset D\}$, or

$$\mathcal{M}_\varphi = \{D : D \cap \text{gr} \varphi = \emptyset\} \cup \{D : \text{gr} \varphi \subset D\}.$$ 

Further, let $F \in \mathcal{T}^\Phi$ with $\emptyset \neq F \subset X^T$, and put

$$\Psi = \{\varphi \in \Phi : F(\varphi) = T\},$$ 

so that $\Phi \setminus \Psi = \{\varphi \in \Phi : F(\varphi) = \emptyset\}$. Then by (12)

$$M(\Phi, F) = \left\{D : \bigcup_{\varphi \in \Psi} \text{gr} \varphi \subset D \subset (T \times X) \setminus \bigcup_{\varphi \in \Phi \setminus \Psi} \text{gr} \varphi\right\},$$ 

where $M(\Phi, F) \neq \emptyset$ if and only if

$$\forall (\psi \in \Psi, \varphi \in \Phi \setminus \Psi) \quad \text{gr} \varphi \cap \text{gr} \psi = \emptyset.$$ 

Evidently, (14) determines (in considered situation, when $\mathcal{T} = \{\emptyset, T\}$) a bijection between $\mathcal{T}^\Phi$ and $\mathcal{P}(\Phi)$. Therefore, by (11) and (15),

$$\mathcal{M}_\varphi = \bigcup_{\Psi \subset \Phi} \left\{D : \bigcup_{\varphi \in \Psi} \text{gr} \varphi \subset D \subset (T \times X) \setminus \bigcup_{\varphi \in \Phi \setminus \Psi} \text{gr} \varphi\right\},$$ 

where only those $\Psi$ are present actually that satisfy the condition (16) (in particular, $\Psi = \emptyset$ and $\Psi = \Phi$ are such).

5. Certain remarks.

Remark 1. Let us fix a function $\varphi : T \to X$ and note that if $A \in \mathcal{T}$ and $B \in \mathcal{M}_\varphi$, then $A \times B \in \mathcal{M}_\varphi$, since

$$G_\varphi^{-1}(A \times B) = \{t : (t, \varphi(t)) \in A \times B\} = A \cap \varphi^{-1}(B) \in \mathcal{T}.$$
From this fact it follows that $\mathcal{F} \times \mathcal{M}_\varphi \subset \mathcal{M}_\varphi$, where $\mathcal{F} \times \mathcal{M}_\varphi$ is product of $\sigma$-algebras $\mathcal{F}$ and $\mathcal{M}_\varphi$, i.e. a $\sigma$-algebra (on $T \times X$) generated by the family

$$\Pi := \{A \times B : A \in \mathcal{F}, B \in \mathcal{M}_\varphi\}.$$ 

Show that in general $\mathcal{F} \times \mathcal{M}_\varphi \neq \mathcal{M}_\varphi$. To this end consider the case when $\mathcal{F} = \{\emptyset, T\}$. Then, evidently, $\Pi = \{T \times B : B \in \mathcal{M}_\varphi\}$. Hence, $\Pi$ is a $\sigma$-algebra on $T \times X$, so that $\mathcal{F} \times \mathcal{M}_\varphi = \Pi$.

Let us suppose that each of the sets $T$ and $X$ contains more than one element and take a point $(t_0, x_0) \in T \times X$ such that $x_0 \neq \varphi(t_0)$. Put $D_0 = \{(t_0, x_0)\}$. Since $G_{\varphi}^{-1}(D_0) = \emptyset \in \mathcal{F}$, $D_0 \in \mathcal{M}_\varphi$. On the other hand, since $\{t_0\} \notin \mathcal{F}$, $D_0 \notin \Pi$. Thus $\mathcal{M}_\varphi \neq \mathcal{F} \times \mathcal{M}_\varphi$.

Note also that $\mathcal{F} \times \mathcal{M}_\Phi \subset \mathcal{M}_\Phi$ for any nonempty $\Phi \subset X^T$.

**Remark 2.** Put $J = \{A \in \mathcal{F} : (E \subset A) \Rightarrow (E \in \mathcal{F})\}$. It is easy to check that $J$ is a $\sigma$-ideal (i.e. $\emptyset \in J$, $J$ is countably additive, and if $A_1 \in J$ and $A_2 \subset A_1$, then $A_2 \in J$). Moreover, $J$ is the maximal from $\sigma$-ideals (even from ideals), contained in $\mathcal{F}$.

For example, if $T = \mathbb{R}$ and $\mathcal{F}$ is the $\sigma$-algebra of all Lebesgue measurable sets in $\mathbb{R}$, then $J$ consists of all sets in $\mathbb{R}$ with null Lebesgue measure.

We call functions $\varphi, \psi \in X^T J$-equivalent (notation: $\varphi \sim_J \psi$) if $\{t \in T : \varphi(t) \neq \psi(t)\} \in J$. It is easy to check that relation $\sim_J$ is an equivalence relation on $X^T$.

Let $\varphi \sim_J \psi$ and $B \subset X$. It is easily to verify that $\varphi^{-1}(B) \in \mathcal{F}$ if and only if $\psi^{-1}(B) \in \mathcal{F}$. It means that $\mathcal{M}_\varphi = \mathcal{M}_\psi$. Hence, $\mathcal{M}_\varphi = \mathcal{M}_\psi$ as well. It follows that by consideration of $\mathcal{M}_\Phi$ and $\mathcal{M}_\Phi$ one can assume that any two functions from $\Phi$ are not $J$-equivalent. In other words, it is possible to say that all these $\sigma$-algebras are generated by $J$-equivalence classes.

**REFERENCES**


THE SHORT TIME ASYMPTOTIC EXPANSION FOR THE TRACE OF THE HEAT KERNEL BY RAY METHOD*  

A. SPIVAK † AND Z. SCHUSS †

Abstract. The problem of recovering geometric properties of a domain from the trace of the heat kernel for an initial-boundary value problem was considered. It is similar to the problem of "hearing the shape of a drum", for which a Poisson type summation formula relates geometric properties of the domain to the eigenvalues of the Dirichlet or Neumann problem for the Laplace equation. It is well known that the area, circumference, and the number of holes in a planar domain can be recovered from the short time asymptotics of the solution of the initial-boundary value problem for the heat equation. It is also known that the length spectrum of closed billiard ball trajectories in the domain can be recovered from the eigenvalues or from the solution of the wave equation. This spectrum can also be recovered from the heat kernel for a compact manifold without boundary. We show that for a planar domain with boundary, the length spectrum can be recovered directly from the short time expansion of the trace of the heat kernel. The results can be extended to higher dimensions in a straightforward manner.

1. Introduction. The problem of recovering geometric properties of a domain from NMR measurements arises in oil explorations and in non-invasive microscopy of cell structure [1]. In these measurements the trace of the heat kernel for the initial value problem with reflecting (Neumann) boundary conditions is measured directly. The problem is analogous to "hearing the shape of a drum", where the solution of the wave equation in the domain is measured directly (it is "heard").

The problem of recovering geometrical properties of a domain from the eigenvalues of the Dirichlet or Neumann problem for the Laplace equation

* Partially supported by a research grant from the Foundation for Basic Research administered by the Israel Academy of Science and by a research grant from the US-Israel Binational Science Foundation.
† Department of Sciences, Academic Institute of Technology, POB 305, Holon 58102, Israel
‡ Department of Mathematics, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel
in a domain has attracted much attention in the literature (see [2]-[7] for some history and early results; for more recent work see [8], [9] an references therein).

The mathematical statement of the problem is as follows. Green's function for the heat equation in a smooth planar domain $\Omega$, with homogeneous Dirichlet boundary conditions, satisfies

\begin{align}
(1.1) \quad \frac{\partial G(y, x, t)}{\partial t} &= D \Delta_y G(y, x, t) \quad \text{for } y, x \in \Omega, \ t > 0 \\
(1.2) \quad G(y, x, 0) &= \delta(y - x) \\
(1.3) \quad G(y, x, t) &= 0 \quad \text{for } y \in \partial\Omega, \ x \in \Omega, \ t > 0.
\end{align}

The function $G(x, x, t) \, dx$ is the probability of return to $x \, dx$ at time $t$ of a free Brownian particle that starts at the point $x$ at time $t = 0$ and diffuses in $\Omega$ with diffusion coefficient 1, with absorption at the boundary $\partial\Omega$. If it is reflected at $\partial\Omega$, rather than absorbed, the Dirichlet boundary condition (1.3) is replaced with the Neumann condition [10]

\begin{align}
(1.4) \quad \frac{\partial G(y, x, t)}{\partial \nu(y)} &= 0 \quad \text{for } y \in \partial\Omega, \ x \in \Omega, \ t > 0,
\end{align}

where $\nu(y)$ is the unit outer normal at the boundary point $y$. The trace of the heat kernel is defined as

\begin{align}
(1.5) \quad P(t) &= \int_{\Omega} G(x, x, t) \, dx
\end{align}

and can be represented by the Dirichlet series

\begin{align}
(1.6) \quad P(t) &= \sum_{n=1}^{\infty} e^{-\lambda_n t},
\end{align}

where $\lambda_n$ are the eigenvalues of Laplace equation with the Dirichlet or Neumann boundary conditions (1.3) or (1.4), respectively.

It has been shown by Kac [2] that for a domain $\Omega$ with smooth boundary $\partial\Omega$, the leading terms in the expansion of $P(t)$ in powers of $\sqrt{t}$ are

\begin{align}
P_{Kac}(t) &\sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{6} (1 - r) + O\left(\sqrt{t}\right), \quad \text{for } t \to 0,
\end{align}
where \(|\Omega|\) denotes the area of \(\Omega\), \(|\partial \Omega|\) denotes the arc-length of \(\partial \Omega\), and \(r\) is the number of holes in \(\Omega\). The full short time asymptotic power series expansion of \(P(t)\) in the form

\[
P(t) \sim \sum_{n=0}^{\infty} a_n t^{n/2-1},
\]

can be deduced from the large \(s\) expansion of the Laplace transform

\[
g(s) = \int_0^\infty \exp\{-s^2 t\} \left( P(t) - \frac{a_0}{t} \right) dt, \quad \left( a_0 = \frac{|\Omega|}{4\pi} \right).
\]
in inverse powers of \(s\). Such an expansion was given by Stewartson and Waechter [3] in the form

\[
\hat{g}(s) \sim \sum_{n=1}^{\infty} \frac{c_n}{s^n},
\]

where

\[
c_n = a_n \Gamma\left(\frac{n}{2}\right).
\]

The constants \(c_n\) are computable functionals of the curvature of the boundary. The full expansion is denoted

\[
(1.7) \quad P_{SW}(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial \Omega|}{8\sqrt{\pi} t} + \frac{1}{6}(1 - r) + \sum_{n=3}^{\infty} a_n t^{n/2-1}, \quad \text{for } t \to 0.
\]

If the boundary is not smooth, but has cusps and corners, the expansion contains a term of the order \(t^{-\nu}\), where \(\nu\) is a number between 0 and 1/2.

The Stewartson-Waechter expansion was used in [8] to deduce further geometric properties of \(\Omega\) by extending \(g(s)\) into the complex plane. Examples were given in [8] of the resurgence of the length spectrum of closed billiard ball trajectories in the domain.

The full length spectrum of closed geodesics on a compact Riemannian manifold without boundary \(\Omega\) appeared in the short time asymptotic expansion given in [6],

\[
(1.8) \quad P(t) \sim \frac{1}{\sqrt{\pi t}} \sum_{n=0}^{\infty} P_n(\sqrt{t}) e^{-\delta_n^2/t}, \quad \text{for } t \to 0,
\]

where \(\delta_n\) are the lengths of closed geodesics on \(\Omega\) and \(P_n(x)\) are power series in \(x\).
In this paper, we construct an expansion of the form (1.8) for the trace of the heat kernel for the initial-boundary value problem (1.1)-(1.3) or (1.4) in a smooth bounded domain $\Omega$ in $\mathbb{R}^2$. The results can be generalized to higher dimensions in a straightforward manner.

The point of departure for our analysis is the observation that transcendentally small terms are not included in the expansion (1.7). These terms have been neglected in [3] and [8] even in the case of a circular domain, where the Laplace transform of $G(y,x,t)$ can be expressed explicitly in terms of modified Bessel functions. In [8] this Laplace transform is expanded in inverse powers of $s$ and the coefficients $c_n$ are evaluated asymptotically for large $n$.

A generalization of the asymptotic expansion “beyond all orders” (1.8) has the form

$$(1.9) \quad P(t) \sim P_{SW}(t) + \frac{1}{\sqrt{\pi t}} \sum_{n=1}^{\infty} P_n(\sqrt{t}) e^{-\delta_n^2/t}, \quad \text{for } t \to 0,$$

where $\delta_n$, ordered by magnitude, are constants to be determined, and $P_n(x)$ are power series in $x$. Transcendentally small terms may be, in fact, quite large and make a finite contribution to the expansion (1.9).

To recover the geometrical information from the expansion (1.9), given the (measured) function $P(t)$, we note that

$$(1.10) \quad |\Omega| = \lim_{t \to 0} 4\pi t P(t), \quad |\partial \Omega| = -\lim_{t \to 0} 8\sqrt{\pi t} \left[ P(t) - \frac{|\Omega|}{4\pi t} \right],$$

and so on. This way the entire expansion (1.7) can be determined.

Once the coefficients of the expansion (1.7) have been determined, the exponent of the dominant term of the transcendentally small part, $\delta_1$, is found as

$$\delta_1 = -\lim_{t \to 0} t \log \left[ P(t) - P_{SW}(t) \right].$$

Proceeding this way, we can recover the entire expansion (1.9) if $P(t)$ is known (e.g., from measurements).

In this paper, we use the “ray method”, as developed in [11], to construct a short time asymptotic expansion of the heat kernel. We use it to expand the trace asymptotically beyond all orders (the so called “hyperasymptotic expansion”) and show that the exponents $\delta_i$ are the squares of half the lengths of the periodic orbits in the domain. The exponentially small terms in the expansion (1.9) are due to rays reflected in the boundary, much like in the
geometric theory of diffraction. This recovers the length spectrum of closed billiard ball trajectories in the domain. In particular, the smallest exponent $\delta_1$ is the width of the narrowest bottleneck in the domain.

2. The one-dimensional case. The solution of the heat equation in an interval can be constructed by the method of images. Specifically, the Green function of the problem satisfies

\begin{equation}
\frac{\partial G(y, x, t)}{\partial t} = \frac{\partial^2 G(y, x, t)}{\partial y^2} \quad \text{for } 0 < x, y < a, \ t > 0
\end{equation}

\begin{equation}
G(y, x, 0) = \delta(y - x) \quad \text{for } 0 < x, y < a
\end{equation}

\begin{equation}
\left(\frac{\partial}{\partial y}\right)^k G(0, x, t) = \left(\frac{\partial}{\partial y}\right)^k G(a, x, t) = 0 \quad \text{for } 0 < x < a, \ t > 0,
\end{equation}

when $k = 0, 1$. The method of images gives the representation

\begin{equation}
G(y, x, t) = \frac{1}{2\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} \left[ \exp \left\{ -\frac{(y - x + 2na)^2}{4t} \right\} - (-1)^k \exp \left\{ -\frac{(y + x + 2na)^2}{4t} \right\} \right],
\end{equation}

for $k = 0, 1$. Note that if the infinite series is truncated after a finite number of terms, the boundary conditions are satisfied only in an asymptotic sense as $t \to 0$. That is, the boundary values of the truncated solution decay exponentially fast in $t^{-1}$ as $t \to 0$ and the exponential rate increases together with the number of retained terms.

The trace is given by

\begin{equation}
\int_0^a G(x, x, t) \, dx = \frac{a}{2\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{(na)^2}{t} \right\} + \frac{(-1)^k}{2} =
\end{equation}

\begin{equation}
\frac{a}{2\sqrt{\pi t}} + \frac{(-1)^k}{2} + \frac{a}{2\sqrt{\pi t}} \sum_{n\neq0} \exp \left\{ -\frac{(na)^2}{t} \right\}, \quad (k = 0, 1).
\end{equation}

On the other hand,

\begin{equation}
\int_0^a G(x, x, t) \, dx = \sum_{n=1}^{\infty} e^{-\lambda_n t},
\end{equation}
where \( \{\lambda_n\} \) are the eigenvalues of the homogeneous Dirichlet or Neumann problem for the operator \( d^2/dx^2 \) in the interval \([0, a]\). Thus

\[
(2.7) \quad \sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{a}{2\sqrt{\pi t}} + \frac{(-1)^k a}{2\sqrt{\pi t}} \sum_{n\neq 0} \exp \left\{ -\frac{(na)^2}{t} \right\},
\]

for \( k = 0, 1 \).

If instead of a single interval of length \( a \), we consider the heat equation in a set \( \Omega \) consisting of \( K \) disjoint intervals of lengths \( l_j \), \( (j = 1, ..., K) \), respectively, the resulting expansion is

\[
(2.8) \quad \sum_{n=1}^{\infty} e^{-\lambda_n t} = \sum_{j=1}^{K} \frac{l_j}{2\sqrt{\pi t}} + \frac{(-1)^k 2K}{4} + \sum_{j=1}^{K} \frac{l_j}{2\sqrt{\pi t}} \sum_{n\neq 0} \exp \left\{ -\frac{(nl_j)^2}{t} \right\}.
\]

The numerator in the first term on the right hand side of eq.(2.8) can be interpreted as the "area" of \( \Omega \), so we denote it \( \sum_{j=1}^{K} l_j = |\Omega| \). The number \( 2K \) is the number of boundary points of \( \Omega \), which can be interpreted as the "circumference" of the boundary, so we denote it \( |\partial\Omega| = 2K \). The exponents in the sum on the right hand side of eq.(2.8) can be interpreted as the "widths" of the components of \( \Omega \). Clearly, for small \( t \), the term containing the smallest width, \( r = \min_{1 \leq j \leq K} l_j \), will dominate the sum. Thus we can rewrite eq.(2.8) as

\[
\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{|\Omega|}{2\sqrt{\pi t}} - \frac{|\partial\Omega|}{4} + \frac{mr}{\sqrt{\pi t}} \exp \left\{ -\frac{r^2}{t} \right\} + \sum_{l_j > r} \frac{l_j}{2\sqrt{\pi t}} \sum_{n\neq 0} \exp \left\{ -\frac{(nl_j)^2}{t} \right\},
\]

where \( m \) is the number of the shortest intervals in \( \Omega \).

Equation (2.9) can be viewed as the short time asymptotic expansion of the sum on the left hand side of the equation. The algebraic part of the expansion consists of the first two terms and all other terms are transcendently small. The geometric information in the various terms of the expansion consists of the "area" of \( \Omega \) and the "circumference" \( |\partial\Omega| \) in the algebraic part of the expansion. The transcendental part of the expansion is dominated by the term containing the smallest "width" of the domain, \( r \).

The geometric information about \( \Omega \) contained in the algebraic part is the information given in the "Can one hear the shape of a drum" expansions [2], [3]. The geometric information contained in the transcendently small
terms in (2.9) can be understood as follows. The terms \( n l_j \) in the exponents are the lengths of closed trajectories of billiard balls in \( \Omega \), or the lengths of closed rays reflected at the boundaries, as in [5].

The representation (2.4) can be constructed as a short time approximation to the solution of the heat equation (2.1)-(2.3) by the ray method [11]. In this method the solution is constructed in the form

\[
G(y, x, t) = e^{-S^2(y, x)/4t} \sum_{n=0}^{\infty} Z_n(y, x)t^{n-1/2}.
\]

Substituting the expansion (2.10) into the heat equation (2.1) and ordering terms by orders of magnitude for small \( t \), we obtain at the leading order the ray equation, also called the eikonal equation,

\[
\left| \frac{\partial S(y, x)}{\partial y} \right|^2 = 1,
\]

and at the next orders the transport equations

\[
2 \frac{\partial S(y, x)}{\partial y} \frac{\partial Z_n(y, x)}{\partial y} + Z_n(y, x) \left( \frac{\partial^2 S(y, x)}{\partial y^2} + \frac{2n}{S(y, x)} \right) = \frac{2}{S(y, x)} \frac{\partial^2 Z_{n-1}(y, x)}{\partial y^2}, \quad n = 0, 1, \ldots.
\]

Denoting

\[
p(y, x) = \frac{\partial S(y, x)}{\partial y},
\]

we write the equations of the characteristics, or rays of the eikonal equation (2.11) as in [12]

\[
\frac{\partial y(\tau, x)}{\partial \tau} = 2p, \quad \frac{dp(\tau)}{d\tau} = 0, \quad \frac{dS(\tau)}{d\tau} = 2p^2(\tau)
\]

with the initial conditions

\[
y(0, x) = x, \quad p(0) = \pm 1, \quad S(0) = 0.
\]

The condition \( S(0) = 0 \) is implied by the initial condition \( G(x, y, 0) = \delta(x - y) \). The solutions are given by

\[
y(\tau, x) = x + 2p\tau, \quad p(\tau) = \pm 1, \quad S(\tau) = 2\tau = \pm(y - x).
\]
Thus $S(y, x)$ is the length of the ray from $y$ to $x$. We denote this solution by $S_0(y, x)$. It is easy to see that the solution of the transport equations corresponding to $S_0(y, x)$ is given by $Z_0(y, x) = \text{const}$, and $Z_n(y, x) = 0$ for all $n \geq 1$. The initial condition (2.2) implies that

$$Z_0(y, x) = \frac{1}{2\sqrt{\pi}}.$$ 

Combined in eq.(2.10) this solution gives Green's function for the heat equation on the entire line,

$$G_0(y, x, t) = \frac{1}{2\sqrt{\pi t}} \exp \left\{ -\frac{(y-x)^2}{4t} \right\},$$ 

which is the positive term corresponding to $n = 0$ in the expansion (2.4).

The ray from $x$ to $y$ is not the only one emanating from $x$. There are rays emanating from $x$ that end at $y$ after reflection in the boundary. Thus the ray from $x$ that reaches $y$ after it is reflected at the boundary 0 has length $y + x$. Therefore there is another solution of the eikonal equation, $S_1(y, x)$, which is the length of the reflected ray, given by

$$S_1(y, x) = y + x.$$ 

The ray from $x$ that reaches $y$ after it is reflected at the boundary $a$ has length $2a - x - y$. The ray from $x$ to 0, then to $a$, and then to $y$ has length $2a + x - y$. Thus the lengths of all rays that reach $y$ from $x$ after any number of reflections in the boundary generate solutions of the eikonal equation, which are the lengths of the rays, which in turn generate solutions of the heat equation. We denote them by $S_k(y, x)$ with some ordering. The corresponding solutions of the transport equation are

$$Z_{0,k}(y, x) = \frac{C_k}{2\sqrt{\pi}},$$ 

where $C_k$ are constant. They are chosen so that the sum of all the ray solutions,

$$G_k(y, x, t) = \frac{Z_{0,k}(y, x)}{\sqrt{t}} e^{-S_k^2(y, x)/4t},$$

satisfies the boundary conditions (2.3). Note that for all $k \neq 0$

$$G_k(y, x, t) \to 0 \quad \text{as} \ t \to 0.$$ 

This construction recovers the solution (2.4).
3. The ray method for short time asymptotics of Green’s function. The ray method consists in the construction of Green’s function $G(y, x, t)$ in the asymptotic form

$$G(y, x, t) \sim e^{-S^2(y, x)/4t} \sum_{n=0}^{\infty} Z_n(y, x)t^{n-1}. \tag{3.1}$$

The function $S(y, x)$ is the solution of the eikonal equation

$$\left| \nabla_y S(y, x) \right|^2 = 1 \tag{3.2}$$

and the functions $Z_n(y, x)$ solve the transport equations

$$2\nabla_y S(y, x) \cdot \nabla_y Z_n(y, x) + Z_n(y, x) \left( \Delta_y S(y, x) + \frac{2n-1}{S(y, x)} \right) = 0, \quad \text{for } n = 0, 1, 2, \ldots \tag{3.3}$$

The eikonal equation (3.2) is solved by the method of characteristics [12]. The characteristics, called rays, satisfy the differential equations

$$\frac{dy(\tau, x)}{d\tau} = 2\nabla_y S(y(\tau, x), x), \quad \frac{d\nabla_y S(y(\tau, x), x)}{d\tau} = 0, \quad \frac{dS(y(\tau, x), x)}{d\tau} = 2.$$

The initial condition (1.2) implies that the rays emanate from the point $x$. Thus we choose the initial conditions

$$y(0, x) = x, \quad \nabla_y S(y(0, x), x) = \nu, \quad S(y(0, x), x) = 0, \tag{3.4}$$

where $\nu$ is a constant vector of unit length. The solution is given by

$$y(\tau, x) = x + 2\nu \tau, \quad S(y, x) = |y - x| = 2\tau, \quad \nabla_y S(y, x) = \nu. \tag{3.5}$$

The pair $(\tau, \nu)$ determines uniquely the point $y = y(\tau, x)$ and the value of $S(y, x)$ at the point. The parameter $\tau$ is half the distance from $y$ to $x$ or half the length of the ray from $x$ to $y$. The vector $\nu$ is the unit vector in the direction from $x$ to $y$.

The function $Z_0(y, x)$ is easily seen to be a constant, $1/4\pi$, and $Z_n(y, x) = 0$ for all $n > 0$. This construction recovers the solution of the heat equation in the entire plane and disregards the boundary $\partial \Omega$, because in the plane every point can be seen from every other point by a straight ray. Note that to
calculate the function $P(t)$ in eq. (1.5) only the values of $S(x, x)$ and $Z_0(x, x)$ are needed. Thus $S(x, x) = 0$ and the first approximation to $G(x, x, t)$ is

$$G(x, x, t) = \frac{1}{4\pi t},$$

hence the first approximation to $P(t)$ is

$$P_0(t) = \frac{\Omega}{4\pi t}.$$

There is another solution of the eikonal equation (3.2) constructed along rays that emanate from $x$, but reach $y$ after they are reflected in $\partial \Omega$ [11]. The law of reflection is determined from the boundary conditions. Dirichlet and Neumann boundary conditions imply that the angle of incidence equals that of reflection [11]. Similarly, there are solutions of the eikonal equation that are the lengths of rays that emanate from $x$ and reach $y$ after any number of reflections in $\partial \Omega$. We denote these solutions $S_k(y, x)$ with some ordering. Thus the full ray expansion of Green’s function has the form

$$G(y, x, t) \sim \sum_{k=1}^{\infty} e^{-S^2_k(y, x)/4t} Z_k(y, x, t),$$

where

$$Z_k(y, x, t) = \sum_{n=0}^{\infty} Z_{n,k}(y, x)t^{n-1}.$$

As above, each one of the series

$$e^{-S^2_k(y, x)/4t} Z_k(y, x, t)$$

is called a ray solution of the diffusion equation. The boundary values of $Z_k(y, x, t)$ are chosen so that $G(y, x, t)$ in eq. (3.6) satisfies the imposed boundary condition. In particular, the values of $S_k(x, x)$ are the lengths of all rays that emanate from $x$ and are reflected from the boundary back to $x$. Note that sums of ray solutions satisfy boundary conditions only at certain points.

To fix the ideas, we consider first simply connected domains. We denote

$$S_0(y, x) = |x - y|$$

and

$$G_0(y, x, t) = \frac{1}{4\pi t} e^{-S^2_0(y, x)/4t}.$$
We consider first solutions corresponding to rays that are reflected only once at the boundary, and in particular, rays that are reflected back from the boundary to the points of their origin. Such rays hit the boundary at right angles (see Fig. 1 and [11]). If there is only one minimal eikonal $S_1(x, x) > 0$, we say that $x$ is a regular point of $\Omega$. If there is more than one minimal eikonal $S_1(x, x)$, we say that $x$ is a critical point of $\Omega$. We denote by $\Gamma$ the locus of critical points in $\Omega$. The eikonal $S_1(y, x)$ is the length of the shortest ray from $x$ to $y$ with one reflection in the boundary such that the ray from $x$ to the boundary does not intersect $\Gamma$. For $x = y$ the eikonal $S_1(x, x)$ is twice the distance of $x$ to the boundary. We denote by $x'$ the orthogonal projection of $x$ on the boundary along the shortest normal from $x$ to the boundary. When $y = x'$

\[(3.7) \quad S_1(x', x) = S_0(x', x) = |x - x'|.\]

The function

$$G_1(y, x, t) = e^{-S_1^2(y, x)/4t}Z_1(y, x, t)$$

has to be chosen so that $G_0(x', x, t) - G_1(x', x, t) = 0$. In view of (3.7), we have to choose

$$Z_1(x', x, t) = \frac{1}{4\pi t}.$$

When $y''$ is the other boundary point on the normal from $x'$ to $x$, we have

$$G_0(y'', x, t) - G_1(y'', x, t) =$$

\[(3.8) \quad \frac{1}{4\pi t}e^{-|x-y''|^2/t} - e^{-|x-y''+y''-y'|x-x'|^2/t}Z_1(y'', x, t).\]

Next, we consider in $\Omega - \Gamma$ the minimal among the remaining eikonals $S_k(x, x) > S_1(x, x)$ and denote it $S_2(x, x)$. This eikonal is twice the length of a ray that emanates from $x$, intersects $\Gamma$ once, and intersects the boundary $\partial \Omega$ at right angles at a point, denoted $x''$. The eikonal $S_2(y, x)$ is the length of the ray from $x$ to $y$ with one reflection in the boundary such that the ray from $x$ to the boundary intersects $\Gamma$ once. When $y = x''$

\[(3.9) \quad S_2(x'', x) = S_0(x'', x) = |x - x'|.\]

When $y'$ is the other boundary point on the normal that emanates from $x''$ (see Fig.2), we have

$$S_2(y', x) = |x - x''| + |y' - x''|.$$
In general $x' \neq y'$ and $x'' \neq y''$. However, if the ray is a 2-periodic orbit (that hits the boundary at only 2 points), $x' = y'$ and $x'' = y''$ so that

$$S_2(y'', x) = S_0(y'', x) = |x - y''|$$

and

$$S_2(y', x) = |x - x''| + |y'' - x'|.$$

**Figure 1.** The locus of critical points, $\Gamma$, is the segment $AB$. The first eikonal is $S_1(y, x) = |x - c| + |c - y|$. It is defined as the shortest reflected ray from $x$ to $y$, such that $x - c$ does not intersect $\Gamma$. For $x = y$ the diagonal values are $S_1(x, x) = 2|x - x'|$. The diagonal values of the second eikonal are $S_2(x, x) = 2|x - x''|$. The vectors $x - x'$ and $x - x''$ are orthogonal to the boundary. For $x_1 \in \Gamma$ the two eikonals are equal.

Since

$$|x - y''| < |x - x'| + |y'' - x'| < |x - x''| + |y'' - x'|$$

for all regular points $x$, the order of magnitude of the boundary error (3.8) decreases if we use the approximation

$$G_0(y, x, t) \sim G_0(y, x, t) - G_1(y, x, t) - G_2(y, x, t)$$
with

\[ Z_2(y'', x, t) = Z_1(y'', x, t) = Z_0(t). \]

Figure 2. The second eikonal \( S_2(y, x) = |x - d| + |d - y| \). It is defined as the shortest reflected ray such that \( x - d \) intersects \( \Gamma \). The eikonals \( S_3(x, x) \) and \( S_4(x, x) \) are ordered according to magnitude.

4. The trace. To find the short time asymptotics of the Dirichlet series (1.6), as given in eq.(1.5),

\[ P(t) = \int_\Omega G(x, x, t) \, dx, \]

we use the ray expansion (3.6) for the evaluation of the integral. We retain in the resulting expansion only terms that are transcendentally small, since all algebraic terms are contained in the expansion (1.7).

We note that according to Sard's theorem, \( \Gamma \) is a set of measure zero and that all points in the domain \( \Omega - \Gamma \) are regular. For any point \( x \in \Omega \), we denote by \( r_1(x) \) its distance to the boundary and note that \( S_1(x, x) = 2r_1(x) \). We also denote by \( s_1(x) \) the arclength at the boundary point \( x' \) (the
orthogonal projection of $x$ on $\partial \Omega$ along the shortest normal from $x$ to $\partial \Omega$), measured from a boundary point where the arclength is set to 0 (see Figure 3). It follows that the change of variables in $\Omega - \Gamma$, given by

\[(4.1) \quad x \rightarrow (r_1(x), s_1(x)),\]

is a one-to-one mapping of $\Omega - \Gamma$ onto a strip $0 \leq r_1 \leq r_1(s_1), 0 \leq s_1 \leq L$, where $r_1(s_1)$ is the distance from the boundary point corresponding to arclength $s_1$ to $\Gamma$.

![Figure 3. The arclength $s_1(x)$ is measured from the point E. Both transformations $x \rightarrow (r_1(x), s_1(x))$ and $x \rightarrow (r_2(x), s_1(x))$ are one to one mappings of $\Omega - \Gamma$. The images are given in Figure 4.](image)

We evaluate the integral over $\Omega$ separately for each summand $k$ in the expansion (3.6). In this notation, we can write

\[(4.2) \quad \int_\Omega G_1(x, x, t) \, dx = \int_\Omega e^{-[S_1(x, x)]^2/4t} \sum_{n=0}^\infty Z_{n,1}(x, x)t^n \, dx = \]
\[
\int_0^L ds \int_0^{r_1(s_1)} e^{-r_1^2/t} J_1(r_1, s_1) Z_1(r_1, s_1, t) \, dr_1,
\]
where \( J_1(r_1, s_1) \) is the Jacobian of the transformation and
\[
Z_1(r_1, s_1, t) = \sum_{n=0}^{\infty} Z_{n,1}(x, x)t^{n-1}.
\]

Figure 4. The domain \( \Omega \) is the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \). The domain enclosed between the \( s_1 \)-axis and the lower curve is the image of the ellipse under the transformation \((4.1)\) and the domain enclosed between the upper and the lower curves is its image under \((4.3)\).

Note that the Jacobian vanishes neither inside \( \Omega - \Gamma \) nor at \( r_1 = 0 \), because the transformation is one-to-one in \( \Omega - \Gamma \), however, it does on \( \Gamma \).

We set \( S_2(x, x) = 2r_2(x) \) and use it as a coordinate. We use \( s_1(x) \) as the other coordinate of the point \( x \in \Omega - \Gamma \). Note that while \( r_2(x) \) is the length of the longer normal from \( x \) to \( \partial \Omega \) (the one that intersects \( \Gamma \)), the other coordinate is the arclength corresponding to the shorter normal from \( x \) to \( \partial \Omega \) (the one that does not intersect \( \Gamma \)). The transformation
\[
(4.3) \quad x \to (r_2(x), s_1(x))
\]
maps \( \Omega - \Gamma \) onto the strip \( r(s_1) \leq r_2 \leq l(s_1), \ 0 \leq s_1 \leq L \), where \( l(s_1) \) is the length of the segment of the normal that starts at the boundary point \( r_1 = 0, s_1 \) and ends at its other intersection point with the boundary. This mapping is one-to-one as well. It follows that

\[
\int_{\Omega} G_2(x, x, t) \, dx =
\]

\[
\int_{\Omega} e^{-[S_2(x, x)]^2/4t} \sum_{n=0}^{\infty} Z_{n,2}(x, x)t^{n-1} \, dx =
\]

\[
\int_{0}^{L} ds_1 \int_{r(s_1)}^{l(s_1)} e^{-r_2^2/t} J_2(r_2, s_1) Z_2(r_2, s_1, t) \, dr_2,
\]

where

\[
Z_2(r_2, s_1, t) = \sum_{n=0}^{\infty} Z_{n,2}(x, x)t^{n-1}.
\]

Note that for \( x \) on \( \Gamma \) both transformations (4.1) and (4.3) are identical and

\[
J_2(r_2, s_1) Z_2(r_2, s_1, t) = J_1(r_1, s_1) Z_1(r_1, s_1, t).
\]

It follows that the two equations (4.2) and (4.4) combine together to give

\[
\int_{\Omega} [G_1(x, x, t) + G_2(x, x, t)] \, dx =
\]

\[
\int_{0}^{L} \int_{0}^{l(s)} e^{-r^2/t} J(r, s) Z(r, s, t) \, dr \, ds,
\]

where \( s = s_1, r = r_1, J = J_1, \) and \( Z = Z_1 \) for \( 0 < r < r_1(s_1) \), and \( s = s_1, r = r_2, J = J_2, \) and \( Z = Z_2 \) for \( r_2(s_1) < r < l(s_1) \). Thus the domain of integration of the function \( e^{-r^2/t} J(r, s) Z(r, s, t) \) in eq. (4.5) is the domain enclosed by the \( s_1 \)-axis and the upper curve in Figure 4. Now, for \( t \ll 1 \), we write the inner integral on the right hand side of eq. (4.5) as

\[
\int_{0}^{l(s)} e^{-r^2/t} J(r, s) Z(r, s, t) \, dr =
\]

\[
\sqrt{\frac{\pi t}{2}} \text{erf} \left( \frac{l(s)}{\sqrt{t}} \right) J(0, s) Z(0, s, t) \left( 1 + O \left( \sqrt{t} \right) \right) =
\]

\[
\sqrt{\frac{\pi t}{2}} \left( 1 - \frac{\exp \left\{ - \frac{l^2(s)}{t} \right\} \sqrt{t} }{l(s)} \right) J(0, s) Z(0, s, t) \left( 1 + O \left( \sqrt{t} \right) \right).
\]
THE SHORT TIME ASYMPTOTIC EXPANSION

Recall that \( J(0, s)Z(0, s, t) \neq 0 \). Only the exponentially small terms have to be considered, because the algebraic terms are included in the SW expansion. Thus

\[
\int_0^L \int_0^{t(s)} e^{-r^2/t} J(r, s)Z(r, s, t) \, dr \, ds -
\]
\[
- \int_0^L \sqrt{\frac{\pi t}{2}} (0, s)Z(0, s, t) \left(1 + O \left(\sqrt{t}\right)\right) \, ds =
\]
\[
= - \int_0^L \exp \left\{ - \frac{l^2(s)}{t} \right\} \frac{J(0, s)Z(0, s, t)}{l(s)} O(t) \, ds \quad \text{for } t \ll 1.
\]

Evaluating the last integral by the Laplace method, we find that each point \( s_i \) that is an extremum point of \( l(s) \) contributes an exponential term of the form

\[
\exp \left\{ - \frac{l^2(s_i)}{t} \right\} \frac{J(0, s_i)Z(0, s_i, t)}{l(s_i)} O(t^\nu).
\]

The expression (4.6) means that some of the \( \delta_n \)-s in the expansion eq.(1.9) are the extremal values \( l(s_i) \) and their multiples. These are half the lengths of the 2-periodic orbits of a billiard ball in \( \Omega \) (see Figure 5). The 2-periodic orbits of the ellipse are the major axes, which correspond to the lowest and highest points of the top curve in Figure 4. There are other exponents as well, as discussed below.

The pre-exponential terms in the expression (4.6) influence the factors \( P_n(\sqrt{t}) \) in eq.(1.9). For example, if \( l'(s_i) = 0, l''(s_i) \neq 0 \), then \( \nu = 3/2 \). If the boundary is flatter, then \( 1 \leq \nu < 3/2 \). In addition to the 2-periodic orbits, there are ray solutions corresponding to rays from \( x \) to \( y \) that are reflected any number of times in the boundary. There are eikonals from \( x \) to \( y \) in \( \Omega \) with \( N - 1 \) different vertices on the boundary, which have \( N \) vertices on \( \partial \Omega \) if \( x = y \) and \( x \in \partial \Omega \) (this is a periodic orbit with \( N - 1 \) reflections).
Among these periodic orbits there are eikonals $S_N(x, x)$ with extremal length, denoted $S_{N,j}$, ($j = 1, \ldots$). At points $x \in \Omega$ on a 2-periodic orbit the eikonal $S_N(x, x)$, which now has $N - 1$ vertices on the boundary, may reduce to the 2-periodic orbit with $N$ reflections. Therefore the change of variables $x \to (S_N(x, x), s(x))$ will map the domain into a strip with extremal widths that are the differences between the lengths $S_{N,j}$ and the length of a 2-periodic orbit with $N$ reflections. It follows that the evaluation of the trace by the Laplace method leads to exponents which are the extremal lengths of periodic orbits with any number of reflections.

For example, there is an eikonal in a circle (centered at the origin) that is the ray from $x$ to $y$ with 2 reflections in the boundary (see Figure 6).

Figure 5. The rays emanating from the boundary points $s_1$ and $s_2$ are orthogonal to the boundary at both ends. They are 2-periodic orbits.
Figure 6. The eikonal $S_3(y, x)$ with two reflections in the circle.

For $x = y$ it is the equilateral triangle (see Figure 7) with circumference

$$S(x, x) = R \left( 2 \frac{\sqrt{2|x|^2+1+\sqrt{8|x|^2+1}}}{\sqrt{4|x|^2+1+\sqrt{8|x|^2+1}}} + \sqrt{4|x|^2+2+2\sqrt{8|x|^2+1}} \right).$$

The eikonal $S_3(x, y)$ reduces to a 2-periodic orbit with two reflections if $x = y = 0$ (the center of the circle). If $x$ is on the circumference, the eikonal becomes the isosceles triangle with one vertex at $x$. To evaluate the contribution of the corresponding ray solution to the trace, we use this eikonal as a coordinate that varies between $4R$, the length of the 2-periodic orbit with two reflections, and $3\sqrt{3}R$, the circumference of the inscribed isosceles triangle. The contribution of this integral to the exponential sum in eq.(1.9) contains exponents that are both lengths.
Figure 7. The eikonal $S_3(x, x)$ with two reflections, where $|x| = OC$.

Similarly, the 2-periodic orbit with 3 reflections has length $6R$ while the periodic orbit with 3 reflections at 3 different points has length $4\sqrt{2}R < 6R$.

REFERENCES


SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS IN SOBOLEV SPACES *

VICTOR V. VLASOV† AND JIANHONG WU †

Dedicated to Professor A.D. Myshkis on the occasion of his 85th-birthday

Abstract. In this paper, we obtain sharp estimates for the growth of strong solutions of difference differential equations of neutral type. Our work is based on existing results for the initial value problem for a related homogeneous neutral equation, by using the Riesz basis consisting of exponential solutions of the homogeneous equation.

Key Words. Functional differential equation, Sobolev space, Riesz Basis, sharp estimates, neutral type

1. Introduction. Despite its profound importance in the theory of control and the theory of dynamical systems and their applications, obtaining the sharpest estimates for solutions to functional differential equations remains to be a challenging task although some significant progress has been achieved, see, for example, [1]-[7], [22].

In this paper, we obtain some sharp estimates of the strong solutions for difference differential equations of neutral type. This is of course a well-known classical problem, and our result (estimate) is based on some previous work by one of the authors about the initial value problem for a corresponding homogeneous equation (see [8]-[14] for more details). As such, our work

* The research of V. Vlasov was accomplished with financial support of Russian Foundation for Basic Research (grants N 02-01-00790, N 04-01-00618). The work of J. Wu is supported by Natural Sciences and Engineering Research Council of Canada and by Canada Research Chair Program.
† Department of Mechanics and Mathematics, Moscow State University, Moscow, 117234, Russia
† Laboratory for Industrial and Applied Mathematics, York University, Toronto, Canada, M3J 1P3
depends heavily on the Riesz basisness of the system of exponential solutions (see [8]-[14]).

The remaining part of this paper consists of four sections. Section 2 provides the statement of the problem and the formulation of the results. The basic results are proved in Section 3. Section 4 gives examples to demonstrate that our estimates are sharp, and this section also contains some remarks, comments, and the comparison of our results with some earlier relevant works. Section 5, the Appendix, is devoted to the proof of the Riesz basis.

2. The Problems and Results. Denote by $W^p_{2,r}((a,b),C^r)$, $(-\infty < a < b \leq +\infty)$, weighted Sobolov spaces of vector valued functions with values in $C^r$, endowed with the norms:

$$
\|V\|_{W^p_{2,r}(a,b)} = \left( \int_a^b \exp(-2rt)\left( \sum_{j=0}^{p} \|V^{(j)}(t)\|^2 \right) dt \right)^{1/2}, r \geq 0.
$$

Here and in what follows, $W^p_{2,0} = W^p_2$, $V^{(j)}(t) = \frac{d^j}{dt^j}V(t)$, $p, j = 1, 2, \cdots$; $\|\cdot\|$ is the norm in the space $C^r$.

We consider the following nonhomogeneous equation

$$
Du \equiv \sum_{j=0}^{n}(B_j(t-h_j) + D_j \frac{du}{dt}(t-h_j)) + \int_0^h B(s)u(t-s)ds \\
+ \int_0^h D(s)u^{(1)}(t-s)ds = f(t), \ t > 0;
$$

subject to the usual Cauchy initial condition

$$
u(t) = g(t), \ t \in [-h,0].
$$

Here $B_j, D_j(j=0,1,\cdots,n)$ are square ($\tau \times \tau$) matrices with constant complex elements; the real numbers $h_j$ are such that $0 = h_0 < h_1 < \cdots < h_n = h$; the elements $B_{ij}(s), D_{ij}(s)(i,j = 1,2,\cdots,\tau)$ of matrices $B(s)$ and $D(s)$ belong to the space $L_2(0,h)$. Moreover, the non-homogeneous term $f$ belongs to $L_2((0,T),C^r)$ for an arbitrarily given $T > 0$, and the initial function $g$ belongs to $W^1_2((-h,0),C^r)$.

**Definition 2.1.** A vector-valued function $u \in W^1_2((-h,T),C^r)$ for arbitrary $T > 0$ is called a strong solution of the problem (1), (2), if $u(t)$ satisfies equation (1) almost everywhere on the semiaxis $\mathbb{R}_+ = (0, +\infty)$ and $u$ satisfies the initial condition (2).

In order to describe our main results we have to introduce certain notations.
We denote by $L(\lambda)$ the matrix-valued function

$$L(\lambda) = \sum_{j=0}^{n} (B_j + \lambda D_j) \exp(-\lambda h_j) + \int_{0}^{h} B(s) \exp(-\lambda s) ds + \lambda \int_{0}^{h} D(s) \exp(-\lambda s) ds.$$  

Let $l(\lambda) = \det L(\lambda)$ be the characteristic quasipolynomial (see [1] for more details) of the equation (1). We shall let $\lambda_q$ denote a typical zero (referred as characteristic number) of the function $l(\lambda)$ with multiplicities $\nu_q$, and we shall arrange these zeros, counting multiplicities, in increasing order of their modulus, and we will denote by $\Lambda$ the set of all zeroes of the function $l(\lambda)$.

We denote the eigenvectors from the canonical system of eigen and associated (root) vectors, corresponding to the characteristic number $\lambda_q$, by $x_{q,j,0}, (j = 1, 2, \ldots, \nu_q)$, their adjoint vectors of order $s$ by $x_{q,j,s} (s = 1, 2, \ldots, p_{qj})$. Here, the index $j$ shows what is the number of the vector $x_{q,j,0}$ in a specially chosen basis of the subspace of the solutions of the equation $L(\lambda_q)x = 0$. See [16], [17] for more details.

Therefore, we have the following set of exponential solutions of the equation (1) in homogeneous case ($f(t) \equiv 0$)

$$y_{q,j,s}(t) = \exp(\lambda_q t) \left( \frac{t^s}{s!} x_{q,j,0} + \frac{t^{s-1}}{(s-1)!} x_{q,j,1} + \cdots + x_{q,j,s} \right).$$

Now we formulate two technical results which will be used in what follows.

**Lemma 2.2.** Suppose that $\det D_0 \neq 0, \det D_n \neq 0$. Then each of the following values

$$\Re\lambda_q = \sup_{\lambda_q \in \Lambda} \Re \lambda_q, \; \Im\lambda_q = \inf_{\lambda_q \in \Lambda} \Re \lambda_q, N = \max_{\lambda_q \in \Lambda} \nu_q$$

are finite.

We denote by $D(\lambda_q, \rho)$ the disk of radius $\rho > 0$ with center in the point $\lambda_q$, define a domain $G(\Lambda, \rho)$ in the following way

$$G(\Lambda, \rho) = C \setminus \bigcup_{\lambda_q \in \Lambda} D(\lambda_q, \rho).$$

**Lemma 2.3.** Suppose that $\det D_0 \neq 0, \det D_n \neq 0$. Then there exists $\beta > 0$ so that each of the following set of contours $\Gamma_n = \{ \lambda \in C: \Re \lambda = \Re_{+} + \beta, \gamma_n \leq \Im \lambda \leq \gamma_{n+1} \} \cup \{ \lambda \in C: \Re_{-} - \beta \leq \Re \lambda \leq \Re_{+} + \beta, \Im \lambda = \gamma_{n+1} \} \cup \{ \lambda \in C: \Re \lambda = \Re_{-} - \beta, \gamma_n \leq \Im \lambda \leq \gamma_{n+1} \} \cup \{ \lambda \in C: \Re_{-} - \beta \leq \Re \lambda \leq \Re_{+} + \beta, \Im \lambda = \gamma_n \}, \beta > 0$, belong to the domain $G(\Lambda, \rho)$ for sufficiently small $\rho > 0$, and the following conditions are satisfied: there exist
positive constants \( \delta, \Delta \) such that the sequence of real numbers \( \{ \gamma_n \} \) satisfy the inequalities: \( 0 < \delta \leq \gamma_{n+1} - \gamma_n \leq \Delta < +\infty, n \in \mathbb{Z} \), and the number \( N(\Gamma_n) \) of zeroes of the function \( l(\lambda) \) (counting multiplicities), which lie in the domains \( G_n \) bounded by the contours \( \Gamma_n \), are uniformly bounded with respect to \( n \), i.e. there exists a constant \( M > 0 \) so that

\[
\max_n N(\Gamma_n) \leq M.
\]

The above two lemmas can be reduced from the results of [15], and we refer to [9] for their proofs.

The next theorem is the main result of this article.

**Theorem 2.4.** Let \( \det D_0 \neq 0, \det D_n \neq 0 \). Suppose also that \( f \in L^2((0,T), C^r) \) for an arbitrary given \( T > h \), and \( g \in W^1_2((-h,0), C^r) \). Then the problem \( (1), (2) \) has a unique solution and this solution \( u(t) \) satisfies

\[
||u(t)||_{W^1_2(-h,t)} \leq d_0 (t+1)^{M-1} \exp(\mathbb{R}_+ t)||g||_{W^1_2(-h,0)}
+ d_1 t (t-s+1)^{2(M-1)} \exp(2\mathbb{R}_+(t-s)) \||f(s)||^2 ds^{1/2}
\]

for \( t \in [h,T] \), with constants \( d_0 \) and \( d_1 \) independent of functions \( g \) and \( f \) as well as the constant \( T \).

We add a few remarks to illustrate the significance of the above result.

**Remark 2.5.** If the set \( \Lambda \) of zeroes \( \lambda_q \) is separate, i.e. \( \inf_{\lambda_p \neq \lambda_q} |\lambda_p - \lambda_q| > 0 \), then in the estimate (6) constant \( M \) (see (5)) may be replaced by \( N = \max_{\lambda_q \in \Lambda} \lambda_q \).

**Remark 2.6.** The estimate (6) is sharp in the following sense. It is impossible to change constant \( \mathbb{R}_+ \) by \( \mathbb{R}_+ - \epsilon \) for every \( \epsilon > 0 \). Moreover, \( \sqrt{t} \) can't be omitted. In Section 4, will have some examples to illustrate these.

**Corollary 2.7.** Suppose the conditions of the Theorem 2.4 are satisfied. Then we have the following

\[
||u(t)|| \leq d_2 (t+1)^{M-1} \exp(\mathbb{R}_+ t)||g||_{W^1_2(-h,0)}
+ d_3 t (t-s+1)^{2(M-1)} \exp(2\mathbb{R}_+(t-s)) \||f(s)||^2 ds^{1/2}
\]

for \( t \in [h,T] \), with constants \( d_2 \) and \( d_3 \) independent of functions \( g \) and \( f \).

The above corollary immediately follows from Theorem 2.4 and trace theorem (see for example [21]).

We conclude this section with a result about the estimate of solutions of the problem (1)-(2) in the special case when the support of function \( f \) is compact.
Proposition 2.8. Let \( \text{det} D_0 \neq 0, \text{det} D_n \neq 0, \) \( g \in W^1_2((-h, 0), C^\gamma) \) and assume that \( f \) has a compact support \( \Omega \). Then we have

\[
\|u\|_{W^1_2(t-h, t)} \leq d_0(t + 1)^{-M-1} \exp(K \tau t) \|g\|_{W^1_2(-h, 0)} + d_2 \int_{\Omega} (t - s + 1)^{2(M-1)} \exp(2\tau(t - s)) \|f(s)\|^2 ds^{1/2}
\]

for all \( t \geq h \), with constants \( d_0 \) and \( d_2 \) independent of the functions \( f \) and \( g \).

In the proofs to be provided next section, we shall use the solvability result of the problem (1)-(2) in weighted Sobolev space \( W^1_{2,r}((-h, +\infty), C) \) (see for example [8], [10]). In order to make the article self-contained, we state this solvability result below.

Let \( L^2_{r,\gamma}(R^+, C) \) be the Hilbert space of vector-valued functions equipped with the norm

\[
\|f\|_{L^2_{r,\gamma}} = \left( \int_0^{+\infty} \exp(-2rt) \|f(t)\|^2 dt \right)^{1/2}, r \in R.
\]

Lemma 2.9. Suppose \( \text{det} D_0 \neq 0, \) \( g \in W^1_2((-h, 0), C^\gamma), \) \( f \in L^2_{r,0}(R^+, C^\gamma) \) for a certain \( r_0 \in R \). Then there exists constant \( r_*(r_* > r_0) \) such that for every \( r > r_* \), problem (1)-(2) has a unique solution \( u \in W^1_{2,r}((-h, 0), C^\gamma) \) satisfying the following inequality

\[
\|u\|_{W^1_2((-h, +\infty)} \leq c_r (\|g\|_{W^1_2(-h, 0)} + \|f\|_{L^2_{r,0}(R^+)})
\]

with constant \( c_r \) independent of the functions \( f \) and \( g \).

3. Proofs of the Main Results. We should emphasize that the main result Theorem 2.4 is based on the estimates of the solutions for homogeneous equation \( (f(t) \equiv 0) \) obtained earlier in [9], [10].

Also since equation (1) is linear, we shall consider the problem (1)-(2) in the case \( g(t) \equiv 0 \).

In the first step, we consider the problem (1)-(2) where the non-homogeneous term \( f \) has the support contained in \([0, h]\) and initial function \( g = 0 \).

Denote the solution of this problem by \( u_h(t) \). Using the results in [9] and [10], we have

\[
\int_0^{+\infty} \exp(-2rt)(\|u_h^{(1)}(t)\|^2 + \|u_h(t)\|^2) dt \leq c_r \|f\|_{L^2_{[0,h]}}^2
\]

for some constant \( r \) and constant \( c_r \) independent of the function \( f \).

It follows immediately that

\[
\|u_h\|_{W^1_2([0,h])} \leq c_1 \|f\|_{L^2_{[0,h]}}
\]
with the constant $c_1$ independent of the function $f$.

We then consider the following initial-value problem

\begin{equation}
(Dv)(t) = 0, \ t > h;
\end{equation}

\begin{equation}
v(t) = u_h(t), \ t \in [0, h].
\end{equation}

Due to the uniqueness of the Cauchy initial value problem, we have $v(t) = u_h(t)$ for all $t > h$.

An estimate of the solution for the homogeneous equation (1) ($f(t) \equiv 0$) with initial data (2) was established earlier in [9], [10] and it has exactly the same form (6) with $f(t) \equiv 0$. Substituting $g$ by $u_h$, $u$ by $v$ and segment $[-h, 0]$ by $[0, h]$, we obtain

\begin{equation}
\|v\|_{W^2_1[t-h,t]} \leq d_1 (t+1)^{M-1} \exp(N_h t) \|u_h\|_{W^2_1[0,h]}, \ t > h,
\end{equation}

with constant $d_1$ independent of the function $u_h$.

From estimates (10)-(13) the following inequality follows

\begin{equation}
\|u\|_{W^2_1[t-h,t]} \leq d_2 (t+1)^{M-1} \exp(N_h t) \|f\|_{L_2[0,h]}, \ t > h,
\end{equation}

with constant $d_2$ independent of the function $f$.

For the second step of the proof of Theorem 2.4, we need the following technical lemma whose proof is deferred to the end of this section.

**Lemma 3.1.** Suppose that $\det D_0 \neq 0$, $\det D_n \neq 0$, the support of $f$ is contained in $[jh-h, jh]$ for some integer $j > 1$, the initial function $g = 0$. Then, for the solution of the problem (1)-(2) we have the following assertions:

(i) $u(t) = 0$, for $t < jh-h, j = 1, 2, \ldots$;

(ii) For each $k > j$ we have

\begin{equation}
\|u\|_{W^2_1[kh-h,kh]} \leq d((k-j+1)h)^{M-1} \exp(N_h (k-j)h) \|f\|_{L_2[jh-h,jh]},
\end{equation}

with constant $d$ independent of the function $f$.

For the proof of the estimate (6) on the whole interval, we need, in addition to Lemma 3.1, the following representation of the function $f$:

\begin{equation}
f(t) = \sum_{j=1}^{\infty} f_j(t), f_j(t) = \mathcal{X}(jh-h, jh) f(t),
\end{equation}

where $\mathcal{X}(jh-h, jh)$ is a characteristic function of the interval $(jh-h, jh)$.
The essential part of the remaining part of the proof for the estimate (6) is the fact that the functions $f_j(t)$ with $j > k$ do not influence on the solution $u$ on the segment $[0, kh]$. Also, as mentioned earlier, the estimates for solutions of the homogeneous equation are based on the fact that exponential solutions give a Riesz basis for the spaces involved. More precisely, we have the following

**Theorem 3.2.** Let $\det D_0 \neq 0$ and $\det D_n \neq 0$. Then the system of the subspace $W_n = \text{Span}_{\lambda_q \in \mathbb{C}} \{y_{q,j,s}(t)\}$ forms the Riesz basis (unconditional basis) in the space $W^1_2((-h, 0), C^r)$. If, in addition, we have $\inf_{\lambda_p \neq \lambda_q}|\lambda_p - \lambda_q| > 0$, then the system of subspaces $V_{\lambda_q} = \text{Span}_{\lambda_q} \{y_{q,j,s}(t)\}$ forms the Riesz basis (unconditional basis) in the space $W^1_2((-h, 0), C^r)$.

The proofs of this results and estimates of the solutions of the homogeneous equation (1) can be found in [8], [9] (in the case $B(s) \equiv 0, D(s) \equiv 0$).

In our situation, the proofs are completely the same. Only certain technical details have to be added. We give the proof of Riesz basisness in the Appendix.

We should mention that more general results for the equations of arbitrary order $m$ with matrix coefficients can be found in [13] and [14]. The most complete results about Riesz basisness of the system of exponential solutions in the scalar case ($\tau = 1$) for equations of neutral type and of arbitrary order were obtained in [12] and [26]. These articles also proved results about Riesz basisness of the system of divided differences constructed by the system of exponential solutions, these are rather useful in the situation when the set $\Lambda$ is not separate. All results in [12] and [26] were obtained in the scale of Sobolev spaces of arbitrary orders ($s \geq m, s \neq l + 1/2, l \in \mathbb{N}$). We should also mention that Riesz basisness of exponential solutions $\{y_{q,j,s}(t)\}$ of homogeneous equation (1) in the space $M_2 = C^r \oplus L_2((-h, 0), C^r)$ was obtained in [27] under an additional condition (the set $\Lambda$ is separate). In the particular case for the equation with one delay (in our notations: $\det D_n \neq 0, D_j \neq 0, j = 1, \ldots, n - 1, D_0 = 1, B_j \equiv 0, j = 0, n$), we can find results about Riesz basisness of exponential solutions in [28].

Before getting into the details of the proofs of our main results, let us briefly indicate how this Riesz basis is used in obtaining the sharp estimates for the solutions of equation (1), in the simplest case of a scalar equation when all roots $\lambda_q$ are simple. In this case, the system of the exponential solutions

$$z_q(t) = \frac{\exp(\lambda_q t)}{|\lambda_q| + 1}, \lambda_q \in \Lambda,$$

forms the Riesz basis in the space $W^1_2((-h, 0), C^r)$. 
Denote by $V$ the orthogonalizer, i.e. the bounded operator, acting in the space $W^1_2((-h,0), C^r)$ and turning the system $\{z_q(t)\}$ into orthonormal basis of the space $W^1_2((-h,0), C^r)$. Decompose the initial function $g$ by the basis $\{z_q(t)\}$ as

$$g(s) = \sum_{\lambda_q \in \Lambda} c_q z_q(s), s \in [-h, 0].$$

The desired estimate now follows from the following chain of the inequalities:

$$\|u(t + .)\|^2_{W^1_2(-h,0)} = \|V^{-1}\sum_{\lambda_q \in \Lambda} c_q e^{\lambda_q t}(Vz_q)(.)\|^2_{W^1_2(-h,0)} \leq \|V^{-1}\|^2 \exp(2N + t) \sum_{\lambda_q \in \Lambda} |c_q|^2 \leq \|V^{-1}\|^2 \|V\|^2 \exp(2N + t) \|g\|^2_{W^1_2((-h,0)}.$$

Note that we need the aforementioned Riesz basis in order to ensure that the bounded operator $V$ has a bounded inverse $V^{-1}$ (see [17] for more details).

Now we are ready to give

**Proof of Lemma 3.1** The first assertion of Lemma 3.1 is valid due to the unique solvability of the problem (1)-(2) in the space $W^1_{2,r}((-h, +\infty), C^r)$ for some $r \in R$(see [8] for more details).

This means that if $u(t) = 0$ for $t \in [-h, 0]$ and if the right-hand side $f(t) = 0$ for $t \in [0, T]$, then the solution $u(t) = 0$ for $t \in [0, T]$. In other words, the operator $D$ is a causal operator (an operator of Volterra type).

In order to obtain the assertion (ii), we change the variables by $t = (jh - h) + \tau$ (note we are abusing $\tau$ here, and hope this will not cause any confusion) and set

$$\tilde{u}(\tau) = u(jh - h + \tau), \tilde{f}(\tau) = f(jh - h + \tau).$$

It is clear that $\tilde{u}(\tau)$ is a solution of the following problem

$$(D\tilde{u}(\tau)) = \tilde{f}(\tau), \tau > 0;$$

and

$$\tilde{u}(\tau) = 0, \tau \in [-h, 0].$$

Taking into account that the support of $\tilde{f}$ is contained in $[0, h]$ due to the inequality (14), we have

$$\|\tilde{u}\|_{W^1_2(t-h,t)} \leq c_1 (t + 1)^{M-1} \exp(N_+ t) \|\tilde{f}\|_{L^2(0,h)}, t > h,$$

with constant $c_1$ independent of the function $f$. 
SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS

Therefore, from (20) we obtain

\[
\|u\|_{W^2_T(jh-2h,t,jh-h,t)} \leq c_1(t+1)^{M-1}\exp(N_+(t))\|f\|_{L_2(jh-h,jh)}.
\]

Using the change of variables \( T = jh - h + t \) once more, we obtain from (21) the following inequality

\[
\|u\|_{W^2_T(T-h,T)} \leq c_1(1 + T - (jh - h))^{M-1}\exp(N_+(T - (jh - h)))\|f\|_{L_2(jh-h,jh)}.
\]

Setting \( T = kh \), we obtain from (22) the assertion (ii) of Lemma 2.2.

**Proof of Theorem 2.4:** First of all, we consider the case \( T = kh, k \in \mathbb{N} \).

It is rather clear that functions \( f_j(t) \) (see (16)) for \( j > k \) do not have any influence to the solution \( u(t) \) on the interval \([0, kh]\).

Denote by \( u_j(t) \) the solution of the problem (1)-(2) for the right parts of \( f = f_j \) and \( g \equiv 0 \). Then, for \( t \leq kh \), we have the representation

\[
u(t) = \sum_{j=1}^{k} u_j(t).
\]

Using well-known inequality

\[
(a_1 + a_2 + \cdots + a_k)^2 \leq k(a_1^2 + a_2^2 + \cdots + a_k^2), a_j \in \mathbb{R},
\]

we obtain the estimate

\[
\|u\|^2_{W^2_T(kh-h,kh)} \leq \left( \sum_{j=1}^{k} \|u_j\|^2_{W^2_T(kh-h,kh)} \right)^2 \leq k\left( \sum_{j=1}^{k} \|u_j\|^2_{W^2_T(kh-h,kh)} \right).
\]

From the inequalities (15) and (24) for \( t = kh \), we deduce the following estimate

\[
\|u\|^2_{W^2_T(kh-h,kh)} \leq c_1 k \sum_{j=1}^{k} \int_{jkh-h}^{jkh} [f_j(t+h-kh)^2 + \exp(2N_+(kh-h-t))]\||f_j(t)||^2 dt
\]

\[
= c_1 k \int_{0}^{kh} [kh-h+1)^{2(M-1)}\exp(2N_+(kh-h-t))]\||f(t)||^2 dt
\]

\[
= c_1 k \int_{0}^{kh} (t+h)^2 + \exp(2N_+(t-h))\||f(t)||^2 dt
\]

with constant \( c_1 \) independent of the function \( f \).

Now, we consider the case of arbitrary real \( t > h \). Let us choose such \( k \) that \( kh \geq t \). It is evident that

\[
\|u\|^2_{W^2_T(t-h,t)} \leq 2\|u\|^2_{W^2_T(kh-2h,kh-h)} + \|u\|^2_{W^2_T(kh-h,kh)}.
\]
Using the inequality (25), we have the following estimate

\[
\|u\|_{W^1_2(t-h,t)}^2 \leq c_2(k-1) \int_0^{kh-h} (kh - h - \tau + 1)^{2(M-1)} \exp[2N_+(kh - h - \tau)] \|f(\tau)\|^2 d\tau \\
+ c_2 k \int_0^{kh} (kh - \tau + 1)^{2(M-1)} \exp[2N_+(kh - \tau)] \|f(\tau)\|^2 d\tau \\
\leq c_3 k \int_0^{kh} (kh - \tau + 1)^{2(M-1)} \exp[2N_+(kh - \tau)] \|f(\tau)\|^2 d\tau
\]

(27)

with constant \(c_2, c_3\) independent of the function \(f\).

The solution \(u(\tau)\) on the segment \([0, t]\) does not depend on the function \(f(\tau)\) for \(\tau > t\), thus we can substitute the \(L_2\)-norm of the right-hand side of (27) on the interval \((0, kh)\) by the \(L_2\)-norm on the interval \((0, t)\).

Finally, we note that for \(t \approx kh\) (for sufficiently large \(k\)), we have

\[
\exp(N_+(kh - \tau)) \approx \exp(N_+(t - \tau)),
\]

\[
(kh - \tau + 1)^{M-1} \approx (t - \tau + 1)^{M-1}.
\]

From the inequality (27), we then obtain the assertion (6) of Theorem 2.4, completing the proof.

We conclude this section with a

**Proof of Proposition 2.8.** Due to the compactness of the support of the function \(f\), there are only a finite number of nonzero terms in the representation (16). Thus there will be only a finite number functions \(u_j(t)\) in the representation (23) of the solution, and hence the multiplier \(k\) in the estimate (24) may be substituted by a constant independent of \(k\). As a result, the term \(\sqrt{t}\) in the inequality (6) can be dropped.

**4. Examples, Remarks and Comments.** The following example shows that our estimates of solutions to the homogeneous equation \((f \equiv 0)\) is sharp in the sense that it is impossible to replace the constant \(N_+ - \epsilon\) for an arbitrary positive \(\epsilon\).

**Example 1.** We consider the following difference differential equation

\[
\frac{du}{dt} - au(t) - \frac{du}{dt}(t-1) + au(t-1) = 0, \ t > 0,
\]

(28)

It is known (see [20], and also [2], [3] (chapter 9)) that each root \(\lambda_q\) of the characteristic quasipolynomial

\[
L_0(\lambda) = \lambda + a - e^{-\lambda}(\lambda - a)
\]
of the equation (28) is on the imaginary axis ($Re \lambda_q = 0$) with multiplicity $\mu_q = 1$. Therefore, $\mathcal{K}_+ = \mathcal{K}_- = 0$, $N = \max_{\lambda_q \in \Lambda} \mu_q = 1$.

Under our definition of the solution for the initial value problem associated with the equation (28), the following estimate holds

$$
\|u\|_{W^1(t-1,t)} \leq d\|g\|_{W^1(-1,0)} , t > 1,
$$

with the constant $d$ independent of the function $g$.

We should emphasize here that our conclusion does NOT contradict the result of [20], as we consider solutions from Sobolev space $W^1_2$, while in the article [20] the initial function $g$ does not belong to the space $W^1_2(-1, 0)$.

The following example shows the sharpness of the estimate for the non-homogeneous equation.

**Example 2** Consider the following problem (in the scalar case with $m = 1$):

$$
u^{(1)}(t) + u^{(1)}(t - 1) = 1, t > 0,
$$

with

$$
u(t) = 0, t \in [-1, 0].
$$

The solutions can be constructed step by step as follows:

$$
u(t) = \begin{cases} 
  k, & t \in [2k - 1, 2k], \\
  t - k, & t \in [2k, 2k + 1].
\end{cases}
$$

Therefore, we have

$$
\|u\|_{W^1_2[n-1,n]} \approx n, \\
\|f\|_{L^2[0,n]} \approx \sqrt{n}, n \in \mathbb{N}.
$$

For the characteristic quasipolynomial

$$
L_1(\lambda) = \lambda(1 + e^{-\lambda}),
$$

the following assertions $\mathcal{K}_- = \mathcal{K}_+ = 0$, $M = N = \max_{\lambda_q \in \Lambda} \nu_q$ hold true.

For $t = n$, the right-hand side of the inequality (6) for our example likes

$$
\sqrt{t}(\int_0^t (t - \tau + 1)^2(\mathcal{K}_+ - \nu)(t - \tau)|f(\tau)|^2)^{1/2} \approx n
$$

Thus it is impossible to substitute $\mathcal{K}_+$ by $\mathcal{K}_+ - \epsilon$ with some $\epsilon > 0$ and to omit $\sqrt{t}$.
We now make a few comments about existing results and the significance of our findings. First of all, the estimates similar to (6) for which the quantity $N_+ \pm \epsilon$ with some $\epsilon > 0$ are well-known (see [1]-[7] for more details). In the so called critical and supercritical cases (the situations where the roots $\lambda_q$ of the quasipolynomial $l(\lambda)$ approach or lie on the imaginary axis, more refined estimates are needed and can be found in [2], [3], [20]). A natural and important question is whether one can refine these estimates by setting $\epsilon = 0$. Theorem 2.4 gives, in a certain sense, a positive answer to this question.

We should emphasize the spectral character of our approach and we show how effective this approach is. Due to the fact that equation (1) has a convolution type, it is rather natural to use the method based on a Laplace transform. But then it is impossible to obtain the estimate (6), since any method based on the Laplace transform must involve the inverse of the Laplace transform that involves integration along lines parallel to the imaginary axis with a positive distance $\epsilon > 0$ to the spectra (the set of all zeroes $\Lambda$ of the quasipolynomial $l(\lambda)$).

It is also relevant to remark that our main purpose here is to obtain the sharpest estimates for solutions of functional differential equations of neutral type. For the retarded type equations, the structure of the set of roots $\Lambda$ is different and in particular, there is a dominating (with the most real part) zero $\lambda_q$ of the characteristic quasipolynomial. In this case, Laplace transform method should be rather effective.

5. Appendix: Proof of Riesz Basisness of Exponential Solutions. The proof below of the Theorem 5.4 about Riesz basisness of exponential solutions is rather technical, so we start with a short discussion of the main steps involved in the proof.

To prove Theorem 5.4, we need to verify the conditions (Lemma 5.8 obtained in [16]) about Riesz basisness of root subspaces in terms of resolvent of an operator in an abstract Hilbert space. In order to do so, we must obtain the representation of the resolvent of the differential operator $A$ subject to the nonlocal boundary conditions (32). It is thus important to note that this operator $A$ is the generator of a $C^0$-semigroup of the shift operator along the trajectories of strong solutions of the homogeneous equation (1) ($f(t) \equiv 0$) (this construction is similar to the well-known one presented in [2] and[3] for the space of continuous functions.) Another important step in our long technical proof is to verify that the resolvent of the operator $A$ satisfies the inequalities formulated in Lemma 5.8, and to obtain these estimates we will need the lower estimates of quasipolynomials (see [1] and [15] for more
SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS 449

details).

We start by recalling some results characterizing the system of exponential solutions of equation (1).

**Proposition 5.1.** Let $\det D_0 \neq 0$. Then exponential solutions (4) form a minimal system in the space $W^1_2((-h,0),C^r)$.

**Lemma 5.2.** Let $\det D_0 \neq 0$, $\det D_n \neq 0$. Then there exist constants $\alpha$ and $\beta$ such that the set $\Lambda$ lies in the vertical strip $\{ \lambda : \alpha < \Re \lambda < \beta \}$ and the system of exponential solutions $y_{q,j,s}(t)$ is complete in the space $W^1_2((-h,0),C^r)$.

In the next Lemma we give the estimates of matrix-valued function $L^{-1}(\lambda)$,

**Lemma 5.3.** Let $\det D_0 \neq 0$ and $\det D_n \neq 0$. Then there exists the system of contours $\Gamma_n = \{ \lambda \in C : \Re \lambda = \alpha, \gamma_n \leq \Im \lambda \leq \gamma_{n+1} \} \cup \{ \lambda \in C : \alpha \leq \Re \lambda \leq \beta, \gamma_n \leq \Im \lambda \leq \gamma_{n+1} \} \cup \{ \lambda \in C : \alpha \leq \Re \lambda \leq \beta, \Im \lambda = \gamma_n \}$, in the set $G(\Lambda,\rho)$ for sufficiently small $\rho > 0$, that satisfies the following conditions:

(i) $0 < \delta \leq \gamma_{n+1} - \gamma_n \leq \Delta < +\infty$ with some positive constants $\delta$ and $\Delta$;

(ii) there exists a constant $K$ such that:

$$\sup_{\lambda \in \Gamma_n} |\lambda|||L^{-1}(\lambda)|| \leq K, \quad n \in \mathbb{Z}.$$ 

Recall that we denote by $W_n$ subspaces of the space $W^1_2((-h,0),C^r)$ which are the span of all exponential solutions $y_{q,j,s}(t)$ corresponding to the numbers $\lambda_q$ lying in the domains bounded by contours $\Gamma_n$; by $V_{\lambda_q}$ subspaces of the space $W^1_2((-h,0),C^r)$ which are the span of all exponential solutions $y_{q,j,s}(t)$, corresponding to the number $\lambda_q$.

We recall the formulation of Theorem 3.2 about Riesz basisness. For the convenience of reference, we reformulate this as follows.

**Theorem 5.4.** Let $\det D_0 \neq 0$ and $\det D_n \neq 0$. Then the system of subspaces $\{W_n\}_{n \in \mathbb{Z}}$ forms a Riesz basis of subspaces in the space $W^1_2((-h,0),C^r)$.

The following theorem makes the previous one more precise in the case of a separate set $\Lambda$.

**Theorem 5.5.** Let $\det D_0 \neq 0$ and $\det D_n \neq 0$. Assume also the set $\Lambda$ is separate, that is, $\inf_{\lambda_p \neq \lambda_q} |\lambda_p - \lambda_q| > 0$. Then the system of subspaces $\{V_{\lambda_q}\}_{\lambda_q \in \Lambda}$ forms a Riesz basis of subspaces in the space $W^1_2((-h,0),C^r)$.

Let us consider operator $Ay = y^{(1)}$, acting in the space $W^1_2([-h,0],C^r)$.
with the following domain:

\[ \text{Dom} A = \{ y : \ y \in W^2_{2}([-h, 0], C^r), \sum_{k=0}^{n} (B_k y(-h_k) + D_k y^{(1)}(-h_k)) \]
\[ + \int_{0}^{h} (B(s)y(-s) + D(s)y^{(1)}(-s))ds = 0 \}. \tag{32} \]

We denote by \( \{ P_n \}_{n \in \mathbb{Z}} \) a system of Riesz spectral projectors of the operator \( A \), corresponding to contours \( \Gamma_n \):

\[ (P_n f) = -\frac{1}{2\pi i} \int_{\Gamma_n} R_A(\lambda) f d\lambda, \tag{33} \]

where \( R_A(\lambda) \) is the resolvent of the operator \( A \).

It can be easily shown that the system of exponential solutions (4) of equation (1) coincides with the system of eigen and associated vectors of the operator \( A \) defined in (32). Therefore, Theorem 3.2 (5.4) is a corollary of the following

**THEOREM 5.6.** Let \( \det D_0 \neq 0, \ det D_n \neq 0. \)

Then the system of subspaces \( W_n = P_n W^2_{2}([-h, 0], C^r) \), corresponding to the system of projectors (33) with contours \( \Gamma_n \), satisfying conditions of Lemma 5.3, form Riesz basis of subspaces in the space \( W^2_{2}([-h, 0], C^r) \).

We shall prove Theorem 5.6 based on the following

**PROPOSITION 5.7.** Suppose that \( \det D_0 \neq 0, \ det D_n \neq 0. \) Then the matrix-valued function \( L^{-1}(\lambda) \) satisfies the following estimates

\[ ||L^{-1}(\lambda)|| \leq c(|\lambda| + 1)^{-1}, \ \lambda \in G(\Lambda, \rho) \cup \{ \text{Re} \lambda > 0 \}, \tag{34} \]
\[ ||L^{-1}(\lambda)|| \leq c_0(|\lambda| + 1)^{-1} \exp(\text{Re} \lambda h), \ \lambda \in G(\Lambda, \rho) \cup \{ \text{Re} \lambda < 0 \}, \tag{35} \]

where \( c, c_0 \) are constants.

This proposition can be reduced from the results of chapter 12 of monograph [1] and the results of article [15]. The proof is based on the lower estimates of quasipolynomials. We give the proof of Proposition 5.7 at the end of the Appendix, for the sake of completeness.

We will also need the following result, and we refer to [16] for a proof.

**LEMMA 5.8.** If for every elements \( f, g \in W^1_{2}((-h, 0), C^r) \equiv H \)

\[ \sum_{n \in \mathbb{Z}} | \int_{\Gamma_n} (R_A(\lambda)f, g)H d\lambda | \leq \text{const} ||f||_H ||g||_H, \tag{36} \]
then the system of the subspaces $W_n$ forms unconditional (Riesz) basis in the closure of its span; and if additionally the system $W_n$ is complete in $H$ then it forms unconditional (Riesz) basis in the whole space $H \equiv W_2^1((-h, 0), C')$.

Let us now calculate the resolvent of the operator $A : R_A(\lambda)z = y$ or $y^{(1)} = \lambda y + z$. We have

$$y(t) = e^{\lambda t}(C + \int_0^t e^{-\lambda s}z(s)ds),$$

where constant vector $C$ may be found from the conditions (32). Namely, using the equality

$$y^{(1)} = \lambda y + z,$$

we have from condition (32) the following

$$\sum_{k=0}^{n} B_k e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s}z(s)ds] + D_k(\lambda e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s}z(s)ds] + z(-h_k)) + \int_0^h (B(s)e^{-\lambda s}[C + \int_0^{-s} e^{-\lambda \tau}z(\tau)d\tau] + D(s)(\lambda e^{-\lambda s}[C + \int_0^{-s} e^{-\lambda \tau}z(\tau)d\tau] + z(-s))ds = 0.$$

As $z \in W_2^1([-h, 0), C')$, we obtain

$$\int_0^{-t} e^{-\lambda s}z(s)ds = \frac{1}{\lambda}(z(0) - z(-t))e^{\lambda t} + \frac{1}{\lambda} \int_0^{-t} e^{-\lambda s}z^{(1)}(s)ds.$$  

Hence condition (32) takes the form

$$\sum_{k=0}^{n} (B_k e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s}z(s)ds] + D_k\lambda e^{-\lambda h_k}[C + \frac{z(0)}{\lambda} + \frac{1}{\lambda} \int_0^{-h_k} e^{-\lambda s}z^{(1)}(s)ds]) + \int_0^h B(s)e^{-\lambda s}(C + \int_0^{-s} e^{-\lambda \tau}z(\tau)d\tau) + D(s)\lambda e^{-\lambda s}(C + \frac{z(0)}{\lambda} + \frac{1}{\lambda} \int_0^{-s} e^{-\lambda \tau}z^{(1)}(\tau)d\tau)ds = 0.$$
or

\[ -L(\lambda)C = \sum_{k=0}^{n} D_k e^{-\lambda h_k} z(0) + \sum_{k=0}^{n} (B_ke^{-\lambda h_k} \int_{0}^{h} e^{-\lambda s} z(s)ds + D_k e^{-\lambda h_k} \int_{0}^{h} e^{-\lambda s} z^{(1)}(s)ds + \int_{0}^{h} (D(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} z(\tau)d\tau + D(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} z^{(1)}(\tau)d\tau)ds. \]

Let us denote by \( F(\lambda) \) the following vector-valued function:

\[ F(\lambda) = L^{-1}(\lambda)[\sum_{k=0}^{n} B_ke^{-\lambda h_k} + \int_{0}^{h} B(s)e^{-\lambda s}ds]z(0) \]

\[ -L^{-1}(\lambda)[\sum_{k=0}^{n} (B_ke^{-\lambda h_k} \int_{0}^{h} e^{-\lambda s} z(s)ds + D_k e^{-\lambda h_k} \int_{0}^{h} e^{-\lambda s} z^{(1)}(s)ds + \int_{0}^{h} (B(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} z(\tau)d\tau + D(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} z^{(1)}(\tau)d\tau)ds]. \]

Then the resolvent of the operator \( A \) may be rewritten as

\[ R_A(\lambda)f = -e^{\lambda t}F(\lambda) - e^{\lambda t}f(0) + \int_{0}^{t} e^{\lambda(t-s)} f(s)ds. \]

We now rewrite the vector-valued function \( F(\lambda) \) as

\[ F(\lambda) = Q(\lambda) + L^{-1}(\lambda)P(\lambda), \]

where

\[ Q(\lambda) = L^{-1}(\lambda)[\sum_{k=0}^{n} B_ke^{-\lambda h_k} + \int_{0}^{h} B(s)e^{-\lambda s}ds]f(0), \]

\[ P(\lambda) = \sum_{k=0}^{n} e^{-\lambda h_k} G_k(\lambda) + G_{n+1}(\lambda), \]

\[ G_k(\lambda) = \int_{0}^{h} e^{-\lambda s}(B_kf(s) + D_kf^{(1)}(s))ds, \]

\[ G_{n+1}(\lambda) = \int_{0}^{h} (B(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} f(\tau)d\tau + D(s)e^{-\lambda s} \int_{0}^{s} e^{-\lambda \tau} f^{(1)}(\tau)d\tau)ds. \]

Now let us give the estimate of the vector-valued function \( F(\lambda) \). Note that the vector-valued functions \( G_k(\lambda) \) are entire functions of exponential type.
SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS

(not more than \( h_k \)) and belong to Hardy space in every strip \( \{ \lambda : A \leq \text{Re}\lambda \leq B \} \). Moreover, the following inequalities hold:

\[
\sup_{A \leq x \leq B} \int_{-\infty}^{+\infty} \|G_k(x + iy)\|^2 dy \leq c_1 \|f\|^2_{W^1_2(-h,0)}
\]

with a constant \( c_1 \) independent of the function \( f(t) \). Hence, we obtain

\[
\sup_{A \leq x \leq B} \int_{-\infty}^{+\infty} \|P(x + iy)\|^2 dy \leq c_2 \|f\|^2_{W^1_2(-h,0)}
\]

with a constant \( c_2 \) independent of the function \( f(t) \).

By Proposition 5.7 and the trace theorem we derive the following estimates of the function \( Q(\lambda) \) in the domain \( \Pi_\rho(\alpha_1, \beta_1) = G(\Lambda, \rho) \cap \{ \lambda : \alpha_1 < \text{Re}\lambda < \beta_1 \} \)

\[
\|Q(\lambda)\| \leq c_3 (|\lambda| + 1)^{-2} \|f\|_{W^1_2(-h,0)}, \quad c_3 = \text{const} > 0.
\]

Here \( \alpha_1, \beta_1 \) are constants with \( \alpha_1 \leq \alpha, \beta \leq \beta_1 \).

Using representation (38), Proposition 5.7 and estimates (40) and (41) (for \( \text{Re}\lambda = \alpha, \ \text{Re}\lambda = \beta \)), we conclude that

\[
\int_{-\infty}^{+\infty} (1 + |\xi + i\mu|^2) \|F(\xi + i\mu)\|^2 d\mu \leq c_4 \|f\|^2_{W^1_2(-h,0)}, \quad \xi = \alpha, \beta;
\]

with a constant \( c_4 \) independent of the function \( f(t) \).

By Lemma 5.2, the system \( \{W_n\}_{n \in \mathbb{Z}} \) is complete in the space \( W^1_2((-h,0), C^\tau) \). So we need only to verify inequality (36).

Due to Lemma 5.8 and (37), we need only to prove that

\[
\sum_{n \in \mathbb{Z}} | \int (e^{\lambda t} F(\lambda), g(t))_{W^1_2} d\lambda | \leq \text{const} \|f\|_{W^1_2} \|g\|_{W^1_2}.
\]

This is due to Cauchy theorem - an integral of holomorphic functions along closed contours \( \Gamma_n \) is equal to zero. The second and third items in (37) are holomorphic functions (except one simple pole). The third item is holomorphic everywhere in arbitrary bounded domain; the second item is holomorphic everywhere except simple pole \( \lambda = 0 \). So if we substitute resolvent (37) in the expression (36) the integrals of the second and third
items will be equal to zero. Therefore, the inequality (36) will have form (43).

We have

\[(e^{\lambda t}F(\lambda), g(t))w^2_{\frac{1}{2}}((-h,0),C^r) = (\lambda F(\lambda), g_1(\lambda))_C^r + (F(\lambda), g_0(\lambda))_C^r,\]

where

\[g_1(\lambda) = \int_{-h}^{0} e^{\lambda t}g^{(1)}(t)dt, \quad \hat{g}_0(\lambda) = \int_{-h}^{0} e^{\lambda t}g(t)dt.\]

Thus (43) holds if

\[\sum_{n \in \mathbb{Z}} |\int \lambda^j F(\lambda), g_j(\lambda))d\lambda| \leq \text{const} \|f\|_w^2 \|g\|_w^2, \quad j = 0, 1.\]

We note that the vector-valued function \(g_1\) and \(g_0\) are entire functions of exponential type, belong to the Hardy space \(H_2(A, B)\) in every strip \(\{\lambda : A \leq \text{Re}\lambda \leq B\}\), since Hardy theorem ensures that the Laplace transform of a function belonging to the space \(L_2(0, +\infty)\) is an element of Hardy space \(H_2(\mathbb{C}_+)\) in the right half plane \(\{\lambda : \text{Re}\lambda > 0\}\).

In the integral representation of function \(g_0(\lambda)\) and \(g_1(\lambda)\) we change the variables from \(t\) to \((-t)\):

\[\hat{g}_1(\lambda) = \int_{0}^{h} e^{-\lambda t}g^{(1)}(-t)dt;\]

\[\hat{g}_0(\lambda) = \int_{0}^{h} e^{-\lambda t}g(-t)dt.\]

Functions \(g^{(1)}(-t)\) and \(g(-t)\) have compact support belonging to segment \([0,h]\). These functions are elements of the space \(L_2((0,h),\mathbb{C}^r)\). Moreover, functions \(\exp(at)g^{(1)}(-t), \exp(at)g(-t)\) are also elements of space \(L_2((0,h),\mathbb{C}^r)\) for arbitrary \(a\). Due to this fact the Laplace transforms \(g_1(\lambda)\) and \(g(\lambda)\) belong to Hardy space \(H_2(\text{Re}\lambda > -a)\) for arbitrary \(a \in \mathbb{R}\). So these functions belong to Hardy space \(H_2(\lambda : A \leq \text{Re}\lambda \leq B)\).
It is well-known that for Hardy space $H_2(\mathbb{C}_+)$ the following equality is valid
\[ \|f\|_{L_2(\mathbb{R}_+)} = \left( \sup_{x > 0} \int_{-\infty}^{+\infty} |\hat{f}(x + iy)|^2 dy \right)^{1/2}, \]
where $\hat{f}(x + iy)$ is the Laplace transform of the function $f(t)$. From this equality we easily deduce the following estimate:

\[ \sup_{A \leq x \leq B} \int_{-\infty}^{+\infty} |g_j(x + iy)|^2 dy \leq k_j \|g^{(j)}\|_{L_2((-h,0))}^2, \quad j = 0, 1, \]

with constants $k_0$ and $k_1$ independent of the function $g$.

So, for $A = \alpha$, $B = \beta$ we obtain
\[ \sum_{\xi+i\gamma_n} | \int_{\xi+i\gamma_n}^{+\infty} (\lambda^j F(\lambda), g_j(\overline{\lambda})) d\lambda | \]
\[ \leq \int_{-\infty}^{+\infty} \frac{(|(\xi + i\mu)^j F(\xi + i\mu), g_j(\xi - i\mu))| d\mu}{c_5(1 + |\xi + i\mu|^2)^{1/2} \|g^{(j)}\|_{L_2((-h,0))}} \]

with constant $c_5$ independent of the function $g(t)$, $(j = 0, 1, \xi = \alpha, \beta)$.

Therefore, from inequalities (42), (45) we have
\[ \sum_{\xi+i\gamma_n} | \int_{\xi+i\gamma_n}^{+\infty} (\lambda^j F(\lambda), g_j(\overline{\lambda})) d\lambda | \leq c_7 \|f\|_{w_2^j((-h,0))} \|g^{(j)}\|_{L_2((-h,0))} \]
with constant $c_6$ independent of the functions $f$ and $g$ $(j = 0, 1, \xi = \alpha, \beta)$.

Thus, we obtain a part of the estimate (43) on the vertical sides of contours $\Gamma_n$. In order to prove the part of estimate (43) on the horizontal sides of contours $\Gamma_n$, we need the following proposition which is a significant modification of theorem 3.3.1 from [24].

Denote by $M_{\nu^2}(R)$ the set of all entire functions of exponential type $\nu$, which belong to the space $L_2(R)$ as functions of real argument $t \in R$.

**Lemma 5.9.** Let $\nu(z) \in M_{\nu^2}(R)$, and let sequence of real numbers \{\(t_n\)\}_{n \in \mathbb{Z}} satisfy the condition: $0 < \delta \leq t_{n+1} - t_n \leq \Delta < +\infty$, with certain positive constants $\delta$ and $\Delta$. Then the following inequality takes place:

\[ \left( \sum_{n \in \mathbb{Z}} |\nu(t_n)|^2 \right)^{1/2} \leq \delta^{-1/2}(1 + \nu \Delta)(\int_{-\infty}^{+\infty} |\nu(t)|^2)^{1/2}. \]
We shall give the proof of Lemma 5.9 at the end of the Appendix. According to representation (38), we have

\[(\lambda^j F(\lambda), g_j(\lambda)) = (\lambda^j Q(\lambda), g_j(\lambda)) + (\lambda^j L^{-1}(\lambda)P(\lambda), g_j(\lambda)).\]

Due to Lemma 5.3 and estimate (41), we have

\[\|\lambda Q(\lambda)\|_{Im\lambda = \gamma} \leq c_8 \sup(|\lambda| + 1)^{-1} \|\lambda\|_{W^2_2(-h, 0)}.\]

From the latter inequality we obtain the estimate

\[
\int_{\alpha + i\gamma}^{\beta + i\gamma} \|\lambda Q(\lambda)\|^2 \, d\lambda \leq c_9 (|n| + 1)^{-2} \|f\|^2_{W^2_2(-h, 0)}
\]

with a constant \(c_9\) independent of the function \(f(t)\).

In turn, applying Lemma 5.9 to the vector-valued function \(g_j(\lambda)\) and to \(t_n = \gamma_n, t = y\), we have

\[
\sum_{n=-\infty}^{+\infty} \left\langle (g_j(x + iy), e_i) \right\rangle^2 \leq c_{10} \int_{-\infty}^{+\infty} \|g_j(x + iy)\|^2 \, dy \leq c_{11} \int_{-\infty}^{+\infty} \|g_j(x + iy)\|^2 \, dy,
\]

for \(x \in [\alpha, \beta]\), \(j = 0, 1\), where \(\{e_i\}_{i=1}^r\) is an orthonormal basis of the space \(C^r\). Then, due to (44), we have

\[
\sum_{n=-\infty}^{+\infty} \int_{\alpha}^{\beta} \|g_j(x + i\gamma_n)\|^2 \, dx \leq c_{12} \sup_{\alpha \leq x \leq \beta} \int_{-\infty}^{+\infty} \|g_j(x + iy)\|^2 \, dy \leq c_{13} \|g^{(j)}\|_{L^2(-h, 0)}, j = 0, 1,
\]

with constants \(c_{10}, c_{11}, c_{12}, c_{13}\) independent of the function \(g(t)\).

Taking into account that function \((P(\lambda), e_i), l = 1, 2, ..., r, \lambda = iz\), also satisfy the conditions of Lemma 5.9, by analogy with estimate \(g_j(\lambda)\), we obtain the inequality

\[
\sum_{n=-\infty}^{+\infty} \|P(x + i\gamma_n)\|_{C^r}^2 \leq c_{14} \int_{-\infty}^{+\infty} \|P(x + iy)\|_{C^r}^2 \, dy, \quad x \in [\alpha, \beta].
\]
Then, according to estimates (49) and (40), we have

\[ \sum_{n=-\infty}^{n=+\infty} \int_{\alpha}^{\beta} ||P(x + i\gamma_n)||^2 dx \leq c_{15} ||f||^2_{W_2^1(-h,0)} \]

with a constant \(c_{15}\) independent of the function \(f(t)\).

From the latter inequality and estimate (Lemma 5.3)

\[ \sup_{\lambda \in l_n} |\lambda| |L^{-1}(\lambda)|| \leq K_0 = \text{const, } n \in \mathbb{Z}, \]

we obtain inequality

\[ \sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||\lambda L^{-1}(\lambda)P(\lambda)||^2 d\lambda | \leq c_{16} ||f||^2_{W_2^1(-h,0)}, \]

where \(l_n = \{ \lambda \in C : \text{Im} \lambda = \gamma_n, \alpha \leq \text{Re} \lambda \leq \beta \}. \)

Taking into account representation (38) and estimates (47) and (50), we have

\[ \sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||\lambda F(\lambda)||^2 d\lambda | \leq c_{17} ||f||^2_{W_2^1(-h,0)} \]

with a constant \(c_{17}\) independent of the function \(f(t)\).

Hence from estimates (48), (51) and inequality

\[ \sum_{n=-\infty}^{n=+\infty} |\int (\lambda^j F(\lambda), g_j(\lambda))d\lambda | \]

\[ \leq ( \sum_{n=-\infty}^{n=+\infty} \int ||\lambda^j F(\lambda)||^2 |d\lambda| )^{1/2} ( \sum_{n=-\infty}^{n=+\infty} \int ||g_j(\lambda)||^2 |d\lambda| )^{1/2}, \quad j = 0, 1, \]

the following estimate follows:

\[ \sum_{n=-\infty}^{n=+\infty} |\int (\lambda^j F(\lambda), g_j(\lambda))d\lambda | \leq c_{18} ||f||_{W_2^1(-h,0)} ||g^{(j)}||_{W_2^1(-h,0)}, j = 0, 1, \]

with constant \(c_{18}\) independent of the functions \(f(t)\) and \(g(t)\).

So, according to Lemma 5.8, the sequence of subspaces \(\{W_n\}_{n \in \mathbb{Z}}\) forms an unconditional basis (Riesz basis) of the space \(W_2^1((-h,0), C^\tau)\).
The estimate (6) of homogeneous equation (1) \( f(t) \equiv 0 \) is a corollary of our results in this Appendix about Riesz basisness of the system \( \{W_u\} \) and Theorem 1 in [23]. For another independent proof on the estimate of the homogeneous equation (in the scalar case \( m = 1 \)), see [12], [26].

The proof of Theorem 5.5 is similar and thus is omitted.

**Proof of Lemma 5.7.** Let \( v(z) \in M_\nu \) be an entire function of \( \nu \)-type, such that \( v(t) \in L_2(-\infty, +\infty) \). Then the following holds:

\[
\int_{-\infty}^{+\infty} |v(t)|^2 dt = \sum_{k \in \mathbb{Z}} \int_{t_k}^{t_{k+1}} |v(t)|^2 dt = \sum_{k \in \mathbb{Z}} |v(\xi_k)|^2 \Delta t_k,
\]

where \( \xi_k \in [t_k, t_{k+1}] \). Due to generalized Bernstein inequality (see [24]) and Holder inequality, and using \( \|a\| - \|b\| \leq \|a - b\| \), we have

\[
|\sum_{k \in \mathbb{Z}} |v(\xi_k)|^2 \Delta t_k|^{1/2} - (\sum_{k \in \mathbb{Z}} |v(t_k)|^2 \Delta t_k)^{1/2}|
\leq \sum_{k \in \mathbb{Z}} \left| v(\xi_k) - v(t_k) \right|^2 \Delta t_k^{1/2} = \sum_{k \in \mathbb{Z}} \left( \int_{t_k}^{t_{k+1}} |v(1)(t)|^2 dt \right) \Delta t_k^{1/2}
\leq \sum_{k \in \mathbb{Z}} \left( \int_{t_k}^{t_{k+1}} |v(1)(t)|^2 dt \right)^{1/2} \sup_{k \in \mathbb{Z}} (\Delta t_k)
\leq \Delta \left( \int_{-\infty}^{+\infty} |v(1)(t)|^2 dt \right)^{1/2}
\leq \Delta \nu \|v(t)\|_{L^2}.
\]

From this, we obtain

\[
\left( \sum_{k \in \mathbb{Z}} |v(t_k)|^2 \Delta t_k \right)^{1/2} = \left[ \left( \sum_{k \in \mathbb{Z}} |v(t_k)|^2 \Delta t_k \right)^{1/2} - (\sum_{k \in \mathbb{Z}} |v(\xi_k)|^2 \Delta t_k)^{1/2} \right] + (\sum_{k \in \mathbb{Z}} |v(\xi_k)|^2 \Delta t_k)^{1/2}
\leq (1 + \Delta \nu) \|v\|_{L^2}.
\]

Hence, we have

\[
\delta^{1/2} \left( \sum_{k \in \mathbb{Z}} |v(t_k)|^2 \right)^{1/2} \leq \left( \sum_{k \in \mathbb{Z}} |v(t_k)|^2 \Delta t_k \right)^{1/2}.
\]

So from (53) and (54) we get the desired inequality.

We now turn to the proof of Proposition 5.7, that is a consequence of several results in [1] and [15]. To be more specific, we add a few comments here. The estimate of the matrix function \( \mathcal{L}^{-1}(\lambda) \) outside the band...
SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS 459

\[ \lambda : \leq \text{Re} \lambda \leq B, A < 0, B > 0, \]
can be established by a straightforward verification (see also [1], Chap.12). The estimates of the matrix function \( L^{-1}(\lambda) \)
on the set \( G(\Lambda, \rho) \cap \{ \lambda : A < \text{Re} \lambda < B \} \) can be derived using the lower estimate of the quasipolynomial \( l(\lambda) \). Indeed, by virtue of the fact that \( \text{det} D_0 \neq 0 \) and \( \text{det} D_n \neq 0 \) the coefficients at \( \lambda^\tau \) and \( \lambda^\tau \exp(-\lambda \tau h) \) are determined by the quantities \( \text{det}(\lambda D_0 + B_0) \) and \( \text{det}((\lambda D_n + B_n) \exp(-\lambda h)) \) (see [1],p.429, formula 12.2.12) that differ from zero if \( |\lambda| \) is large. Then in correspondence with inequality (3.12) in [25], we obtain the estimate

\[
|l(\lambda)| \geq c_20(|\lambda|^\tau + 1), \lambda \in G(\Lambda, \rho) \cap \{ \lambda : A \leq \text{Re} \lambda \leq B \}.
\]

Since the entries of the matrix function \( L^{-1}(\lambda) \) are composed of cofactors of \( L(\lambda) \) and quasipolynomial \( l(\lambda) \), from estimate (55) we get inequalities (34) and(35) in the band \( \{ \lambda : A \leq \text{Re} \lambda \leq B \}, A < 0, B > 0. \)

Therefore, the proof of the existence of the sequence \( \{ \gamma_n \}_{n \in \mathbb{Z}} \), mentioned in Lemma 5.3, follows from Lemma 4 and 5 in [15] as follows: Consider \( \tau \)
different zeros \( \lambda_{q_j} (j = 1, 2, \cdots, \tau) \) of the quasipolynomial \( l(\lambda) \) and introduce the function

\[ \eta(\lambda) = \exp(\frac{\lambda \tau h}{2})l(\lambda)/(\lambda - \lambda_{q_1}) \cdots (\lambda - \lambda_{q_\tau}). \]

Using the fact that \( \text{det} D_0 \neq 0 \) and \( \text{det} D_n \neq 0 \), and repeating literally the corresponding arguments in [11], we get that the function \( \eta(\lambda) = \eta(-\frac{2\pi i}{ht}) \)
satisfies the conditions of Lemma 5 in [15]. Also for sufficiently small \( \rho > 0 \), on every interval of unit length we can determine a number \( \gamma_n \) such that the straight line \( \text{Im} \lambda = \gamma_n \) does not intersect the exceptional set \( \bigcup_{\lambda_q \in \Lambda} D(\lambda_q, \rho) \).

Hence we have the assertion of Lemma 5.3.

REFERENCES


<< vlasov@mech.math.msu.su >>
<< wujh@mathstat.yorku.ca >>
ON HOMOGENIZATION OF VARIATIONAL INEQUALITIES WITH OBSTACLES ON $\varepsilon$ - PERIODICALLY SITUATED INCLUSIONS *

M.N. ZUBOVA † AND T.A. SHAPOSHNIKOVA ‡

Abstract. The main goal of this paper is to prove the weak convergence as $\varepsilon \to 0$ of the sequence $u_\varepsilon$, where $u_\varepsilon$ is the solution of the variational inequality

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon) \, dx \geq \int_\Omega f(v - u_\varepsilon) \, dx,$$

here $u_\varepsilon$ and $v$ are elements of $K_\varepsilon = \{ g \in H^1_0(\Omega) | g(x) \geq \phi(x) \text{ a.e. in } G_\varepsilon \}$ in a domain $\Omega$, which contains $\varepsilon$ - periodically situated small inclusions $G_\varepsilon$ with radius $a_\varepsilon$. We consider three cases of asymptotic behavior of the solution $u_\varepsilon$ as $\varepsilon \to 0$, the choice of a particular case depending on asymptotic behavior of $a_\varepsilon$ as $\varepsilon \to 0$.

Key Words. Homogenization, convergence, variational inequality, boundary conditions, small parameter, asymptotic behavior

AMS(MOS) subject classification. 35B27

In the present paper we study a behavior of variational inequalities for Laplace operator, which satisfy one - side conditions on $\varepsilon$ - periodically situated subsets of the domain $\Omega$. Also we suppose, that these subsets can be

---

* This research was carried out with the financial support of the Russian Foundation for Fundamental Research (grant $\mathcal{N}^0$ 04-01-00618, $\mathcal{N}^0$ 1464.2003.1)

† Department of Differential equations, Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia

‡ Department of Differential equations, Faculty of Mechanics and Mathematics, Moscow State University, GSP-2, Moscow 119992, Russia
described as \( a_\varepsilon G_0 \), where \( 0 < a_\varepsilon \leq d \varepsilon, \ d = \text{const} > 0 \). Three types of asymptotic behavior of solutions of variational inequalities \( u_\varepsilon \) as \( \varepsilon \to 0 \) depending on conditions for \( a_\varepsilon \) are possible.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \); \( Q = \{ x \in \mathbb{R}^n | 0 < x_j < 1, \ j = 1, \ldots, n \} \); and \( G_0 \) be a domain in \( Q \) such that \( \overline{G_0} \subset Q \) and \( G_0 \) is diffeomorphic to a ball. We set \( G_\varepsilon = \bigcup_{z \in Z} (a_\varepsilon G_0 + \varepsilon z) \cap \Omega, \ Omega_\varepsilon = \{ x \in \Omega | \rho(x, \partial \Omega) > \varepsilon \} \), where \( \varepsilon \) is a positive parameter, \( 0 < a_\varepsilon \leq d \varepsilon, \ a_\varepsilon G_0 \subset \varepsilon Q \), \( Z \) is a set of vectors \( z \) with integer coordinates; \( aB = \{ x | a^{-1}x \in B \} \).

We consider a following problem in a domain \( \Omega \): let us find an element

\[
(1) \quad u_\varepsilon \in K_\varepsilon = \{ g \in H^1_0(\Omega) | g(x) \geq \phi(x) \ \text{a.e. in} \ G_\varepsilon \},
\]

which satisfies the following variational inequality

\[
(2) \quad \int_\Omega \nabla u_\varepsilon \nabla (v - u_\varepsilon) dx \geq \int_\Omega f(v - u_\varepsilon) dx,
\]

here \( v \) is an arbitrary function from \( K_\varepsilon \) and \( \phi(x) \in C^2(\overline{\Omega}), \phi(x) \leq 0 \) in \( \Omega \), \( f \in L^2(\Omega) \).

Existence and uniqueness of the weak solution of the problem (1), (2) follows from assertions which are proved in [1].

Let us study the asymptotic behavior of the solution \( u_\varepsilon \) as \( \varepsilon \to 0 \).

1. Suppose, that

\[
(3) \quad \lim_{\varepsilon \to 0} \alpha_\varepsilon^{2-n} \varepsilon^n = 0, \ \text{if} \ n \geq 3,
\]

and

\[
(4) \quad \lim_{\varepsilon \to 0} \varepsilon^2 | \ln \alpha_\varepsilon | = 0, \ \text{if} \ n = 2.
\]

It is easy to see that \( (u_\varepsilon - \phi(x))^+ \in H^1_0(\Omega) \) and \( (u_\varepsilon - \phi)^- = 0 \) on \( G_\varepsilon \). Let us apply inequalities, which are proved in [2]: for any function \( v \in H^1_0(\Omega) \), which is equal to zero a.e. on \( G_\varepsilon \), the following estimates are valid:

\[
(5) \quad \| v \|_{L^2(\Omega)}^2 \leq K \alpha_\varepsilon^{2-n} \varepsilon^n \| \nabla v \|_{L^2(\Omega)}^2, \ \text{if} \ n \geq 3,
\]

and

\[
(6) \quad \| v \|_{L^2(\Omega)}^2 \leq K \varepsilon^2 | \ln \frac{\varepsilon}{\alpha_\varepsilon} | \| \nabla v \|_{L^2(\Omega)}^2, \ \text{if} \ n = 2.
\]
Here and in what follows $K$ will be some constant, which does not depend on $\varepsilon$.

It follows from inequalities (5), (6) and conditions (3), (4) that

\begin{equation}
\| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)}^2 \leq K a_\varepsilon^{2-n \varepsilon^n} \| \nabla (u_\varepsilon - \phi)^- \|_{L^2(\Omega)}, \text{ if } n \geq 3,
\end{equation}

\begin{equation}
\| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)}^2 \leq K \varepsilon^2 \ln \frac{\varepsilon}{a_\varepsilon} \| \nabla (u_\varepsilon - \phi)^- \|_{L^2(\Omega)}, \text{ if } n = 2.
\end{equation}

From (2) and the Friedrichs inequality we deduce

\begin{equation}
\| u_\varepsilon \|_{H^1_0(\Omega)} \leq K.
\end{equation}

From (7)-(9) we obtain the following estimates

\begin{equation*}
\| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)} \leq K \sqrt{a_\varepsilon^{2-n \varepsilon^n}}, \text{ if } n \geq 3,
\end{equation*}

\begin{equation*}
\| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)} \leq K \varepsilon \sqrt{\ln \frac{\varepsilon}{a_\varepsilon}}, \text{ if } n = 2.
\end{equation*}

Taking $v = (u_\varepsilon - \phi)^+ + \phi \in K_\varepsilon$ as a test function in the inequality (2) we get

\begin{equation*}
\int_\Omega \nabla ((u_\varepsilon - \phi)^+ + (u_\varepsilon - \phi)^- + \phi) \nabla (u_\varepsilon - \phi)^- dx \leq \int_\Omega f (u_\varepsilon - \phi)^- dx.
\end{equation*}

It follows from the last inequality that

\begin{equation}
\int_\Omega | \nabla (u_\varepsilon - \phi)^- |^2 dx \leq \int_\Omega f (u_\varepsilon - \phi)^- dx - \int_\Omega \nabla \phi \nabla (u_\varepsilon - \phi)^- dx.
\end{equation}

If we integrate by parts the last integral in the right-hand part of (10) and taking into account, that $(u_\varepsilon - \phi)^- \in H^1_0(\Omega)$, we deduce

\begin{equation*}
\int_\Omega | \nabla (u_\varepsilon - \phi)^- |^2 dx \leq \int_\Omega f (u_\varepsilon - \phi)^- dx + \int_\Omega \Delta \phi (u_\varepsilon - \phi)^- dx \leq \| f \|_{L^2(\Omega)} \| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)} + \| \Delta \phi \|_{L^2(\Omega)} \| (u_\varepsilon - \phi)^- \|_{L^2(\Omega)} \leq
\end{equation*}
and, similarly,

$$\left\| \nabla (u_\varepsilon - \phi)^- \right\|_{L^2(\Omega)}^2 \leq K\varepsilon \sqrt{\varepsilon \ln \frac{1}{\varepsilon}} \left\| \nabla (u_\varepsilon - \phi)^- \right\|_{L^2(\Omega)}, \text{ if } n = 2.$$ 

Therefore we have

\begin{align}
(11) & \quad \left\| \nabla (u_\varepsilon - \phi)^- \right\|_{L^2(\Omega)} \leq K\varepsilon \sqrt{\varepsilon \ln \frac{1}{\varepsilon}}, \text{ if } n \geq 3, \\
(12) & \quad \left\| \nabla (u_\varepsilon - \phi)^- \right\|_{L^2(\Omega)} \leq K\varepsilon \sqrt{\varepsilon \ln \frac{1}{\varepsilon}}, \text{ if } n = 2.
\end{align}

We define the function $u_0$ as a solution of the following problem: let us find the function

\begin{equation}
(13) \quad u_0 \in K_0 = \{ v \in H_0^1(\Omega) \mid v(x) \geq \phi(x) \text{ a.e. in } \Omega \}
\end{equation}

such that the following inequality

\begin{equation}
(14) \quad \int_{\Omega} \nabla u_0 \cdot \nabla (v - u_0) \, dx \geq \int_{\Omega} f(v - u_0) \, dx,
\end{equation}

is valid for any $v$ from $K_0$.

Let us deduce that $\|(u_\varepsilon - \phi)^+ + \phi\|_{H_1(\Omega)} \to u_0$ as $\varepsilon \to 0$ and derive the estimate for the rate of this convergence. We take $v = (u_\varepsilon - \phi)^+ + \phi \in K_0$ as a test function in the inequality (16) and $v = u_0 \in K_0$ as a test function in the inequality (2) and add these inequalities. We have

$$\int_{\Omega} |\nabla((u_\varepsilon - \phi)^+ + \phi - u_0)|^2 \, dx \leq -\int_{\Omega} \nabla(u_\varepsilon - \phi)^- \cdot \nabla((u_\varepsilon - \phi)^+ + \phi - u_0) \, dx + \int_{\Omega} f(u_\varepsilon - \phi) \, dx - \int_{\Omega} \nabla u_\varepsilon \cdot \nabla(u_\varepsilon - \phi) \, dx.$$

From estimates (11), (12) we derive the following inequality

$$\left\| \nabla((u_\varepsilon - \phi)^+ + \phi - u_0) \right\|_{L^2(\Omega)}^2 \leq K \left\{ \left\| \nabla(u_\varepsilon - \phi)^- \right\|_{L^2(\Omega)}^2 + \|(u_\varepsilon - \phi)^-\|_{H_1(\Omega)} \right\}.$$
Hence we have proved the following theorem.

**Theorem 1.** Let \( u_\varepsilon \) be a solution of the problem (1), (2); let \( u_0 \) be a solution of the problem (13), (14). Then the following estimates are valid:

\[
\|(u_\varepsilon - \phi)^+ + \phi - u_0\|_{H_1(\Omega)} \leq \begin{cases} 
K a_\varepsilon^{2-n} \varepsilon^n, & \text{if } n \geq 3, \\
K \varepsilon^2 |\ln \frac{\varepsilon}{a_\varepsilon}|, & \text{if } n = 2, 
\end{cases}
\]

and

\[
\|(u_\varepsilon - \phi)^-\|_{H_1(\Omega)} \leq \begin{cases} 
K a_\varepsilon^{2-n} \varepsilon^n, & \text{if } n \geq 3, \\
K \varepsilon^2 |\ln \frac{\varepsilon}{a_\varepsilon}|, & \text{if } n = 2, 
\end{cases}
\]

2. Suppose that

\[
(15) \quad \lim_{\varepsilon \to 0} a_\varepsilon^{2-n} \varepsilon^n = C_1 \neq 0, \text{ if } n \geq 3,
\]

and

\[
(16) \quad \lim_{\varepsilon \to 0} \varepsilon^2 |\ln a_\varepsilon| = C_2 \neq 0, \text{ if } n = 2.
\]

In this section we assume that \( G_0 = \{x| |x| < a\} \) is a ball in \( \mathbb{R}^n \) with radius \( a \), \( Q = \{x| -1/2 \leq x_j < 1/2, j = 1, \ldots < n\} \) is a cube with unity edge. For easiness of calculations we discuss the \( n \geq 3 \) case. Let us numerate all sets \( a_\varepsilon G_0 \) which compose \( G_\varepsilon \) and denote the \( j \)-th ball as \( T^j_{a_\varepsilon a} \); the center of this ball we denote \( P^j \), and a ball with radius \( \tau \) which is concentric to the ball \( T^j_{a_\varepsilon a} \) we denote by \( T^j_\tau \).

Let us introduce notations

\[
w^j_\varepsilon = \frac{|x - P^j|^2-n - (a_\varepsilon a)^2-n}{(\varepsilon b)^2-n - (a_\varepsilon a)^2-n}, j = 1, \ldots, N_\varepsilon,
\]
if \( x \in T^j_{eb} \setminus T^j_{ae,a} \), \( b = \text{const} \), \( a_\varepsilon \alpha < \varepsilon b < \varepsilon/2 \), where \( N_\varepsilon \) is a number of balls which compose \( G_\varepsilon \).

We introduce the function \( w_\varepsilon \in H^1_{1,\text{loc}}(\mathbb{R}^n) \) assume that \( w_\varepsilon = w_\varepsilon^j \) for \( x \in T^j_{eb} \setminus T^j_{ae,a} \) and \( w_\varepsilon = 0 \) for \( x \in T^j_{ae,a} \), \( w_\varepsilon = 1 \) if \( x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{N_\varepsilon} T^j_{eb} \).

Therefore, using the Mintie lemma [3], we deduce that the function \( u_\varepsilon \in K_\varepsilon \) is a solution of the problem (1), (2) if and only if for this function the following inequality holds

\[
\int_\Omega \nabla v \nabla (v - u_\varepsilon) \, dx \geq \int_\Omega f(v - u_\varepsilon) \, dx,
\]

here \( v \) is an arbitrary function of \( K_\varepsilon \).

Suppose, that \( g \in C^\infty_0(\Omega) \). We take \( v = (g - \phi)^+ + \phi + (g - \phi)^- w_\varepsilon(x) \in K_\varepsilon \) as a test function in the inequality (17). We obtain

\[
\int_\Omega |\nabla ((g - \phi)^+ + \phi) \nabla ((g - \phi)^- w_\varepsilon)|^2 \, dx + 2 \int_\Omega \nabla ((g - \phi)^+ + \phi) \nabla ((g - \phi)^- w_\varepsilon) \, dx -
\]

\[
- \int_\Omega \nabla ((g - \phi)^+ + \phi) \nabla u_\varepsilon \, dx + \int_\Omega |\nabla ((g - \phi)^- w_\varepsilon)|^2 \, dx - \int_\Omega \nabla u_\varepsilon \nabla ((g - \phi)^- w_\varepsilon) \, dx \geq
\]

\[
\int_\Omega (f((g - \phi)^+ + \phi + (g - \phi)^- w_\varepsilon - u_\varepsilon) \, dx.
\]

Taking into account that \( \|u_\varepsilon\|_{H^1(\Omega)} \leq K \), we choose a subsequence of a sequence (using the same notation for this subsequence) such that \( u_\varepsilon \to u_0 \) as \( \varepsilon \to 0 \) weakly in \( H^1_1(\Omega) \).

From above mentioned we get

\[
\lim_{\varepsilon \to 0} \int_\Omega \nabla ((g - \phi)^+ + \phi) \nabla u_\varepsilon \, dx = \int_\Omega \nabla ((g - \phi)^+ + \phi) \nabla u_0 \, dx.
\]

From the definition of \( w_\varepsilon \) and conditions (15),(16) we have that \( w_\varepsilon \to 1 \) as \( \varepsilon \to 0 \) weakly in \( H^1_{1,\text{loc}}(\mathbb{R}^n) \). Hence, we have

\[
\lim_{\varepsilon \to 0} \int_\Omega (\nabla u_\varepsilon, \nabla (g - \phi)^-) w_\varepsilon \, dx = \int_\Omega \nabla u_0 \nabla (g - \phi)^- \, dx,
\]
We pass to the limit as $\varepsilon \to 0$ in the inequality (18) and, taking into account (19)-(21), we conclude that

\begin{align}
\int_\Omega |\nabla \{(g - \phi)^+ + \phi\}|^2 dx &+ 2 \int_\Omega \nabla \phi \nabla (g - \phi)^- dx - \\
- \int_\Omega \nabla u_\varepsilon \nabla (g - \phi)^- dx &+ \lim_{\varepsilon \to 0} \int |(g - \phi)^-| \nabla w_\varepsilon|^2 dx - \lim_{\varepsilon \to 0} \int (g - \phi)^- \nabla u_0 \nabla w_\varepsilon dx \geq \\
&\geq \int_\Omega f(g - u_0) dx.
\end{align}

Let us find a limit of two expressions which are in the left-hand part of (22) as $\varepsilon \to 0$. Taking into account that

\begin{align}
\int_\Omega (g - \phi)^- \nabla u_\varepsilon \nabla w_\varepsilon dx = \\
= \int_\Omega (g - \phi)^- \nabla (u_\varepsilon - \phi) \nabla w_\varepsilon dx + \int_\Omega (g - \phi)^- \nabla \phi \nabla w_\varepsilon dx = \\
(23) = \int_\Omega \nabla ((u_\varepsilon - \phi)(g - \phi)^-) \nabla w_\varepsilon dx - \int_\Omega (u_\varepsilon - \phi) \nabla (g - \phi)^- \nabla w_\varepsilon dx + 
\end{align}
\[ + \int_{\Omega} (g - \phi)^{n-2} \nabla \phi \nabla w_{\varepsilon} \, dx = \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon}^j} (u_{\varepsilon} - \phi)(g - \phi) \frac{\partial w_{\varepsilon}}{\partial \nu} \, ds - \int_{\Omega} (u_{\varepsilon} - \phi) \nabla (g - \phi)^{n-2} \nabla w_{\varepsilon} \, dx + \int_{\Omega} (g - \phi)^{n-2} \nabla \phi \nabla w_{\varepsilon} \, dx. \]

From equality (23) we deduce that

\[ \lim_{\varepsilon \to 0} \int_{\Omega} (g - \phi)^{n-2} \nabla u_{\varepsilon} \nabla w_{\varepsilon} \, dx = \lim_{\varepsilon \to 0} \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon}^j} (u_{\varepsilon} - \phi)(g - \phi) \frac{\partial w_{\varepsilon}}{\partial \nu} \, ds. \]

Let us note that \( \frac{\partial w_{\varepsilon}}{\partial \nu} \bigg|_{\partial T_{\varepsilon}^j} \leq 0 \), and, consequently

\[ \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon}^j} (u_{\varepsilon} - \phi)(g - \phi) \frac{\partial w_{\varepsilon}}{\partial \nu} \, ds \geq 0. \]

Therefore, we deduce

\[ - \lim_{\varepsilon \to 0} \int_{\Omega} (g - \phi)^{n-2} \nabla u_{\varepsilon} \nabla w_{\varepsilon} \, dx \leq - \lim_{\varepsilon \to 0} \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon}^j} (g - \phi)^{n-2} (u_{\varepsilon} - \phi) \frac{\partial w_{\varepsilon}}{\partial \nu} \, ds. \]

Taking into account that \( \frac{\partial w_{\varepsilon}}{\partial \nu} \bigg|_{\partial T_{\varepsilon}^j} = \frac{(n-2)(b^2-1)_{\varepsilon}}{1-(a^2-1)_{\varepsilon}^2} \) and using the following equality (see [4])

\[ \lim_{\varepsilon \to 0} \sum_{j=1}^{N_\varepsilon} \int_{\partial T_{\varepsilon}^j} (u_{\varepsilon} - \phi)(g - \phi)^{n-2} \, ds = b^{n-1} \omega(n) \int_{\Omega} (u_0 - \phi)(g - \phi)^{n-2} \, dx, \]

we conclude that

\[ - \lim_{\varepsilon \to 0} \int_{\Omega} (g - \phi)^{n-2} \nabla u_{\varepsilon} \nabla w_{\varepsilon} \, dx \leq \]
(24) \[ \leq -\omega(n)(n-2)C_1^{-1}a^{n-2} \int_{\Omega} (g - \phi)^-(u_0 - \phi) \, dx, \]

here \( \omega(n) \) is the area of the unit sphere in \( \mathbb{R}^n \).

Similarly,

\[
\lim_{\varepsilon \to 0} \int_{\Omega} |(g - \phi)^-|^2 |\nabla w_\varepsilon|^2 \, dx =
\]

\[
= \lim_{\varepsilon \to 0} \sum_{j=1}^{N_\varepsilon} \int_{\partial \Omega_{\varepsilon}^j} |(g - \phi)^-|^2 \frac{\partial w_\varepsilon}{\partial \nu} \, ds =
\]

\[
= \omega(n)(n-2)C_1^{-1}a^{n-2} \int_{\Omega} |(g - \phi)^-|^2 \, dx.
\]

From (22), (24) and (25) we deduce that

\[
\int_{\Omega} |\nabla \{(g - \phi)^+ + \phi\}|^2 \, dx + 2 \int_{\Omega} \nabla \phi \nabla (g - \phi^-) \, dx -
\]

\[- \int_{\Omega} \nabla ((g - \phi)^+ + \phi) \nabla u_0 \, dx + \int_{\Omega} |\nabla (g - \phi^-)|^2 \, dx -
\]

\[
(26) \quad - \int_{\Omega} \nabla u_0 \nabla (g - \phi^-) \, dx + \omega(n)(n-2)a^{n-2}C_1^{-1} \int_{\Omega} |(g - \phi^-)|^2 \, dx -
\]

\[- \omega(n)(n-2)a^{n-2}C_1^{-1} \int_{\Omega} (u_0 - \phi)(g - \phi^-) \, dx \geq \int_{\Omega} f(g - u_0) \, dx.
\]

From (26) we deduce for \( u_0 \) the following inequality

\[
(27) \quad \int_{\Omega} \nabla g \nabla (g - u_0) \, dx + \tilde{C} \int_{\Omega} (g - \phi^-)(g - u_0) \, dx \geq \int_{\Omega} f(g - u_0) \, dx
\]

for any function \( g \in H_1^0(\Omega) \); \( \tilde{C} = \omega(n)(n-2)a^{n-2}C_1^{-1} \).
We set in inequality (27) \( g = u_0 + \lambda h, \ h \in C_0^\infty(\Omega) \) and pass to the limit as \( \lambda \to 0, \ \lambda > 0, \) and \( \lambda \to 0, \ \lambda < 0. \) We derive that \( u_0 \) satisfies the integral identity

\[
\int_{\Omega} \nabla u_0 \nabla hdx + \tilde{C} \int_{\Omega} (u_0 - \phi)^- hdx = \int_{\Omega} f hdx,
\]

where \( h \) is an arbitrary function from \( C_0^\infty(\Omega). \) Thus, the function \( u_0 \in H_1^0(\Omega) \) is a weak solution of the problem

(28) \( -\Delta u_0 + \tilde{C} (u_0 - \phi)^- = f, \ x \in \Omega; \ u_0 = 0, \ x \in \partial \Omega. \)

Existence and uniqueness of the solution of the problem (28) were proved in [5].

**Theorem 2.** Suppose that conditions (15), (16) are satisfied and \( u_0 \) is a weak solution of the problem (28). Then \( u_\varepsilon \to u_0 \) as \( \varepsilon \to 0 \) weakly in \( H_1^0(\Omega). \)

3. Suppose that

(29) \( \lim_{\varepsilon \to 0} a_\varepsilon^{2-n} \varepsilon^n = \infty, \ \text{if} \ n \geq 3, \)

and

(30) \( \lim_{\varepsilon \to 0} \varepsilon^2 \ln a_\varepsilon = \infty, \ \text{if} \ n = 2. \)

The set \( G_\varepsilon \) has the same form as in the part 1. Let \( u_0 \in H_1^0(\Omega) \) be a weak solution of the problem

(31) \( -\Delta u_0 = f, \ x \in \Omega; \ u_0 = 0, \ x \in \partial \Omega. \)

Let us construct functions \( \Psi_j(x), \ j = 1, \ldots, N_\varepsilon. \) Assume that: \( \Psi_j(x) = 0, \ \text{if} \ |x - P_j| \leq \rho_0 a_\varepsilon; \ \Psi_j(x) = 1, \ \text{if} \ |x - P_j| \geq \rho_1 a_\varepsilon; \ \Psi_j(x) = \frac{1}{\ln(\frac{\rho_0 a_\varepsilon}{\rho_0 a_\varepsilon})}, \) if \( \rho_0 a_\varepsilon < |x - P_j| < \rho_1 a_\varepsilon; \ \rho_i = \text{const} > 0, \ i = 0, 1; \ \rho_0 < \rho_1; \ P_j \) is a point, which is contained in the \( j-\)th cell, such that a set \( a_\varepsilon G_j \) is contained in a ball \( |x - P_j| \leq \rho_0 a_\varepsilon. \) We denote \( \Psi_j(x) = \Pi_{j=1}^{N_\varepsilon} \Psi_j(x). \)

We take \( v = (g - \phi)^+ + \phi + \Psi_\varepsilon (g - \phi)^- \in K_\varepsilon; \) here \( g \in C_0^\infty(\Omega) \) is the same as a test function in the variational inequality (17), and get the following inequality
\[
\int_{\Omega} \nabla((g - \phi)^+ + \phi + \Psi_\varepsilon(g - \phi^-)) \nabla((g - \phi)^+ + \phi + \Psi_\varepsilon(g - \phi^-) - u_\varepsilon) \, dx \geq \\
\geq \int_{\Omega} f((g - \phi)^+ + \phi + \Psi_\varepsilon(g - \phi^-) - u_\varepsilon) \, dx.
\]

We note that
\[
\int_{\Omega} (\Psi_\varepsilon - 1)^2 \, dx \leq K(a_\varepsilon \varepsilon^{-1})^n \to 0, \varepsilon \to 0,
\]
\[
\int_{\Omega} |\nabla \Psi_\varepsilon|^2 \, dx \leq K \sum_{j=1}^{N_\varepsilon} \int_{\rho_0 < |x - \rho_j| < \rho_1} |\nabla \Psi_\varepsilon|^2 \, dx \leq \\
\leq KN_\varepsilon \int_{\rho_0 < |x - \rho_j| < \rho_1} r^{n-3} \, dr \leq Ka_\varepsilon^{-n} \varepsilon^{-n}, \varepsilon \to 0, \text{ if } n \geq 3.
\]

Therefore, passing to the limit as \( \varepsilon \to 0 \) in (32), we get
\[
\int_{\Omega} |\nabla g|^2 \, dx - \int_{\Omega} \nabla g \nabla u_0 \, dx \geq \int_{\Omega} f(g - u_0) \, dx.
\]

Taking into account that \( g \) is an arbitrary function from \( C_0^\infty(\Omega) \) we deduce that \( u_0 \) is a weak solution of the problem (31).

**Theorem 3.** Suppose that conditions (29), (30) satisfied; \( u_0 \) is a weak solution of the problem (31). Then \( u_\varepsilon \to u_0 \) weakly in \( H_1^0(\Omega) \) as \( \varepsilon \to 0 \).

**Remark.** Using inequalities which were given earlier, we can estimate the rate of convergence of \( u_\varepsilon \) to \( u_0 \). We didn't give this estimation here, because it would require additional calculations.

**References**
