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Alexander Domoshnitsky
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A VARIATIONAL APPROACH TO SOLVE THE AXISYMMETRIC MAXWELL EQUATIONS IN A NON CONVEX GEOMETRY

F. ASSOUS * AND I. RAICHIK †

Abstract. We propose a new numerical method to solve the axisymmetric static Maxwell equations in singular domains, as for example non convex polygonal domains. We focus on the computation of the static magnetic field, and show that the key point to solve this problem is related to the solution of a Laplace-like operator in a singular domain. We then introduce a new approach, that consists in decomposing the domain into 2 subdomains, and to derive an ad hoc variational formulation, in which the interface conditions are imposed with a method deduced from a Nitsche approach. Numerical examples to illustrate our method will be shown.

Key Words. Maxwell equations, Singular geometries, Laplace operator, Nitsche method, Domain decomposition

AMS(MOS) subject classification. 65P25; 77C10.

1. Introduction. Electromagnetic phenomena play a very important role in the modern age. Maxwell’s equations (see [9],[10],[11],[20],[24]) represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. Today there are an ever-growing number of engineering problems requiring to model and simulate them numerically. Moreover, many structures that are to be modeled have a complex three-dimensional geometry and often present a surface with reentrant edges and/or corners. They are called geometrical singularities since they can generate very strong fields that have to be taken into account. However,

* Ariel University Center, 40700, Ariel, Israël
† Bar Ilan University, 52900, Ramat Gan, Israël

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three-dimensional computations are very expensive. Reducing the three-dimensional problem to two-dimensional computations, by assuming that the geometry, the data and the initial conditions are independent of one of the coordinates, reduces the cost of computations, and often gives a quite good approximation of the real three-dimensional solution.

Various approaches have been suggested for solving Maxwell equations in a singular domain. To our knowledge, first theoretical works for solving Maxwell equations in singular domain have been proposed by M. Birman and M. Solomyak (see [7],[8]). The principle of their approach is to decompose the space of electromagnetic fields into a simple sum of a regular field subspace and the subspace made of gradients of solutions to the Laplace problem. However, from a numerical point of view, this decomposition is not very useful since it is not a direct one.

In 1990s, a new vision on the characterizations of the singularity of the electromagnetic fields was given by M. Costabel and M. Dauge (see [12],[13] and references therein), called afterwards electromagnetic singularities. They also accomplished a thorough description of the Maxwell operator in 2D and 3D domain and proved density results. Lately (see [14]), they proposed that the electric field be fully taken into account without any splitting thanks to a technique which consists in introducing suitable weights in the definition of functional spaces: it allows to capture numerically strong electric fields.

We refer to the work of Bonnet-Ben Dhia, Hazard and Lohrengel (see [15]), for solving the time-harmonic, divergence-free Maxwell equations. But, in practice, they use a truncation function that leads to very slow numerical convergence. Let us also mention comparisons of different existing approaches for solving the 2D Maxwell equations in (see [18],[21]).

In this paper, we are interested in the axisymmetric singular geometry case \((r, \theta, z)\). Two-dimensional geometry is achieved by assuming that the domain, the data and the initial conditions are independent of \(\theta\), namely \(\frac{\partial}{\partial \theta} = 0\). This can be viewed as an intermediate between a full three-dimensional problem and a two-dimensional one. Since geometric singularities of the domain (like reentrant corners) have basically an influence on the space part of the equations, we will restrict ourselves to the static problem, by assuming \(\frac{\partial}{\partial t} = 0\). Then, it will be shown that the key point to compute the singular solution of the static Maxwell equations in an axisymmetric singular geometry, is re-
lated to solving a Laplace-like operator in a singular domain. The core of this paper will be devoted to derive an *ad hoc* numerical method to solve this Laplace-like problem. This method will be based on a domain decomposition method, so that it may be supported by the huge bibliography of the well-known domain decomposition methods. It will require the introduction of extended Nitsche method (see [5],[25]) with an "exchange" approach in order to handle the transmission conditions.

2. From Maxwell to a Laplace-like operator.

2.1. The Maxwell equations. Let Ω be a bounded and simply connected Lipschitz domain, Γ its boundary, and n the unit outward normal to Γ. The time-dependent Maxwell equations are a set of four partial differential equations that relate the electric field \( E \) and the magnetic induction \( B \) to the charge density \( \rho \) and the current density \( J \). If we let \( c \) and \( \varepsilon_0 \) be the speed of light and the dielectric permittivity of the vacuum, the Maxwell equations in vacuum read

\[
(1) \quad \frac{\partial E}{\partial t} - c^2 \text{curl} B = -\frac{1}{\varepsilon_0} J,
\]

\[
(2) \quad \frac{\partial B}{\partial t} + \text{curl} E = 0,
\]

\[
(3) \quad \text{div} E = \frac{\rho}{\varepsilon_0},
\]

\[
(4) \quad \text{div} B = 0,
\]

these quantities depending on the space variable \( x \) and on the time variable \( t \).

These equations are supplemented with appropriate boundary conditions. In this paper, we assume that the boundary is a perfect conducting medium or perfect conductor, in which the conductivity is assumed to be "infinite": all electromagnetic fields \( E \) and \( B \) are uniformly equal to zero in such a medium. This ideal situation is often used to model metals. Of course, extension to other boundary conditions is possible. Classically, one models the perfect conductor boundary conditions by

\[
E \times n = 0 \quad \text{and} \quad B \cdot n = 0 \quad \text{on the boundary } \Gamma.
\]

Last, initial conditions are provided, for instance at initial time \( t = 0 \)

\[
E(t = 0) = B(t = 0) = 0.
\]
2.2. Reduction to two-dimensional problems. We make the supplementary assumption that the bounded domain \( \Omega \) is an axisymmetric one, limited by the surface of revolution \( \Gamma \). We denote by \( \omega \) and \( \gamma_b \) their intersections with a meridian half-plane (see Fig. 1). One has \( \partial \omega := \gamma = \gamma_a \cup \gamma_b \), where either \( \gamma_a = \emptyset \) when \( \gamma_b \) is a closed contour (i.e. \( \Omega \) does not contain the axis), or \( \gamma_a \) is the segment of the axis lying between the extremities of \( \gamma_b \). We denote \( \nu \) is outward normal, and by \( \tau \) the unit tangential vector such that \((\tau, \nu)\) is direct. The natural coordinates for this domain are the cylindrical coordinates \((r, \theta, z)\), with the basis vectors \((e_r, e_\theta, e_z)\). A meridian half-plane is defined by the equation \( \theta = \text{constant} \), and \((r, z)\) are Cartesian coordinates in this half-plane.

Assuming symmetry of revolution, namely \( \partial / \partial \theta = 0 \), means that the fields are entirely determined by their trace in \( \omega \), that is the datum of their value in a meridian half-plane. Applying \( \partial / \partial \theta = 0 \) in the classical formulae for the gradient, divergence, and curl operators in cylindrical coordinates (see for instance [3]), we get the following expressions:

\[
\text{grad} f = \frac{\partial f}{\partial r} e_r + \frac{\partial f}{\partial z} e_z, \quad \text{div} \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z},
\]
\[
\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial z^2},
\]

whereas the curl operator is given by the formula

\[
\text{curl} \mathbf{u} = \left( -\frac{\partial u_\theta}{\partial z} \right) e_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) e_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} (ru_\theta) \right) e_z.
\]

Now, if \( \mathbf{u}_m = (u_r, u_z) \) is a meridian vector field, \( \text{curl} \mathbf{u}_m \) is azimuthal. It is thus convenient to introduce the roman type notation

\[
\text{curl} \mathbf{u}_m = \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right).
\]

In the same way, if \( \mathbf{u}_\theta \) is azimuthal, \( \text{curl} \mathbf{u}_\theta \) is meridian and we set, with the bold type notation,

\[
\text{curl} \mathbf{u}_\theta = -\frac{\partial u_\theta}{\partial z} e_r + \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) e_z.
\]

For this reason, it is possible to decouple the Maxwell system (1-4) into two systems of equations. From now on, let us denote by \( \mathbf{E} = (E_r, E_z) \), \( \mathbf{J} = (J_r, J_z) \) the meridian component of the electric field and of the current...
density. Also denote by $B_\theta$, $J_\theta$ the azimuthal component of the magnetic induction and of the current density, the axisymmetric Maxwell equations can be written first with a system in $(E, B_\theta)$

$$
\begin{align*}
\frac{\partial E}{\partial t} - c^2 \text{curl } B_\theta &= -\frac{1}{\varepsilon_0}J, \\
\frac{\partial B_\theta}{\partial t} + \text{curl } E &= 0, \\
\text{div } E &= \frac{\rho}{\varepsilon_0}.
\end{align*}
$$

(8)

where the perfect conductor boundary condition is expressed

$$
E \cdot \tau = 0 \text{ on } \gamma_b,
$$

and the symmetry conditions on the axis $\gamma_a$ becomes

$$
E \cdot \nu = E_r = 0, \quad B_\theta = 0 \text{ on } \gamma_a.
$$

Then, with similar notations, a second system in $(B, E_\theta)$ reads:

$$
\begin{align*}
\frac{\partial E_\theta}{\partial t} - c^2 \text{curl } B &= -\frac{1}{\varepsilon_0}J_\theta, \\
\frac{\partial B}{\partial t} + \text{curl } E_\theta &= 0, \\
\text{div } B &= 0.
\end{align*}
$$

(9)

with the perfect conductor boundary condition

$$
B \cdot \nu = 0, \quad E_\theta = 0 \text{ on } \gamma_b,
$$

and the symmetry conditions on the axis $\gamma_a$

$$
B \cdot \nu = B_r = 0, \quad E_\theta = 0 \text{ on } \gamma_a.
$$

The two systems (8) and (9) being of the same nature, we will focus in this paper on the second one in which $(E_\theta, B)$ are involved.

At this point, let us introduce some Sobolev functional spaces and their respective norms, which will be useful for the following of the article. Note that, as we consider an axisymmetric domain $\omega$, the weight $r$ plays a important
role in these definitions. We define

\[ L^2_r(\omega) = \{ f : \omega \to \mathbb{R} ; \int_\omega f^2 r dr dz < +\infty \} ; \| f \|_{0, \omega} = (\int_\omega f^2 r dr dz)^{1/2} ; \]

\[ H^1_r(\omega) = \{ f \in L^2_r(\omega) ; \int_\omega |\text{grad} f|^2 r dr dz < +\infty \} , \| f \|_{1, \omega} = (\| f \|^2_{0, \omega} + \| \text{grad} f \|^2_{0, \omega})^{1/2} ; \]

\[ H^1_m(\omega) = \{ f \in H^1_r(\omega) ; \int_\omega \frac{f^2}{r} dr dz < +\infty \} , \| f \|_{m, \omega} = (\| f \|^2_{1, \omega} + \| \frac{f}{r} \|^2_{0, \omega})^{1/2} ; \]

\[ H^1_{m0}(\omega) = \{ f \in H^1_m(\omega) ; f = 0 \text{ on } \gamma_b \} . \]

This allows to define the usual vector Sobolev spaces

\[ L^2_r(\omega) = L^2_r(\omega) \times L^2_r(\omega) ; \]

\[ H^1(\omega) = \{ \mathbf{v} = (v_r, v_z) \in L^2_r(\omega) ; v_r \in H^1_m(\omega) \text{ and } v_z \in H^1_r(\omega) \} ; \]

and the classical spaces for the Maxwell equations

\[ H(\text{curl}, \omega) = \{ \mathbf{v} \in L^2_r(\omega) ; \text{curl} \mathbf{v} \in L^2_r(\omega) \} , \]

\[ H(\text{div}, \omega) = \{ \mathbf{v} \in L^2_r(\omega) ; \text{div} \mathbf{v} \in L^2_r(\omega) \} . \]

This allows to define the corresponding subspaces with the vanishing tangential trace \( H_0(\text{curl}, \omega) \) and normal trace \( H_0(\text{div}, \omega) \), and finally the spaces of solutions

\[ \mathbf{X} := H_0(\text{curl}, \omega) \cap H(\text{div}, \omega) \quad \mathbf{Y} := H(\text{curl}, \omega) \cap H_0(\text{div}, \omega) . \]

Now, considering the azimuthal component \( E_\theta \), one can easily show that it is solution to a scalar wave equation. Indeed, it is sufficient to take the time derivative of the first equation of (9), to apply the \text{curl} operator to the second one, and to use the identity \text{curl} \text{curl} = -\Delta. In these conditions, even in a singular domain, \( E_\theta \) is always regular, namely belongs to \( H^1_m(\omega) \) (see for a proof [3]) and there is no difficulty to compute it.

For this reason, we focus on the computation of magnetic field \( \mathbf{B} \). Moreover, as we handle the geometrical singularity of the domain, we are basically concerned with the space-dependent part of the model (not the time-dependent one). Thus, we simplify the problem by considering the stationary problem associated with equations (9), by performing \( \frac{\partial}{\partial t} = 0 \), which becomes

\[ \begin{cases} \text{curl} \mathbf{B} = \frac{1}{c^2 \varepsilon_0} \mathbf{J}_\theta , \\ \text{div} \mathbf{B} = 0 , \end{cases} \]
This equation appears as a particular case of the following (\( \text{div}, \text{curl} \)) problem, that we will examine next:

For a given scalar function \( f \), find \( u = (u_r, u_z) \) a divergence-free vector solution in a meridian singular two dimensional domain \( \omega \), solution to

\[
\begin{align*}
\text{curl} \ u &= f, \\
u \cdot \nu &= 0.
\end{align*}
\]

2.3. A decomposition in regular and singular parts. As proved in [3], problem (11) is singular in the sense that for a non-convex domain \( \omega \) - typically \( \omega \) containing a reentrant corner - the space of solutions, says \( W \), is a subspace of \( Y \) but not a subspace of \( H^1(\omega) \). More precisely, \( W \) is defined by

\[
W = \{ w \in Y : \text{div} \ w = 0 \}, \text{with norm } ||\text{curl} \ w||_{0, \omega}.
\]

Nevertheless, \( W \) can be decomposed into two subspaces, where \( \oplus \) denotes a direct sum

\[
W = W_R \oplus W_S
\]

with

\[
W_R = W \cap H^1(\omega)
\]

Hence, \( W_R \) is a regular subspace, as a subspace of \( H^1(\omega) \), and one can compute a numerical approximation by a standard method, for instance a \( P_1 \)-conforming finite element method. The difficulty comes from the singular subspace \( W_S \), that has been proved to be a finite-dimensional subspace (see [1]), the dimension being the number of reentrant corners of the domain \( \omega \). Moreover, one has also proved in [2] that a basis function \( w_S \) of \( W_S \) can be characterized as the solution to

\[
\begin{align*}
\text{curl} \ w_S &= P_S \quad \text{in } \omega, \\
\text{div} \ w_S &= 0 \quad \text{in } \omega, \\
w_S \cdot \nu &= 0 \quad \text{on } \gamma.
\end{align*}
\]

Here, the right-hand side \( P_S \) is singular in the sense that it belongs to \( L^2(\omega) \), but not to \( H^1(\omega) \), and is solution to the problem

\[
\begin{align*}
\Delta' P_S &= 0 \quad \text{in } \omega, \\
P_S &= 0 \quad \text{on } \gamma,
\end{align*}
\]

where \( \Delta' \) is a Laplace-like operator defined in axisymmetric coordinates \((r, z)\) by:

\[
\Delta' P_S := \frac{\partial^2 P_S}{\partial r^2} + \frac{\partial^2 P_S}{\partial z^2} + \frac{1}{r} \frac{\partial P_S}{\partial r} - \frac{P_S}{r^2}.
\]
Hence, the key point is to compute $P_S$, which cannot be solved by a standard finite element method, which would give $P_S = 0$ as a solution. However, following [22], [23], [16], [17] in the cartesian case, or [1] in the axisymmetric case, there exists a non-vanishing singular solution $P_S$ which tends to infinity near the reentrant corner, and which does not belong to $H^1_r(\omega)$ but belongs to $L^2_r(\omega)$. From now on, our aim is to propose a numerical method to compute $P_S$, and then $w_S$ will be computed.

3. The numerical method. We will introduce now a numerical method based on a domain decomposition method, to compute $P_S$. For the sake of simplicity, we will consider a non convex domain $\omega$ with only one reentrant corner.

3.1. A domain decomposition method. The principle of a domain decomposition method is to split the computational into several subdomains. Here, we split the domain $\omega$ into two subdomains $\omega_1$ and $\omega_2$. As shown on Figure 2, the subdomain $\omega_1$ is taken as the "external" domain and the subdomain $\omega_2$ is the vicinity of the reentrant corner, with the interface $\gamma = \omega_2 \cap \omega_1$. For simplicity, we choose $\omega_2$ to be an open angular sector in the neighborhood of the reentrant corner. This leads us to use local polar coordinates centered at the reentrant corner. In these conditions, $\omega_1$ is the "regular" subdomain, since it does not contain the singularity, whereas $\omega_2$ is the "singular" subdomain that contains the singularity.

Our aim is now to solve the problem in each subdomain separately, and to take into account the continuity of the solution at the interface $\gamma$. Since we deal with the second order operator $\Delta^\prime$, it is necessary to ensure the continuity of the solution $P_S$

$$P_S|_{\omega_1} = P_S|_{\omega_2} \text{ across the interface } \gamma,$$

and the continuity of the normal derivative of $P_S$

$$\frac{\partial P_S}{\partial \nu_1}|_{\omega_1} + \frac{\partial P_S}{\partial \nu_2}|_{\omega_2} = 0 \text{ across the interface } \gamma,$$

where $\nu_1$ and $\nu_2$ are respectively the outgoing normals of $\omega_1$ and $\omega_2$ at the interface $\gamma$. Denoting by $P_1$ the "external" solution, that is the restriction of $P_S$ to $\omega_1$, and by $P_2$ the "singular" solution, that is the restriction of $P_S$ to
In the system on \( P_1 \) (left), \( P_2 \) is assumed to be a data, whereas in the system on \( P_2 \) (right), \( P_1 \) is assumed to be a data. In the regular subdomain \( \omega_1 \), \( P_1 \) is regular everywhere and can be solved by standard finite element method. However, the problem set in \( \omega_2 \) remains singular, since \( \omega_2 \) still contains the reentrant corner, and it would also give \( P_2 = 0 \) as solution with a standard numerical method. To overcome this difficulty, one uses as in [2] that \( P_2 \) can be decomposed into

\[
P_2 = Q_2 + P^S_2,
\]

where \( Q_2 \) is the regular part everywhere in \( \omega_2 \) of \( P_2 \), that is \( Q_2 \) belongs to \( H^1_r(\omega_2) \). \( P^S_2 \) denoting the singular part of \( P_2 \), that is the part that belongs to \( L^2_r(\omega_2) \) but not to \( H^1_r(\omega_2) \). Following [1], we have an analytic expression of the singular part \( P^S_2 \), given by

\[
P^S_2 = \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi),
\]

where \( a \) is the distance from the reentrant corner to \( z \)-axis, and \((\rho, \phi)\) denote the local polar coordinates centered at the reentrant corner. Decomposition (18)-(19) will be used to derive the final variational formulation. Moreover, we have also to achieve first continuity of the solution. This will be performed by a Nitsche method. Secondly, the continuity of the normal derivative of the solution which will be imposed by an exchange technique in the variational formulations, that will be explained below. Let us introduce now the classical Nitsche method.

**3.2. A variational formulation deduced from a Nitsche method.**

In 1970, J. Nitsche proposed in [25] a variational approach to enforce weakly Dirichlet boundary condition. In a certain sense, the method resembles Lagrange multiplier method, but it possesses better convergence property, ensures the existence and uniqueness of the solution.
Nitsche’s method can be extend for imposing essential boundary conditions weakly in the finite element method for approximation of elliptic problems. Roughly speaking, essential boundary conditions, like the Dirichlet boundary conditions, are conditions that have to be imposed in the functional space in which the solution belongs to. At the opposite, natural boundary conditions, like the Neumann conditions, are conditions that appear naturally in the variational formulation and that can not be handled by a Nitsche approach. The strong advantage of Nitsche’s method is that it keeps the convergence rate of the finite element method [26], as opposed to the standard penalty method. Essentially, Nitsche’s method imposes the boundary conditions via three boundary terms. Two of them contain the weak form of the normal derivatives of the solution and test functions. These two terms cause the method to be symmetric and consistent. A third term depends on the domain triangulation and causes the method to be stable.

To illustrate the method, we consider the standard Laplace problem. For a given bounded domain $\Omega$ of boundary $\Gamma$, let $H^{1/2}(\Gamma)$ be equal to the trace on $\Gamma$ of elements of $H^1(\Omega)$ and introduce the classical problem

$$
\begin{align*}
-\Delta u &= f \text{ in } \Omega, \\
u &= g \text{ on } \Gamma.
\end{align*}
$$

We first introduce a shape regular finite element partition $T_h = \bigcup K$ of the domain $\Omega$. For any element $K$ of the mesh $T_h$, let $P_k(K)$ be the space of polynomials of degree $k \geq 1$ on $K$. We denote by $E$ an edge of an element of $T_h$ and by $C_h$ the trace mesh induced by $T_h$ on the boundary $\Gamma$, that is

$$
C_h = \{ E; E = K \cap \Gamma, K \in T_h \}.
$$

Moreover, we assume that the elements of $C_h$ verify the regularity condition

$$
h_E \leq C \rho_E
$$

where $h_E$ is the diameter element $E \in C_h$ and $\rho_E$ is the maximum diameter of circle inscribed in $E$. Finally, let us introduce the finite element space

$$
V_h = \{ v \in H^1(\Omega); v|_K \in P_k(K) \}.
$$

According to these definitions, Nitsche’s method for the above problem consists in the following steps:
• Derivation of the classical variational formulation

\[ \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma} \frac{\partial u}{\partial n} v d\Gamma = \int_{\Omega} f v \, d\Omega, \quad \forall v \in V_h. \]

• Symmetrization by adding the expression

\[ -\int_{\Gamma} \frac{\partial v}{\partial n} u \, d\Gamma \]

to left-hand side of equation and, since \( u = g \) on \( \Gamma \), adding the expression

\[ -\int_{\Gamma} \frac{\partial v}{\partial n} g \, d\Gamma \]

to the right-hand side of equation, that yields

\[ \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma} \frac{\partial u}{\partial n} v d\Gamma - \int_{\Gamma} \frac{\partial v}{\partial n} u d\Gamma = \int_{\Omega} f v \, d\Omega - \int_{\Gamma} \frac{\partial v}{\partial n} g \, d\Gamma \]

• Stabilization to ensure stability and coerciveness of the above variational formulation, by adding the expression (\( \beta \) being a given constant)

\[ \beta \sum_{E \in C_h} \frac{1}{h_E} \int_E uv \, dE \]

to the left-hand side of equation, and in parallel the expression

\[ \beta \sum_{E \in C_h} \frac{1}{h_E} \int_E gv \, dE \]

to the right-hand side of equation.

Hence, the Nitsche formulation for the Laplace operator is finally written as

\[ \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma} \frac{\partial u}{\partial n} v d\Gamma - \int_{\Gamma} \frac{\partial v}{\partial n} u d\Gamma + \beta \sum_{E \in C_h} \frac{1}{h_E} \int_E uv \, dE \]

\[ = \int_{\Omega} f v \, d\Omega - \int_{\Gamma} \frac{\partial v}{\partial n} g \, d\Gamma + \beta \sum_{E \in C_h} \frac{1}{h_E} \int_E gv \, dE \]

(21)

The method is proved to be consistent and stable for a \( \beta \) sufficient large see [26]. In other words, \( u \) solution to (20) is also solution to (21).
3.3. Computation of the basis $P_S$. We apply the Nitsche method here to impose $P_1 = P_2$ on $\gamma$ for subdomain $\omega_1$, $P_2$ being considered as a data, and reciprocally. We get the two following variational formulations:

**Find** $P_1 \in H_{m}^1(\omega_1)$ **solution to**, $\forall \varphi_1 \in H_{m0}^1(\omega_1)$

\[
\iint_{\omega_1} \left( \nabla P_1 \nabla \varphi_1 + \frac{P_1 \varphi_1}{r^2} \right) \, rdr - \int_{\gamma} \frac{\partial P_1}{\partial \nu_1} \varphi_1 \, rd\gamma - \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} P_1 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_1 \varphi_1 \, rdE.
\]

**Find** $P_2 \in H_{m}^1(\omega_2)$ **solution to**, $\forall \varphi_2 \in H_{m0}^1(\omega_2)$

\[
\iint_{\omega_2} \left( \nabla P_2 \nabla \varphi_2 + \frac{P_2 \varphi_2}{r^2} \right) \, rdr - \int_{\gamma} \frac{\partial P_2}{\partial \nu_2} \varphi_2 \, rd\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} P_2 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_2 \varphi_2 \, rdE.
\]

We have now to impose the continuity of the normal derivatives across $\gamma$. To deal with this problem, we introduce an exchange method that consists in exchanging the term $\partial P_1/\partial \nu_1$ with $-\partial P_2/\partial \nu_2$ in the variational formulation on $\omega_1$, and reciprocally for the variational formulation on $\omega_2$. The above variational formulations become:

**Find** $P_1 \in H_{m}^1(\omega_1)$ **solution to**, $\forall \varphi_1 \in H_{m0}^1(\omega_1)$

\[
\iint_{\omega_1} \left( \nabla P_1 \nabla \varphi_1 + \frac{P_1 \varphi_1}{r^2} \right) \, rdr + \int_{\gamma} \frac{\partial P_2}{\partial \nu_2} \varphi_1 \, rd\gamma - \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} P_1 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_1 \varphi_1 \, rdE.
\]

**Find** $P_2 \in H_{m}^1(\omega_2)$ **solution to**, $\forall \varphi_2 \in H_{m0}^1(\omega_2)$

\[
\iint_{\omega_2} \left( \nabla P_2 \nabla \varphi_2 + \frac{P_2 \varphi_2}{r^2} \right) \, rdr + \int_{\gamma} \frac{\partial P_1}{\partial \nu_1} \varphi_2 \, rd\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} P_2 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_2 \varphi_2 \, rdE.
\]

As explained above, we must at this point use the substitution (18), so that

\[
\frac{\partial P_2}{\partial \nu_2} = \frac{\partial Q_2}{\partial \nu_2} + \frac{\partial}{\partial \nu_2} \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right).
\]
Moreover, using that
\[
\Delta' \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) = -\frac{3\alpha}{a} \rho^{-\alpha-1} \sin((\alpha + 1) \phi),
\]
we are now looking for \( Q_2 \) - instead of \( P_2 \) - satisfying
\[
\begin{cases}
-\Delta Q_2 = -\frac{3\alpha}{a} \rho^{-\alpha-1} \sin((\alpha + 1) \phi) \text{ in } \omega_2, \\
Q_2 = 0 \text{ on } \partial \omega_2, \\
Q_2 = P_1 - \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \text{ on } \gamma, \\
\frac{\partial Q_2}{\partial \nu_2} = -\frac{\partial P_1}{\partial \nu_1} - \frac{\partial}{\partial \nu_2} \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) \text{ on } \gamma.
\end{cases}
\]

In these conditions, we obtain a new variational formulation with \((P_1, Q_2)\) as unknowns, that reads

Find \( P_1 \in H^1_m(\omega_1) \) such that, \( \forall \varphi_1 \in H^1_{m0}(\omega_1) \)
\[
\int_{\omega_1} \left( \nabla P_1 \cdot \nabla \varphi_1 + \frac{P_1 \varphi_1}{r^2} \right) rd\omega + \int_{\gamma} \frac{\partial Q_2}{\partial \nu_2} \varphi_1 rd\gamma - \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} P_1 rd\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_1 \varphi_1 r dE = -\int_{\gamma} \frac{\partial}{\partial \nu_2} \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) \varphi_1 rd\gamma \\
- \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} Q_2 r d\gamma - \int_{\gamma} \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \varphi_1 r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E \left( Q_2 \varphi_1 r + \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) \varphi_1 r \right) dE,
\]

Find \( Q_2 \in H^1_m(\omega_2) \) such that, \( \forall \varphi_2 \in H^1_{m0}(\omega_2) \)
\[
\int_{\omega_2} \left( \nabla Q_2 \cdot \nabla \varphi_2 + \frac{Q_2 \varphi_2}{r^2} \right) rd\omega + \int_{\gamma} \frac{\partial P_1}{\partial \nu_1} \varphi_2 r d\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} Q_2 r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E Q_2 \varphi_2 r dE = -\int_{\gamma} \frac{\partial}{\partial \nu_2} \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) \varphi_2 rd\gamma \\
- \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_1} P_1 r d\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \varphi_2 r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E P_1 \varphi_2 r dE - \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_E \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \varphi_2 r dE \\
- \int_{\omega_2} \frac{3\alpha}{a} \rho^{-\alpha-1} \sin((\alpha + 1) \phi) \varphi_2 r d\omega,
\]
To get the variational formulations which are of practical interest for numerical computations, we first remark that due the geometry of domain $\omega_2$, which is a angular sector, we have $\frac{\partial}{\partial \nu_2} = \frac{\partial}{\partial \rho}$. Since $r = a + \rho \cos \phi$, we easily get that
\[
\frac{\partial}{\partial \rho} \left( \frac{r}{a} \right) = \frac{1}{a} \frac{\partial}{\partial \rho} \left( a + \rho \cos \phi \right) = \frac{1}{a} \cos \phi
\]
which leads to
\[
\frac{\partial}{\partial \rho} \left( \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \right) = \left( \frac{1}{a} \cos \phi \rho^{-\alpha} - \alpha \frac{r}{a} \rho^{-\alpha-1} \right) \sin(\alpha \phi)
\]
Substituting these expressions in the above expressions, the variational formulations that will be the basis of the numerical scheme read

Find $P_1 \in H^1_{m_1}(\omega_1)$ such that, $\forall \varphi_1 \in H^{1}_{m_0}(\omega_1)$
\[
\int_{\omega_1} \left( \nabla P_1 \nabla \varphi_1 + \frac{P_1 \varphi_1}{r^2} \right) r d\omega + \int_{\gamma} \frac{\partial Q_2}{\partial \nu_1} \varphi_1 r d\gamma - \int_{\gamma} \frac{\partial P_1}{\partial \nu_1} P_1 r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} P_1 \varphi_1 r dE = - \int_{\gamma} \left( \frac{1}{a} \cos \phi \rho^{-\alpha} - \alpha \frac{r}{a} \rho^{-\alpha-1} \right) \sin(\alpha \phi) \varphi_1 r d\gamma \\
- \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} Q_2 r d\gamma - \int_{\gamma} \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} Q_2 \varphi_1 r dE + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \varphi_1 r dE.
\]

Find $Q_2 \in H^1_{m_2}(\omega_2)$ such that, $\forall \varphi_2 \in H^{1}_{m_0}(\omega_2)$
\[
\int_{\omega_2} \left( \nabla Q_2 \nabla \varphi_2 + \frac{Q_2 \varphi_2}{r^2} \right) r d\omega + \int_{\gamma} \frac{\partial P_2}{\partial \nu_2} \varphi_2 r d\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} Q_2 r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} Q_2 \varphi_2 r dE = - \int_{\gamma} \left( \frac{r}{a^2} \cos \phi \rho^{-\alpha} - \alpha \frac{r}{a} \rho^{-\alpha-1} \right) \sin(\alpha \phi) \varphi_2 r d\gamma \\
- \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} P_2 r d\gamma - \int_{\gamma} \frac{\partial \varphi_2}{\partial \nu_2} r a \rho^{-\alpha} \sin(\alpha \phi) r d\gamma \\
+ \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} P_2 \varphi_2 r dE - \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} \frac{r}{a} \rho^{-\alpha} \sin(\alpha \phi) \varphi_2 r dE \\
- \int \int_{\omega_2} 3 \alpha \frac{\rho^{-\alpha-1}}{a} \sin((\alpha + 1) \phi) \varphi_2 r d\omega.
\]

Numerical applications of this method will be given in section 4.
3.4. Computation of the basis $w_S$. Assuming that $P_S$ has been obtained, our aim is now to compute a basis function $w_S$ of $W_S$ characterized by (12-14). If we want to use the fact that $w_S$ belongs to $W_S \subset H(\text{curl}, \omega)$ to solve this problem, we can for example transform it into a problem in $\text{curl} \text{curl}$. A second and more convenient way to find $w_S$ is to use its so-called ”stream function” $\psi$. Using the following property (see [1]):

$$\forall w_S \in W_S; \exists! \psi \in H^1_m(\omega) \text{ such that } w_s = \text{curl} \psi,$$

where $\psi$ is the unique solution to the Laplace-like problem

$$\begin{cases} -\Delta' \psi = P_S \text{ in } \omega, \\ \psi = 0 \text{ on } \gamma. \end{cases} \quad (23)$$

As it is more easy to invert a Laplace-like operator than a $\text{curl}$ or a $\text{curl} \text{curl}$ one, instead of solving (12-14), we chose to solve first the system (23) to get $\psi$, then to compute $w_S \in W_S$ solution to $w_S = \text{curl} \psi$. Note also that $\psi$ being equal to $\Delta' - \frac{1}{\beta}P_S \in L^2(\omega)$, it is quite regular and thus easy to compute.

3.5. Computation of $\psi$. Using the same domain decomposition as for $P_S$, we introduce for $\psi_S$ the two systems

Find $\psi_1 \in H^1_m(\omega_1)$ solution to

$$\begin{cases} -\Delta' \psi_1 = P_1 \text{ in } \omega_1, \\ \psi_1 = 0 \text{ on } \partial \omega_1, \\ \frac{\partial \psi_1}{\partial \nu_1} = -\frac{\partial \psi_2}{\partial \nu_2} \text{ on } \gamma. \end{cases}$$

Find $\psi_2 \in H^1_m(\omega_2)$ solution to

$$\begin{cases} -\Delta \psi_2 = P_2 \text{ in } \omega_2, \\ \psi_2 = 0 \text{ on } \partial \omega_2, \\ \frac{\partial \psi_2}{\partial \nu_2} = -\frac{\partial \psi_1}{\partial \nu_1} \text{ on } \gamma. \end{cases}$$

As previously, one can solve this problem with the variational formulations

Find $\psi_1 \in H^1_m(\omega_1)$ such that

$$\iint_{\omega_1} \left( \nabla \psi_1 \nabla \varphi_1 + \frac{\psi_1 \varphi_1}{r^2} \right) rd\omega - \int_{\gamma} \frac{\partial \psi_1}{\partial \nu_1} \varphi_1 rd\gamma - \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} \psi_1 rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} \psi_1 \varphi_1 rdE = - \int_{\gamma} \frac{\partial \varphi_1}{\partial \nu_1} \psi_2 rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_{\gamma} \psi_2 \varphi_1 rdE$$

$$+ \iint_{\omega_1} (P_1 \varphi_1) rd\omega, \quad \forall \varphi_1 \in H^1_{m_0}(\omega_1)$$
Find $\psi_2 \in H^1_m(\omega_2)$ such that
\[
\iint_{\omega_2} \left( \nabla \psi_2 \nabla \varphi_2 + \frac{\psi_2 \varphi_2}{r^2} \right) \, rd\omega - \int_\gamma \frac{\partial \psi_2}{\partial n_2} \varphi_2 \, rd\gamma - \int_\gamma \frac{\partial \varphi_2}{\partial n_2} \psi_2 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_\gamma \psi_2 \varphi_2 \, rdE \\
+ \iint_{\omega_2} P_2 \varphi_2 \, rd\omega, \quad \forall \varphi_2 \in H^1_m(\omega_2)
\]

As we already explained in subsection 3.3, after taking into account that $\frac{\partial \psi_1}{\partial n_1} + \frac{\partial \psi_2}{\partial n_2} = 0$ and using the exchange method, we get the following final variational formulation

Find $\psi_1 \in H^1_m(\Omega_1)$ such that:
\[
\iint_{\omega_1} \left( \nabla \psi_1 \nabla \varphi_1 + \frac{\psi_1 \varphi_1}{r^2} \right) \, rd\omega + \int_\gamma \frac{\partial \psi_1}{\partial n_1} \varphi_1 \, rd\gamma - \int_\gamma \frac{\partial \varphi_1}{\partial n_1} \psi_1 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_\gamma \psi_1 \varphi_1 \, rdE \\
+ \iint_{\omega_1} (P_1 \varphi_1) \, rd\omega, \quad \forall \varphi_1 \in H^1_m(\omega_1)
\]

Find $\psi_2 \in H^1_m(\Omega_2)$ such that:
\[
\iint_{\omega_2} \left( \nabla \psi_2 \nabla \varphi_2 + \frac{\psi_2 \varphi_2}{r^2} \right) \, rd\omega + \int_\gamma \frac{\partial \psi_2}{\partial n_2} \varphi_2 \, rd\gamma - \int_\gamma \frac{\partial \varphi_2}{\partial n_2} \psi_2 \, rd\gamma + \beta \sum_{E \in \Sigma_h} \frac{1}{h_E} \int_\gamma \psi_2 \varphi_2 \, rdE \\
+ \iint_{\omega_2} (P_2 \varphi_2) \, rd\omega, \quad \forall \varphi_2 \in H^1_m(\omega_2)
\]

After $\psi$, the last step consists in computing $w_S$ which satisfies the equation $w_S = \text{curl} \, \psi$.

Note that with this approach, we ensure that the divergence of $w_S = 0$, due to the relation $\text{div} \, \text{curl} \equiv 0$. Here again, we use the same domain decomposition of $\omega = \omega_1 \cup \omega_2$.

A $P_1$-conforming finite element method has been developed, based on the FreeFem++ package [19], to solve this problem. Numerical illustrations are given in the following section.
4. Numerical results. In this section, we present some numerical results to illustrate the efficiency of the proposed method. We consider the 3-D top hat domain $\Omega$ with a reentrant circular edge, that corresponds to an L-shaped 2-D domain $\omega$ with reentrant corner (see Fig. 2). Our aim is to compute the magnetic basis $w_S$ of $W_S$.

We first introduce an unstructured mesh of the L-shaped domain $\omega$ made up of triangles, with no particular mesh refinement near the corner. Following the method exposed in previous sections, to begin with, we compute the singular function $P_S$. The numerical results are depicted in Figures 3-5. When representing functions or fields with a singular behavior, we have chosen to truncate the results in the singular (infinite) node. One can see that the method is able to compute $P_S$, whereas a usual method would give $P_S = 0$ as solution. Moreover, as expected, the continuity of the solution and of its normal derivative are well handled.

The second step consists in computing the stream function $\psi$, where $P_S$ is used as a right-hand side. Results are shown in Figures 6-8, and illustrate that the method is also efficient for more regular functions or fields like $\psi$. Again, the continuity conditions are properly taken into account.

Finally, we get the basis magnetic basis $w_S$ of $W_S$ by taking the curl of $\psi$, that ensures that the relation $\text{div} w_S = 0$. The two component $(w'_S, w'_z)$ obtained are depicted in Figure 9. This still shows that the method captures well the field $w_S$ near the edge (and far away from it). A conforming $P^1$ Finite Element Method can not yield such a result. In addition, the results are not noisy, even though the mesh is not particularly refined near the edge.

5. Conclusion. In this paper, we presented a new method to solve the Maxwell equations in an axisymmetric singular domain. It is based on a decomposition of the computational domain into two subdomains: an internal one, close to the singularity, and an external one.

Since geometric singularities of the domain have basically an influence on the space part of the equations, we restricted ourselves to the static problem, by assuming $\frac{\partial}{\partial r} = 0$. Then, we considered as an example the static magnetic field $\mathbf{B} = (B_r, B_z)$.

We have shown that to compute a basis of this singular subspace, the key point is to compute $P_S$, which can not be solved by a standard finite element method, which would give $P_S = 0$. We proposed a new method to efficiently
compute $P_S$ and consequently $w_S$. It consists in decomposing the domain $\omega$ into 2 subdomains, and to derive an ad hoc variational formulation, in which the interface conditions have to be handled.

This method also uses the local expression of the singularity, and an extended version of the Nitsche method coupled with an "exchange" original approach. This allows us to handle continuity interface conditions, both of the Dirichlet and the Neumann type. Numerical results have been shown to illustrate the efficiency of the method.

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Fig. 1. The axisymmetric domain $\Omega$ and the corresponding meridian domain $\omega$.

Fig. 2. L-shape domain with one reentrant corner
Fig. 3. Singular $P_S$ solution in the internal domain $\omega_2$: 2-d view (left) and 3-d view (right).

Fig. 4. Singular $P_S$ solution in the external domain $\omega_1$: 2-d view (left) and 3-d view (right).

Fig. 5. Singular $P_S$ solution in the whole domain $\omega$: 2-d view (left) and 3-d view (right).
Fig. 6. Stream function $\psi$ in the internal domain $\omega_2$: 2-d view (left) and 3-d view (right).

Fig. 7. Stream function $\psi$ in the external domain $\omega_3$: 2-d view (left) and 3-d view (right).

Fig. 8. Stream function $\psi$ in the whole domain $\omega$: 2-d view (left) and 3-d view (right).
Fig. 9. Singular $w_S$ solution: $r$-component (left) and $z$-component (right).
A MATHEMATICAL MODEL OF IMATINIB AND INTERFERON–ALPHA COMBINED TREATMENT OF CHRONIC MYELOID LEUKEMIA

L. BEREZANSKY *, S. BUNIMOVICH-MENDRAZITSKY † AND A. DOMOSHNITSKY ‡

Abstract.

In this paper, we propose and analyze a mathematical model for the treatment of chronic myelogenous (or myeloid) leukemia (CML), a cancer of the blood. We introduce combined treatment of CML based on Imatinib therapy and Immunotherapy. Imatinib therapy is a molecular targeted therapy that inhibits the cell containing the oncogenic protein BCR-ABL, involved in the chronic CML pathogenesis. Immunotherapy based on interferon alfa-2a (IFN-α) effects on the cancer cells mortality and leads to improvement outcome of the combined therapy. We model the interaction between CML cancer cells in the body and effector cells of the immune system, using a system of differential equations. The proposed model belongs to a special class of nonlinear nonautonomous systems of ordinary differential equations (ODEs) with time-varying delays in the treatment.

For this system the following results were obtained: existence of a unique global positive solution, existence of a unique nontrivial equilibrium, explicit local and global stability conditions for the nontrivial equilibrium.

Key Words. CML, mathematical model, imatinib, Interferon alpha, time delay.

* Department of Mathematics, Ben-Gurion University of Negev, Beer-Sheva 84105, Israel
† Department of Computer Science and Mathematics and Department of Physics, Ariel University Center of Samaria, Ariel 40700, Israel, Email: SvetlanaBu@ariel.ac.il
‡ Department of Computer Science and Mathematics, Ariel University Center of Samaria, Ariel 40700, Israel
1. Introduction. Leukemia, a cancer of the white blood cells, is classified clinically on the basis of the character of the disease (acute or chronic), and from the type of mutant cells (myeloid, lymphoid or monocytic). The character of leukemia defines the disease progression rate where no treatment is applied: a patient with acute leukemia will die within months, while chronic leukemia will kill within years. In this paper, we will analyze chronic myeloid leukemia (CML), one of the most common types of leukemia. CML is usually diagnosed by the presence of a specific chromosomal abnormality named the Philadelphia chromosome, where due to a reciprocal translocation between chromosome 9 and chromosome 22 those two become longer and shorter than normal, respectively [10]. Myelogenous leukemia progresses through four distinct phases. After an initial period 3-5 years where the Abnormal Cell counts rise to a relatively steady state, called the Chronic Phase. At this stage the disease can be diagnosed. After several years of the chronic phase, the period of the order of months gives rise to oscillatory instability (Acceleration Phase). Ultimately, this leads to usually fatal Acute Phase with sharp increase in the cell count. The last is known also as the blast crisis.

The first therapeutic options for CML included the administration of cytotoxic drugs such as busulfan, hydroxyurea, interferon-α and allogeneic stem cell transplantation. In year 2000, when the imatinib treatment was introduced, the results revealed its substantial superiority over classic treatments [8]. Imatinib (Gleevec) inhibits the oncogenic protein BCR-ABL tyrosine kinase created by the Philadelphia chromosome at CML [5]. Note, that 10-20% of patients do not respond to imatinib after prolonged therapy [27, 15]. Imatinib resistance has been associated to BCR-ABL point mutations [24, 26]. Less favorable Imatinib function is inhibition of proliferation of both immune T-cells and dendritic cells [7, 11]. While it is widely agreed that imatinib does not represent a true cure for CML, it does provide, however, an effective control measure available during immunotherapy.

Interferon alfa-2α (IFN-α) immunotherapy was firstly introduced to produce cytogenetic response in the CML patients. IFN-α is physiologically produced by a variety of cells upon viral infection. Recent data suggests that IFN-α activation of host immune cells, including T, B, natural killer cells and tumor antigen-presenting dendritic cells contributes to the killing of leukemic cells [21]. Moreover, it is already proved that IFN-α, which has long been considered as the standard conservative therapy in CML, may exert its life-prolonging effect by activating immunological effector functions [3, 12, 25].

Based on these studies, we can assume that combination treatment of ima-
tinib and IFN-\(\alpha\) can be good candidate to improve the CML therapy. In this paper we develop a mathematical model validating the goodness of such a combination treatment. The currently accepted model of CML is developed by Moore and Li [20]. In their model the interaction between the immune system (naive T cells and effector T cells) and CML cancer cells in the body is described by a system of ordinary differential equations. The change rates of these cells populations can be obtained from this model. In our work we focus on the understanding of the contribution of IFN-\(\alpha\) in eliciting strong CTL-cell responses against CML cells in addition to imatinib therapy by examine of two cells populations only, effector T cells and CML cancer cells namely.

The mathematical description of the above model is described in Section 2. In Section 3 we discuss the existence of a positive and unique equilibrium; prove local and global stability of this equilibrium. Section 4 validates the model results using published data taken from \textit{in vitro}, mouse and human studies.

2. Model description. Following the Moore and Li model, we omitted from their ODE system one equation describing the naive effector T-cells behavior. The additional terms describing the imatinib influence on cancer cells and the INF-\(\alpha\) influence on the effector T-cells were added to each one of the remaining equations. The obtained system of two ordinary non-linear differential equations characterizes the dynamics of the interaction between two biological components in the following form:

1. \(x(t)\) presents the population of cancer cells of CML;
2. \(y(t)\) presents the effector T cells specific to CML (CTL).

\[
\begin{align*}
\dot{x}(t) &= \beta_1 x(t) \frac{K}{x(t)} - \gamma_1 x(t) y(t) - \omega \gamma_3 x(h(t)) \\
\dot{y}(t) &= \beta_2 \frac{x(t)}{\eta_1 + x(t)} y(t) - \gamma_2 x(t) y(t) + \ln(\alpha) \gamma_4 \frac{y(t)}{\eta_2 + y(t)} y(t - \tau) - \mu y(t),
\end{align*}
\]

where \(K > 0, \beta_i > 0, \gamma_i > 0, \omega > 0, \ln(\alpha) > 0, h(t) \leq t\) is a continuous function such that \(t - h(t) \leq \theta, \lim_{t \to \infty}(t - h(t)) = 0\).

Consider the system (1) with following general initial conditions:

\[
\begin{align*}
x(t) &= \varphi(t), y(t) = \phi(t), t \leq 0.
\end{align*}
\]

We suppose that \(\varphi(t)\), where \(-\theta \leq t \leq 0\) and \(\phi(t)\), where \(-\tau \leq t \leq 0\) are nonnegative continuous functions where \(x(0) = \varphi(0) > 0, y(0) = \phi(0) > 0\).

The first term on the right-hand side of the first equation of the system (1) describes the growth of CML cancer cells population in the form of a Gompertz law with the growth rate \(\beta_1\). The Gompertz curve provides a
significantly better fit for leukemic cancer data than logistic, exponential or polynomial curves [20]. The constant K in the first term estimates the maximum carrying capacity of CML cells, taking account the birth and death rate of cancer cells [18, 23]. The second term of the first equation of the system (1) represents the loss of CML cancer cells due to CTL cells influence with rate $\gamma_1$. The third term of this equation represents inhibition of cancer cells due to imatinib administered therapy, where $\omega$ is a dose of imatinib given every day and $h(t)$ is the time-varying function, showing delay of imatinib treatment impact on the number cancer cells in the blood. It was shown by Volpe [27], that there is a delay from the imatinib treatment start ($t = 0$), when the number of cancer cells is reduced. The maximum delay time is about 3 weeks. The influence of drugs tends to zero over the time so the delay function looks like: $h(t) = t - \theta e^{-\lambda t}$.

The second equation describes the dynamic balance between stimulatory and inhibitory effects of CTL cells. The first term represents the growth of the effector CTL cells population $y(t)$ due to CML antigen in the lymph nodes influence, where $\beta_2$ is the rate of this growth and $\eta_1$ is the standard half-saturation concentration in a Michaelis–Menten term, taking into account the saturation effect of CML cells in the lymph nodes. The second term describes the loss of CTL cells due to CTL and CML cancer cells interaction with rate $\gamma_2$. The third term corresponds to the stimulatory augmentation of the CTL cells due to IFN-$\alpha$ immunotherapy, where $in_\alpha$ is the dose of IFN-$\alpha$. Interferon-$\alpha$ leads to increase the expression of other cytokines, such as interferon-$\gamma$ that increase the pro-inflammatory environment with delay $\tau$ of about 7 days [3]. $\eta_2$ is the standard half-saturation concentration of CTL immune cells in a Michaelis-Menten term. The last term in the second equation describes the loss of CML cells due a natural death, where $\mu_y$ is their death rate. The summary of parameter values used in this paper is presented in the Table 1.

3. The local and global stability of combined model for tumor immunotherapy.

3.1. Positivity and uniqueness of solution system (1)-(2).

In this section we will discuss the positivity and uniqueness of the solution $(x(t), y(t))$ of system (1)-(2).

Lemma 1. [13] Consider initial value problem for a scalar linear delay equation

$$\dot{u}(t) = -a(t)u(h(t)), \quad t \geq 0, \; u(t) = \varphi(t), \; t < 0, \; u(0) = u_0.$$

If

$$a(t) \geq 0, \; t - h(t) \leq \delta, \; \delta \sup a(t) \leq \frac{1}{e}, \; 0 \leq \varphi(t) \leq u_0, \; u_0 > 0,$$

then

$$\dot{u}(t) = -a(t)u(h(t)), \quad t \geq 0, \; u(t) = \varphi(t), \; t < 0, \; u(0) = u_0.$$
then \( u(t) > 0, t > 0 \) and for the solution of the inequality

\[ v(t) \geq -a(t)v(h(t)), \quad t \geq 0, \quad v(t) = u(t), t \leq 0 \]

one has \( v(t) \geq u(t) \).

**Theorem 1.** Suppose that \( \omega \geq \theta \leq \frac{1}{e} \). Then system (1)-(2) has the unique global solution \( (x, y) \) such that

\[ 0 < x(t) \leq \max\{x(0), K\}, \quad 0 < y(t) \leq y(0)e^{(|\beta_2 - \mu_y| + \eta_\alpha \gamma_4)t}. \]

The theorem was proven in [2].

**Theorem 2.** If

\[ \mu_y > \beta_2 + \eta_\alpha \gamma_4. \]

there is the unique equilibrium \( (X^*, 0) \) of system (1) where \( X^* = Ke^{-\gamma_3 \eta_1} \).

**Proof.** It is obvious that \( (X^*, 0) \) is an equilibrium of system (1). We have to prove the uniqueness only of this equilibrium or in another words we have to prove there is no another equilibrium \( (X_0, Y_0) \), where \( X_0 > 0, Y_0 > 0 \).

Suppose that \( (X_0, Y_0) \), where \( X_0 > 0, Y_0 > 0 \) is equilibrium of system (1). Then from the second equation in (1) we have

\[ \beta_2 \frac{X_0}{\eta_1 + X_0} - \gamma_2 X_0 + \eta_\alpha \gamma_4 \frac{Y_0}{\eta_2 + Y_0} - \mu_y = 0. \]

Hence

\[ Y_0 = \frac{\eta_2 \left[ \mu_y + \gamma_2 X_0 - \beta_2 \frac{X_0}{\eta_1 + X_0} \right]}{\eta_\alpha \gamma_4 - \mu_y - \gamma_2 X_0 + \beta_2 \frac{X_0}{\eta_1 + X_0}}. \]

Thus, we have

\[ \mu_y + \gamma_2 X_0 - \beta_2 \frac{X_0}{\eta_1 + X_0} \geq \mu_y - \beta_2 > 0, \]

\[ \eta_\alpha \gamma_4 - \mu_y - \gamma_2 X_0 + \beta_2 \frac{X_0}{\eta_1 + X_0} \leq \eta_\alpha \gamma_4 - \mu_y + \beta_2 < 0. \]

Then \( Y_0 < 0 \) contradicts to our assumption and, therefore, the system (1) has only one equilibrium \( (X^*, 0) \). \( \Box \)
3.2. Local stability of the equilibrium \((X^*, 0)\). To analyse local stability of the equilibrium \((X^*, 0)\) for the system (1) we will use the lemma defined below. Consider a scalar linear equation

\[
\dot{x}(t) + \sum_{k=1}^{m} a_k x(h_k(t)) = f(t),
\]

where \(a_k > 0\), \(\lim_{t \to \infty} h_k(t) = \infty\).

LEMMMA 2. \cite{16} If \(\limsup_{t>0} \sum_{k=1}^{m} a_k(t - h_k(t)) < \frac{3}{2}, \lim_{t \to \infty} f(t) = 0\), then for any solution \(x\) of (5) \(\lim_{t \to \infty} x(t) = 0\).

THEOREM 3. If

\[
\omega \gamma_3 < \beta_1, \quad \gamma_2 X^* + \mu_y > \beta_2 \frac{X^*}{\eta_1 + X^*},
\]

then the equilibrium \((X^*, 0)\) is locally asymptotically stable.

Proof. Look \cite{2}.

3.3. Global stability of the equilibrium \((X^*, 0)\). To analyse the global stability of the equilibrium \((X^*, 0)\) for the system (1) we will use the following lemmas:

LEMMMA 3. \cite{14} Consider the scalar linear delay differential equation

\[
\dot{x}(t) = -a(t) + b(t)x(h(t)).
\]

If \(a(t) \geq a_0 > 0\), \(|b(t)| \leq qa(t), 0 < q < 1\), then the above equation is asymptotically stable.

LEMMMA 4. \cite{1} Consider the following equation and inequalities

\[
\begin{align*}
\dot{x}(t) + a(t)x(t) - \sum_{k=1}^{m} a_k(t)x(h_k(t)) &= 0, \quad t \geq 0, \\
\dot{y}(t) + a(t)y(t) - \sum_{k=1}^{m} a_k(t)y(h_k(t)) &\leq 0, \quad t \geq 0, \\
\dot{z}(t) + a(t)z(t) - \sum_{k=1}^{m} a_k(t)z(h_k(t)) &\geq 0, \quad t \geq 0,
\end{align*}
\]

where \(a_k(t) \geq 0\). Denote by \(X(t, s)\) the fundamental function of equation (7). Then \(X(t, s) > 0, 0 \leq s \leq t\). Moreover, for any \(t_0\) equality \(x(t) = y(t) = z(t), t \leq t_0\) implies \(y(t) \leq x(t) \leq z(t), t > t_0\), were \(x, y, z\) are solutions of (7), (8), (9) respectively.

**Lemma 5.** Consider ODE and corresponding differential inequalities:

\[
\dot{x}(t) = f(t, x(t)), \quad t \geq t_0,
\]
\[
\dot{y}(t) \leq f(t, y(t)), \quad t \geq t_0,
\]
\[
\dot{z}(t) \geq f(t, z(t)), \quad t \geq t_0,
\]

where \( f(t, u) \) is a continuous function. If \( y(t_0) \leq x(t_0) \leq z(t_0) \), then \( y(t) \leq x(t) \leq z(t), t > t_0 \).

**Theorem 4.** If condition (4) holds then for any solution \((x, y)\) of the system (1) we have \( \lim_{t \to \infty} x(t) = X^* \), \( \lim_{t \to \infty} y(t) = 0 \), which means that \((X^*, 0)\) is a global attractor for all solutions of system (1).

The theorem was proven in [2].

By definition local stability and global attractivity imply global stability. Hence we have the following result.

**Theorem 5.** Suppose conditions (4) and (6) hold. Then the equilibrium \((X^*, 0)\) is globally asymptotically stable.

### 4. Results.

In order to verify the mathematical model of the biological system, it is widely accepted to estimate the conditions defined in the presented theorems based on the parameters set taken from the biological and medical literature (these parameters are taken from the Table 1).

System (1) has an equilibrium \((X^*, 0)\) where \( X^* = Ke^{-\frac{\gamma\alpha}{\beta}} \). Substituting the relevant parameters from the Table 1, we receive that \( X^* = 1.5 \times 10^5 \exp(-\frac{4000 \times 0.000014}{0.03}) = 23196 \), which means that the cancer cells number will not exceed the 23196 cells at a constant daily dose of 400 mg imatinib.

With increasing the daily dose of imatinib to 600 mg, the amount of cancer cells decreases to almost 10000. These numerical results are consistent with the data given in [22].

In order to check the local and global stability conditions, we substitute parameters to the expression \( \gamma_1 + \beta_2 < \mu_y \). The following numerical inequality is received: \( 13 \times 0.005 + 4.1 \times 10^{-4} < 0.05 \). The obtained result supports that the model (1)-(2) has a positive stable local and global equilibrium if condition (4) is satisfied.

### 5. Conclusion.

We have proposed mathematical model for the treatment of Chronic Myelogenous Leukemia with imatinib and IFN-α to overcome immune suppressive side effects of imatinib, prolonging the chronic phase of the disease. Our model is based on the work of Moore&Li [20]. We modified their system of equations, omitting the equation for the naive T-cells
and analyzed obtained two-compartment model consisting of cancer CML cells and immune killer CTL cells only. This model consists of a system of two nonlinear delay differential equations with logarithmic and rational nonlinearities.

In order to describe the influence of two types of the treatment component (imatinib and IFN-α) on the model compartments, we introduced delays and explained a biological motivation for it. The proposed two-compartment model allows to evaluate directly the expected steady states of the system. The existence of unique global solutions for CML model was defined. Explicit local and global stability conditions for the unique nontrivial equilibrium were obtained by applying the method of delays in differential inequalities and linear stability theory of nonlinear delay differential equations. The numerical results show that our model replicates the averaged behavior of the combined treatment.

Mathematical expressions obtained in this paper represent the interaction between cell populations of cancer cells (CML) and immune cells (CTL). The dose of IFN-α has inhibitory effect on the value of \( x(t) \), as seen from the formula (4). As a result of calculations shown in the Section 4, we observe that \( y(t) \) inhibits \( x(t) \), destroying itself to 0 in spite of constant addition of IFN-α.

The prolonged treatment with imatinib confers resistance to the induction of cancer cells apoptosis. This resistance should be corrected by inhibition of cancer cells with Bcr-Abl. It is very important to note, that IFN-α acts synergically with imatinib, which forms the basis for using it in the combination with imatinib in clinical trials. In the [22], Nanda et al. suggest combined therapy of imatinib and cytarabine onto cancer blood cells, naive T cells and effector T cells and used optimal control theory to analyze the effects of this treatment.

In the future, in order to avoid its resistance of IFN-α, it looks reasonable to change the IFN-α to another type of the treatment as a combination to the imatinib [6]. The best form of combination therapy that leads to improved survival in patients, remains to be seen in future by mathematical models and clinical trials.

REFERENCES


M. Montoya et al., Type I interferons produced by dendritic cells promote their phenotypic and functional activation, Blood 99 (2002) 3263 – 3271.


6. Parameter estimation. Table 1. List of all parameters.

<table>
<thead>
<tr>
<th>Param</th>
<th>Physical Interpretation</th>
<th>Estimated value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau)</td>
<td>the delay for development CTL cells</td>
<td>7 [days]</td>
<td>[20]</td>
</tr>
<tr>
<td>(\theta)</td>
<td>maximal period to react to imatinib</td>
<td>20 [days]</td>
<td>[27]</td>
</tr>
<tr>
<td>(\mu_y)</td>
<td>death rate of effector T cells</td>
<td>0.06 [0,0.5] [days(^{-1})]</td>
<td>[17]</td>
</tr>
<tr>
<td>(\eta_1)</td>
<td>the saturation effect of CML cells in the lymph nodes</td>
<td>100 [cells/ml(^{-1})]</td>
<td>[20]</td>
</tr>
<tr>
<td>(\eta_2)</td>
<td>the saturation effect of immune cell recruitment by cancer cells</td>
<td>(2 \times 10^7) [cells/ml(^{-1})]</td>
<td>[17]</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>growth rate of CML cancer cells in the form of a Gompertz law</td>
<td>0.03 [0;0.5] [days(^{-1})]</td>
<td>[20]</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>the change in the effector T cell ((y(t))) population due to encounters with CML antigen</td>
<td>(0.41 \times 0.001) [days(^{-1})]</td>
<td>[9]</td>
</tr>
<tr>
<td>(\gamma_1)</td>
<td>loss of CML cancer cells due to encounters with the effector T cells</td>
<td>0.005 [days(^{-1}) [cells/ml(^{-1})]</td>
<td>[20]</td>
</tr>
<tr>
<td>(\gamma_2)</td>
<td>loss of CTL cells due to these encounters between CTL and CML cancer cells</td>
<td>0.005 [days(^{-1}) [cells/ml(^{-1})]</td>
<td>[9]</td>
</tr>
<tr>
<td>(\gamma_3)</td>
<td>factor using Imatinib treatment</td>
<td>0.00014 [mg](^{-1})</td>
<td>Estimated</td>
</tr>
<tr>
<td>(\gamma_4)</td>
<td>factor using IFN-a treatment</td>
<td>0.005 [mg](^{-1})</td>
<td>Estimated</td>
</tr>
<tr>
<td>(\omega)</td>
<td>once-daily dose of imatinib</td>
<td>400 – 800 [mg/day]</td>
<td>[8]</td>
</tr>
<tr>
<td>(\ln a)</td>
<td>the IFN-a dose</td>
<td>13 [mg/days] (90 [mg] weekly)</td>
<td>[25]</td>
</tr>
<tr>
<td>(K)</td>
<td>Constant, the maximum possible concentration of CML</td>
<td>([1.5 \times 10^5, 4 \times 10^5]) [cells/ml]</td>
<td>[20]</td>
</tr>
</tbody>
</table>
**Abstract.** In the paper the initial value problem for the scalar linear functional differential equation (LFDE) of delay type

\[
\dot{x}(t) = (a_0 + a_1 t)x(t - (1 - \varepsilon t)) + (b_0 + b_1 t)x(t) + \bar{f}(t), \quad t \in \mathbb{R}, \quad x(0) = x_0,
\]

where \(\bar{f}(t) = \sum_{n=0}^{P} \bar{f}_n t^n\) is considered. This equation is investigated with the use of the method of polynomial quasisolutions based on the representation of an unknown function in the form of polynomial \(x(t) = \sum_{n=0}^{N} x_n t^n\). As a result of substitution of this function into equation (\(*\)), there appears a residual \(\Delta(t) = O(t^N)\), for which an exact analytical representation has been obtained. In turn, this allows one to find the unknown coefficients \(x_n\) and consequently the polynomial quasisolution \(x(t)\).

**Key Words.** Functional differential equations, variable delay, initial value problem, polynomial quasisolutions.

**AMS(MOS) subject classification.** 34K06, 34K10

1. **Introduction.** In the process of investigation of many applied problems, especially related to such problem areas as electric radio communication, automatics, biology, immunology, economics, etc., differential-delay equations (or, in other terms, functional differential equations (FDEs)) are used as the mathematical models. The principal difference of FDEs from ordinary differential equations is that FDE allow one to take account of the influence of the previous state of the process on its subsequent development.

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* Irkutsk State Technical University, Irkutsk 664074, Russia
† Baikal National University of Economics and Law, Irkutsk 664003
As a rule, such equations are nonlinear. But since it is comparatively easier to investigate linear FDEs, and the theory of such equations is developed rather well, in the process of solving various theoretical and especially applied problems, nonlinear equations are replaced with linear ones. Nowadays, the class of FDEs with constant deviation of the argument – linear differential-difference equations (LDDEs) – may be considered as the most investigated. In this respect, we have to note fundamental works of A.D. Mishkis [7], E. Pinni [8], R. Bellman and K.L. Cook [2], N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina [1].

In the process of investigation of LDDEs, we consider mainly the following two initial-value problems: the initial-value problem with the initial function, when the initial function, which generates the solution of the object problem, is somehow given on the initial set, and the initial-value problem with the initial point, when the classical solution is to be found whose substitution into the initial equation turns it into an identity.

It is known that the solution of the initial-value problem with the initial function at the points multiple to the delay has discontinuities of the derivative. But in course of investigation of several applied problems described by LDDEs the structure of solution observed possesses sufficient smoothness. So, investigation of the initial-value problem with the initial point for such equations represents an urgent problem of substantial importance from the application viewpoint. When coefficients of LDDE are constant or have some special representations, application of Euler’s method allows one to obtain analytical solutions, which are generated by roots of the characteristic quasi-polynomial, for the initial-value problem with the initial point. The authors have no information on the solvability conditions for LDDEs in the class of analytical functions in the case when parameters of the equation are variable. When coefficients of the equation are represented by polynomials, a method of polynomial quasi-solutions (PQ-solutions) has been proposed in [3]-[6]. It implies that a formal solution of the form of polynomial $x(t) = \sum_{n=0}^{N} x_n t^n$ is introduced into the investigation. Hence the term ”polynomial quasi-solution” (PQ-solution) is understood in the sense that after its substitution into the initial-value problem there appears a residual $\Delta(t) = O(t^N)$, for which an exact analytical formula has been obtained. On the other hand, this allows one to find unknown coefficients $x_n$ and, consequently, the PQ-solution itself. The method of PQ-solutions has initially been oriented to investigation of LDDEs. Further investigations has given evidence that this method may be applied also to some functional differential equations with variable delay.

The paper is devoted to finding out the conditions of existence of PQ-
solutions for the initial-value problem with the initial point for some linear FDEs of delay type with slowly varying delay, as well as to techniques of their obtaining.

2. Statement of the problem. Consider the following initial value problem for a delay type linear FDE

\[(1) \; \dot{x}(t) = (a_0 + a_1 t)x(t - (1 - \varepsilon t)) + (b_0 + b_1 t)x(t) + \bar{f}(t), \; t \in R, \; x(0) = x_0,\]

where

\[(2) \; a(t) = a_0 + a_1 t, \; b(t) = b_0 + b_1 t, \; \bar{f}(t) = \sum_{n=0}^{F} \bar{f}_n t^n; \; |\varepsilon| \ll 1.\]

Representing

\[(3) \; x(t - (1 - \varepsilon t)) = x(\sigma t - 1), \; \sigma = 1 + \varepsilon,\]

rewrite problem (1) as follows

\[(4) \; \dot{x}(t) = (a_0 + a_1 t)x(\sigma t - 1) + (b_0 + b_1 t)x(t) + \bar{f}(t), \; t \in R, \; x(0) = x_0,\]

Introduce the polynomial

\[(5) \; x(t) = \sum_{n=0}^{N} x_n t^n, \; t \in R.\]

In this case,

\[(6) \; \dot{x}(t) = \sum_{n=0}^{N} nx_n t^{n-1},\]

\[(7) \; x(\sigma t - 1) = x_0 + x_1 (\sigma t - 1) + x_2 (\sigma t - 1)^2 + \ldots + x_N (\sigma t - 1)^N = \sum_{n=0}^{N} \tilde{x}_n t^n;\]

\[(8) \; \tilde{x}_n = \sigma^n \sum_{i=0}^{N-n} \tilde{C}_n^{n+i} x_{n+i}, \; \tilde{C}_n^{n+i} = (-1)^i \frac{(n+i)!}{i! n!}.\]

Let us perform analysis of dimensions of the polynomials obtained as a result of substitution of (2), (5)–(7) into (4). The derivative \(\dot{x}(t)\) is represented as a polynomial of degree \(N - 1\), and the function \(\bar{f}(t)\) has the degree
Hence, by analyzing powers for the equal powers of $t$ in the equality (4), we conclude: the last coefficient $x_N$ in (5) is defined by the last given coefficient $\bar{f}_F$ in (2) if $N = F + 1$.

Define the function $f(t)$ in the form

$$f(t) = \sum_{n=0}^{F} f_n t^n + \Delta(t),$$

where $f_i = \bar{f}_i$, $i = 0, F = N - 1$ are known coefficients (furthermore, the case when $f_i = 0$, $i = 0, F$ is not excluded), and the residual $\Delta(t) = f_N t^N + f_{N+1} t^{N+1}$, $f_N$ and $f_{N+1}$ are some unknown coefficients.

Having put $N = F + 1$ in (5), consider the following initial-value problem in terms of denotations of problem (4):

$$\dot{x}(t) = a(t)x(\sigma t - 1) + b(t)x(t) + f(t), \ t \in R; \ x(0) = x_0$$

**Definition 1.** The problem (10) is said to be coordinated with respect to the dimension of the polynomials with respect to problem (1) or (4).

When substituting (5)–(7) and (9) into (4) and equating the coefficients of the same powers of $t^n$, we have

$$nx_n = \begin{cases} 
a_0 \ddot{x}_0 + b_0 x_0 + f_0, & n = 1; \\
\sum_{i=0}^{1} (a_i \dddot{x}_{n-1-i} + b_i x_{n-1-i}) + f_{n-1}, & 2 \leq n \leq N; \\
0 = \sum_{i=0}^{1} (a_i \dddot{x}_{N-i} + b_i x_{N-i}) + f_N, & n = N + 1; \\
0 = a_1 \dddot{x}_N + b_1 x_N + f_{N+1}, & n = N + 2. \end{cases}$$

**Remark 1.** Since the degree of the polynomial $x(t)$ is $F + 1$, it is possible to choose the degree of the polynomial $\bar{f}(t)$ in (2) depending on the desired degree of the polynomial $x(t)$, by adding the corresponding number of zero terms to $\bar{f}(t)$.

**Definition 2.** If there exists a polynomial of degree $N = F + 1$

$$x(t) = \sum_{n=0}^{N} x_n t^n, \ t \in R,$$

which identically satisfies problem (10), then this polynomial will be called the polynomial quasi-solution (PQ-solution) of problem (1).
Let us find the conditions under which unknown coefficients \( f_N \) and \( f_{N+1} \) may be defined so that the polynomial (5) represent the identical solution of the initial-value problem (10) and, consequently, on account of Definition 1, this polynomial is the polynomial quasi-solution of the initial-value problem (4).

3. Principal results. Let us express unknown coefficients \( x_n; n = 1, N \) of polynomial (4) via unknown coefficients \( f_N \) and \( f_{N+1} \). To this end let us substitute into (11) the values of \( \tilde{x}_n \), defined according to (8) and rewrite formulas (11), while assuming that \( n = N + 2 - l, l = 0, N + 1 \), as follows:

\[
\begin{align*}
  a_{N,N}x_N + f_{N+1} &= 0, \\
  a_{N-1,N-1}x_{N-1} + a_{N-1,N}x_N + f_N &= 0, \\
  a_{N-2,N-2}x_{N-2} + a_{N-2,N-1}x_{N-1} + a_{N-2,N}x_N + f_{N-1} &= 0, \\
  \vdots \\
  a_{N-s,N-s}x_{N-s} + a_{N-s,N-s+1}x_{N-s+1} + \ldots + a_{N-s,N}x_N + f_{N-s+1} &= 0,
\end{align*}
\]

(13)

where

\[
\begin{align*}
  a_{N,N} &= a_1\sigma^N + b_1; \\
  a_{N-1,N-1} &= a_1\sigma^{N-1} + b_1, \\
  a_{N-1,N} &= a_0\sigma^N + a_1\sigma^{N-1}\bar{C}_N^1 + b_0; \\
  a_{N-2,N-2} &= a_1\sigma^{N-2} + b_1, \\
  a_{N-2,N-1} &= a_0\sigma^{N-1} + a_1\sigma^{N-2}\bar{C}_N^1 + b_0, \\
  a_{N-2,N} &= a_0\sigma^{N-1}\bar{C}_N^1 + a_1\sigma^{N-2}\bar{C}_N^2 - N; \\
  a_{N-s,N-s} &= a_1\sigma^{N-s} + b_1, \\
  a_{N-s,N-s+1} &= a_0\sigma^{N-s+1} + a_1\sigma^{N-s}\bar{C}_N^{s+1} + b_0, \\
  a_{N-s,N-s+2} &= a_0\sigma^{N-s+1}\bar{C}_N^{s+1} + a_1\sigma^{N-s}\bar{C}_N^{s+2} - (N - s + 2), \\
  a_{N-s,N-s+k} &= a_0\sigma^{N-s+1}\bar{C}_N^{s+k-1} + a_1\sigma^{N-s}\bar{C}_N^{s+k}, \ 3 \leq k \leq s.
\end{align*}
\]

Hence the following lemma is valid.
Lemma 1. [4] Let $a_1\sigma^n + b_1 \neq 0$, $n = \overline{1,N}$. Hence the common term of the sequence $\{x_n\}^N_{n=1}$, which is generated by relations (13), is defined by the formula

\begin{equation}
 x_{N-s} = \sum_{i=0}^{s} K_{N-s,N-s+i} f_{N-s+1+i},
\end{equation}

where

\begin{equation}
 K_{N-s,N-s} = -\frac{1}{a_{N-s,N-s}};
\end{equation}

\begin{equation}
 K_{N-s,N-r} = -\frac{1}{a_{N-s,N-s}} \sum_{i=1}^{s-r} a_{N-s,N-s+i} K_{N-s+i,N-r}, \ s \geq r + 1.
\end{equation}

Now turn back to formulas (11). For $n = 1$ on account of (8) we have

\[ x_0 = a_0 \tilde{x}_0 + b_0 x_0 + f_0 = a_0 \sigma \sum_{i=0}^{N} (-1)^i x_i + b_0 x_0 + f_0. \]

Rewrite this equality in the form

\[ (a_0 \sigma + b_0) x_0 + f_0 = (a_0 \sigma + 1) x_1 - a_0 \sigma x_2 + \ldots + (-1)^{N+1} a_0 \sigma x_N. \]

Introduce the following denotations:

\[ Q_0 = (a_0 \sigma + b_0) x_0 + f_0, \ Q_1 = a_0 \sigma + 1, \ldots, Q_n = (-1)^{n+1} a_0 \sigma, \ n = 2, N; \]

Hence

\begin{equation}
 Q_0 = Q_1 x_1 + Q_2 x_2 + \ldots + Q_N x_N.
\end{equation}

While putting $s = N, N-1, \ldots, 0$, express relations (14) in the form

\[ x_0 = \sum_{i=0}^{N} \sum_{k=0}^{N-1} K_{0,i} f_{i+1} = \sum_{k=0}^{N} f_{N-k} K_{0,N-k}, \]

\[ x_1 = \sum_{i=0}^{N-1} \sum_{k=0}^{N-2} K_{1,i} f_{i+2} = \sum_{k=0}^{N-2} f_{N-k} K_{1,N-k}, \]

\[ x_2 = \sum_{i=0}^{N-2} \sum_{k=0}^{N-3} K_{2,i} f_{i+3} = \sum_{k=0}^{N-3} f_{N-k} K_{2,N-k}, \]

\[ \ldots \]

\[ x_m = \sum_{i=0}^{N-m} \sum_{k=0}^{N-m} K_{m,i} f_{i+m+1} = \sum_{k=0}^{N-m} f_{N-k} K_{m,N-k}, \]

\[ \ldots \]

\[ x_N = K_{N,N} f_{N+1}. \]
Substitute the coefficients $x_n$ obtained into formula (17)

$$Q_0 = Q_1 \sum_{k=0}^{N-1} f_{N+1-k}K_{1,N-k} + Q_2 \sum_{k=0}^{N-2} f_{N+1-k}K_{2,N-k} + \cdots$$

$$\cdots + Q_m \sum_{k=0}^{N-m} f_{N+1-k}K_{m,N-k} + \cdots + Q_N K_{N,N} f_{N+1}.$$  

Now transform this formula, while grouping the addends at similar coefficients $f_j$.  

For $f_{N+1}$ we have

$$Q_1 K_{1,N} + Q_2 K_{2,N} + \ldots + Q_m K_{m,N} + \ldots + Q_N K_{N,N};$$

for $f_N$

$$Q_1 K_{1,N-1} + Q_2 K_{2,N-1} + \ldots + Q_m K_{m,N-1} + \ldots + Q_{N-1} K_{N-1,N-1};$$

for $f_{N-1}$

$$Q_1 K_{1,N-2} + Q_2 K_{2,N-2} + \ldots + Q_m K_{m,N-2} + \ldots + Q_{N-2} K_{N-2,N-2};$$

for $f_{N-m}$

$$Q_1 K_{1,N-m-1} + Q_2 K_{2,N-m-1} + \ldots + Q_{N-m-1} K_{N-m-1,N-m-1};$$

for $f_2$

$$Q_1 K_{1,1}.$$  

These transformations lead to the formula

$$Q_0 = \sum_{k=0}^{N-1} \left( \sum_{i=1}^{N-k} Q_i K_{i,N-k} \right) f_{N+1-k}.$$  

While introducing the denotation

$$(18) \quad \tilde{K}_{0,N-k} = \sum_{i=1}^{N-k} Q_i K_{i,N-k},$$

we obtain

$$(19) \quad Q_0 = \sum_{k=0}^{N-1} \tilde{K}_{0,N-k} f_{N+1-k}.$$  

The results obtained allow us to formulate and prove the following theorem.
Theorem 1. Let for the initial-value problem
\( (20) \quad \dot{x}(t) = a(t)x(t - (1 - \varepsilon t)) + b(t)x(t) + \bar{f}(t), \ t \in \mathbb{R}, \ x(0) = x_0 \)
conditions (2) hold, and \( a_1(1 + \varepsilon)^n + b_1 \neq 0, \ n = 1, N. \)

Hence this problem has a unique PQ-solution of the form \( x(t) = \sum_{n=0}^{N} x_n t^n, \)
\( N = F + 1 \) with the residual \( \Delta(t) = f_N t^N + f_{N+1} t^{N+1}, \) when the determinant
\[ D = \begin{vmatrix} K_{0,N} & K_{0,N-1} \\ \tilde{K}_{0,N} & \tilde{K}_{0,N-1} \end{vmatrix}, \]
where \( K_{0,N-s} \) and \( \tilde{K}_{0,N-s}, \ s = 0, 1 \) are defined, respectively, by formulas (15), (16) and (18), is nonzero.

Proof. On account of (3) and the condition of the theorem, the expression \( a_1(1 + \varepsilon)^n + b_1 \neq 0, \ n = 1, N. \) In this case, Lemma 1 is valid. Hence for \( s = N \) we can write down formula (14) as follows:
\[ x_0 = \sum_{k=0}^{N} K_{0,N-k} f_{N+1-k}. \]

Let us jointly write down equalities (19) and (21), while identifying the addends, which contain unknown coefficients \( f_N \) and \( f_{N+1}, \)
\[ x_0 = K_{0,N} f_{N+1} + K_{0,N-1} f_N + \sum_{k=2}^{N} K_{0,N-k} f_{N+1-k}, \]
\[ Q_0 = \tilde{K}_{0,N} f_{N+1} + \tilde{K}_{0,N-1} f_N + \sum_{k=2}^{N-1} \tilde{K}_{0,N-k} f_{N+1-k}. \]
Now rewrite these relations in the form of a linear system of algebraic equations with respect to the coefficients \( f_{N+1} \) and \( f_N \)
\[ \begin{cases} 
K_{0,N} f_{N+1} + K_{0,N-1} f_N = W_0, \\
\tilde{K}_{0,N} f_{N+1} + \tilde{K}_{0,N-1} f_N = W_1,
\end{cases} \]
where
\[ W_0 = x_0 - \sum_{k=2}^{N} K_{0,N-k} f_{N+1-k}, \quad W_1 = Q_0 - \sum_{k=2}^{N-1} \tilde{K}_{0,N-k} f_{N+1-k}. \]
With respect to the linear system (22) let us write down the determinants

\[ D_1 = \begin{vmatrix} K_{0,N} & W_0 \\ \tilde{K}_{0,N} & W_1 \end{vmatrix} \quad D_2 = \begin{vmatrix} W_0 & K_{0,N-1} \\ W_1 & \tilde{K}_{0,N-1} \end{vmatrix}, \]

Since – according to the condition of the theorem – the determinant \( D \) of system (22) is nonzero, when solving this system by method, we obtain the coefficients \( f_N \) and \( f_{N+1} \):

\[ f_N = \frac{D_1}{D}, \quad f_{N+1} = \frac{D_2}{D}. \]

In this case, the PQ-solution’s coefficients \( x_n \) are sequentially determined from the chain of equalities (13), and, consequently, the PQ-solution of the initial-value problem (20) is determined in the form (12). The coefficients \( f_N \) and \( f_{N+1} \) obtained allow us to explicitly write out the expression for the residual, i.e.

\[ \Delta = f_N t^N + f_{N+1} t^{N+1} = \frac{D_1}{D} t^N + \frac{D_2}{D} t^{N+1}. \]

This proves the theorem.

4. Example. Consider the initial-value problem

\[ \dot{x}(t) = x(t - (1 - 0.2t)), \quad t \in R, \quad x(0) = x_0 = 1, \]

which – on account of (4) – may be rewritten in the form:

\[ \dot{x}(t) = x(1.2t - 1), \quad t \in R, \quad x(0) = x_0 = 1. \]

Let us try to find the PQ-solution of this problem in the form

\[ x(t) = \sum_{n=0}^{N} x_n t^n. \]

Due to Definition 1 the initial-value problem

\[ \dot{x}(t) = x(1.2t - 1) + \Delta(t), \quad t \in R, \quad x(0) = x_0 = 1, \]

where \( \Delta(t) = f_N t^N \), is coordinated with problem (23) as far as the dimension of the polynomials is concerned.

On the basis of results of Theorem 1 we have obtained the following formulas defining the PQ-solutions and the residuals for \( N = 4, 5, 6 \):

\[ x_4(t) = 1 + 0.5738t + 0.1854t^2 + 0.0448t^3 + 0.0071t^4, \]
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\[ \Delta_4(t) = -0.0147t^4; \]
\[ x_5(t) = 1 + 0.5740t + 0.1848t^2 + 0.0445t^3 + 0.0089t^4 + 0.0012t^5; \]
\[ \Delta_5(t) = -0.0030t^5; \]
\[ x_6(t) = 1 + 0.5740t + 0.1849t^2 + 0.0443t^3 + 0.0089t^4 + 0.0015t^5 + 0.0001t^6; \]
\[ \Delta_6(t) = 0.0005t^6. \]

Fig. 1 shows the PQ-solutions and the residuals of the initial-value problem (23), which correspond to these PQ-solutions. Obviously, side by side with growth of the polynomial’s degree, the PQ-solutions obtained have the tendency of mutual attraction when the residual decreases.

\[ F_{t} \]

Now introduce the following definition.

**Definition 3.** \( \varepsilon \)-attractability of PQ-solutions on some interval \([t_0, t_1]\) is understood as the property of reciprocal attraction of the sequence of PQ-solutions generated by some increase in degree \(N\) of the polynomial of the PQ-solutions, in the sense that there exists \(N_\varepsilon\) such that for \(N \geq N_\varepsilon\) and for a given \(\varepsilon\)

\[ |x_{N+i}(t) - x_{N+i-1}(t)| < \varepsilon, \ i = 1, 2 \ldots k, \ \forall t \in [t_0, t_1]. \]

According to this definition, for \(\varepsilon = 0.005\) PQ-solutions \(x_n(t)\) of problem (23) on the interval \([-1, 1]\) possess the property of \(\varepsilon\)-attractability.
REFERENCES

STABILITY OF EQUILIBRIUM OF DIFFERENTIAL SYSTEMS WITH QUADRATIC RIGHT-HAND SIDE

I. A. DZHALLADOVA

Abstract. A non-linear system of differential equation with quadratic right-hand part is considered. One of the basic methods for stability analysis of trivial solution of non-linear systems are methods of liberalization and stability studying based on stability results for linear approximation system. If the trivial solution of linear approximation system is asymptotically stable, then trivial solution of the initial non-linear system will be also stable in a sufficiently small neighborhood of the equilibrium. Differential systems with the quadratic right-hand side are considered in the report. The systems are presented in the uniform vector-matrix form. An algorithm of estimation of stability region in the phase space of trivial equilibrium of the system is proposed.

Key Words. Non-linear stochastic differential delay equations; asymptotic stability of zero solution; stochastic Lyapunov functionals

AMS(MOS) subject classification. 34K20; 34K40; 34K50; 34K6; 34K12

1. Introduction. We consider the non-linear system of differential equations with quadratic non-linearity and the asymptotically stable linear part. One of the first methods in researching the stability of the zero solution of non-linear systems should be considered as methods of liberalization and stability analysis based on the stability of a system of linear approximation. Such types of works were carried out during the second half of the last century, for example, in [1-3]. If the zero solution of the linear approximation is asymptotically stable then in a sufficiently small neighborhood of the equilibrium will be stable and trivial solution of the original non-linear system. In this article we pay attention to systems with a quadratic term. System is written in a special uniform vector-matrix form. An algorithm for evaluation

* Chair of Higher Mathematics Kiev National University of Economics Kyiv, Ukraine
of the stability region in the phase space of the zero equilibrium system with a quadratic right-hand part. The algorithm is based on the second method of Lyapunov quadratic function of the form [4]. Further the system with a quadratic part with one fixed delay is studying. We study the stability of the stationary state of the system with delay. The apparatus of investigation we choose the Lyapunov second method of quadratic functions of the form with the B.S. Razumihin condition. Theory and applications of functional differential equations form an important part of modern non-linear dynamics. Those equations are natural mathematical models for various real life phenomena where the after-effects are intrinsic features of their functioning. In recent years functional differential equations have been used to model processes in diverse areas such as population dynamics and ecology, physiology and medicine, economics and other natural sciences. In many of the models the initial data and parameters are subject to random perturbations, or the dynamical systems themselves represent stochastic processes. This leads to consideration of stochastic functional differential equations.

2. Main Results.

2.1. Systems without delay. Let’s consider the following a system with non-linearity of the particular (individual) form, namely systems with quadratic right-hand side, written in vector-matrix form [7-9]

\[
x(t) = 3D Ax(t) + X^T(t)Bx(t)
\]

where \( B = 3D \{ B_1, B_2, ..., B_n \}^T \) rectangle \( n^2 \times n \) matrix consisting symmetric matrix \( n \times n \) matrix \( B_i \)

\[
B_i := 3D \begin{bmatrix}
b_{11} & b_{12} & \ldots & b_{1n} \\
b_{21} & b_{22} & \ldots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \ldots & b_{nn}
\end{bmatrix},
\]

\( X^T = 3D \{ X_1(t), X_2(t), ..., X_n(t) \} \)- rectangular \( n \times n^2 \) - matrix consisting of square \( n \times n \) matrix \( X_i(t) \), in which the \( i \)-x rows are vectors \( x(t) \), the other elements are zero, i.e.

\[
X_{ii}(t) := 3D \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
x_{1}(t) & x_{2}(t) & \ldots & x_{n}(t)
\end{bmatrix},
\]
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\[
X_1(t) := 3D \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.
\]

Vector and matrix norms define next form

\[
| x(t) | = 3D \left\{ \sum x_i^2(t) \right\}^{\frac{1}{2}},
\]

\[
| B | = 3D \left\{ \lambda_{\text{max}}(B^T B) \right\}^{\frac{1}{2}},
\]

where \( \lambda_{\text{max}}(\bullet) \) and \( \lambda_{\text{min}}(\bullet) \) extreme eigenvalues of the corresponding symmetric matrices.

Let the matrix of the linear part of (1) is asymptotically stable. Then, as it follows from the theory of the stability of the linear approximation [10], the zero solution of a non-linear system is also asymptotically stable. If it’s taken as a quadratic form \( V(x) = 3D x^T H x \) of the Lyapunov function, then its derivative with respect to the system (1) has the form

\[
(2) \quad \frac{dV(x(t))}{dt} = 3D x^T (A^T H + HA) + (B^T X(t)H + HX^T B) x(t).
\]

If matrix \( A \) is asymptotically stable, then for any positive definite matrix, there is Lyapunov matrix equation

\[
A^T H + HA = 3D - C
\]

has a unique solution - a positive definite matrix [10]. Taking in account that is the solution of the this Lyapunov equation, we obtain

\[
\frac{dV(x(t))}{dt} = 3D - x^T(t) [C - (B^T X(t)H + HX^T(t)B)] x(t).
\]

The stability domain of the zero equilibrium is the interior surface of the level of the Lyapunov function, which lies within the area

\[
G_0 = 3D \left\{ x \in \mathbb{R}^n : C - B^T X H - HX^T B > \Theta \right\},
\]

where the symbol denotes

\[
C - B^T X H - HX^T B > \Theta,
\]

is positive definition of matrix.
Let’s replace this condition with more ”rough.” Since, by the selected matrix and vector norms it will be implemented

\[ |X(t)| = 3D |x(t)| \]

then the total derivative of the Lyapunov function is satisfied

\[ \frac{dV(x(t))}{dt} \leq [\lambda_{\text{min}}(C) - 2 |H| |B| |x(t)|] |x(t)|^2, \]

Let’s denote

\[ G_0 = 3D \left\{ x \in \mathbb{R}^n : |x| < \frac{\lambda_{\text{min}}(C)}{2 |H| |B|} \right\}, \]

Then the regions of ”guarantee” stability has the form

\[ G_{r_0} = 3D \max \{ G_r : G_r \subset G_0 \}, \]

\[ G_r = 3D \{ x \in \mathbb{R}^n : x^T H x < r^2 \}. \]

If it follows from this dependence, to determine the ”maximum” of sustainability it should be placed inside a sphere of radius

\[ R = 3D \frac{\lambda_{\text{min}}(C)}{2 |H| |B|} \]

e ellipse \( x^T H x = 3Dr^2 \) and ”stretch” \( r \to \infty \) as long as the ellipse touches the sphere.

We obtain an estimate of the convergence of solutions, the initial position of which is to ”guarantee of stability.”

**Theorem 1.** Suppose the matrix of the linear part of system (1) asymptotically stable. Then the trivial solution of this system equation is asymptotically stable and for your solutions with initial conditions

\[ (x(0)) < \frac{\gamma(H)}{2|B| \varphi(H)}, \varphi(H) = 3D \frac{\lambda_{\text{max}}(H)}{\lambda_{\text{min}}(H)}, \gamma(H) = 3D \frac{\lambda_{\text{min}}(C)}{\lambda_{\text{max}}(H)}, \]

are holds next estimates convergence

\[ |x(t)| \leq \frac{\gamma(H) |x(0)|}{[= \gamma(H) - 2 |B| \varphi(H) | |x(0)|] e^{0.5\gamma(H)t} + 2 |B| \varphi(H) |x(0)|}. \]

**Proof.** Assessment of convergence solutions, which are in the field of sustainability, will be obtained using a quadratic Lyapunov function \( V(x) = 3Dx^T H x \). Its total derivative with respect to the system has the form (2). As for the quadratic function \( V(x) = 3Dx^T H x \) holds two-sided inequality
\[
\lambda_{\min}(H)|x|^2 \leq V(x) \leq \lambda_{\max}(H)|x|^2,
\]
then (2) can be rewritten as
\[
\frac{dV(x(t))}{dt} \leq -\frac{\lambda_{\min}(C)}{\lambda_{\max}(H)} V(x(t)) + 2\lambda_{\max}(H)|B| V^{\frac{3}{2}}(x(t)) \frac{\lambda_{\min}(H)}{\lambda_{\min}(H)}.
\]
Using (3), we rewrite this expression in the form of
\[
\frac{dV(x(t))}{dt} \leq -\gamma(H) V(x(t)) + 2|B| \frac{\varphi(H)}{\lambda_{\min}(H)}.
\]
Divide it by \(V^{\frac{3}{2}}(x)\) and get
\[
V^{\frac{3}{2}}(x(t)) \frac{dV(x(t))}{dt} \leq -\gamma(H) V^{\frac{3}{2}}(x = (t)) + 2|B| \frac{\varphi(H)}{\lambda_{\min}(H)}.
\]
Denoting \(V^{\frac{3}{2}}(x(t)) = 3Dz(t)\), we get
\[
-2z(t) \frac{dt}{dt} \leq -\gamma(H) z(t) + 2|B| \frac{\varphi(H)}{\lambda_{\min}(H)},
\]
hence
\[
\frac{dz(t)}{dt} \geq \frac{1}{2} \gamma(H) z(t) - \frac{|B| \varphi(H)}{\lambda_{\min}(H)}.
\]
Solving the inequality (similar to the linear non-homogeneous equation), we obtain
\[
|z(t)| \geq |z(0) - 2 \frac{|B| \varphi(H)}{\sqrt{\lambda_{\min}(H)}} e^{0.5 \gamma(H)t} + 2 \frac{|B| \varphi(H)}{\gamma(H) \sqrt{\lambda_{\min}(H)}}.
\]
Substituting \(V^{\frac{3}{2}}(x(t)) = 3Dz(t)\), we get
\[
V^{-0.5}(x(t)) \geq [V^{-0.5}(x(0)) - 2 \frac{|B| \varphi(H)}{\sqrt{\gamma(H) \lambda_{\min}(H)}}] e^{0.5 \gamma(H)t} + 2 \frac{|B| \varphi(H)}{\sqrt{\lambda_{\min}(H)}}.
\]
Hence
\[
V^{-0.5}(x(t)) \geq \{[V^{-0.5}(x(0)) - 2 \frac{|B| \varphi(H)}{\gamma(H) \sqrt{\lambda_{\min}(H)}}] e^{0.5 \gamma(H)t} + 2 \frac{|B| \varphi(H)}{\sqrt{\lambda_{\min}(H)}}\}^{-1}.
\]
Using a two-sided inequalities of quadratic forms, we obtain
\[
\sqrt{\lambda_{\min}(H)} = x(t) \leq \frac{\gamma(H) \sqrt{\lambda_{\min}(H)} |x(0)|}{\{\gamma(H) - 2|B| \varphi(H)|x(0)|\} e^{0.5 \gamma(H)t} + 2|B| \varphi(H)|x(0)|}.
\]
Thus, for the solutions \(x(t)\) of (1) with initial conditions that are in the area \(x_0 \in G_0\), will estimate the convergence of (4).
2.2. Model problem. Let’s consider the scalar equations
\[ x(t) = 3D - ax(t) + bx^2(t) \]
with solutions
\[ x(t) = 3D \frac{ax(0)e^{-at}}{a - bx(0)[1 - e^{-at}]} . \]
Let’s consider the use of Lyapunov functions with function \( V(x) = 3Dx^2 \). For this function \( \lambda_{\text{min}}(H) = 3D\lambda_{\text{max}}(H) = 3D1 \). Full derivative has form
\[ \frac{dV(x(t))}{dt} = 3D - 2ax^2(t). \]
Estimates convergence (4) for solutions with initial conditions \( x(0) < \frac{a}{b} \) has a similar form
\[ x(t) \leq \frac{ax(0)}{[a - bx(0)] e^{at} + bx(0)} \rightarrow 0. \]
Therefore for this equations exact solutions coincide with estimates = given quadratic Lyapunov functions.

2.3. System on the plane. More positive results of the assessment of convergence of systems with quadratic right-hand side can be obtained by considering a system of the form (1) on the plane. It has the form
\[
\begin{align*}
x_1'(t) &= 3Da_{11}x_2(t) + a_{12}x_2(t) + b_{11}^1x_1^2(t) + 2b_{12}^1x_1x_2 + b_{22}^2x_2^2(t), \\
x_2'(t) &= 3Da_{21}x_1(t) + a_{22}x_2(t) + b_{11}^2x_1^2(t) + 2b_{12}^2x_1x_2 + b_{22}^2x_2^2(t).
\end{align*}
\]
By using notations
\[ A := 3D \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} = 09 \end{bmatrix}, \]
\[ B_1 := 3D \begin{bmatrix} b_{11}^1 & b_{12}^1 \\ b_{21} & b_{22}^2 \end{bmatrix}, \]
\[ B_2 := 3D \begin{bmatrix} b_{11}^2 & b_{12}^2 \\ b_{21} & b_{22}^2 \end{bmatrix}. \]
The last system can be rewritten in the following vector-matrix form
\[ x(t) = 3D \quad Ax(t) + X^T(t)Bx(t). \]
Then full derivative of the Lyapunov function in the system (8) has the form
\[
\frac{dV(x(t))}{dt} \leq \left[ \lambda_{\min}(C) - 2 \right] \left| H \right| \left| B \right| \left| x(t) \right| |x(t)|^2,
\]
where
\[
\left| H \right| = 3D\lambda_{\max}(H) = 3D\frac{1}{2} \left\{ (h_{11} + h_{22} + \sqrt{(h_{11} - h_{22}) + 4h_{12}^2}) \right\},
\]
\[
\lambda_{\min}(C) = 3D\frac{1}{2} \left\{ (c_{11} + c_{22} + \sqrt{(c_{11} - c_{22}) + 4c_{12}^2}) \right\},
\]
\[
\left| B \right| = 3D \left\{ \lambda_{\max}(B^TB) \right\}.
\]

Guaranteed stability region, as it follows from (4), will be the interior of the ellipse
\[
h_{11}x^2 + 2h_{12}xy + h_{22}y^2 \leq r_0^2.
\]

2.4. System with a dedicated linear part. In general, the system (1) with the linear part can be written as [4]
\[
x_i(t) = 3D \left[ a_i - \sum b_{ij}x_j(t) \right] x_i(t).
\]

Here $A$ - the square diagonal matrix with constant coefficients $A = 3D \left\{ a_n \right\}$, $B = 3D \left\{ B_1, B_2, ..., B_n \right\}^T$ - the rectangular matrix consisting of symmetric square matrices $B_i$, in which to place $i$ column is the vector $b_i^T = 3D(b_{i1}, b_{i2}, ..., b_{in})$.

$X^T = 3D \left\{ X_1(t), X_2(t), ..., X_n(t) \right\}$ - rectangular $n \times n^2$ - matrix, which consists of matrix $X_i(t)$, in which the $i$ rows are vectors $x(t)$ other elements zero.

Let’s suppose, that $\det B_0 \neq 0$ and
\[
B := 3D \begin{bmatrix}
    b_{11} & b_{21} & \cdots & b_{n1} \\
    b_{12} & b_{22} & \cdots & b_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{1n} & b_{2n} & \cdots & b_{nn},
\end{bmatrix},
\]

Then, as a rule, the interest for the searching is a singular point $x_0^T = 3D(x_1^0, x_2^0, ..., x_n^0)$ that is the solution of algebraic equations
\[
B_0x = 3Da, a^T = 3D(a_1, a_2, ..., a_n)
\]
and located in the first quadrant, i.e. $x_i^0 > 0$. 

After replacing $x(t) = 3Dy(t) + x_0$ and transformation, we get the system of equations of perturbations

\[
y'(t) = 3D\bar{A}y(t) + Y^T(t)By(t).
\]

Let’s suppose that the matrix defined in (10) is asymptotically stable, i.e. $\text{Re}\lambda_i(\bar{A}) < 0$. Then the singular point $x_0^T = 3D(x_0^1, x_0^2, ..., x_0^n)$ is asymptotically stable and the region of its stability can be assessed using a quadratic Lyapunov function $V(y) = 3Dy^THy$, which is symmetric positive definite matrix is a solution of the Lyapunov equation [5]

\[
\bar{A}^TH + H\bar{A} = 3DC.
\]

Here $C$ is an arbitrary, symmetric, positive definite matrix.

Taking the total derivative of a function in the system (10), considering that the matrix is a solution of the Lyapunov equation, we obtain

\[
\frac{dV(y(t))}{dt} \leq -[\lambda_{\text{min}}(C) - 2 \| H \| B \| y(t) \| \| y(t) \|^2],
\]

and in the case of asymptotic stability of the matrix $\bar{A}$, the guaranteed stability of the equilibrium area of the singular point is inside the ellipse $y^THy = 3Dr^2$ is inside the sphere $|y| 3DR$.

Denoting

\[
G_0 = 3D \left\{ y \in \mathbb{R}^n : |y| < \frac{\lambda_{\text{min}}(C)}{2 \| H \| B} \right\}
\]

we find that the area “guarantee” stability has the form [4]

\[
G_{r_0} = 3D\max \left\{ G_r : G_r \subset G_0 \right\},
\]

\[
G_r = 3D \left\{ y \in \mathbb{R}^n : x^THx < r^2 \right\}.
\]

As it follows from the relation ellipse $y^THy = 3Dr^2$. As it follows from the relation (10), ellipse should be placed inside a sphere of radius

\[
R = 3D\frac{\lambda_{\text{min}}(C)}{2 \| H \| B}
\]

and “stretched” $r \rightarrow \infty$ as long as the ellipse touches the sphere.

We obtain estimates convergence of solution initial state lay in ‘guaranty’ stability regions.
3. System with delay. Typically, the models of the economy and the environment inherent lag factor, defined "the time of puberty," or "the time of the decision." Are therefore more appropriate mathematical model describing the system of functional differential equations with delay [6-7]. One of the first mathematical models described by differential equations with constant delay, were the equation Hutchison and Voltaire [1-3].

3.1. Verhulst equation with delay. Verhulst equation displays the dynamics of population growth with saturation. Limited growth due to "internal competition." Note the following factor, specifying the model Verhulst. Competition generally occurs between the new population and the population born with retardation. In this case, population dynamics is determined by the equation Hutchison (1948), which has the form of a differential equation with delay

\[
\frac{dx(t)}{dt} = 3D \ ax(t) \left(1 - \frac{x(t-\tau)}{k}\right).
\]

The delay is due to the finite time required to achieve the "time of puberty."

Dynamical system described by equation (11) has two equilibrium $x(t) \equiv 0$ and $x(t) \equiv k$. It is easy to see that the linear approximation is given by the equation at the point $x = 3D0$ and points to the instability of the zero equilibrium. Consider the second point of rest $x(t) \equiv k$.

Draw linearization

\[
\frac{dx(t)}{dt} = 3D \ f(x(t), x(t-\tau))
\]

in the neighborhood $x(t) \equiv k$.

After transformation and substituting the corresponding values we get

\[
\frac{dx(t)}{dt} = 3D \ -a [x(t-\tau) - k],
\]

and in the neighborhood of a singular point $x(t) = 3Dk$ of the equation of the linear approximation of the form

\[
\frac{y(t)}{dt} = 3D - a [y(t-\tau) - k]
\]

where $y(t) = 3Dx(t) - k$, the characteristic equation is

\[
\lambda + ae^{\lambda \tau} = 3D0,
\]
and, as a consequence of [6,7], with
\[ 0 < a\tau < \frac{\pi}{2} \]
equilibrium \( x(t) = 3Dk \) is locally asymptotically stable.

Estimate the stability region in the phase space of the equilibrium position \( x(t) = 3Dk \) of the original non-linear system (11). After the transformation \( x(t) = 3Dy(t) + k \) point \( x(t) = 3Dk \) to the origin, we obtain the equation
\[
\frac{y(t)}{dt} = 3D - a\left[ y(t) + k \right]y(t - \tau).
\]

We use a quadratic Lyapunov function \( V(y) = 3D\frac{1}{2}y^2 \). Since we consider the delay equation, the total derivative of the evaluation will be used B.S. Razumihin’s condition [4]. This condition means geometrically that the total derivative is calculated, subject approach solutions from the inside surface of the level of the Lyapunov function. For the function \( V(y) = 3D\frac{1}{2}y^2 \) it has the form
\[
|y(t) - y(t) - \tau)| < |y(t)|.
\]

And the total derivative of the Lyapunov function has the form
\[
\frac{dV(x(t))}{dt} \leq -a\left[ 1 - \frac{1}{k} |y(t)| \right] |y(t)|^2 + \frac{a}{k} |y(t)| |y(t)| + k |y(t) - y(t - \tau)|.
\]

We estimate the phase coordinates with delay and without. Rewrite (12) in the integral form, using (8), and for the total derivative of the Lyapunov function will be
\[
\frac{dV(x(t))}{dt} \leq -a\left[ 1 - \frac{1}{k} |y(t)| \right] |y(t)|^2 + \frac{a}{k} |y(t)| |y(t)| + k |y(t) - y(t - \tau)|.
\]

Thus, when
\[
a\left[ 1 - \frac{1}{k} |y(t)| \right] > \left( \frac{a}{k} \right)^2 |y(t)| + k |y(t)| \tau
\]
the total derivative of the Lyapunov function is negative definite. Thus, the stability conditions are determined by the inequalities
\[
|y(0)| < k,
\]
\[
\tau < \frac{k[k - |y(0)|]}{a[|y(0)| + k]^2},
\]
3.2. General quadratic model with delay: main results. In the universal vector-matrix form, the quadratic model with delay is written as

\[ \dot{x}(t) = 3D \left[ Ax(t) + X^T(t - \tau) \right] Bx(t). \]

Suppose, as in the case without delay \( x_0^T = 3D(x_0^1, x_0^2, ..., x_0^n) \) is the solution of algebraic equations

\[ B_0x = 3Da, a^T = 3D(a_1, a_2, ..., a_n)^T. \]

Making the substitution in \( x(t) = 3Dy(t) + k \), we obtain the system of equations of the perturbation which, after transformation, takes the form

\[ \dot{y}(t) = 3D\tilde{A}y(t) + Y^T(t - \tau)By(t), \]

\[ \tilde{A} := 3D \begin{bmatrix} b_{11}x_0^1 & b_{21}x_0^1 & \ldots & b_{n1}x_0^1 \\ b_{12}x_0^2 & b_{22}x_0^2 & \ldots & b_{n2}x_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}x_0^n & b_{2n}x_0^n & \ldots & b_{nn}x_0^n \end{bmatrix}. \]

We’ll study the stability of the zero equilibrium state of the system = (15) using the method of Lyapunov functions quadratic form \( V(y) = 3Dy^THy \). In assessing the total derivative is used B.S. = Razumihin’s condition [8]. For the function \( V(y) \) it has the form

\[ | y(t - \tau) | \leq \sqrt{\varphi(H)} | y(t) |. \]

The set of points \( y \in R^n \) that are within the level surfaces \( V(y) = 3D\alpha \) of a Lyapunov functions \( V(y) = 3Dy^THy \) by \( V^\alpha \), and its boundary by \( \partial V^\alpha \), i.e.

\[ V^\alpha = 3D \{ y \in R^n : V(y) < \alpha \}. \]

**Theorem 2.** Let the solutions \( y(t) \) of (15) is performed \( y(T) \in \partial V^\alpha \) at the time \( t = 3DT > \tau \), and at \(-\tau \leq t < T \) will \( y(t) \in V^\infty \). Then holds the inequality

\[ |y(T) - y(T - \tau)| \leq \left[ \tilde{A} + B = \sqrt{\varphi(H)}y(T) \right] \sqrt{\varphi(H)}y(T)\tau. \]

**Theorem 3.** Let the \(-\tau \leq t \leq 0 \) initial conditions \( \varphi(t) \) for the solutions \( y(t) \) is implemented \( \varphi(t) < \delta \). Then this solutions \( y(t) \) on interval \( 0 \leq t \leq \tau \), be implemented

\[ |y(t)| \leq \delta \exp \left[ \tilde{A} + B\delta \right] \tau. \]
Theorem 4. Let the matrix $\bar{A}$ is asymptotic stability: $\text{Re} \lambda_i(\bar{A}) < 0$. Then as $\tau < \tau_0$, where
\[
\tau_0 = 3D \frac{\lambda_{\text{min}}(C)}{2HB\sqrt{\varphi(H)}},
\]
equilibrium is asymptotic stable.

REFERENCES

Abstract. In the paper optimal guaranteed control problems for linear nonstationary dynamical systems under set-membership uncertainty with use of feedforward-feedback (combined) closable loop on output are considered. The preposterior analysis of the observation subsystem and the control subsystem introduced lay the foundation for solving the problem. On the basis of preposterior analysis sets of closure and optimal closable programs are introduced and a positional solution of the optimal control problem is determined. A method of the quasirealization of optimal combined closable loop by Optimal Estimators (OE) and Optimal Regulator (OR) generating control actions in real time is suggested. The examples are given.

Key Words. observation, optimal control, set-membership uncertainty, preposterior analysis, closable loop, on-line control, algorithms, positional solution.

AMS(MOS) subject classification. 49J15, 34H05, 93C15.

Introduction. Any control process proceeds under uncertainties of different types and structure (inaccuracies in system parameters, initial states, input parameters; external disturbances, etc.). Information theory and communication constraints affect the performance of control systems and demand new tools to handle the uncertainty. The first approaches to treating uncertainty in the framework of optimal control theory were based on their stochastic representation. Since the late 1960s for description of disturbances set-membership constraints have been used as well (D. Bertsekas, F. Schweppe, H.S. Witsenhausen). Ellipsoidal techniques developed in [8] have led to set-valued solutions to estimation and control problems in terms...
of approximating ellipsoidal-valued functions. Many problems of control under uncertainty have been originated in [7]. One approach to investigating control problems under set-membership uncertainties has been elaborating in Minsk since early 1990s. This approach is based on original fast (on-line) algorithms of optimal control and the use of computer tools in the course of control process. The main principle of the approach is to use procedures of correcting of optimal open-loop controls in real time ([5]).

The paper is closely related to the previous results of the authors ([6, 2]). We shall describe a new method and algorithms of real-time implementing feedforward-feedback loop by use of OE and OR.

1. Problem statement. Let \( T = [t_s, t^*] \) be a time interval; \( T_u = \{t_s, t_s + h, \ldots, t^* - h\}, \ h = (t^* - t_s)/N \) (\( N \) is a positive integer); \( T_w = \{\theta^w_i \in T_u, i = 1, N_w\}, \ t_s = \theta^w_0 < \theta^w_1 < \ldots < \theta^w_{N_w}, \) is a set of measurement instants of input signals; \( T_x = \{\theta^x_i \in T_u, i = 1, N_x\}, \ t_s = \theta^x_0 < \theta^x_1 < \ldots < \theta^x_{N_x}, \) is a set of measurement instants of output signals; \( A(t) \in \mathbb{R}^n_x \times n_x, A_w(t) \in \mathbb{R}^{n_x \times n_w}, B(t) \in \mathbb{R}^{n_w} \times n_x, M(t) \in \mathbb{R}^{n_x \times n_w}, M_w(t) \in \mathbb{R}^{n_x \times n_w}, \ t \in T, \) are given piecewise continuous functions; \( C_w(t) \in \mathbb{R}^{q_w \times n_z}, C_x(t) \in \mathbb{R}^{q_x \times n_z}, \ t \in T, \) are continuous functions; \( h_i \in \mathbb{R}^{n_x}, h'_i \in \mathbb{R}^{n_x} \); \( \Xi_w = \{\xi \in \mathbb{R}^{q_w} : \xi_w \leq \xi \leq \xi^w\}, \) \( \Xi_x = \{\xi \in \mathbb{R}^{q_x} : \xi_x \leq \xi \leq \xi^x\} \) are bounded sets; \( X^* = \{x \in \mathbb{R}^{n_x} : g_{si} \leq h'_i x \leq g'_i, \ i \in I\} \) is a bounded body; \( u(t) \equiv u(t) = (u(t), t \leq t < t^*). \)

A function \( u(\cdot) = u(t_s : t^*) \) is called discrete (with a quantization period \( h \)) if \( u(t) \equiv u(s), \ t \in [s, s + h], \ s \in T_u. \)

In the class of discrete control actions \( u(\cdot) \) we consider the optimal control problem:

1. \( J(u) = c^t x(t^*) \rightarrow \max; \)

2. \[ \dot{x} = A(t)x + B(t)u + M(t)w(t), \ x(t_s) = x_0; \]
\[ x(t^*) \in X^*; \ u(t) \in U; \]

3. \[ y_w(\theta^w) = C_w(\theta^w)z(\theta^w) + \xi_w(\theta^w), \ \xi_w(\theta^w) \in \Xi_w, \ \theta^w \in T_w; \]
\[ \dot{z} = A_w(t)z + M_w(t)w(t), \ z(t_s) = z_0, \ t \in T; \]

4. \[ y_x(\theta^x) = \int_{\theta^x_{i-1}}^{\theta^x_i} C_x(v)x(v)dv + \xi_x(\theta^x_i), \ \xi_x(\theta^x_i) \in \Xi_x, \quad i = 1, N_x. \]

Here \( x = x(t) \in \mathbb{R}^{n_x} \) is a state of control object (2) at time instant \( t; \)
\( u = u(t) \in \mathbb{R}^r \) is a value of control action; \( z = z(t) \in \mathbb{R}^{n_z} \) is a state of
mathematical model of a measuring device for input signals (3); $y_w(\theta^w)$, $\theta^w \in T_w; y_x(\theta^x), \theta^x \in T_x$, are signals of devices measuring disturbance and state (3), (4); $\xi_w(\theta^w) \in \mathbb{R}^{q_w}, \theta^w \in T_w; \xi_x(\theta^x) \in \mathbb{R}^{q_x}, \theta^x \in T_x$, are unknown errors of measurements (3), (4); $w(t) \in \mathbb{R}^{n_w}, t \in T$, is unknown disturbance, which is a finite-parametric function

$$w(t) = L(t)w, \ t \in T,$$

with a given piecewise continuous function $L(t) \in \mathbb{R}^{n_w \times l}, t \in T$, and an unknown vector $w \in \mathbb{R}^l$ from a bounded set $W = \{w \in \mathbb{R}^l : \omega_* \leq w \leq \omega^*\}$.

Problem (1)-(5) is to generate in real time bounded discrete control actions $u(t) \in U; t \in T$, by inexact and incomplete measurements of devices signals (3), (4). This actions have to steer system (2) to the terminal set $X^*$ at time $t^*$ with guarantee and provide the maximum guaranteed value of the cost function $J(u)$.

At first we investigate an auxiliary optimal observation problem.

### 2. Optimal preposterior observation.

#### 2.1. Initial preposterior distribution.

Let us single out from (1)-(4) an observation subsystem:

$$\dot{x} = A(t)x + M(t)w(t), \ t \in T; \ x(t_*) = x_0;$$

$$y_w(\theta^w) = C_w(\theta^w)z(\theta^w) + \xi_w(\theta^w), \ \xi_w(\theta^w) \in \Xi_w, \ \theta^w \in T_w;$$

$$\dot{z} = A_w(t)z + M_w(t)w(t), \ z(t_*) = z_0, \ t \in T;$$

$$y_x(\theta^x) = \int_{\theta^x_i-1}^{\theta^x_i} C_x(v)x(v)dv + \xi_x(\theta^x_i), \ \xi_x(\theta^x_i) \in \Xi_x, \ i = 1, N_x.$$

The terminal state $x(t^*|w)$ of (6) can be established only within the set

$$X_o^* = \{x \in \mathbb{R}^{n_x} : x = F(t^*, t_*)x_0 + \int_{t_*}^{t^*} F(t^*, t)M(t)L(t)dw, w \in W\}.$$

Here $F(t, \tau) = F(t)F^{-1}(\tau); F(t) \in \mathbb{R}^{n_x \times n_x}, t \in T: \dot{F} = A(t)F, F(t_*) = E$. Set (9) is called a priori distribution of the terminal state of the observation subsystem.

The set $X_o^*$ is determined by the a priori information about system (6) without regard for a priori information about devices (7), (8). Let us take into consideration a priori information about the whole observation subsystem
The set can generate and measurement errors only those terminal states \( x \) they satisfy: 

\[
T \subseteq \Xi
\]

For \( \left| y \right| \) (for direction \( d \)) subsystem and by the virtual signals \( \tilde{y} \) is said to be an initial preposterior distribution (on totality \( Q \)).

Let \( \tilde{y} \) be the set of all possible virtual signals \( \tilde{y} \). The number

\[
(10) \quad d^0(q) = \max d(\tilde{y} | q), \quad \tilde{y} \in \tilde{Y}
\]

is called the maximal width or the initial preposterior estimate of sets \( X^o_t (\tilde{y}) \), \( \tilde{y} \in \tilde{Y} \), along \( q \).

Calculation of estimate (10) is called the optimal initial preposterior observation problem (for direction \( q \)), its result is the initial preposterior solution (for direction \( q \)).

Let \( Q \) be a finite totality of identity \( n_x \)-vectors (directions), in which each set from \( n_x \) vectors are linear-independent.

**Definition 1.** The set

\[
X^o_t = \{ x \in \mathbb{R}^{n_x} : -d^0(q)/2 \leq q \cdot x \leq d^0(q)/2, \quad q \in Q \}
\]

is said to be an initial preposterior distribution (on totality \( Q \)) at instant \( t^* \).

For \( \left| Q \right| \leq n_x \) each of the sets \( X^o_t (\tilde{y}) \), \( \tilde{y} \in \tilde{Y} \), can be placed in \( X^o_t \).

It can be proved that the set \( X^o_t (\tilde{y}) \) consists of all \( x \in \mathbb{R}^{n_x} \) such that they satisfy:

\[
\begin{align*}
  x &= F(t^*, t_*)x_0 + \int_{t_*}^{t^*} F(t^*, t)M(t)L(t)dtw, \\
  \xi_{tw} &\leq D_w(t_*)w - C_w(t_*)F_w(t_*)x_{0} + \tilde{y}_w(t_*) \leq \xi_{w}, \\
  \xi_{sx} &\leq D_x(t_*)w - \int_{t_*}^{t^*} C_x(v)F(v, t_*)x_{0}dv + \tilde{y}_x(t_*) \leq \xi_{s}, \\
  \omega_* &\leq w \leq \omega^*, \quad \theta^w \in T_{clw}, \quad \theta^x \in T_{clx},
\end{align*}
\]
Let us carry out the pre-
which together with some
(10) is a linear programming problem (LP)

where

\[ D_w(\theta^w) = - \int_t^{\theta^w} C_w(\theta^w) F_w(\theta^w, t) M_w(t) L(t) \, dt; \]

\[ D_x(\theta^x) = - \int_{\theta_{x-1}^x}^{\theta^x} C_x(v) \int_t^{\theta^x} F(v, t) M(t) L(t) \, dt \, dv. \]

\[ (F_w(t, \tau) = F_w(t) F_w^{-1}(\tau); F_w(t) \in \mathbb{R}^{n_x \times n_w}, t \in T : \hat{F}_w = A_w(t) F_w, \]

\[ F_w(t_s) = E). \]

Denote \( q^{\tau'} = q' \int_{t_s}^{\tau'} F(t', t) M(t) L(t) \, dt. \)

We can conclude that optimal initial preposterior observation problem (10) is a linear programming problem (LP)

\[
\left\{ \begin{array}{l}
    d^{\theta}(q) = \max_{\tilde{w}, \tilde{w}; \tilde{x}(\cdot), \tilde{x}(\cdot)} q^{x'}(\tilde{w} - \tilde{w}), \\
    \xi_{xw} \leq D_w(\theta^w)(\tilde{w} - \tilde{w}) + \tilde{c}_w(\theta^w) \leq \xi^w, \\
    \xi_{xw} \leq D_x(\theta^x)(\tilde{w} - \tilde{w}) + \tilde{c}_x(\theta^x) \leq \xi^x, \\
    \xi_{xw} \leq D_x(\theta^x)(\tilde{w} - \tilde{w}) + \tilde{c}_x(\theta^x) \leq \xi^x, \\
    \omega_s \leq \tilde{w} \leq \omega^s, \omega_s \leq \tilde{w} \leq \omega^s; \\
    \xi_{xw} \leq \tilde{c}_w(\theta^w) \leq \xi^w, \xi_{xw} \leq \tilde{c}_x(\theta^x) \leq \xi^x, \\
\end{array} \right.
\]

Here \( \tilde{c}_w(\cdot) = \{\tilde{c}_w(\theta^w), \theta^w \in T_{clw}\}, \tilde{c}_x(\cdot) = \{\tilde{c}_x(\theta^x), \theta^x \in T_{clx}\}. \) The preposterior analysis is carried out before the observation process starting.

2.2. Current preposterior distribution. Let us carry out the pre-posterior analysis for current instant \( \tau \in T_w \cup T_x, \) assuming that observation has been accomplished during the interval \( T_{w+\tau} = [t_s, \tau] \) and by logged signals \( y^*_w(\tau) = \{y^*_w(\theta^w), \theta^w \in T_w \cap T_{w+\tau}; y^*_x(\theta^x), \theta^x \in T_x \cap T_{w+\tau}\} \) the current distribution \( W(\tau, y^*_w(\cdot)) \) of vector \( w, \) corresponding to the position \( (\tau, y^*_w(\cdot)) \) has been determined. It consists of those and only those \( w \in W \) that together with some measurement errors \( \xi_w(\theta^w) \in \Xi_w, \theta^w \in T_w \cap T_{w+\tau}; \xi_x(\theta^x) \in \Xi_x, \theta^x \in T_x \cap T_{w+\tau} \) are able to generate \( y^*_w(\cdot). \)

We choose arbitrary \( \tilde{w} \in W(\tau, y^*_w(\cdot)), \tilde{c}_w(\theta^w) \in \Xi_w, \theta^w \in T_{clw} \cap T_{-\tau}; \tilde{c}_x(\theta^x) \in \Xi_x, \theta^x \in T_{clx} \cap T_{-\tau}; \tilde{x}(\tau) = F(\tau, t_0) x_0 + \int_{t_0}^{T} F(\tau, t) M(t) L(t) \, dt \tilde{w}, \) at the observation sub-system. By the logged \( y^*_w(\cdot) \) and virtual \( \tilde{y}_w(\cdot) = \{\tilde{y}_w(\theta^w), \theta^w \in T_{clw} \cap T_{-\tau}; \tilde{y}_x(\theta^x), \theta^x \in T_{clx} \cap T_{-\tau}\} \) measurements we determine the a posteriori distribution of terminal state \( X^*_w(\tilde{y}_w(\cdot) | \tau, y^*_w(\cdot)) \) for the position \( (\tau, y^*_w(\cdot)). \) The set \( X^*_w(\tilde{y}_w(\cdot) | \tau, y^*_w(\cdot)) \) is composed of those terminal states of system (6) which together with some \( w \in W \) and errors \( \xi_w(\theta^w) \in \Xi_w, \theta^w \in (T_w \cap T_{w+\tau}) \cup (T_{clw} \cap T_{-\tau}); \xi_x(\theta^x) \in \Xi_x, \theta^x \in (T_x \cap T_{w+\tau}) \cup (T_{clx} \cap T_{-\tau}), \) can generate \( y^*_w(\cdot), \)

\( \tilde{y}_w(\cdot). \)
Introduce the current width of set $X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))$ in the direction $q$:

$$d(\tilde{y}^\tau(\cdot)|q, (\tau, y^*_\tau(\cdot))) = \max_{x \in X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))} q'x - \min_{x \in X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))} q'x = \max_{\bar{x} \in X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))} q'(\bar{x} - x).$$

Suppose that $\tilde{Y}(\tau, y^*_\tau(\cdot))$ is a set of all possible virtual signals $\tilde{y}^\tau(\cdot)$ for $(\tau, y^*_\tau(\cdot))$. Introduce the number

$$(13) \quad d^0(q|\tau, y^*_\tau(\cdot)) = \max \{d(\tilde{y}^\tau(\cdot)|q, (\tau, y^*_\tau(\cdot))), \tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot))\},$$

and call it the current maximal width (current preposterior estimate) of the sets $X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))$, $\tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot))$ along the direction $q$ or the current preposterior solution to the optimal observation problem (preposterior solution to the optimal observation problem for the current position).

**Definition 2.** The set

$$X_{t^*}^o(\tau, y^*_\tau(\cdot)) = \{x \in \mathbb{R}^{n_x} : -d^0(q|\tau, y^*_\tau(\cdot))/2 \leq q'x \leq d^0(q|\tau, y^*_\tau(\cdot))/2, q \in Q\},$$

is said to be the current preposterior distribution of the terminal sets at time instant $t^*$. For $|Q| \leq n_x$ each of sets $X_{t^*}^o(\tilde{y}^\tau(\cdot) | \tau, y^*_\tau(\cdot))$, $\tilde{y}^\tau(\cdot) \in \tilde{Y}(\tau, y^*_\tau(\cdot))$, can be placed in $X_{t^*}^o(\tau, y^*_\tau(\cdot))$.

By analogy with problem (10) it is easy to obtain the analytical form of problem (13) that is LP problem.

**2.3. Positional solution to the optimal preposterior observation problem.** In order to generate current control actions (with aim of obtaining sufficiently complete information about uncertainty) estimates $d^0(q|\tau, y^*_\tau(\cdot))$ are calculated for several vectors (directions) $q \in Q$. A vector

$$d^0(\tau, y^*_\tau(\cdot)) = (d^0(q|\tau, y^*_\tau(\cdot)), q \in Q)$$

is called the vector of sufficient estimates for $(\tau, y^*_\tau(\cdot))$.

Let $Y^*_\tau$ be a collection of all possible signals $y^*_\tau(\cdot)$ of measuring devices (7), (8) which can be obtained by $\tau$.

**Definition 3.** A function

$$(14) \quad d^0(\tau, y^*_\tau(\cdot)), y^*_\tau(\cdot) \in Y^*_\tau, \tau \in T_w \cup T_x,$$

is called the positional solution to the optimal preposterior observation problem.
Knowledge of positional solution (14) enables one to obtain the sufficient estimates for each possible position \((\tau, y^*_{\tau}(\cdot))\) and generate on the basis of them optimal control actions in the system a) with optimal output closable loop if the partial preposterior analysis is used in the course of observation; b) with optimal output closed loop if the full preposterior analysis is used. At present such method of synthesis of optimal systems is very difficult to realize due to “the curse of dimensionality”.

2.4. Optimal preposterior real-time observation. As can be seen from the foregoing, positional solution (14) is constructed for all possible positions before observation process starting that requires to remember huge amount of information. In contemporary era of rapid development of computers it is natural to resort to another way of optimal observation where function (14) is not constructed but its current values required for control are calculated in the course of processes.

To describe this method, we first find out how the positional solution is used in a particular observation process. Let positional solution (14) was constructed. Consider some particular observation process where unknown \(w^*, \xi^*_w(\theta^w), \theta^w \in T_w; \xi^*_x(\theta^x), \theta^x \in T_x\) are realized. This collection generates in subsystem (6)-(8) the transient \(x^*(t), t \in T\), and the signals \(y^*_w(\theta^w), \theta^w \in T_w; y^*_x(\theta^x), \theta^x \in T_x\). Knowing positional solution (14), by this signals it is easy to establish the current estimates \(d^*(\tau) = d^0(\tau, y^*_\tau(\cdot)), \tau \in T_w \cup T_x\). Hence it follows that in a particular observation process positional solution (14) is not used as a whole and only its values along a separate sequence of signals \(y^*_\tau(\cdot), \tau \in T_w \cup T_x\) are required.

**Definition 4.** A function

\[
d^*(\tau), \quad \tau \in T_w \cup T_x,
\]

is called a realization of positional solution in a particular observation process. On account of stated above reasons it is impossible to implement such method of observation. We describe another way of optimal observation which we call the optimal real-time observation. Assume that for each \(\tau \in T_w \cup T_x\) there exists a method to calculate the values \(d^0(\tau, y^*_\tau(\cdot))\) during a time \(s^0(\tau)\) not exceeding \(h\).

**Definition 5.** The function

\[
d^{**}(t) = \begin{cases} 
(d^0(q), q \in Q), t \in [t_*, \bar{t}_* +s^0(\bar{t}_*)]; \\
(d^*(\tau), t \in [\tau + s^0(\tau), \bar{\tau} + s^0(\bar{\tau})], \tau \in T_w \cup T_x; \\
(d^*(\bar{t}^*)), t \in [\bar{t}_* + s^0(\bar{t}^*), \bar{t}^*],
\end{cases}
\]

\(\bar{t} = \min \{\tau \in T_w \cup T_x : \tau > t\}, \bar{t}_* = \max \{\tau \in T_w \cup T_x : \tau < t\}\), is said to be a
quasirealization of positional solution, and the device able to construct it the optimal estimator.

In other words, quasirealization is a realization of the positional solution with regard for expenditure of time of calculation of its current values.

Thus, the synthesis problem to the optimal observation system is reduced to constructing OE algorithm.

Below we suggest the following OE algorithm.

Since calculations for each direction \( q \in Q \) can be carried out parallel, an algorithm of work of OE will be described only for one OE. Before the observation process starting OE solves problem (12) using the dual method ([5]) and, thus, calculating the initial preposterior estimate \( d_0^*(q) \) and corresponding optimal support \( K_0^b(q, t_\star) \) which is the main element of the method mentioned.

Let OE have worked during the interval \( T + \tau \). It constructed the optimal support \( K_0^b(q, \tau) \) for the position \( (\tau, y_\tau^*(\cdot)) \) and calculated the current preposterior estimate \( d_0^*(q|\tau, y_\tau^*(\cdot)) \) from the signals \( y_\tau^*(\cdot) \). At the nearest next instant \( \tau \) of measurements the signals a) \( y_w^*(\cdot) \) if \( \tau \in T_w \); b) \( y_x^*(\cdot) \) if \( \tau \in T_x \); c) both signals a), b) if \( \tau \in T_w \cap T_x \) become known.

During interval \( [\tau, \tau + s^0(\tau)] \) OE solves problem (13) for \( (\tau, y_\tau^*(\cdot)) \). This problem differs from the solved at the previous step for \( (\tau, y_\tau^*(\cdot)) \) only by some constraints.

OE solves the new problem by the dual method, correcting the optimal support \( K_0^b(q, \tau) \) of the problem solved at the previous step till constructing the optimal \( K_0^b(q, \tau) \). As these problems differ from each other insignificantly, the current support \( K_0^b(q, \tau) \) can be corrected fastly.

3. Example 1. Consider the example:

\[
\ddot{x} + 2.7x = 0.5w(t); \quad x(0) = 0.8, \quad \dot{x}(0) = -1.0, \quad t \in T = [0, 12];
\]
\[
y_w = z + \xi_w(t), \quad |\xi_w(t)| \leq \xi_w^*;
\]
\[
\dot{z} + 1.8z = w(t), \quad z(0) = -3.0;
\]
\[
y_x(\theta_i^*) = \int_{\theta_i^*-1}^{\theta_i^*} (x + \dot{x}) ds + \xi_x(\theta_i^*), \quad |\xi_x(t)| \leq \xi_x^*;
\]
\[
w(t) = w_1 \sin(t) + w_2 \sin(3t) + w_3 \sin(5t), \quad t \in T;
\]
\[
(w_1, w_2, w_3) \in W = \{w \in R^3 : |w_i| \leq 1.6, \ i = \{1,3\}\};
\]
\[
Q = (q(i) = (\cos(\pi i/12), \sin(\pi i/12)), \ i = \{1,2,\ldots,24\}).
\]

The aim of experiments is to construct initial preposterior distributions at terminal moment \( T = 12 \).
In the first series of experiments it was assumed that $\xi^*_w = 0.1$, $\xi^*_x = 0.1$, and distributions were constructed for the following cases (Fig. 1a): 1. $X^o_t = \{x \in \mathbb{R}^{n_x} : x = \int_t^{t_p} F(t^*, t)M(t)L(t)dw, w \in W\}$; 2. $X^o_{t^*}, T_{clw} = T_{clx} = \{9\}$; 3. $X^o_{t^*}, T_{clw} = T_{clx} = \{6\}$; 4. $X^o_{t^*}, T_{clw} = T_{clx} = \{3, 6, 9\}$; 5. $X^o_{t^*}, T_{clw} = T_{clx} = \{1, 3, 6, 9, 11\}$; 6. $X^o_{t^*}, T_{clw} = T_{clx} = \{1, 2, ..., 11\}$.

In the second series it was supposed that $T_{clw} = T_{clx} = \{3, 6, 9\}, \xi^*_x = 0.1$ and the following cases are considered (Fig. 1b): 1. $X^o_{t^*}$; 2. $X^o_{t^*}$, $\xi^*_w = 0.6$; 3 $X^o_{t^*}$, $\xi^*_w = 0.3$; 4. $X^o_{t^*}$, $\xi^*_w = 0.1$; 5. $X^o_{t^*}$, $\xi^*_w = 0.05$; 6. $X^o_{t^*}$, $\xi^*_w = 0.005$.

![Fig. 1](image)

In Fig. 1 one can see that the larger and more accurate information we have, then the preposterior distributions are smaller.

4. Optimal output closable loop.

4.1. Optimal initial closable program. Let $t_j \in T_{cl}, j = 1, p$, be the instants of closure for problem (1)-(5). We continue preposterior analysis and for each closing instants construct the closure sets $X^p, X^{p-1}, ..., X^1$.

Denote: $X^o_{t_{+0}}$ is an initial preposterior distribution for the observation subsystem at the instant $t$ constructed by virtual signals along the interval $[t_s, t]$; $X^o_{t_{-0}}$ - by signals along the interval $[t_s, t]$. Let $X^o_{t_{p+0}}, X^o_{t_{-0}}$ be initial preposterior distributions at the instants $t_p, t^*$ respectively. Let $X^c_{t_{p+0}}(z) = z + X^o_{t_{p+0}}$. We introduce a set $Z^p$ composed of all such vectors $z \in \mathbb{R}^{n_z}$ for which there exist available control actions $u(t_p : t^* | X^c_{t_{p+0}}(z))$ that $X^c_{t^*}(u(t_p : t^* | X^c_{t_{p+0}}(z))) = \{F(t^*, t_p)z + \int_t^{t_p} F(t^*, s)B(s)u(s | X^c_{t_{p+0}}(z))ds + X^o_{t_{-0}}\} \subset X^*$. Family of the sets $X^p = \{X^c_{t_{p+0}}(z), z \in Z^p\}$ is called a closure set of the control system at instant $t_p$.

Let nonempty sets $X^p, X^{p-1}, ..., X^{j+1}$ are determined. Using the sets
Now we introduce a transient instant and the control process is carried out during the time interval $t_j$ to $t_{j+1}$. Let $X^c_{t_j+1}(u(t_j : t_{j+1}|X^c_{t_j+0}(z))) \subset \mathcal{X}^{j+1}$, where $X^c_{t_j+1}(u(t_j : t_{j+1}|X^c_{t_j+0}(z))) = F(t_{j+1}, t_j)z + \int_{t_j}^{t_{j+1}} F(t_{j+1}, s)B(s)u(s|X^c_{t_j+0}(z))ds + X^o_{t_j+1}$. Denote

$$
\mathcal{X}^j = \{X^c_{t_j+0}(z), z \in Z^j\}.
$$

Continuing the process we construct $\mathcal{X}^j$, $j = 0, \ldots, p$. Let the inclusion $X^c_{t_1}(u(t_1 : t_2|x_0)) = \int_{t_0}^{t_1} F(t_1, s)B(s)u(s|x_0)ds + X^o_{t_1} \subset \mathcal{X}^1$ holds. Totality $u(\cdot) = \{u(t_1 : t_2|x), X \in \mathcal{X}^1; \ldots; u(t_p : t^*|X), X \in \mathcal{X}^p\}$ is called an initial closable program. It guarantees transferring system (2) to the terminal set for any implementations of uncertainty if measurements are carried out at the instants $t \in T_d$.

Choose $\beta > \min c^* x, x \in X^*$, replace the set $X^*$ by $X^{*\beta} = X^* \cap \{x \in R^n : c^* x \geq \beta\}$ and construct the sets $\mathcal{X}^{\beta^0}, \mathcal{X}^{\beta^0-1^\beta}, \ldots, \mathcal{X}^{1^\beta}$ while following the above rules. The maximum $\beta^0$ for which an initial closable program exists, is equal to the maximum guaranteed value of the cost function of problem (1)-(5).

**Definition 6.** The totality

$$
u^0(\cdot) = \{u^{\beta^0}(t_1 : t_2|x_0); \nu^{\beta^0}(t_1 : t_2|X), X \in \mathcal{X}^{1^\beta^0}; \ldots; \nu^{\beta^0}(t_p : t^*|X), X \in \mathcal{X}^{p^\beta^0}\}
$$

is called an optimal initial closable program (a program preposterior solution).

**4.2. Optimal current closable program.** Now we introduce a positional preposterior solution to the problem. Let $\tau \in T_w \cup T_x$ be a current instant and the control process is carried out during the time interval $T_{-\tau} = [t_\tau, \tau]$, the control actions $u^\tau(\cdot) = u^*(t_\tau : \tau)$ are generated and “pure” from $u^\tau(\cdot)$ signals $y^\tau(\cdot)$ known by the instant $\tau$ are logged (measuring devices signals of the observation object). Denote: $T^\tau_d = T_d \cap T_{-\tau} = \{t_k(\tau), t_k(\tau) + 1, \ldots, t_p\}; t_k(\tau) = \min\{t \in T_d : \tau < t\}; T^\tau_d = \emptyset, \tau \geq t_p$. Having replaced the a priori information $\{t_\tau, W\}$ by current $\{\tau, W(\tau, y^\tau(\cdot))\}$, we perform described above preposterior analysis on the time interval $T_{-\tau}$. As a result, we get the closure sets $\mathcal{X}^{\tau}(\tau, y^\tau(\cdot)), \mathcal{X}^{\tau-1}(\tau, y^\tau(\cdot)), \ldots, \mathcal{X}^{k(\tau)}(\tau, y^\tau(\cdot))$ and determine an optimal current closable program $u^0(t|\tau, y^\tau(\cdot)), t \in T_{-\tau} = [\tau, t^*]$, for the position $(\tau, y^\tau(\cdot))$. Note that for $\tau \geq t_p$ the optimal current closable program turns into disclosable.
4.3. Positional solution to the optimal control problem. Denote $Y_{\theta(\tau)}(\cdot), \tau \in T_u$, a set of all signals $y_{\theta(\tau)}(\cdot)$ such that for the position $(\theta(\tau), y_{\theta(\tau)}(\cdot))$ a closable program exists; $\theta(\tau) = \max\{\theta^w \in T_w \cap T_{+\tau}; \theta^z \in T_z \cap T_{+\tau}; t_s\}$.

**Definition 7.** A functional

\begin{equation}
(15)
\begin{align*}
u^0(\tau, y_{\theta(\cdot)}) &= u^0(\tau|\theta(\tau), y_{\theta(\cdot)}(\cdot)), \\
y_{\theta(\cdot)}(\cdot) &\in Y_{\theta(\cdot)}(\cdot), \tau \in T_u,
\end{align*}
\end{equation}

is called an optimal output closable (combined, discrete) loop (OCOL) (a positional solution to the optimal control problem in the class of output closable loops); contraction of (15) to a set of signals of measuring device (3) is an optimal output closable feedforward loop; contraction of (15) to a set of signals of measuring device (4) is an optimal output closable feedback loop.

4.4. Optimal real-time control. Like of the case of optimal observation (Section 2), we adhere to the principle of optimal real-time control, by which OCOL is not constructed wholly, but in each particular control process its current values (a realization of OCOL) $u^*(\tau) = u^0(\tau, y^*_{\theta(\cdot)})$, $\tau \in T_u$, are generated by the optimal regulator for the time $s^c(\tau)$, and $s^o(\tau) + s^c(\tau) < h$.

**Definition 8.** The function

\begin{equation}
u^*(t) = \begin{cases} u^*(t_s), t \in [t_s, t_s + h + s^o(t_s + h) + s^c(t_s + h)]; \\
u^*(\tau), t \in [\tau + s^o(\tau) + s^c(\tau), \tau + h + s^o(\tau + h) + s^c(\tau + h)]; \\
u^*(t^* - h), t \in [t^* - h + s^o(t^* - h) + s^c(t^* - h), t^*], \end{cases}
\end{equation}

constructed by OE and OR, is called a quasirealization of OCOL.

Before the control process starts, OR carries out initial preposterior analysis and constructs the closure sets $X^{p\beta}$, $X^{p-1\beta}$, ..., $X^{1\beta}$, where $\beta = \min c'x, x \in X^*$. At first, we describe a method of constructing $X^{p\beta}$ (the other closure sets are constructed similarly). Let $\tilde{\eta}_{t}(q) = \max q'x, x \in X^{\tilde{\eta}_{t}}; \tilde{\eta}_{t}(q) = \min q'x, x \in X^{\tilde{\eta}_{t}}$, be estimates of the initial preposterior distribution in the direction $q$ at terminal instant. Then $X^{p\beta} = X^{\tilde{\eta}_{t}} + Z^{p\beta}$, where $Z^{p\beta}$ consists of all $z \in \mathbb{R}^{n_z}$ such that

\begin{equation}
\begin{cases} g_s - \tilde{\eta}_{t}(h_i) \leq h_i \int_{t_p}^{t^*} F(t^*, s)B(s)u(s)ds \leq \\
\leq g^{*}_{t} - \tilde{\eta}_{t}(h_i), \ i \in I; \\
\beta - \tilde{\eta}_{t}(c) \leq c'F(t^*, t_p)z + c' \int_{t_p}^{t^*} F(t^*, s)B(s)u(s)ds;
\end{cases}
\end{equation}

$u(t) \in U, t \in [t_p, t^*]$. 

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We note that estimates are calculated along the directions \( h_i, i \in I; c \), because the set \( X^{i\beta} \), by which \( X^{p\beta} \) is constructed, is determined on this directions.

Let sets \( X^{p\beta}, X^{p-1\beta}, \ldots, X^{1\beta} \) have been constructed; \( H^{1\beta} \) are directions, by which the outer approximation of set \( X^{1\beta} \) is under construction; \( \gamma^{1\beta}(q) = \max q^i x, x \in X^{1\beta} \) is an estimate of the closure set at instant \( t_1 \) in direction \( q \); \( \gamma^{0}(q) = \max q^i x, x \in X^{0}_{t_1} \) is an estimate of the cost function, the optimal closable program \( q^* \) solves the problem: \( \gamma^{0}(q) \), \( q \in H^{1\beta}; u(t) \in \mathcal{U}, t \in [t_s, t_1] \).

\[
\begin{align*}
\alpha & \rightarrow \min_{\alpha, u(t_s, t_1)} \\
q^i \int_{t_1}^{t_0} F(t_1, s)B(s)u(s)ds - \alpha \leq \gamma^{1\beta}(q) - \gamma^{0}(q), \\
q & \in H^{1\beta}; u(t) \in \mathcal{U}, t \in [t_s, t_1].
\end{align*}
\]

Step by step increasing \( \beta \) and solving problem (16) by the dual method (iterations begin with the empty support), OR generates the optimal initial closable program \( u^{0}(\cdot) \), computes the maximum value \( \beta^{0}(t_s) = \beta^{0} \) of the cost function, the optimal support \( K^{0}_{t_s}(t_s) \) and constructs a set \( S^{0}_{t_s}(t_s) \) of supporting control action indices which are to “freeze” at the nearest next measurement instant \( \tilde{t}_s \). As initial supports for solving LP problems the empty supports are taken. Since operations are implemented in advance, the time expence is not significant.

To the input of the control object OR sends the control action \( u^*(t) = u^{0}(t), t \in [t_s, \tilde{t}_s + s^0(\tilde{t}_s) + s^c(\tilde{t}_s)], \) where \( s^0(\tilde{t}_s) \) is a time of operating \( \mathcal{O} \) and OR correspondingly.

Suppose that OR has operated on the interval \( T_{w} \cup T_{x}, \tau < t_p \), computed the optimal closable program \( u^{0}(\cdot|\tau, y^{*}(\cdot)) \), the sets \( K^{0}_{\tau}(\tau) \), \( S^{0}_{\tau}(\tau) \) and calculated the maximum value \( \beta^{0}(\tau) \) of the cost function for the nearest previous measurement instant \( \tau \). At the instant \( \tau \) OR receives a new signal from measuring devices and for the current position \( (\tau, y^{*}(\cdot)) \) solves the optimal preposterior observation problem. By results of OE work, OR computes the closure sets \( X^{p\beta}(\tau, y^{*}(\cdot)) \), \( X^{p-1\beta}(\tau, y^{*}(\cdot)), \ldots, X^{k(\tau)\beta}(\tau, y^{*}(\cdot)) \) and the optimal current closable program \( u^{0}(\cdot|\tau, y^{*}(\cdot)) \). The algorithm of constructing the optimal current closable program \( u^{0}(\cdot|\tau, y^{*}(\cdot)) \) begins with value \( \beta = \beta^{0}(\tau) \) of the cost function and the initial support \( K^{0}_{\tau}(\tau) \). Solving the problem for \( \beta = \beta^{0}(\tau) \), OR computes the set \( S^{0}_{\tau}(\tau) \) and sends the control action \( u^*(t) = u^{0}(t|\tau, y^{*}(\cdot)), t \in [\tau + s^0(\tau) + s^c(\tau), \tilde{\tau} + s^0(\tilde{\tau}) + s^c(\tilde{\tau})] \) to the input of the control object.
5. Example 2. Consider the example which demonstrates the efficiency in comparison with the known data (see [3]):

\[
x(t^*) + \dot{x}(t^*) \to \max;
\]
\[
\ddot{x} + 2.7x = 0.5u + 0.5w(t); \ x(0) = -1.0, \ \dot{x}(0) = -1.7,
\]
\[(x(12), \dot{x}(12)) \in X^* = \{x \in \mathbb{R}^2 : |x_1| \leq 0.5, |x_2| \leq 0.5\}; \ |u(t)| \leq 1.0, \ t \in T = [0, 12];
\]
\[
y_w = z + \xi_w(t), \ |\xi_w(t)| \leq 0.1;
\]
\[
\dot{z} + 1.8z = w(t), \ z(0) = -3.0;
\]
\[
y_x(t) = \int_{t-3}^{t} (x(t) + \dot{x}(t))ds + \xi_x(t), \ |\xi_x(t)| \leq 0.1;
\]
\[
w(t) = w_1 + w_2 \sin(t) + w_3 \sin(3t), \ t \in T;
\]
\[
(w_1, w_2, w_3) \in W = \{w \in \mathbb{R}^3 : |w_1| \leq 2.4, |w_2| \leq 0.8, |w_3| \leq 0.8\};
\]
\[
w^* = (1.0, -0.1, -0.5),
\]
\[
\xi_w^*(t) = 0.1 \cos(t), \ \xi_x^*(t) = 0.1 \sin(t), \ t \in T;
\]
\[
Q = (q_{i(1)} = (\cos(\pi i/12), \sin(\pi i/12)), \ i = \{1, 2, \ldots, 24\});
\]
\[
h = 1; \ T_w = T_x = \{3, 6, 9\}; \ T_{cl} = T_{clw} = T_{clx} = \{6\}.
\]

In a particular control process its guaranteed value of the cost function with use of the optimal output disclosable loop is \(J(w^*(\cdot)) = 0.7028\). If we use the optimal output closable loop, then \(J(\bar{u}^*(\cdot)) = 0.7920\). It appears that on the realized trajectory \(\bar{x}(t), t \in T\), corresponding to the optimal output closable loop, value of the cost function is equal to 0.9250; on the realized trajectory \(x(t), t \in T\), that corresponds to the optimal output disclosable two-phase loop with parameter \(\varepsilon = 0.001\), value of the cost function is equal to 0.8358.

The phase trajectories that correspond that two types of loops (optimal output closable loop – solid line, optimal disclosable – dashed line) are depicted in Fig. 2a; Fig. 2b contains on enlarge scale fragments of the phase trajectories at the final control stage; \(\bar{X}, X\) are the posteriori distributions of terminal states of the control system.

Conclusions. In the class of discrete control actions optimal guaranteed control problems to linear nonstationary dynamical systems under set-membership uncertainty are studied. It is assumed that in the course of control processes states of the system and disturbances are unknown and signals of two imperfect measurement devices are available for use. Pre-posterior analysis of the observation subsystem is carried out which makes possible to obtain sufficient estimates for a position. The algorithm of positional solving the optimal observation problem is suggested. A new type
of positional control called optimal closable loop on output is introduced. An algorithm of the quasirealization of the optimal output closable loop is described and tested on computer.

Following [1], one can develop the results presented to nonlinear dynamical systems via piecewise linear approximations. More interesting results elaborated by the extention of the technique can be obtained at investigating optimal rendezvous (approaching) problems of a dynamical object with a moving target, optimal aiming problems of a dynamical object at a moving target and optimal counteraction problems. The first results in this direction were presented in [4].

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REFERENCES


MODAL SUBSPACE LQR AND $\mathcal{H}_\infty$ CONTROL OF PROPORTIONALLY DAMPED MODELS

I. HALPERIN *, G. AGRANOVICH † AND Y. RIBAKOV ‡

Abstract. According to the seismic modal approach, optimal modal control design should derive a control signal affecting only the dominant modal coordinates. However, existence of an optimal state feedback, affecting only a selected set of modes, is questionable. This study proves the existence of optimal modal feedback for the case of infinite horizon LQR and $\mathcal{H}_\infty$ control methods. It also formulates a modal design method, based on a corresponding selection of LQR weighting and $\mathcal{H}_\infty$ output matrices. New state-space similarity transformations are introduced.

Key Words. modal control, $\mathcal{H}_\infty$, LQR, optimal structural control, algebraic Riccati’s equation.

AMS(MOS) subject classification. 93C05, 93C15, 93C95, 49K15, 70E55

1. Introduction. In the last decades, there is an increasing interest in the field of structural control. It suggests the use of approaches and tools from the control theory for analysis and manipulation of structures dynamic behavior. A dynamic response of a structure can be described by the displacements in a selected set of degrees of freedom, also known as nodal coordinates [9]. These displacements are calculated by solving the corresponding equations of motion.

Equations of motion for a seismically excited controlled structure can be described by a second order, ordinary differential matrix equation, governing
structural accelerations, velocities, displacements as well as the earthquake and control inputs in determined directions and locations in the structure. When the displacements are small enough, the constructive elements behavior remains elastic. Therefore, it is common to describe the motion by a linear differential equation. This equation is denoted as the nodal model.

The equations of motion in the nodal coordinates are internally coupled through the structure inertia, damping and stiffness properties. Moreover, structural models are quite large, compared to other controlled systems. It creates computational difficulties in finding an efficient control design for large structures. A common solution to this problem is known as modal decomposition [6]. It leads to a more simple and convenient design process [9]. Apart from this it is also physically interpreted and adequate from structural engineering viewpoint.

In modal decomposition the equations of motion are written as a set of scalar equations in a new, internally uncoupled modal coordinates. Each modal coordinate is related to the original coordinates by vibration frequency and by a vector denoted as mode shape. The contribution of each modal coordinate to the overall dynamic response is different, therefore, coordinates with small contribution may be neglected [6, 16, 12, 8]. For structures, subjected to earthquake excitation, it is known that a seismic design, taking in account merely the dominant, low frequency, modal coordinates set - is mostly adequate.

In order to carry out modal decomposition, the nodal model should be diagonalized so that \( l \) internally independent equations are achieved. The conditions that a dynamic system should satisfy in order to be diagonalized by a real and orthogonal transformation are clarified in [4]. If such transformation does not exist, sometimes it is possible to find a real and nonsingular diagonalizing transformation [2]. The latter is the case for a Rayleigh damped model [6].

An optimal control design that relies on the structure modal properties is denoted as optimal modal control. According to the seismic modal approach, optimal modal control design should derive a control signal that affects only the dominant modal coordinates [5]. The problem is that in general case, when a control design is done, each single control signal generates multiple modal control signals and therefore couples the structural modal coordinates [15]. In other words, the control signals affect multiple modes simultaneously.
This coupling also happens for feedback control signals. It is unwanted since the modal control design tries to increase the control efficiency by manipulating only the dominant modes.

In independent modal space control (IMSC) strategy a feedback control signal is designed in such a way that only the desired modes are affected. According to this method, modal control signals are designed in the modal space for each modal coordinate separately. Next, the implemented control is derived from the modal control signals [7].

Though in the general case the feedback couples the modes of the system, in the case of independent modal space control, where a modal control design deals merely the designed modal coordinates, the closed loop remains decoupled [14]. A problem that can be troublesome is that a feedback, designed by the IMSC approach, requires estimation of the modal coordinates. An approach, requiring no modal state estimation is direct feedback control, whereby the sensors are collocated with the actuators and a given actuator force is a linear function of the sensor output [13]. Another problem is that the use of IMSC might affect the uncontrolled modes such that their contribution to the overall response might increase [8].

An active direct feedback control method denoted as Modal Truncated Output Feedback (MTOF) is presented and applied for the vibration control of adaptive truss structures in [12]. The method allows the control feedback gain to be determined in a decoupled and truncated modal space in which only the critical vibration modes are retained.

A design method, based on fitting dampers parameters to steady state LQR (denoted here as LQR) design results, was studied [10]. A full state feedback gain matrix was derived by the LQR and a new feedback gain matrices, whose structure correspond to a physical dampers distribution, was approximated by using a least squares cost function.

Design of dampers, based on modal LQR was also studied [16]. The dominance of the first set of modes from energy viewpoint was emphasized and control design approach of the dominant modes by LQR method was suggested.

Many studies have been done in the field of optimal modal control design [16]. It is logically that such approach is more effective and economical since
the design is not concerned in mitigating mode shapes with low contribution to the overall vibration. However, existence of a direct optimal state feedback, affecting only a selected set of modes, is questionable. This study proves the existence of such direct optimal modal feedback for the case of infinite horizon LQR and $H_\infty$ control design methods. Additionally, it offers a corresponding verified design approach, based on the weighting and output matrices selection.

2. Equations of Motion in Nodal Model Form.

2.1. Second Order ODE. The equation of motion for a controlled structure, subjected to seismic excitation is:

\[ M \ddot{z}(t) + C \dot{z}(t) + Kz(t) = -M\gamma \ddot{z}_g(t) + Wu(t) \]

were $z : \mathbb{R} \to \mathbb{R}^l$ is a vector function of the nodal displacements; $M \in \mathbb{R}^{l \times l}$, $C \in \mathbb{R}^{l \times l}$, $K \in \mathbb{R}^{l \times l}$ are the structural mass, damping and stiffness matrices, respectively; $\ddot{z}_g : \mathbb{R} \to \mathbb{R}$ represents the earthquake ground acceleration signal; $u : \mathbb{R} \to \mathbb{R}^{nu}$ is a vector of the forces produced by the actuators; $W \in \mathbb{R}^{l \times nu}$ is the dampers distribution matrix that relates the input forces in the dampers to the forces in the nodal coordinates; $\gamma \in \mathbb{R}^l$ is the ground acceleration distribution vector. It should be noted that $M$, $C$ and $K$ are symmetrical positive definite matrices [6].

In this study, the damping of the structure is modeled as Rayleigh damping, i.e. mass and stiffness proportional [6]:

\[ C = \alpha_1 M + \alpha_2 K \]

where $\alpha_1$ and $\alpha_2$ are nonnegative scalars. Such approach allows simultaneous diagonalization of $M$, $C$, $K$, if the pencil $(M, K)$ can be diagonalized. The main known tool in finding such transformation is eigenvalue and eigenvector decomposition.

2.2. State-Space Form and LQR Control. In order to use the LQR method, a state-space formulation for (1) is used. As LQR design does not take in account disturbance signals, the ground acceleration is omitted. In the state space (1) takes the following form [9]:

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad x(t) \triangleq \begin{bmatrix} z \\ \dot{z} \end{bmatrix} \]
where \( x : \mathbb{R} \rightarrow \mathbb{R}^{2l} \) is the states vector of the model and \( A \in \mathbb{R}^{2l \times 2l}, B \in \mathbb{R}^{2l \times n_u} \) are the state matrices:

\[
A = \begin{bmatrix}
0 & I \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}; \quad B = \begin{bmatrix}
0 \\
M^{-1}W
\end{bmatrix}
\]

Writing an infinite horizon, LQ cost function for the model, described by (2), yields [3]:

\[
J(x, u) = \frac{1}{2} \int_{0}^{\infty} x^T Q x + u^T R u \, dt
\]

The time dependency notation in this equation was removed for simplicity. According to optimal control approach, (2) is used as a constraint equation and the following augmented cost function is derived:

\[
J(x, u, p) = \frac{1}{2} \int_{0}^{\infty} x^T Q x + u^T R u + p^T [Ax + Bu - \dot{x}] \, dt
\]

(3)

here \( p : \mathbb{R} \rightarrow \mathbb{R}^{2l} \) is the Lagrange multiplier also known as co-state. According to the LQR design method, the optimal closed loop control signals vector is

\[
u(t) = -Gx(t)
\]

Here \( G \in \mathbb{R}^{n_u \times 2l} \) is a matrix of the optimal feedback coefficients, derived by the LQR method [3] so that \( J(x, u) \) is minimized.

\( G \) is derived by

\[
G \triangleq R^{-1} B^T P
\]

where \( P \in \mathbb{R}^{2l \times 2l} \) is the solution of the algebraic matrix Riccati equation [3]:

\[-PA - A^T P - Q + PSP = 0; \quad S \triangleq BR^{-1} B^T\]

2.3. \( H_\infty \) Control. Consider the state space model

\[
\dot{x}(t) = Ax(t) + Bu(t) + E\ddot{z}_g(t)
\]

\[
y(t) = Cx(t) + D\ddot{z}_g(t)
\]
\( \mathcal{H}_\infty \) control yields a feedback signals that minimize the criterion [3]:

\[
J(x, u) = \|y\|^2_2 - \beta^2\|\ddot{z}_g(t)\|^2_2 = \int_0^\infty y^T y - \beta^2 \ddot{z}_g^2 dt
\]

for the worst disturbance signal \( \ddot{z}_g \) that exists. \( \beta \) is some arbitrary constant that should be bigger than the closed loop system \( \mathcal{H}_\infty \) norm [3].

The control signals are

\[
u(t) = -Gx(t)
\]

where

\[
G \triangleq B^T P
\]

and \( P \) is a solution of the algebraic Riccati’s equation:

\[
-PA - A^T P - C^T C + P(BB^T - \beta^2 EE^T)P = 0
\]

2.4. Modal Form and Orthogonality of the Undamped Model

Mode Shapes.

**Definition 1.** In structural dynamics, modal decomposition refers to simultaneous decoupling of the equations of motion. That is, the simultaneous diagonalization of \( M, C \) and \( K \).

The symmetric matrix pencil \((M, K)\) can be diagonalized by a real and orthogonal transformation matrix \( \Phi \) if and only if \( M \) and \( K \) commute [4]:

\[
MK = KM
\]

Otherwise, there is no real and orthogonal diagonalizing matrix \( \Phi \).

A dynamic model, whose mass matrix is diagonal with equal values, always satisfies the necessary and sufficient condition, given in (4):

\[
MK = (m_0 I)K = m_0 K
\]

\[
KM = K(m_0 I) = m_0 K
\]

Therefore, such model can be decoupled by a real and orthogonal eigenvectors matrix \( \Phi \). Nevertheless, it should be noted that in practical structural
control applications, (4) is rarely met.

When the mass matrix is diagonal with unequal values, or when it is not diagonal, the existence of orthogonal $\Phi$ is questionable. However, it is known that a real and nonsingular $\Phi$ that diagonalizes $M$ and $K$, such that $\Phi^T M \Phi$ and $\Phi^T K \Phi$ are diagonal matrices, can be found [2]. Alternatively, it can be obtained by solving the undamped structure eigenproblem [6].

The generalized eigenvalue problem for an undamped model is:

$$(K - \lambda_i M) \phi_i = 0$$

As $M$ is nonsingular, that problem can be rewritten as:

$$(M^{-1} K - \lambda_i I) \phi_i = 0$$

The sets $\{\lambda_i\}$ and $\{\phi_i\}$, satisfying (5), are the undamped eigenvalues and right eigenvectors of the structure, respectively. $\Phi$ is defined by:

$$\Phi \triangleq [\phi_1 \ldots \phi_l]$$

$\Phi$ diagonalizes both $M$ and $K$, i.e.:

$$M^* \triangleq \Phi^T M \Phi; \quad K^* \triangleq \Phi^T K \Phi$$

where $M^*$ and $K^*$ are diagonal matrices.

2.5. Modal Decomposition. Modal decomposition can be applied to (1) if there is a nonsingular transformation that diagonalizes $M$, $C$ and $K$. For a Rayleigh damped model $\Phi$ decouples the left hand side of (1) into $l$ internally independent modal coordinates $\{z_i^*(t)\}$ [6]. However, it does not decouple the control inputs. That makes the modal coordinates externally coupled. Each modal equation has the form:

$$m_i^* \ddot{z}_i^*(t) + c_i^* \dot{z}_i^*(t) + k_i^* z_i^*(t) = -\phi_i^T M \gamma \ddot{z}_g(t) + \phi_i^T W u(t)$$

where:

$$m_i^* \triangleq \phi_i^T M \phi_i; \quad c_i^* \triangleq \phi_i^T C \phi_i; \quad k_i^* \triangleq \phi_i^T K \phi_i$$

are the modal mass, damping and stiffness for modal coordinate $i$. 
2.6. State Space Formulation of Modal Decomposed Model.

Here the terms similarity is defined as follows.

**Definition 2.** Two square matrices, \( A \in \mathbb{R}^{2 \times 2l} \) and \( A^* \in \mathbb{R}^{2 \times 2l} \) are said to be similar if exists a nonsingular matrix \( T \) such that:

\[
T^{-1}AT = A^*
\]

**Lemma 1.** Similar matrices have the same set of eigenvalues.

**Proof.** See [1].

**Definition 3.** Two state space models - \( \dot{q} = A^*q + B^*u \) and \( \dot{x} = Ax + Bu \), are said to be similar if \( A^* \) and \( A \) are similar and \( B^* = T^{-1}B \).

The nodal state space model can be transformed from (2) into a modal state space model, composed from \( l \) internally independent modal subsystems, by using similarity transformation. Here a modal subsystem is described by the state-space formulation of (7)

\[
\frac{d}{dt} \begin{bmatrix} z_i^*(t) \\ \dot{z}_i^*(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_i^*/m_i^* & -c_i^*/m_i^* \end{bmatrix} \begin{bmatrix} z_i^*(t) \\ \dot{z}_i^*(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_i^T W/m_i^* \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\phi_i^T M\gamma/m_i^* \end{bmatrix} \ddot{z}_g(t) \tag{8}
\]

A state space similarity transformation from the nodal to the modal space is defined as follows. Consider the diagonal block matrix

\[
\Phi_0 \triangleq \text{diag}(\Phi, \Phi)
\]

where \( \Phi \) is defined in (6). As \( \Phi \) is non-singular, \( \Phi_0 \) is non-singular too. Denoting \( e_k \) as \( k^{th} \) column in the identity matrix \( I \in \mathbb{R}^{n \times n} \). A permutation matrix \( T_0 \) is:

\[
T_0 \triangleq [e_1 \ e_3 \ \ldots \ e_{n-1} \ e_2 \ e_4 \ \ldots \ e_n]^T
\]

It reorders the modal displacement and velocity of each mode in pairs so that

\[
\begin{bmatrix} z_1^*(t) & \ldots & z_l^*(t) & \dot{z}_1^*(t) & \ldots & \dot{z}_l^*(t) \end{bmatrix}^T = T_0 \begin{bmatrix} z_1^*(t) & \dot{z}_1^*(t) & \ldots & z_l^*(t) & \dot{z}_l^*(t) \end{bmatrix}^T
\]

This permutation matrix is orthonormal.
The non-singular similarity transformation matrix is
\[ T \triangleq \Phi_0 T_0 \]
and the similarity transformation yields
\[ x(t) \triangleq Tq(t) \leftrightarrow q(t) \triangleq T^{-1}x(t) \]
Therefore \( q(t) = [z_1^*(t) \ z_2^*(t) \ \ldots \ z_l^*(t) \ \dot{z}_1^*(t) \ \ldots \ \dot{z}_l^*(t)]^T \). Applying this transformation to (2) yields the following modal model:
\[ \dot{q}(t) = A^* q(t) + B^* u(t) \]
\[ A^* = T^{-1} A T = \begin{bmatrix} A_1^* & 0 & \ldots & 0 \\ 0 & A_2^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_l^* \end{bmatrix}; \quad A_i^* \triangleq \begin{bmatrix} 0 & 1 \\ -k_i^*/m_i^* & -c_i^*/m_i^* \end{bmatrix} \]
\[ B^* \triangleq T^{-1} B \]
where \( l \) is the number of modal coordinates. The block diagonal form of \( A^* \) implies that it is composed of \( l \) internally independent subsystems. Although the approach for transformation is different, the blocks \( A_i^* \) are the same blocks defined by (8). Examining the form of \( B^* \) shows that these subsystems are externally coupled. It means that each input \( u_i(t) \) affects different modal subsystems simultaneously.

A modal cost function is defined by substituting the similarity relations from (9) into (3):
\[ J = \frac{1}{2} \int_0^\infty q^T T^T Q T q + u^T R u + p^T [A T q + B u - T \dot{q}] \, dt \]
\[ = \frac{1}{2} \int_0^\infty q^T T^T Q T q + u^T R u + p^T T^{-1} A T q + T^{-1} B u - \dot{q} \, dt \]
Denoting
\[ p^* \triangleq T^T \ p; \quad Q^* \triangleq T^T Q \]
(11) takes the following form:
\[ J = \frac{1}{2} \int_0^\infty q^T Q^* q + u^T R u + (p^*)^T [A^* q + B^* u - \dot{q}] \, dt \]
The suitable algebraic Riccati equation for (13) is

\[ -P^*A^* - (A^*)^TP^* - Q^* + P^*S^*P^* = 0 \]

\[ S^* \equiv B^*R^{-1}(B^*)^T = T^{-1}BR^{-1}B^TT^{-T} = T^{-1}ST^{-T} \]

Left and right multiplying each term of this equation by \( T^{-T} \times \ldots \times T \) yields

\[ -T^{-T}P^*A^*T^{-1} - T^{-T}(A^*)^TP^*T^{-1} \]

\[ -T^{-T}Q^*T^{-1} + T^{-T}P^*S^*P^*T^{-1} = 0 \]

Substituting Eqs. (9) and (12) into the last one leads to

\[ -T^{-T}P^*(T^{-1}AT)T^{-1} - T^{-T}(T^TA^TT^{-T})P^*T^{-1} \]

\[ -T^{-T}(T^TQT)T^{-1} + T^{-T}P^*(T^{-1}ST^{-T})P^*T^{-1} = 0 \]

\[ -(T^{-T}P^*T^{-1})A - A^T(T^{-T}P^*T^{-1}) \]

\[ -Q + (T^{-T}P^*T^{-1})S(T^{-T}P^*T^{-1}) = 0 \]

which is the Riccati equation for the original model, described by (2) . Therefore

\[ P = T^{-T}P^*T^{-1} \leftrightarrow P^* = T^TP \]

In the same way the modal feedback is

\[ u(t) = -R^{-1}(B^*)^TP^*q(t) = -G^*q(t) \]

By switching back into the states space

\[ u(t) = -R^{-1}(T^{-1}B)^TP^*(T^{-1}x(t)) = -R^{-1}B^T(T^{-T}P^*T^{-1})x(t) \]

\[ = -R^{-1}B^TPx(t) = -Gx(t) \]

It can be seen that the modal control signal equals to the states space one. The explicit writing of this relation yields

\[ -G^*q(t) = -G^*T^{-1}x(t) = -Gx(t) \]

That concludes that \( G = G^*T^{-1} \).

2.7. Orthonormal Similarity Transformation. When dealing with models, for which \( \Phi_0 \) is an orthonormal matrix, the similarity transformation matrix \( T \) is also orthonormal, i.e. \( TT^T = I \). In that case the previous derivation can be written by using the following relations:

\[ T^{-1} = T^T; \quad T^{-T} = T \]
3. Independent Modal Space Design.

3.1. Independent Modal LQR Control Design.

**Definition 4.** Define \( \tilde{C} \) as some a matrix decomposed from \( Q \) such that
\[
Q = \tilde{C}^T \tilde{C}
\]

**Lemma 2.** If the pair \((A, B)\), defined by (2), is stabilizable and \((A, \tilde{C})\) is detectable, the algebraic Riccati’s equation
\[
-PA - A^T P - Q + PSP = 0
\]
has a unique nonnegative solution \( P \geq 0 \).

**Proof.** See [11].

**Assertion 1.** For a LTI model describing a stable, Rayleigh damped structure, an independent modal LQR control that takes in account only \( k \) modal subsystems can be designed by using matrix \( Q \) of the form:
\[
Q = T^{-T} Q^* T^{-1}; \quad T = T_0 \Phi_0
\]
where
\[
Q^* = \begin{bmatrix} Q_{11}^* & 0 \\ 0 & 0 \end{bmatrix}; \quad Q_{11}^* \in \mathbb{R}^{2k \times 2k}; \quad 1 \leq k < l
\]

**Proof.** Transforming into the modal model yields the states-space equation described in (10). The controlled structure is stable. Therefore \((A^*, B^*)\) is stabilizable.

\( Q^* \) can be decomposed into \( \tilde{C}^* \) by choosing
\[
\tilde{C}^* = \begin{bmatrix} \tilde{C}_{11}^* & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{C}_{11}^* \equiv (Q_{11}^*)^{1/2}
\]

The corresponding partitioning of \( A^* \) is
\[
A^* = \begin{bmatrix} A_{11}^* & 0 \\ 0 & A_{22}^* \end{bmatrix}
\]
Without loss of generality, it can be assumed that \( Q_{11}^* \) is chosen so that the pair \((A_{11}^*, \tilde{C}_{11}^*)\) is observable. Therefore, (18) and (17) constitute a standard form [1] of the pair \((A^*, \tilde{C}^*)\). Such form implies that \( A_{22}^* \) eigenvalues are unobservable, therefore \((A^*, \tilde{C}^*)\) is unobservable. However, the controlled
structure is stable, therefore the pair \((A^*, \tilde{C}^*)\) is detectable [1].

Writing (14) as a block matrix equation yields

\[
\begin{bmatrix}
P_{11}^* & P_{12}^* \\
0 & A_{22}^*
\end{bmatrix}
+ \begin{bmatrix}
A_{11}^* & 0 \\
0 & A_{22}^*
\end{bmatrix}
- \begin{bmatrix}
P_{11}^* & P_{12}^* \\
0 & 0
\end{bmatrix}
= 0
\]

Substituting:

\[
P^* = \begin{bmatrix}
P_{11}^* & 0 \\
0 & 0
\end{bmatrix}
\]

yields:

\[
-\begin{bmatrix}
P_{11}^*A_{11}^* & 0 \\
0 & 0
\end{bmatrix}
- \begin{bmatrix}
A_{11}^*P_{11}^* & 0 \\
0 & 0
\end{bmatrix}
- \begin{bmatrix}
Q_{11}^* & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
P_{11}^*S_{11}^*P_{11}^* & 0 \\
0 & 0
\end{bmatrix}
= 0
\]

That proves that \(P^*\) (19) is a solution for (14). Because \(P^*\) is a nonnegative solution, and because \((A^*, B^*)\) is stabilizable and \((A^*, \tilde{C}^*)\) is detectable, according to Lemma 2, \(P^*\) is the unique nonegative solution.

The non zero terms of (20) results in the following equation:

\[-P_{11}^*A_{11}^* - A_{11}^*P_{11}^* - Q_{11}^* + P_{11}^*S_{11}^*P_{11}^* = 0\]

As \(Q_{11}^*\) is chosen so that the pair \((A_{11}^*, \tilde{C}_{11}^*)\) is observable, there is one strictly positive solution \(P_{11}^* > 0\).

The control signal at the modal space is given by (15). Substituting (19) leads to

\[
u(t) = -R^{-1}[(B_1^*)^T \quad (B_2^*)^T]
\begin{bmatrix}
P_{11}^* & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
q_1(t) \\
q_2(t)
\end{bmatrix}
\]

The matrix left-multiplying \(q(t)\) is the modal feedback gain matrix. It can be seen that only the weighted modal subsystems participate in the feedback.
control signal.

The closed loop system is given by

\[ \dot{x}(t) = A_c x(t); \quad A_c = A - SP \]

The similarity transformation of (21), given by (9), yields the closed loop modal system

\[ \dot{q}(t) = A^\ast q(t) - S^\ast P^\ast q(t) = A_c^\ast q(t) \]

Here \( A_c^\ast \) denotes the closed loop modal state matrix. Its block matrix representation leads to:

\[
A_c = \begin{bmatrix}
A_{11}^\ast & 0 \\
0 & A_{22}^\ast
\end{bmatrix} - \begin{bmatrix}
S_{11}^\ast & S_{12}^\ast \\
S_{21}^\ast & S_{22}^\ast
\end{bmatrix} \begin{bmatrix}
P_{11}^\ast & 0 \\
0 & 0
\end{bmatrix}^\ast
= \begin{bmatrix}
A_{11}^\ast & 0 \\
0 & A_{22}^\ast
\end{bmatrix} - \begin{bmatrix}
S_{11}^\ast P_{11} & 0 \\
S_{21}^\ast P_{11} & 0
\end{bmatrix}
= \begin{bmatrix}
A_{11}^\ast - S_{11}^\ast P_{11}^\ast & 0 \\
- S_{21}^\ast P_{11}^\ast & A_{22}^\ast
\end{bmatrix}
\]

That proves that changes in eigenvalues occur only for the set of modes that was weighted by (16).

3.2. Independent Modal \( \mathcal{H}_\infty \) Control Design.

**Assertion 2.** For a LTI model describing a stable, Rayleigh damped structure, an independent modal \( \mathcal{H}_\infty \) control that takes in account only \( k \) modal subsystems can be designed by using matrix \( C \) of the form:

\[ C = C^\ast T^{-1}; \quad T = T_0 \Phi_0 \]

where

\[ C^\ast = \begin{bmatrix} C_{11}^\ast & 0 \end{bmatrix}; \quad C_{11}^\ast \in \mathbb{R}^{n_u \times 2k}; \quad 1 \leq k < l \]

**Proof.** Denote:

\[ \tilde{S}(\beta) \triangleq BB^T - \beta^2 EE^T; \quad Q = C^T C \]

That makes \( \mathcal{H}_\infty \) Riccati’s equation:

\[ -PA - A^T P - Q + P \tilde{S}(\beta) P = 0 \]
By transforming into the modal form, the modal $\mathcal{H}_\infty$ algebraic Riccati’s equation is:

\begin{equation}
-P^*A^* - (A^*)^TP - Q^* + P^*\tilde{S}^*(\beta)P^* = 0
\end{equation}

where $A^*$ is defined by (10) and

$$
\tilde{S}^*(\beta) = \begin{bmatrix}
\tilde{S}^*_{11}(eta) & \tilde{S}^*_{12}(eta) \\
\tilde{S}^*_{21}(eta) & \tilde{S}^*_{22}(eta)
\end{bmatrix} = T^{-1}S(\beta)T^{-T} = B^*(B^*)^T - \beta^2E^*(E^*)^T
$$

$$
Q^* = T^T(C^TC)T = (C^*)^TC^*
$$

As the controlled structure is stable, the pair $(A^*, B^*)$ is stabilizable.

The partitioning of $A^*$ which corresponds the blocks of $C^*$ is

\begin{equation}
A^* = \begin{bmatrix}
A^*_{11} & 0 \\
0 & A^*_{22}
\end{bmatrix}
\end{equation}

Without loss of generality, it can be assumed that $C^*_{11}$ is chosen so that the pair $(A^*_{11}, \tilde{C}^*_{11})$ is observable. The standard form [1] of the $(A^*, \tilde{C}^*)$ implies that $A^*_{22}$ eigenvalues are unobservable. Therefore, $(A^*, \tilde{C}^*)$ is unobservable. However, the controlled structure is stable, therefore the pair $(A^*, \tilde{C}^*)$ is detectable [1].

(24) is solved by $P^*$ of the form:

\begin{equation}
P^* = \begin{bmatrix}
P^*_{11} & 0 \\
0 & 0
\end{bmatrix}
\end{equation}

which turns out to be the unique nonegative solution, from the same considerations appears in Assertion 1. That reduce (24) into:

$$
-P^*_{11}A^*_{11} - A^*_{11}^TP^*_{11} - Q^*_{11} + P^*_{11}\tilde{S}^*_{11}(\beta)P^*_{11} = 0
$$

As the pair $(A^*_{11}, \tilde{C}^*_{11})$ is observable, there is one strictly positive solution $P^*_{11} > 0$.

The control signals derived by this modal $\mathcal{H}_\infty$ approach are

$$
u(t) = -\begin{bmatrix} (B_1^*)^T & (B_2^*)^T \end{bmatrix} \begin{bmatrix}
P^*_{11} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} q_1(t) \\
q_2(t)
\end{bmatrix}
$$

$$
= \begin{bmatrix} (B_1^*)^TP^*_{11} & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\
q_2(t)
\end{bmatrix}
$$
It can be seen that only the considered modal subsystems participate in the feedback control signal.

The closed loop system is given by

\[ \dot{x}(t) = A_c x(t); \quad A_c = A - BB^T P \]

The transformation of (27) to the modal coordinates yields

\[ \dot{q}(t) = A^* q(t) - B^*(B^*)^T P^* q(t) = A^*_c q(t) \]

Here \( A^*_c \) denotes the closed loop modal state matrix. Its block matrix representation leads to:

\[
A_c = \begin{bmatrix}
A^*_{11} & 0 \\
0 & A^*_{22}
\end{bmatrix} - \begin{bmatrix}
B^*_1 \\
B^*_2
\end{bmatrix} \begin{bmatrix}
(B^*_1)^T & (B^*_2)^T
\end{bmatrix} \begin{bmatrix}
P^*_{11} & 0 \\
0 & 0
\end{bmatrix} 
\]

That proves that changes in eigenvalues occur only for the set of modes that were considered by (23).

**Remark.** The off diagonal term in (28) illustrates the spillover effect [13] in the modal controlled system. It defines the relation between the controlled and the uncontrolled modes.

4. Conclusions. A method for modal LQR and modal \( H_\infty \) control of lumped mass structures was formulated and verified theoretically and numerically. The method includes a procedure for definition of states weighting matrix (LQR) and states output matrix (\( H_\infty \)) such that the control will affect only the desired mode shapes dynamics. It reduces the order of Riccati’s equation that should be solved and formulates the relation between the controlled and uncontrolled modes in the closed loop system. It was proved that reduced modal control does exist and that the closed loop eigenvalues differ from the uncontrolled eigenvalues only for the selected mode shapes.

REFERENCES

ON POSITIVITY OF THE GREEN OPERATOR FOR FUNCTIONAL-DIFFERENTIAL EQUATION

S. LABOVSKIY *

Abstract. A necessary and sufficient condition for positivity of the Green function of the boundary value problem

\[ \mathcal{L}u = -u'' + \int_0^l u(t) r(x, t) \, dt = 0, \quad u(0) = u(l) = 0 \]

is obtained. It is assumed that \( r(x, t) \) is non-decreasing with respect to \( t \).

The second question is study of relation between positivity of the Green function and positive definiteness of the quadratic functional

\[ E(u) = \int_0^l u'^2 \, dx + \int_0^l dx \, u(x) \int_0^l u(t) d_t r(x, t). \]

Key Words. Green function, quadratic functional

AMS(MOS) subject classification. Primary 34K10, 34K06; Secondary 34B27

1. Motivation.

1.1. Positive definiteness and Positivity of the Green function. This work is devoted to

- the relation between Positive definiteness of a quadratic Functional (PF)

\[ E(u) = \int_0^l u'^2 \, dx + \int_0^l dx \, u(x) \int_0^l u(t) d_t r(x, t). \]

* Moscow State University of Economics, Statistics and Informatics
and Positiveness of the Green function (PG) of the problem

\[ \mathcal{L}u = -u'' + \int_0^l u(t)d_t r(x, t) = 0 \]

with boundary conditions

\[ u(0) = u(l) = 0. \]

- necessary and sufficient conditions for positivity of the Green function in the case when the function \( r(x, t) \) is non-decreasing with respect to \( t \)

Under certain symmetry condition (when \( \xi(x, y) = \xi(y, x) \), where \( \xi(x, y) = \int_0^x r(s, y)ds \), if \( r(s, 0) = 0 \)) the equation (2) is the Euler equation for the quadratic functional. Note that deviating equation

\[ -u'' + \sum_i q_i(x)u(h_i(x)) = 0 \]

is a particular case of (2).


Jacobi necessary condition of minimum of functional. Distance between two consecutive zeros.

The problem about positivity of the Green function is studied intensively because of its relation with variational problems. In sufficient conditions of minimum of the functional \( \int_0^l F(x, y, y')dx \) surges the problem of positive definiteness of the quadratic functional

\[ E(u) = \int_0^l (pu'^2 + qu^2)dx. \]

The Euler equation for (5) is the Jacobi equation

\[ -(pu')' + qu = 0, \ x \in [0, l], \]

with boundary conditions (3).

It is well known (see for example [2]) the following assertion.

**Proposition 1 (well known).** Let \( p > 0 \). Following three assertions are equivalent:

1. The quadratic functional (5) is positive definite
2. In the interval $[0, l]$ any nonzero solution of (6) may have at most one zero.

3. The Green function of the boundary value problem (6), (3) is positive.

Because of this theorem, many articles are devoted to estimating of the distance between consecutive zeros. One of basic works is [1].

Note that the equivalence between the first and the third propositions can be expressed as

$$\text{PF} \leftrightarrow \text{PG}.$$ 

1.3. Deviating equation.  For the deviating equation (2) the mentioned above affirmations are not equivalent. The assertion $\text{PF} \rightarrow \text{PG}$ is false. The quadratic functional $E(u)$ can be positive definite, while the Green function of the problem (2), (3) can change sign. See below example 1. Maybe, $\text{PG} \rightarrow \text{PF}$ is true.

2. Positive definiteness and positiveness of the Green function.  $\text{PF}$ is equivalent [8] to unique solvability of the boundary value problem (2),(3) and of all truncated problems on all intervals, lying inside the interval $[0, l]:$

$$-u'' + \int_{x_1}^{x_2} u(t) d_r(x, t) = 0, \; u(x_1) = u(x_2) = 0, \; 0 \leq x_1 < x_2 \leq l.$$ 

Consider two cases. In the first one the function $r(x, t)$ does not increase with respect to $t$. This condition is equivalent to negativity $q_i(x) \leq 0$ for the equation (4). The operator

$$Q u = \int_0^l u(t) d_r(x, t)$$

is negative ($u \geq 0 \rightarrow Qu \leq 0$). In this case the mentioned propositions are equivalent:

**Theorem 1.** If $r(x, t)$ does not increase with respect to $t$ then $\text{PF} \leftrightarrow \text{PG}.$

The situation with non-equivalency of $\text{PF}$ and $\text{PG}$ surges when the operator $Q$ is positive, or when $r(x, t)$ doesn’t decrease with respect to the second argument $t$. Note that for ordinary equation (6) this case corresponds to $q \geq 0$, when the quadratic functional is automatically positive definite. Therefore the Green function is positive too, and any interval $[0, l]$ is a nonoscillation interval. Note that in deviating case $E(u)$ is not automatically positive definite. For example,

$$E(u) = \int_0^l u^2 dx + q \int_0^l u(x)(u(x - \delta) + u(x + \delta)) dx$$
has negative value, if \( u = \sin kx, \ k = \pi/\delta, \ 2l/\delta \) is integer, and
\[
q\delta^2 > \pi^2/2.
\]

**Example 1.** Let’s allow to use the Dirac \( \delta \)-function \( \delta(x) \) for simplicity. The boundary value problem (with conditions \( u(0) = u(l) = 0 \))
\[
\mathcal{L}_0 u + Qu = -u'' + \delta(x-x_0)u(x_1) + \delta(x-x_1)u(x_0) = 0,
\]
where \( x_0 = 2, \ x_1 = 6, \ l = 8 \), has nonzero solution \( u_0 \), changing sign:
\[
u_0(x) = \begin{cases} x & \text{if } x \notin [0, x_0], \\ 4 - x & \text{if } x \in [x_0, x_1], \\ x - 8 & \text{if } x \in [x_1, l]. \end{cases}
\]

For \( \mathcal{L}_0 u + (1 - \varepsilon)Qu \) the corresponding quadratic functional
\[
\int_0^l u^2 \, dx + 2(1 - \varepsilon)u(x_0)u(x_1)
\]
is positive definite but the Green function changes sign.

3. **Positivity of the Green function (in case when \( Q \geq 0 \)).**
In [4],[5] and [7] this problem was considered for a delay equation. Here a necessary and sufficient condition of positivity of the Green function in terms of solvability of auxiliary boundary value problems is obtained.

3.1. **The problem of positivity.** Positivity of the Green function and positivity with respect to boundary conditions.

For the equation (2) represent the corresponding homogeneous equation in the form
\[
\mathcal{L} u = \mathcal{L}_0 u + Qu = f,
\]
where \( \mathcal{L}_0 u = -u'' \), \( Qu = \int_0^l u(t) \mathcal{d}_t r(x, t) \). If the problem for the equation (7) with boundary conditions
\[
u(0) = \alpha_0, \ u(l) = \alpha_1
\]
is uniquely resolvable, then its solution can be represented in the form
\[
u = G f + U \alpha,
\]
where \( U \alpha = \alpha_0 u_0 + \alpha_1 u_1 \) is the solution of the homogeneous equation (2) under boundary conditions (8), \( G \) is integral Green’s operator.
For ordinary equation positivity of $G$ is equivalent to the positivity of $(G,U)$. Thus,

$$G \geq 0 \leftrightarrow (G,U) \geq 0.$$  

For deviating equation these propositions are not equivalent but distinction is not essential.

Below we will consider positivity of both operators $G$ and $U$, i.e. the pair $(G,U)$. This signifies the implication

(10)  
\[ \{f \geq 0, \alpha \geq 0\} \rightarrow u \geq 0. \]

3.2. Assumptions. Assume that the function $r(x,l) - r(x,0)$ (in general case it is the variation in the second argument) and the function $f$ are Lebesgue integrable on $[0,l]$. Solutions of all problems we are seeking in the class $D$ of functions, that have absolutely continuous derivative and satisfying (7) almost everywhere on $[0,l]$. The operator $L$ acts from $D$ to the space $L_1(0,l)$ of Lebesgue integrable on $[0,l]$ functions. It is well known [6], that boundary value problems for the equation (7) under boundary conditions below has the Fredholm property. This problem may be uniquely resolvable, but in contrary case the correspondent homogeneous problem has a nonzero solution.

3.3. Auxiliary problems. Consider two boundary value problems for the equation (7) with boundary conditions

(11) \[ u(0) = \alpha_0, \quad u'(0) = \beta_0 \]

and

(12) \[ u(l) = \alpha_1, \quad u'(l) = \beta_1. \]

Transform the first problem by means of a substitution to an integral equation. If $-u'' = z$, then $u(x) = u(0) + u'(0)x - \int_0^x (x - s)z(s)ds$, or shortly

(13) \[ u = u(0)h + u'(0)h_1 - H_0z. \]

Substituting (13) to (7) we obtain

(14) \[ z - QH_0z = -u(0)Qh - u'(0)Qh_1 + f. \]

If the operator $I - QH_0$ is invertible, then

$$u = u(0)u_1 + u'(0)u_2 + G_0f,$$
where $G_0$ is Green’s operator of the problem (7),(11), $u_1$ and $u_2$ are corresponding solutions of the homogeneous equation (2).

For the problem with boundary conditions (12) we have respectively

\[(15) \quad z - QH_l z = -u(l)Qh + u'(l)Qh_2 + f,\]

and

\[u = u(l)u_3 - u'(l)u_4 + G_l f.\]

3.4. Negativeness of Green functions of problems with conditions (11) and (12). If (14) is uniquely resolvable, and the implication

\[(16) \quad \{u(0) \leq 0, \ u'(0) \leq 0, \ f \geq 0\} \rightarrow z \geq 0,\]

is fulfilled, then operator $G_0$ is negative and functions $u_1$ and $u_2$ are nonnegative.

Let $\rho(QH_0)$ be the spectral radius of the operator $QH_0$, and $\rho(QH_l)$ be the spectral radius of the operator $QH_l$.

Equations (14) and (15) are considered in the space $L_1$. The operator $QH_0$ is positive and compact, therefore the following assertion is fulfilled.

**Proposition 2.** If $\rho(QH_0) < 1$, then $G_0 \leq 0$, $u_1 \geq 0$, $u_2 \geq 0$.

Analogous conclusion can be formulated for the problem with boundary condition (12):

**Proposition 3.** If $\rho(QH_l) < 1$, then $G_l \leq 0$, $u_3 \geq 0$, $u_4 \geq 0$.

Thus, it is stated not only negativeness of Green’s operators but also positiveness in boundary conditions (11),(12).

**Proposition 4 (Nonoscillation).** If

\[(17) \quad \rho(QH_0) < 1 \land \rho(QH_l) < 1,\]

then any solution of homogeneous equation (2) can have at most one simple zero in $[0,l]$.

**Proof.** The solution $u_2$ of homogeneous equation $L_0 u + Qu = 0$ under conditions $u(0) = 0$, $u'(0) = 1$ is positive and has positive derivative on $(0,l]$. The solution $u_4$ under conditions $u(l) = 0$, $u'(l) = -1$ is positive and has negative derivative on $[0,l]$. Arbitrary solution can be represented in the form $c_1 u_2 + c_2 u_4$, and, obviously, does not have multiple zeros. \qed

**Remark 1.** Obviously, propositions 2, 3, 4 remain valid for the equation with parameter $\lambda \in [0,1]$

\[(18) \quad L_0 u + \lambda Qu = 0.\]
3.5. Problem with condition at arbitrary point.

**Proposition 5.** Suppose $\rho(QH_0) < 1 \land \rho(QH_l) < 1$. Then for any $\xi \in [0, l]$ the boundary value problem for the equation (7) under conditions

\[ u(\xi) = u'(\xi) = 0 \tag{19} \]

is uniquely resolvable.

The proof follows from the absence of multiple zeros (proposition 4) and from the Fredholm property. The Green’s operator of this problem denote $G_\xi$ (i.e. $u = G_\xi f$). The solution $L_0 u = z$ under conditions (19) denote $u = -H_\xi z$. Substituting to (7) we will have

\[ z - QH_\xi z = f. \tag{20} \]

Note that if spectral radii of operators $QH_0$ and $QH_l$ are less than unity then it will be true for the operator $QH_\xi$. In fact, consider the equation with parameter

\[ L_0 u + \lambda Q u = f. \tag{21} \]

If suppose that $\rho(QH_\xi) \geq 1$, then for $\lambda = 1/\rho(QH_\xi)$ the equation (18) will have a nonzero solution, satisfying the condition (19). In fact, this problem is equivalent to equation $z - \lambda QH_\xi z = 0$, where $QH_\xi$ acts in $L_1$ and it is positive and compact. Thus, its spectral radius is positive eigenvalue, and the correspondent eigenfunction is positive [3]. But this contradicts to absence of multiple zeros (remark to the proposition 4).

**Proposition 6.** If $\rho(QH_0) < 1 \land \rho(QH_l) < 1$, then the operator $G_\xi$ is negative.

In fact, $f \geq 0 \rightarrow z \geq 0$, where $z$ is the solution of (20). From here $u = G_\xi f \leq 0$, i.e. the operator $G_\xi$ is negative.


**Theorem 2.** Suppose $\rho(QH_0) < 1 \land \rho(QH_l) < 1$. Then the problem (7), (8) is uniquely resolvable, its solution is positive in $(0, l)$, and does not have multiple zeros at the ends of the interval, if $f \geq 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and this triple is not identically zero.

**Proof.** By virtue of Fredholmity and nonoscillation the problem (7),(3) is uniquely resolvable. Suppose $f \geq 0$, $f \not= 0$ and $u$ is the solution of the problem. Let $v$ be the solution of the equation (7) under conditions $v(0) = 0$, $v'(0) = 0$. Then, since $u_2(0) = 0$, $u_2'(0) = 1$,

\[ u = v - \frac{v(l)}{u_2(l)} u_2. \]
From here \( u'(0) = -\frac{v(l)}{w_2(l)} > 0 \).

Let’s show, that \( u(x) > 0 \) on \((0, l)\). In fact, if \( \min u(x) \leq 0 \) on \((0, l)\), then for some \( k \geq 0 \) the function \( u + ku_2 \) will have minimum equal to zero at some point \( \xi \in (0, l) \). At that point \( u' (\xi) = 0 \), therefore \( u \leq 0 \) by virtue of negativeness \( G_\xi \). The obtained contradiction shows positiveness of \( u \).

Note that \( u'(l) < 0 \), analogously as for positiveness of \( u'(0) \).

From the proposition 4 it follows, that the solution of the problem for the equation (2) with boundary conditions (8) is positive on \((0, l)\), if \( \alpha_1 \geq 0 \), \( \alpha_2 \geq 0 \), \( \alpha_1 + \alpha_2 > 0 \).

4. Necessary conditions of positivity of Green’s operator. We will use the following theorem on differential inequality for each problem for the equation (7) with boundary conditions (11) and (12).

**Theorem 3 (On differential inequality).** Suppose there exists a nonnegative solution of the inequalities \( Lu = \psi \leq 0 \), \( u(0) = \alpha \geq 0 \), \( u'(0) = \beta \geq 0 \), and \( \alpha + \beta > 0 \). Then the spectral radius \( \rho(QH_0) < 1 \).

Suppose there exists a nonnegative solution of the inequalities \( Lu = \psi \leq 0 \), \( u(l) = \alpha \geq 0 \), \( -u'(l) = \beta \geq 0 \), and \( \alpha + \beta > 0 \). Then the spectral radius \( \rho(QH_l) < 1 \).

**Proof.** We use the theorem 3 from [9]. Obviously, the operator \( H_0 \) is \( u_0 \)-bounded, where \( u_0(x) = x \). For the operator \( H_l \) the function \( u_0 = l - x \) can be choused.

**Theorem 4.** Suppose the Green’s operator and the operator \( U \) are positive, and, if \( f \geq 0 \), \( \alpha_1 \geq 0 \), \( \alpha_2 \geq 0 \), then

\[ u = Gf + U\alpha \]

is positive on \((0, l)\) and at the ends of the interval does not have multiple zeros.

Then \( \rho(QH_0) < 1 \), \( \rho(QH_l) < 1 \).
Proof. Solutions of problems for the equation (7) with conditions \( u(0) = 0, \) \( u(l) = 1 \) and \( u(0) = 1, \) \( u(l) = 0 \) are positive in \((0,l)\), and at the ends of the interval do not have multiple zeros. These solutions satisfy conditions formulated in the theorem 3. This guaranties the inequalities \( \rho(QH_0) < 1, \) \( \rho(QH_l) < 1. \)

Thus, the condition (17) is necessary and sufficient condition not only for positivity of the Green’s operator, but also for the operator \( U \), and for the strong positivity (absence of multiple zeros at the ends of the interval). Note that nonoscillation is also equivalent to the same condition. In fact, it can be analyzed this proof and the proposition 4.

In summary, we have the assertion.

**Theorem 5.** Following affirmations are equivalent

1. the operator \((G,U)\) is positive, and if \((f,\alpha) \geq 0, (f,\alpha)\) is not identically zero, then \( u = Gf + U\alpha > 0 \) in \((0, l)\) and does not have at the ends of the interval multiple zeros,

2. in \([0, l]\) all solutions of the homogeneous equation (2) can have at most one simple zero,

3. \( \rho(QH_0) < 1 \) and \( \rho(QH_l) < 1. \)

Effective conditions for positivity can be obtained, applying the theorem 3 on differential inequality. Consider the second part of the theorem. Let \( u = e^{-kx}, \) \( k > 0. \) We have the differential inequality

\[
-k^2e^{-kx} + \int_0^l e^{-kt}dr(x,t) \leq 0.
\]

Suppose for some \( \delta > 0 \)

\[
(22) \quad r(x,t) = 0, \quad \text{if } 0 \leq t \leq x - \delta,
\]

Since

\[
\int_0^l e^{-kt}dr(x,t) = e^{-k\delta}r(x,l) + k \int_0^l r(x,t)e^{-kt}dt \\
\leq e^{-k\delta}R(x) + kR(x) \int_{x-\delta}^l e^{-kt}dt = R(x)e^{-k(x-\delta)},
\]

we have

\[
-k^2e^{-kx} + \int_0^l e^{-kt}dr(x,t) \leq e^{-k\delta} \left(-k^2 + R(x)e^{k\delta}\right).
\]
The last expression will be negative if to choose \( k = 2/\delta \)

\[
\delta \sqrt{R(x)} \leq \frac{2}{e}.
\]

Analogously it can be showed that the function \( e^{kt}, \, k > 0 \) satisfies the same differential inequality, if assume that

\[
r(x, t) = r(x, l), \quad \text{for} \quad t \geq x + \delta, \, \delta > 0,
\]

and

\[
\delta \sqrt{R(x)} \leq \frac{2}{e}.
\]

Thus we have following affirmation

**THEOREM 6.** Suppose the conditions (22), (24) and (23), (25) are fulfilled. Then are valid all equivalent assertions of the theorem 5.

In the particular case of the equation \(-u'' + q(x)u(h(x)) = 0, \, q(x) \geq 0,\) the conditions of the last the theorem signify that

\[
x - \delta \leq h(x) \leq x + \delta, \quad \delta \sqrt{q(x)} \leq \frac{2}{e}.
\]

5. The case of non-strict inequalities. Let one of inequalities (17) be non-strict, for instance,

\[
\rho(QH_l) = 1, \quad \rho(QH_0) < 1.
\]

In that case there exists a nonzero nonnegative solution of the equation \( z = QH_l z, \) (see, for example, [3]), and \( u = H_l z \) is a solution of boundary value problem \( Lu = 0, \, u(l) = 0, \, u'(l) = 0. \) Denote it by \( v. \) Then the problem \( Lu = f, \, u(0) = u(l) = 0 \) is uniquely resolvable.

Conjecture: If \( \rho(QH_l) \leq 1 \) and \( \rho(QH_0) \leq 1 \) but for truncated equations these inequalities are strict then positivity is valid except for simplicity of zeros.

REFERENCES


ABOUT SOLVABILITY OF QUASILINEAR SINGULAR
FUNCTIONAL DIFFERENTIAL EQUATION

I. PLAKSINA *

Abstract. This article discusses first order quasilinear functional differential equations with independent variable singularity. Conditions of Cauchy function sign constancy have been obtained. Vallee-Poussin-like theorem and Chaplygin-like theorem have been formulated.

Key Words. Quasilinear equations, functional differential equations, singular equations, Cauchy function, Vallee-Poissin theorem.

AMS(MOS) subject classification. 34K06, 34K26.

This article is devoted to quasilinear singular functional differential equation. Solvability of such equation is studied by the method of differential inequalities. Differential inequalities are considered as Vallee-Poisson theorem. The term 'Vallee-Poissin theorem' was proposed by N. V. Azbelev in 1970ths for theorem about equivalence of list of statements about ordinary differential equation: about differential inequality, about the Cauchy (Green’s) function sign, and others. Later = N. V. Azbelev formulated and proved this theorem for abstract functional = differential equation in the most general form [1].

This article considered singular functional differential equation of special type. It was shown that theory of abstract functional differential equation can be applied to this equation. Then Vallee-Poisson theorem and Chaplygin theorem are applicable to this equation, too.

Singular equations of considered type arise in many practical problems. List of such problems is given in the books [2] and [3].

* Perm National Research Polytechnic University, Perm, Russia
The method of differential inequalities for singular functional differential equation was used for example in the publications [4], [5] and [6].

**Description of researching object.**

1. **Description of researching object.** The main object of research is equation

\[(Lx)(t) \equiv \dot{x}(t) + a(t)x(t) + (Tx)(t) = 3Df(t) \quad t \in [0, b]\]

Here right-hand side \(f\) belongs to the space \(L^p\) of \(p\)-integrable functions \((1 < p < \infty)\) with standard norm \(|z|_{L^p} = 3D\int_0^b |z(t)|^p dt = \frac{1}{p}\).

Solution \(x\) belongs to space \(D^p_0\) of absolutely continuous functions \(x: [0, b] \to \mathbb{R}\) which have \(p\)-integrable derivative and satisfy additional condition \(x(0) = 3D0\). Norm at this space has form \(|x|_{D^p_0} = 3D|x|_{L^p}\).

Coefficient \(a\) is \(p\)-integrable on segment \([0, b]\) for any positive \(\varepsilon\) and satisfies additional condition \(\lim_{t \to 0^+} ta(t) = 3Dk\). Difference \(\dot{a}(t) \equiv a(t) - \frac{k}{t}\) is \(p\)-integrable on = segment \([0, b]\). So coefficient \(a(t)\) is not integrable and therefore operator \(L\) is singular. For example coefficient \(a\) can have form \(a(t) = 3D1\sin t\) on segment \(t \in [0, 1]\).

Operator \(T: D^p_0 \to L^p\) is completely continuous. Let this = operator have form \((T_qx)(t) = 3Dq(t)x_h(t)\). Here \(x_h(t) = 3D\begin{cases} x[h(t)], & \text{if } h(t) \in [0, b] \\ 0, & \text{if } h(t) \notin [0, b] \end{cases}\),

coefficient \(q(t)\) is \(p\)-integrable, delay \(h(t) \leq t\) is = measurable. These conditions guarantee complete continuity and Volterra = property of operator \(T_q\).

Let \(q = 3Dq^+ - q^-\), \(q^+ = 3D\frac{q + |q|}{2}\), \(q^- = 3D\frac{q - |q|}{2}\). Let’s define operators \(T^-: D^p_0 \to L^p\) and \(T^+: D^p_0 \to L^p\) of form \((T^+x)(t) = 3Dq^+(t)x_h(t)\) and \((T^-x)(t) = 3Dq^-(t)x_h(t) = (t)\). So \(T^+\) and \(T^-\) are completely continuous Volterra positive = operators.

Here operators \(T^+: D^p_0 \to L^p\) and \(T^-: D^p_0 \to L^p\) which transform non-negative on segment \([0, b]\) function \(x(t)\) to almost everywhere non-negative function \((T^+x)(t)\) and \((T^- = x)(t)\) will be called positive.

2. **Auxiliary results.** Let \(p'\) be conjugate to \(p\) index \(\left(\frac{1}{p} + \frac{1}{p'} = 3D1\right)\) and \(m = 3Dk + \frac{1}{p'}\) be auxiliary constant.
Let’s define auxiliary operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) of form
\[
(\mathcal{L}_0 x)(t) = 3D^2 x(t) + \left( \frac{k}{t} + \tilde{a}(t) \right) = x(t) \quad \text{and equation} \quad (\mathcal{L}_0 x)(t) = 3D f(t)
\]
on segment \( t \in [0, b] \). The properties of this operator were considered in the paper [7].

**Lemma 1. Proposition.** If \( m > 0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is Fredholm operator of index zero and the equation \((\mathcal{L}_0 x)(t) = 3D f(t)\) has unique solution for any right-hand side.

If \( m < 0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is Fredholm operator of index 1 and the equation \((\mathcal{L}_0 x)(t) = 3D f(t)\) has set of solutions for any right-hand side.

If \( m = 3D0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is not Fredholm operator and the equation \((\mathcal{L}_0 x)(t) = 3D f(t)\) has solution not for any right-hand side.

**Lemma 2.** The operator \( \mathcal{L} \) is Fredholm operator of index zero if and only if \( m > 0 \).

If \( m > 0 \) then the operator \( \mathcal{L}_0 \) is invertible and Cauchy operator \((\mathcal{L}_0)^{-1} = 3DC_0 : L^p \rightarrow D^p_0\) is linear integral Volterra operator (it means \((C_0 z)(t) = 3D \int_0^t C_0(t, s) z(s) \, ds\) and its kernel has form
\[
C_0(t, s) = 3D \left( \frac{t}{s} \right)^{-k} \exp \left\{ - \int_s^t \tilde{a}(\eta) \, d\eta \right\}
\]

Let’s define operator \( \mathcal{L}^+ : D^p_0 \rightarrow L^p \) as sum of the operators \( \mathcal{L}_0 \) and \( T^+ (\mathcal{L}^+ = 3D\mathcal{L}_0 + T^+) \) and consider equation
\[
(\mathcal{L}^+ x)(t) = 3D f(t) \quad t \in [0, b]
\]

**Lemma 3.** The equation (2) has unique solution for any right-hand side \( f \in L^p \) if and only if \( m > 0 \).

**Definition 1.** If \( h(t) > 0 \) for any \( t \in (0, b] \) let’s speak that function \( h(t) \) is not bounded away from zero.

Example of this function is \( h(t) = 3D \frac{t}{2} \).

Let’s formulate conditions of constant sign of the Cauchy function of the equation (2) in case of not bounded away from zero function \( h(t) \).

**Lemma 4.** Let \( m > 0 \). Let also the function \( \left( \frac{t}{h(t)} \right)^k \) be \( p' \)-integrable and the inequality \( \int_t^b \left( \frac{\zeta}{h(\zeta)} \right)^k \exp \left\{ - \int_{h(\zeta)}^{\zeta} \tilde{a}(\eta) \, d\eta \right\} \, d\zeta < 1 \) be fulfilled
for every $t \in (0, b]$. Then the Cauchy function $C^+(t, s)$ of the = equation (2) is strongly positive for $t \geq s$.

Proof of the lemma 4 is based on methods of mathematical analysis.

For example let $h(t) = 3D t^2$. Hence for $\int_0^b |q(\zeta)| d\zeta < \frac{1}{2k} M$ the Cauchy function has constant sign. Here $M = 3D \exp \left\{ \int_0^b |\tilde{a}(\eta)| d\eta = \right\}$.

3. Vallee-Poisson theorem for singular equation (1).

The lemma 4 allows us to formulate Vallee-Poussin theorem for first = order singular functional differential equation.

Let’s write the operator $L$ as difference of operators $= = L^+ - T^-$:

$L = 3D L^+ - T^-.$

Let also define operator $K: C \to C$ as superposition of the = operators $C^+$ and $T^-:$

$K = 3DC^+T^-.$

**Theorem 1.** Let $m > 0$. Let also conditions of the lemma 4 be fulfilled. =

The next conditions are equivalent.

a) There is non-negative on segment $(0, b]$ function $v(t) \in D^p_0$ such that $(L v)(t) > 0$ for almost every $t \in (0, b]$.

b) Spectral radius of the operator $K$ is less than 1: $\rho(K) < 1$.

c) The equation (1) has unique solution for any right-hand = side $f \in L^p$ and also its Green operator is positive: $L^{-1} = 3DC^- \geq 0$.


Let’s consider in the space $D^p_0$ quasilinear equation

(3) 

$$(L x)(t) = 3D f(t, x(t))$$

**Theorem 2.** Let the next conditions be fulfilled.

1. $m > 0$.

2. Conditions of the lemma 4 are fulfilled.

3. There is non-negative on segment $(0, b]$ function $v(t) \in D^p_0$ such that $(L v)(t) > 0$ for almost every $t \in (0, b]$.

4. The function $f: [0, b] \times \mathbb{R} \to \mathbb{R}$ = satisfies Caratheodory conditions.

5. The function $f$ as function of second argument is non-decreasing.

6. Nemytsky operator $N: D^p_0 \to L^p$ of form $(N x)(t) = 3D f(t, x(t))$ is continuous.

7. There are functions $u_0 \in D^p_0$ and $z_0 \in D^p_0$ such that $= u_0 \leq z_0$, 

$(Lu_0)(t) \leq f(t, u_0(t))$ and $(Lz_0)(t) \geq f(t, z_0(t))$. 


Then the equation \((\mathcal{L}x)(t) = 3Df(t, x(t))\) has a solution \(x_0 \in D_0^p\) and \(u_0(t) \leq x_0(t) \leq z_0(t), \ t \in [0, b].\)
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* Perm National Research Polytechnic University, Perm, Russia
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\[(Lx)(t) = \dot{x}(t) + a(t)x(t) + (Tx)(t) = f(t) \quad t \in [0, b]\]

Here right-hand side \( f \) belongs to the space \( L^p \) of \( p \)-integrable functions

\((1 < p < \infty)\) with standard norm \( \|z\|_{L^p} = \left( \int_0^b |z(t)|^p \, dt \right)^{\frac{1}{p}}.\)

Solution \( x \) belongs to space \( D_0^p \) of absolutely continuous functions

\(x: [0, b] \to \mathbb{R}\) which have \( p \)-integrable derivative and satisfy additional condition \( x(0) = 0.\) Norm at this space has form \( \|x\|_{D_0^p} = \|\dot{x}\|_{L^p}.\)

Coefficient \( a \) is \( p \)-integrable on segment \([\varepsilon, b]\) for any positive \( \varepsilon \) and satisfies additional condition \( \lim_{t \to 0^+} ta(t) = k.\) Difference \( \dot{a}(t) \equiv a(t) - \frac{k}{t} \) is \( p \)-integrable on all segment \([0, b].\) So coefficient \( a(t) \) is not integrable and therefore operator \( L \) is singular. For example coefficient \( a \) can have form \( a(t) = \frac{1}{\sin t} \) on segment \( t \in [0, 1].\)

Operator \( T: D_0^p \to L^p \) is completely continuous. Let this operator have form \((T_q x)(t) = q(t)x_h(t).\) Here \( x_h(t) = \begin{cases} x[h(t)], & \text{if } h(t) \in [0, b] \\ 0, & \text{if } h(t) \notin [0, b] \end{cases},\) coefficient \( q(t) \) is \( p \)-integrable, delay \( h(t) \leq t \) is measurable. These conditions guarantee complete continuity and Volterra property of operator \( T_q.\)

Let \( q = q^+ - q^-, \quad q^+ = \frac{q + |q|}{2}, \quad q^- = \frac{q - |q|}{2}.\) Let’s define operators \( T^+: D_0^p \to L^p \) and \( T^-: D_0^p \to L^p \) of form \((T^+ x)(t) = q^+(t)x_h(t)\) and \((T^- x)(t) = q^-(t)x_h(t).\) So \( T^+ \) and \( T^- \) are completely continuous Volterra positive operators.

Here operators \( T^+: D_0^p \to L^p \) and \( T^-: D_0^p \to L^p \) which transform non-negative on segment \([0, b]\) function \( x(t) \) to almost everywhere non-negative function \((T^+ x)(t)\) and \((T^- x)(t)\) will be called positive.

2. Auxiliary results. Let \( p' \) be conjugate to \( p \) index \((\frac{1}{p} + \frac{1}{p'} = 1)\) and \( m = k + \frac{1}{p'} \) be auxiliary constant.

Let’s define auxiliary operator \( L_0: D_0^p \to L^p \) of form

\[(L_0 x)(t) = \dot{x}(t) + \left( \frac{k}{t} + \dot{a}(t) \right) x(t)\] and equation \((L_0 x)(t) = f(t)\) on segment \( t \in [0, b].\) The properties of this operator were considered in paper [7].
Lemma 1. If \( m > 0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is Fredholm operator of index zero and the equation \((\mathcal{L}_0 x)(t) = f(t)\) has unique solution for any right-hand side.

If \( m < 0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is Fredholm operator of index 1 and the equation \((\mathcal{L}_0 x)(t) = f(t)\) has set of solutions for any right-hand side.

If \( m = 0 \) then operator \( \mathcal{L}_0 : D^p_0 \rightarrow L^p \) is not Fredholm operator and the equation \((\mathcal{L}_0 x)(t) = f(t)\) has solution not for any right-hand side.

Lemma 2. The operator \( \mathcal{L} \) is Fredholm operator of index zero if and only if \( m > 0 \).

If \( m > 0 \) then the operator \( \mathcal{L}_0 \) is convertible and Cauchy operator \((\mathcal{L}_0)^{-1} = C_0 : L^p \rightarrow D^p_0\) is linear integral Volterra operator (it means \((C_0 z)(t) = \int_0^t C_0(t, s)z(s) \, ds\)) and its kernel has form

\[
C_0(t, s) = \left( \frac{t}{s} \right)^{-k} \exp \left\{ -\int_s^t \tilde{a}(\eta) \, d\eta \right\}
\]

Let’s define operator \( \mathcal{L}^+ : D^p_0 \rightarrow L^p \) as sum of the operators \( \mathcal{L}_0 \) and \( T^+ (\mathcal{L}^+ = \mathcal{L}_0 + T^+) \) and consider equation

\[
(2) \quad (\mathcal{L}^+ x)(t) = f(t) \quad t \in [0, b]
\]

Lemma 3. The equation (2) has unique solution for any right-hand side \( f \in L^p \) if and only if \( m > 0 \).

Definition 1. If \( h(t) > 0 \) for any \( t \in (0, b] \) let’s speak that function \( h(t) \) is not bounded away from zero.

Example of this function is \( h(t) = \frac{t}{2} \).

Let’s formulate conditions of constant sign of the Cauchy function of the equation (2) in case of not bounded away from zero function \( h(t) \).

Lemma 4. Let \( m > 0 \). Let also the function \( \left( \frac{t}{h(t)} \right)^k \) be \( p' \)-integrable and the inequality \[\int_t^b \left( \frac{\zeta}{h(\zeta)} \right)^k \exp \left\{ \int_{h(\zeta)}^{\zeta} \tilde{a}(\eta) \, d\eta \right\} q^+(\zeta) \, d\zeta < 1 \] be fulfilled for every \( t \in (0, b] \). Then the Cauchy function \( C^+(t, s) \) of the equation (2) is strongly positive for \( t \geq s \).

Proof of lemma 4 is based on methods of mathematical analysis.
For example let $h(t) = \frac{t}{2}$. Hence for $\int_{0}^{b} |q(\zeta)| d\zeta < \frac{1}{2^k \cdot M}$ the Cauchy function has constant sign. Here $M = \exp \left\{ \int_{0}^{b} |\tilde{a}(\eta)| d\eta \right\}$.

3. Vallee-Poisson theorem for singular equation (1). Lemma 4 allows us to formulate Vallee-Poussin theorem for first order singular functional differential equation.

Let’s write the operator $\mathcal{L}$ as difference of operators $\mathcal{L}^+ - T^-:

\mathcal{L} = \mathcal{L}^+ - T^-.

Let also define operator $K : C \to C$ as superposition of the operators $C^+ - T^- : K = C^+ T^-$.

**Theorem 1.** Let $m > 0$. Let also conditions of lemma 4 be fulfilled. The next conditions are equivalent.

a) There is non-negative on segment $(0, b]$ function $v(t) \in D^p_0$ such that $(\mathcal{L} v)(t) > 0$ for almost every $t \in (0, b]$.

b) Spectral radius of the operator $K$ is less than 1: $\rho(K) < 1$.

c) The equation (1) has unique solution for any right-hand side $f \in L^p$ and also its Green operator is positive: $\mathcal{L}^{-1} = C \geq 0$.

4. Chaplygin theorem. Let’s consider in the space $D^p_0$ quasilinear equation

(3) \quad (\mathcal{L} x)(t) = f(t, x(t))

**Theorem 2.** Let the next conditions be fulfilled.

1. $m > 0$.

2. Conditions of lemma 4 are fulfilled.

3. There is non-negative on segment $(0, b]$ function $v(t) \in D^p_0$ such that $(\mathcal{L} v)(t) > 0$ for almost every $t \in (0, b]$.

4. The function $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory conditions.

5. The function $f$ as function of second argument is non-decreasing.

6. Nemysky operator $N : D^p_0 \to L^p$ of form $(N x)(t) = f(t, x(t))$ is continuous.

7. There are functions $u_0 \in D^p_0$ and $z_0 \in D^p_0$ such that $u_0 \leq z_0$, $(\mathcal{L} u_0)(t) \leq f(t, u_0(t))$ and $(\mathcal{L} z_0)(t) \geq f(t, z_0(t))$.

Then the equation $(\mathcal{L} x)(t) = f(t, x(t))$ has solution $x_0 \in D^p_0$ and $u_0(t) \leq x_0(t) \leq z_0(t)$, $t \in [0, b]$. 
REFERENCES


Abstract. This paper presents high-order conservation laws and symmetries for five asymmetric quad-graph equations. Mastersymmetries were presented for four equations.

Key Words. Conservation laws, symmetries, asymmetric, quad-graph, mastersymmetries.

AMS(MOS) subject classification. 35L65, 37K10, 39A14, 70S10, 76M60.

1. Introduction. The first quad-graphs were derived in the works of [4, 5]. Later the theory of integrability of quad-graphs was encouraged by the calculation of Lax pairs for some of them (see [2, 11]). A classification of integrable equations on the quad-graph has been carried out in [1]. There authors have classified all integrable quad-graphs that are consistent on a cube. Conservation laws for quad-graph equations have been presented in [7, 17, 18, 19, 20, 21]. These laws also confirm integrability properties of quad-graph equations. Symmetries for quad-graph equations were researched in [9, 10, 15, 22, 24, 25].

In this article we research conservation laws and symmetries of five quad-graph equations which are asymmetric. These equations are not presented in [1] because they are not invariant under the group $D_4$ of the square symmetries. We shall refer to these equations as to the asymmetric equations. Some of the asymmetric equations have already appeared in literature (see [3, 8, 12, 13, 14]).

All these methods were clearly explained and we do not need to recall them again. To calculate conservation laws we use the direct method (see

* Department of Computer Science and Mathematics, Ariel University Center of Samaria, Ariel 40700, Israel.
2. Asymmetric equations on the quad-graph. In this section we present the asymmetric quad-graph equations and their connection with equations from [1]. Notations which we use here are as follows, \( u \) is a scalar dependent variable that is defined on the domain \( \mathbb{Z}^2 \); we shall use the coordinates \((k, l)\) as the independent variables. For brevity we denote the values of \( u \) at the vertexes of the quad-graph by \( u_{0,0} = u(k, l), \ u_{1,0} = u(k+1, l), \ u_{0,1} = u(k, l+1), \ u_{1,1} = u(k+1, l+1) \). More generally \( u_{i,j} \) denotes \( u(k+i, l+j) \).

Here we use the following operators
\[
\Gamma_{\pm k} : \ P(u(k+i, l+j)) \mapsto \lim_{\epsilon \to 0} P(\epsilon^\pm u(k+i, l+j)), \quad \alpha \mapsto \epsilon^{\mp 1} \alpha,
\]
\[
\Gamma_{\pm l} : \ P(u(k+i, l+j)) \mapsto \lim_{\epsilon \to 0} P(\epsilon^\pm u(k+i, l+j)), \quad \beta \mapsto \epsilon^{\mp 1} \beta.
\]

Application of \( \Gamma_l \) to equation \( H_{3\delta=0} \) (we use names as in ABS) gives
\[
(1) \quad \alpha u_{0,0}u_{1,0} - \beta (u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) = 0.
\]

Equation
\[
(2) \quad \frac{1}{u_{1,0}} - \frac{1}{u_{0,1}} + u_{1,1} - u_{0,0} = 0.
\]
can be obtained from (1) by non-point transformation. This transformation is presented in section 3. Equations
\[
Q1\delta=0 : \quad \alpha (u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta (u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) = 0,
\]
\[
Q3\delta=0 : \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \beta(\alpha^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \alpha(\beta^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) = 0,
\]
transformed by \( \Gamma_l \) respectively
\[
(3) \quad \alpha u_{0,0}u_{1,0} - \beta (u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) = 0,
\]
\[
(4) \quad 3\beta (u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + (\alpha^2 - 1)u_{0,0}u_{1,0} - \alpha \beta u_{0,0}u_{1,0} + u_{1,0}u_{1,1} = 0.
\]

One more equation can be obtained by acting \( \Gamma_l \Gamma_{-k} \) on \( Q3\delta=0 \):
\[
(5) \quad \beta (u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) - u_{0,0}u_{1,0} - \alpha \beta u_{1,0}u_{1,1} = 0.
\]

As we said before we shall refer to equations (1,2,3,4,5) as to the asymmetric equations. Equations (3) and (5) can be transformed to partial cases of the NQC equation (see [20, 12]). Equation (1) was presented in [13]. Equation (2) was presented in [3, 14, 8]. Equation (4) does not seem to appear in the literature before.
3. Conservation laws of the asymmetric equations. In this section we find conservation laws for the asymmetric equations by the direct method, which is fully explained in [7, 19, 20, 21]. Three-point conservation laws have components

\[ F = F(k, l, u_{0,0}, u_{0,1}), \]
\[ G = G(k, l, u_{0,0}, u_{1,0}), \]

that satisfy the following functional equation on solutions of the given quad-graph equation:

\[ F(k+1, l, u_{1,0}, u_{1,1}) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, u_{1,1}) - G(k, l, u_{0,0}, u_{1,0}) = 0. \]

To find these conservation laws we first substitute

\[ u_{1,1} = \omega(k, l, u_{0,0}, u_{1,0}) \]

into (6). Here \( \omega \) denote the result of solving (8) for \( u_{1,1} \). The resulting equation involves \( u_{0,0}, u_{1,0} \) and \( u_{0,1} \), but each instance of \( F \) and \( G \) depends on only two continuous arguments. Therefore we can eliminate terms by repeated differentiation until a partial differential equation for \( F \) is obtained. Having solved that, it is simple to work up the hierarchy of functional-differential consequences of (6) until the general solution is obtained. The same process can also be used to find high order conservation laws.

3.1. Equation (1). Three-point conservation laws for (1) are

\[ F_1 = -\alpha \frac{u_{0,0}}{u_{0,1}}, \quad G_1 = \beta \frac{u_{0,0}}{u_{1,0}} - \beta \frac{u_{1,0}}{u_{0,0}}, \]
\[ F_2 = -(-1)^{k+l} \alpha \frac{u_{0,0}}{u_{1,0}}, \quad G_2 = (-1)^{k+l} \left( \beta \frac{u_{0,0}}{u_{1,0}} + \beta \frac{u_{1,0}}{u_{0,0}} \right). \]

High order conservation laws for (1) are

\[ F_3 = \ln \left( \frac{u_{0,1}}{u_{-1,0}} \right), \quad G_3 = \ln \left( \frac{u_{1,0} - u_{-1,0}}{u_{1,0}} \right), \]
\[ F_4 = \ln \left( \frac{u_{0,-1}}{u_{0,0}} \right), \quad G_4 = -\ln \left( \frac{\beta u_{1,0} - \alpha u_{0,-1}}{u_{0,0}} \right), \]
\[ F_5 = k \ln \left( \frac{u_{0,-1}}{u_{0,0}} \right), \quad G_5 = -(k+1) \ln \left( \frac{\beta u_{1,0} - \alpha u_{0,-1}}{u_{0,0}} \right) + \ln(u_{0,-1}), \]
\[ F_6 = \frac{u_{0,0}}{u_{-1,0}} + \frac{u_{-1,0}}{u_{0,0}}, \quad G_6 = \frac{u_{1,0}}{u_{0,0}} - \frac{u_{-1,0}}{u_{0,0}}. \]
As one can see conservation laws 1,2,3,4,5 for (1) are the reduced cases of conservation laws for $H3_{3=0}$ (see [21]). Conservation law 6 does not correspond to any of five-point conservation laws of $H3_{3=0}$. This law has a very notable form, namely, it can be rewritten as

\begin{equation}
F = v_{0,0} + \frac{1}{v_{0,0}}, \quad G = v_{1,0} - \frac{1}{v_{0,0}},
\end{equation}

where $v_{0,0} = \frac{u_{0,0}}{u_{-1,0}}$. Substituting (7) into (6) we obtain equation (2):

\begin{equation}
\frac{1}{v_{1,0}} - \frac{1}{v_{0,1}} + v_{1,1} - v_{0,0} = 0.
\end{equation}

### 3.2. Equation (2). Equation (2) has five three-point conservation laws

- $F_1 = (-1)^{k+l}(u_{0,1} - u_{0,1}^{-1})$,
- $F_2 = u_{0,1} + u_{0,1}^{-1}$,
- $F_3 = \ln \left( \frac{1 + u_{0,0}u_{0,1}}{u_{0,1}} \right)$,
- $F_4 = \ln (u_{0,1})$,
- $F_5 = k \ln (u_{0,1}) - l \ln \left( \frac{1 + u_{0,0}u_{0,1}}{u_{0,1}} \right)$,

This result was also obtained in [8].

### 3.3. Equation (3). Three-point conservation laws for (3) are

- $F_1 = \frac{\beta u_{0,1}}{u_{0,0}}$,
- $F_2 = (-1)^{k+l}(2 \ln(u_{0,0}) - \ln(\beta))$,
- $F_3 = \ln \left( \frac{u_{0,0} - u_{-1,0}}{u_{0,0} - u_{-1,0}} \right)$,

High order conservation law (3) is

\begin{equation}
F_3 = \ln \left( \frac{u_{0,0} - u_{-1,0}}{u_{0,0} - u_{-1,0}} \right), \quad G_3 = \ln \left( \frac{u_{1,0} - u_{1,0}}{u_{0,0} - u_{1,0}} \right).
\end{equation}

### 3.4. Equation (4). Three-point conservation laws for (4) are

- $F_1 = \ln \left( \frac{u_{0,0} - \beta u_{0,1}}{u_{0,0}} \right)$,
- $F_2 = (-1)^{k+l} \ln \left( \frac{\beta}{u_{0,0}(u_{0,0} - \beta u_{0,1})} \right)$,

\begin{equation}
F_1 = \ln \left( \frac{\alpha u_{0,0} - u_{1,0}}{\alpha u_{1,0} - u_{0,0}} \right), \quad G_1 = \ln \left( \frac{\alpha u_{0,0} - u_{1,0}}{\alpha u_{1,0} - u_{0,0}} \right),
\end{equation}

\begin{equation}
F_2 = (-1)^{k+l} \frac{\beta}{u_{0,0}(u_{0,0} - \beta u_{0,1})} - 1.
\end{equation}
Three-point conservation laws for (5) are
\[ F_3 = - \ln \left( \frac{(\alpha u_{-1,0} - u_{0,0})(\beta u_{0,1} - u_{0,0})}{(\beta u_{0,1} - \alpha u_{-1,0})(u_{0,0} - \alpha u_{-1,0})} \right), \]
\[ G_3 = \ln \left( \frac{(\alpha u_{1,0} - u_{0,0})(u_{0,0} - \alpha u_{-1,0})}{(u_{1,0} - u_{-1,0})^2} \right). \]

High order conservation law for (4) is
\[ F_1 = \ln \left( \frac{\beta u_{0,1} - u_{0,0}}{u_{0,0}} \right), \quad G_1 = - \ln \left( \frac{\alpha u_{1,0} - u_{0,0}}{u_{1,0}} \right), \]
\[ F_2 = (-1)^{k+l} \ln \left( \frac{\beta u_{0,1} - u_{0,0}}{\beta u_{0,1}} \right), \quad G_2 = -(-1)^{k+l} \ln \left( \frac{\alpha u_{1,0} - u_{0,0}}{u_{0,0}} \right), \]
\[ F_3 = (k + l) \ln \left( \frac{\beta u_{0,1} - u_{0,0}}{\beta u_{0,0}} \right) - \ln(u_{0,1}), \quad G_3 = -(k + l) \ln \left( \frac{\alpha u_{1,0} - u_{0,0}}{u_{1,0}} \right) - \ln(u_{0,0}). \]

High order conservation law for (5) is
\[ F_4 = \frac{u_{-1,0} - u_{1,0}}{u_{0,0}} - \frac{u_{-1,0}}{\beta u_{0,1}}, \quad G_4 = \frac{u_{-1,0} - u_{1,0}}{\alpha u_{1,0}} - \frac{u_{-1,0}}{u_{0,0}}. \]

3.5. Equation (5). Three-point conservation laws for (5) are
\[ F_3 = - \ln \left( \frac{(\alpha u_{-1,0} - u_{0,0})(\beta u_{1,0} - u_{0,0})}{(\beta u_{1,0} - \alpha u_{-1,0})(u_{0,0} - \alpha u_{-1,0})} \right), \]
\[ G_3 = \ln \left( \frac{(\alpha u_{1,0} - u_{0,0})(u_{0,0} - \alpha u_{-1,0})}{(u_{1,0} - u_{-1,0})^2} \right). \]

4. Symmetries of the asymmetric equations. The topic of symmetries of quad-graph equations was developed recently in [9, 10, 15, 22, 24, 25]. In this section, we find three-point symmetries for the asymmetric equations by the direct method, which is fully explained in [22]. A characteristic of three-point symmetry for quad-graph equations
\[ P(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha, \beta) = 0, \]
has three components
\[ \eta = \eta_k(k, l, u_{-1,0}, u_{0,0}, u_{1,0}, \alpha, \beta) + \eta_l(k, l, u_{0,-1}, u_{0,0}, u_{0,1}, \alpha, \beta), \]
\[ \xi_1 = \xi_1(\alpha), \]
\[ \xi_2 = \xi_2(\beta). \]
These components satisfy to the linearized symmetry condition
\[ XP = 0 \]
whenever (8) holds. Here
\[ X = \eta \frac{\partial}{\partial u_{0,0}} + (S_k \eta) \frac{\partial}{\partial u_{1,0}} + (S_l \eta) \frac{\partial}{\partial u_{0,1}} + (S_k S_l \eta) \frac{\partial}{\partial u_{1,1}} + \xi_1 \frac{\partial}{\partial \alpha} + \xi_2 \frac{\partial}{\partial \beta}. \]
4.1. Equation (1). By using the direct method we find three-point symmetries for (1)

\[ X_1 = \alpha \partial_{\alpha} + \beta \partial_{\beta}, \]
\[ X_3 = (-1)^{k+l}u_{0,0} \partial_{u_{0,0}}, \]
\[ X_5 = \frac{u_{0,0}(u_{1,0} + u_{-1,0})}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}}, \]
\[ X_7 = \frac{ku_{0,0}(u_{1,0} + u_{-1,0})}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}} + \alpha \partial_{\alpha}. \]

\(X_7\) is mastersymmetry for \(X_6\).

**Definition 1.** A symmetry \(X_m\) is a mastersymmetry for the symmetry \(X\) if it satisfies (see [16, 22, 23])

\[(10) \quad [X_m, X] \neq 0, \quad [[X_m, X], X] = 0.\]

Here \([., .]\) denotes the commutator.

From [22] we know that \(H_{3,0}^{2}\) has two mastersymmetries in the \(k\) and \(l\) directions. Therefore, two infinite hierarchies of local symmetries can be constructed. For (1) we have mastersymmetry only in the \(k\) direction. Therefore, only one infinite hierarchy of local symmetries can be constructed. However we found a higher symmetry for (1) in the \(l\) direction by using the direct method

\[ X_8 = \frac{u_{0,1}(u_{0,2}u_{0,-1}u_{0,-2} + u_{0,0}^2u_{0,-1} + u_{0,0}u_{0,1}u_{0,-2})}{u_{0,-1}u_{0,-2}} \partial_{u_{0,0}}. \]

4.2. Equation (2). Three-point symmetries for (2)

\[ X_1 = (-1)^{k+l}u_{0,0} \partial_{u_{0,0}}, \]
\[ X_2 = \frac{u_{0,0}^2(u_{0,-1} - u_{1,0})}{(u_{0,-1}u_{0,0} + 1)(1 + u_{0,0}u_{1,0})} \partial_{u_{0,0}}, \]
\[ X_3 = \frac{u_{0,0}^2(u_{1,0} - u_{-1,0})}{(u_{-1,0}u_{0,0} - 1)(1 - u_{0,0}u_{1,0})} \partial_{u_{0,0}}, \]
\[ X_4 = \left(\frac{2u_{0,0}^2(u_{0,1} - u_{0,-1})}{(u_{0,-1}u_{0,0} + 1)(1 + u_{0,0}u_{1,0})} + \frac{u_{0,0}(u_{0,0}u_{0,1} - 1)}{1 + u_{0,0}u_{1,0}}\right) \partial_{u_{0,0}}, \]
\[ X_5 = \left(\frac{2u_{0,0}^2(u_{1,0} - u_{-1,0})}{(u_{-1,0}u_{0,0} - 1)(1 - u_{0,0}u_{1,0})} - \frac{u_{0,0}(u_{0,0}u_{1,0} + 1)}{1 - u_{0,0}u_{1,0}}\right) \partial_{u_{0,0}}. \]

Here \(X_4\) and \(X_5\) are mastersymmetries for \(X_2\) and \(X_3\) respectively.
4.3. Equation (3). Three-point symmetries for (3) are

\[ X_1 = \alpha \partial_\alpha + \beta \partial_\beta, \quad X_2 = u_{0,0} \partial_{u_{0,0}}, \quad X_3 = t u_{0,0} \partial_{u_{0,0}} - \beta \partial_\beta, \quad X_4 = \frac{u_{0,1} u_{0,-1} + u_{0,0}^2}{u_{0,-1}} \partial_{u_{0,0}}, \quad X_5 = \frac{(u_{1,0} - u_{0,0})(u_{0,0} - u_{-1,0})}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}}, \]

\[ X_6 = \frac{k(u_{1,0} - u_{0,0})(u_{0,0} - u_{-1,0})}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}} + \alpha \partial_\alpha. \]

Here \( X_6 \) is mastersymmetry for \( X_5 \).

4.4. Equation (4). Three-point symmetries for (4) are

\[ X_1 = u_{0,0} \partial_{u_{0,0}}, \quad X_2 = l u_{0,0} \partial_{u_{0,0}} - \beta \partial_\beta, \quad X_3 = \frac{(\beta u_{0,1} - u_{0,0})(u_{0,-1} - \beta u_{0,0})}{u_{0,-1}} \partial_{u_{0,0}}, \quad X_4 = \frac{(1 + \alpha^2) u_{0,0}(u_{1,0} + u_{-1,0}) - 2 \alpha (u_{0,0}^2 + u_{1,0} u_{-1,0})}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}}, \]

\[ X_5 = \frac{k((1 + \alpha^2) u_{0,0}(u_{1,0} + u_{-1,0}) - 2 \alpha (u_{0,0}^2 + u_{1,0} u_{-1,0}))}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}} + \alpha (\alpha^2 - 1) \partial_\alpha, \]

Here \( X_5 \) is mastersymmetry for \( X_4 \).

4.5. Equation (5). Three-point symmetries for (5) are

\[ X_1 = u_{0,0} \partial_{u_{0,0}}, \quad X_2 = k u_{0,0} \partial_{u_{0,0}} - \alpha \partial_\alpha, \quad X_3 = t u_{0,0} \partial_{u_{0,0}} - \beta \partial_\beta, \quad X_4 = \frac{(\beta u_{0,1} - u_{0,0})(u_{0,-1} - \beta u_{0,0})}{u_{0,-1}} \partial_{u_{0,0}}, \]

\[ X_5 = \frac{(\alpha u_{1,0} - u_{0,0})(\alpha u_{0,0} - u_{-1,0})}{u_{1,0}} \partial_{u_{0,0}}. \]

Equation (5) does not have any five-point mastersymmetry because mastersymmetries of corresponding symmetric equation \( Q_{3=0} \) reduce to point symmetries \( X_2 \) and \( X_3 \). Nevertheless (5) has higher symmetries, for example the higher symmetry in the \( l \) direction is

\[ X_6 = \left( \frac{(\alpha u_{0,0} - u_{-1,0})(\alpha u_{1,0} - u_{0,0})(u_{2,0} u_{-2,0} + u_{0,0}^2)}{u_{2,0} u_{1,0} u_{0,0}} + \frac{(\alpha u_{0,0}^2 + \alpha u_{1,0} u_{-1,0} - u_{0,0} u_{-1,0})^2}{u_{1,0}^2 u_{0,0}} \right) \partial_{u_{0,0}}. \]
5. Conclusion. We presented all three-point conservation laws for the asymmetric equations. For most of the equations we also found higher order conservation laws. Some conservation laws and symmetries can be obtained from the conservation laws found in [21] by action of $\Gamma_{\pm k}$ or $\Gamma_{\pm 1}$. New conservation laws can be obtained by using three-point symmetries as it was shown in [21].

We presented all three-point symmetries for the asymmetric equations. For the equations (1,2,3,4) we found mastersymmetries which allow us to construct infinite hierarchies of local symmetries. Five-point symmetries were found for two equations.

REFERENCES


Homogenization Limit for the Boundary Value Problem With the p-Laplace Operator and a Nonlinear Third Boundary Condition on the Boundary of the Holes in a Perforated Domain

T. Shaposhnikova and A. Podolskiy

Abstract. In the paper, we study homogenization problem for equation with the p-laplace operator in a periodically perforated domain with a nonlinear boundary condition for the flux on the cavity boundaries. The homogenized problem was constructed for all possible values of the parameters that determine scale of the holes and scale of the boundary condition. Under the certain relations on the problem scale, the homogenized equations may have different character of the nonlinearity.

Key Words. Homogenization limit, p-laplace operator, perforated domain, nonlinear third boundary condition, boundary value problem.

AMS(MOS) subject classification. 35B27, 35J66, 35J60, 35J92, 35J62.

Introduction. We study the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u_\varepsilon$ to the boundary value problem for the equation

$$-\Delta_p u_\varepsilon \equiv -\text{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = f$$

in an $\varepsilon$-periodically perforated domain $\Omega_\varepsilon \subset \mathbb{R}^n$, $n \geq 3$, $p \in [2, n)$ with the nonlinear third boundary condition $\partial_{\nu_\varepsilon} u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = \varepsilon^{-\gamma}\sigma(x)$ specified on the boundary of the holes, where $\partial_{\nu_\varepsilon} u_\varepsilon \equiv |\nabla u_\varepsilon|^{p-2}(\nabla u_\varepsilon, \nu)$ and $\nu$ is the outward unit normal vector on the boundary of the holes. It is assumed that the diameter of the holes is equal to $C_0\varepsilon^\alpha$, where $C_0 > 0$, $\alpha > 1$;
γ is arbitrary. The asymptotic behavior of the solution \( u_\varepsilon \) is given for all possible values of the parameters \( \alpha, \gamma \). In particular, it is proved that in case \( \alpha = n/(n - p) \), \( \gamma = \frac{\alpha}{n - p}(p - 1) \) a homogenized problem containing a new nonlinear term, which has to be determined as solution of a functional equation, is constructed.

A theorems are proved stating weak convergence of the original problem solution to the solution of the homogenized problem as \( \varepsilon \to 0 \) for all appropriate coefficients \( \gamma, \alpha \). This problem arising e.g. in modeling diffusion of substances in a domain with inclusions, on the boundary of which we assume nonlinear adsorption. It’s supposed that inclusions are small balls with the radius of order \( \varepsilon \), but the number of such inclusions is large. On the other hand, we assume that the process on the boundary is of order \( \varepsilon^\gamma \). Thereby, we have strong processes, for \( \gamma > 0 \) on the small inclusions.

Passing to the scale limit, effective equations are derived for the considered problem. When \( \alpha \in (1, \frac{n}{n - p}], \gamma = \alpha(n - 1) - n \), adsorption process on the inclusions at the micro-scale gives rise to an effective sink/source term in the macroscopic equation. So, it’s the most interesting case of the problem.

1. Problem statement. Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 3 \), with a smooth boundary \( \partial \Omega \) and let \( Y = (-1/2, 1/2)^n \). Denote by \( G_0 \) the ball of radius 1 centered at the origin of coordinates. For \( \delta > 0 \) and \( \varepsilon > 0 \) we define the sets \( \delta B = \{ x | \delta^{-1}x \in B \} \) and \( \Omega_\varepsilon = \{ x \in \Omega | \rho(x, \partial \Omega) > 2\varepsilon \} \). Define \( a_\varepsilon = C_0 \varepsilon^\alpha \), where \( \alpha > 1 \) and \( C_0 \) is a positive number, and let

\[
G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G^j_\varepsilon,
\]

where \( \Upsilon_\varepsilon = \{ j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\Omega_\varepsilon} \neq \emptyset \} \), \( |\Upsilon_\varepsilon| \cong d \varepsilon^{-n} \), \( d = \text{const} > 0 \), \( \mathbb{Z}^n \) is the set of vectors \( z \) with integer coordinates. Define \( Y^j_\varepsilon = \varepsilon Y + \varepsilon j \), where \( j \in \Upsilon_\varepsilon \) and note that \( \overline{G^j_\varepsilon} \subset Y^j_\varepsilon \) and the center of the ball \( G^j_\varepsilon \) coincides with that of the cube \( Y^j_\varepsilon \). Define

\[
\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, S_\varepsilon = \partial G_\varepsilon, \partial \Omega_\varepsilon = \partial \Omega \cup S_\varepsilon.
\]

Consider the problem

\[
\begin{cases}
-\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\
\partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = \varepsilon^{-\gamma} g(x), & x \in S_\varepsilon, \\
u_\varepsilon = 0, & x \in \partial \Omega,
\end{cases}
\]

(1)

where \( \Delta_p u \equiv \text{div}(|\nabla u|^{p-2}\nabla u), p \in [2, n), \partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu), \nu \) is the outward unit normal to \( S_\varepsilon \) and it is assumed that \( f \in L_q(\Omega_\varepsilon), q = p/(p - 1) \) and \( g \in C(\overline{\Omega}) \). In addition we suppose that \( g(x) \equiv 0 \) if \( \gamma > \alpha(n - 1) - n \).
Assume that $\sigma(x,u)$ is a continuously differentiable function of $x \in \Omega$ and $u \in \mathbb{R}$ such that $\sigma(x,0) = 0$ and there are positive constants $k_1, k_2$ such that

\begin{equation}
(\sigma(x,u) - \sigma(x,v))(u - v) \geq k_1|u - v|^p,
\end{equation}

\begin{equation}
|\sigma(x,u)| \leq k_2|u|^{p-1}.
\end{equation}

**Remark 1.** The homogenization of boundary value problems in perforated domains with a third boundary condition specified on the boundary of the holes has been intensively studied. For $\sigma(x,u) \equiv \varepsilon^{-\gamma} \alpha(x/\varepsilon) u$ and $\gamma \in \mathbb{R}$, $\alpha = 1$, a similar problem for Poisson’s was investigated in [7]. A linear problem in a perforated domain with a third boundary condition was considered in [1], [4], [9]. An analogous problem for an elliptic equation with rapidly oscillating coefficients and a nonlinear boundary condition on the boundary of the holes with $\alpha = 1$ was studied in [2], [3], [5], [10]-[13]. A similar problem for a parabolic equation with $\alpha = 1$ was analyzed in [6].

A function $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial \Omega)$ is the weak solution of problem (1), if it satisfies the integral identity

\begin{equation}
\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla v dx + \varepsilon^{-\gamma} \int_{\partial \Omega_\varepsilon} \sigma(x,u_\varepsilon)v ds = \int_{\Omega_\varepsilon} f v dx + \varepsilon^{-\gamma} \int_{\partial \Omega_\varepsilon} g v ds,
\end{equation}

for an arbitrary function $v \in W^{1,p}(\Omega_\varepsilon, \partial \Omega)$. Let $W^{1,p}(\Omega_\varepsilon, \partial \Omega)$ denote the closure in $W^{1,p}(\Omega_\varepsilon)$ of the set of infinitely differentiable functions in $\Omega_\varepsilon$ that vanish near the boundary $\partial \Omega$.

Applying the methods of [8], [9], we obtain the following result.

**Lemma 1.** Let $u \in W^{1,p}(\Omega_\varepsilon, \partial \Omega)$, $2 \leq p < n$, $n \geq 3$. Then

\begin{equation}
\int_{\Omega_\varepsilon} |u|^p dx \leq K \left\{ a_\varepsilon^{n-1} \varepsilon^n \int_{\partial \Omega_\varepsilon} |u|^p ds + a_\varepsilon^{p-n} \varepsilon^n \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p dx \right\}.
\end{equation}

Here and below, $K$ is a constant independent of $\varepsilon$.

**Lemma 2.** Let $u \in W^{1,p}(Y_\varepsilon)$, $2 \leq p < n$, $n \geq 3$. Then

\begin{equation}
\int_{a_\varepsilon S_0} |u|^p dy \leq K \left\{ a_\varepsilon^{n-1} \varepsilon^{-n} \int_{Y_\varepsilon} |u|^p dy + a_\varepsilon^{p-1} \int_{Y_\varepsilon} |\nabla u|^p dy \right\}.
\end{equation}

Using inequality (5) and the monotonicity method gives the following assertion.
Theorem 1. Problem (1) has a unique weak solution $u_\varepsilon$ that satisfies the estimate:

$$\|u_\varepsilon\|_{W^{1,p}(\Omega, \partial\Omega))} \leq K, \quad \varepsilon^{-\gamma/p}\|u_\varepsilon\|_{L_p(S_\varepsilon)} \leq K$$

By $\tilde{u}_\varepsilon$ we denote an extension of $u_\varepsilon$ to the set $\Omega \setminus \Omega_\varepsilon$ such that $\tilde{u}_\varepsilon \in H^1_0(\Omega)$ and

$$\|	ilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C\|u_\varepsilon\|_{H^1(\Omega)}, \quad \|
abla \tilde{u}_\varepsilon\|_{L^2(\Omega)} \leq C\|
abla u_\varepsilon\|_{L^2(\Omega)}.$$

The possibility of such an extension was proved in [8, 9], so we have that:

$$\|	ilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C.$$

Therefore, there exists a subsequence $\tilde{u}_\varepsilon$ (we preserve the previous notation for it) and a function $u \in H^1_0(\Omega)$ such that:

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega) \quad \text{as } \varepsilon \to 0.$$

Define:

$$z_\varepsilon = \begin{cases} u_\varepsilon, & x \in \Omega_\varepsilon, \\ 0, & x \in G_\varepsilon. \end{cases}$$

$$z^j_\varepsilon = \begin{cases} \frac{\partial u_\varepsilon}{\partial x_j}, & x \in \Omega_\varepsilon, \\ 0, & x \in G_\varepsilon. \end{cases}$$

It follows from (6) that, with respect to some subsequence (denote as the original sequence), as $\varepsilon \to 0$, we have

$$z_\varepsilon \rightharpoonup z \quad \text{in } L^p(\Omega),$$

(8)

$$z^j_\varepsilon \rightharpoonup z^j \quad \text{in } L^p(\Omega).$$

We have that:

$$W = \int_\Omega (\partial_{x_j} \tilde{u}_\varepsilon - z^j_\varepsilon)\varphi dx = \int_{G_\varepsilon} \partial_{x_j} \tilde{u}_\varepsilon\varphi dx \Rightarrow$$

$$\Rightarrow |W| \leq K_1\|
abla \tilde{u}_\varepsilon\|_{L^2(\Omega)}\varepsilon^{(n-1)/2} \to 0,$$

and
\[ \mathcal{V} = \int_{\Omega} (\tilde{u}_\varepsilon - z_\varepsilon) \varphi dx = \int_{G_\varepsilon} \tilde{u}_\varepsilon \varphi dx \Rightarrow |\mathcal{V}| \leq K_2 \| \tilde{u}_\varepsilon \|_{L^2(\Omega)} \varepsilon^{(\alpha-1)n/2} \to 0 \]

for an arbitrary continuous function \( \varphi \in L_p(\Omega) \).

On the other hand we have:

\[ W \to \int_{\Omega} (\partial_{x_j} u - z_j) \varphi dx, \quad \mathcal{V} \to \int_{\Omega} (u - z) \varphi dx \]

for an arbitrary continuous function \( \varphi \in L_p(\Omega) \).

From there we obtain that \( u = z \) and \( \partial_{x_j} u = z_j \) nearly everywhere. So \( u, \partial_{x_j} u \in L_p(\Omega) \).

Next theorems give description of the limit function \( u \) for different \( \alpha \) and \( \gamma \).

1. **The case** \( \alpha = \frac{n}{n-p}, \gamma = \alpha(n-1) - n = \frac{n}{n-p}(p-1) \).

**Theorem 2.** Let \( n \geq 3, 2 \leq p < n, \alpha = n/(n-p), \gamma = \alpha(p-1) \) and \( u_\varepsilon \) be a weak solution of problem (1). Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the problem

\[ \begin{cases} -\Delta_p u + A|H(x,u)|^{p-2}H(x,u) = f(x), \text{ in } \Omega; \\ u = 0, \text{ on } \partial \Omega; \end{cases} \]

Here \( A = \left( \frac{n-p}{p-1} \right)^{p-1} C_0^{n-p}, \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \), and \( H(x,u) \) is the solution of the equation

\[ B_0 |H|^{p-2}H = \sigma(x,u - H) - g(x), \]

where \( B_0 = \left( \frac{n-p}{p-1} \right)^{p-1} C_0^{1-p}. \) Then \( z_\varepsilon \rightharpoonup u \) and \( z_j \rightharpoonup \partial_{x_j} u \) in \( L_p(\Omega) \) as \( \varepsilon \to 0. \)

**Proof.** Integral identity (4) and the monotonicity of \(|\lambda|^{p-2}\lambda \) and \( \sigma(x,u) \) with respect to \( u \) imply that

\[ \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla (v-u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,v)(v-u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} g(v-u_\varepsilon) ds, \]

where \( v \) is an arbitrary function from \( W^{1,p}(\Omega_\varepsilon, \partial \Omega) \).
Denote by $P^j_\varepsilon$ the center of the ball $G^j_\varepsilon = \{ x \in Y^j_\varepsilon : |x - P^j_\varepsilon| < a_\varepsilon \}$. Let $T^j_\varepsilon$ denote the ball of radius $\varepsilon/4$ centered at the point $P^j_\varepsilon$. Let $w^j_\varepsilon (j = 1, \ldots, N(\varepsilon))$ be the solution of the boundary problems

$$
\begin{align*}
\Delta_p w^j_\varepsilon &= 0, \quad x \in T^j_\varepsilon \setminus G^j_\varepsilon; \\
w^j_\varepsilon &= 1, \quad x \in \partial G^j_\varepsilon; \\
w^j_\varepsilon &= 0, \quad x \in \partial T^j_\varepsilon.
\end{align*}
$$

(12)

It is easy to see, that for $p \in [2, n)$,

$$
w^j_\varepsilon = \frac{|x - P^j_\varepsilon|^{\frac{p-n}{p-1}} - (\varepsilon/4)^{\frac{p-n}{p-1}}}{a_\varepsilon^{\frac{p-n}{p-1}} - (\varepsilon/4)^{\frac{p-n}{p-1}}}.
$$

The function $W_\varepsilon \in W_0^{1,p}(\Omega)$ is defined as

$$
W_\varepsilon = \begin{cases} \\
\begin{align*}
w^j_\varepsilon, & x \in T^j_\varepsilon \setminus G^j_\varepsilon, \ j = 1, \ldots, N(\varepsilon) = |\Upsilon_\varepsilon|; \\
1, & x \in G_\varepsilon, \\
0, & x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{N(\varepsilon)} T^j_\varepsilon.
\end{align*} \\
\end{cases}
$$

(13)

We have

$$
W_\varepsilon \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0.
$$

(14)

Let $v = \psi - W_\varepsilon H(x, \psi)$, where $\psi \in C_0^{\infty}(\Omega)$, be a test function in (15). Then

$$
\int_{\Omega_\varepsilon} |\nabla (\psi - W_\varepsilon H(\psi - W_\varepsilon H(\psi - W_\varepsilon H - u_\varepsilon)) + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, \psi - H)(\psi - H - u_\varepsilon)ds \geq \int_{\Omega_\varepsilon} f(\psi - W_\varepsilon H - u_\varepsilon)dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} g(\psi - H - u_\varepsilon)ds.
$$

(15)

Let us pass to the limit in (15) as $\varepsilon \to 0$. Defining

$$
\mathcal{P}_\varepsilon \equiv \int_{\Omega_\varepsilon} (|\nabla (\psi - W_\varepsilon H(x, \psi))|^{p-2} - |\nabla \psi|^{p-2}) \nabla \psi \nabla (\psi - W_\varepsilon H(x, \psi) - u_\varepsilon)dx
$$

we show that $\mathcal{P}_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, taking into account the estimate

$$
\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q dx \leq K \varepsilon^{n(p-q)/(n-p)},
$$

(16)
where $1 \leq q \leq p$ and applying the Minkowski inequality for $p \in [2, 3]$ and the inequality $(a + b)^{p-2} - b^{p-2} \leq (p - 2)a(a + b)^{p-3}$ for $a, b \geq 0$ and $p > 3$, we have, for $p \in [2, 3]$

\[
|\mathfrak{P}_\varepsilon| \leq \int_{\Omega_\varepsilon} \left( (|\nabla \psi | + |\nabla (W_\varepsilon H)|)^{p-2} - |\nabla \psi |^{p-2} ) |\nabla (\psi - W_\varepsilon H - u_\varepsilon) | \right) dx \leq
\]

\[
\leq \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-2} |\nabla \psi | |\nabla (\psi - W_\varepsilon H - u_\varepsilon) | \right) dx \leq
\]

\[
\leq K \left\{ \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-2} dx + \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-1} dx + \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-2} |\nabla u_\varepsilon | dx \right\} \to 0,
\]

and, for $p > 3$

\[
\mathfrak{P}_\varepsilon \leq K \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)| (|\nabla \psi | + |\nabla (W_\varepsilon H)|)^{p-3} (1 + |\nabla (W_\varepsilon H)| + |\nabla u_\varepsilon |) dx \to 0,
\]

as $\varepsilon \to 0$.

Define

\[
\mathfrak{F}_\varepsilon \equiv \int_{\Omega_\varepsilon} \left( |\nabla (\psi - W_\varepsilon H)|^{p-2} - |\nabla (W_\varepsilon H)|^{p-2} \right) |\nabla (W_\varepsilon H)\nabla (\psi - W_\varepsilon H - u_\varepsilon) | dx.
\]

In a similar manner, we find that $\mathfrak{F}_\varepsilon \to 0$ as $\varepsilon \to 0$.

Therefore, the limit of the left-hand side of (15) as $\varepsilon \to 0$ coincides with the limit of the expression

\[
\int_{\Omega_\varepsilon} |\nabla \psi |^{p-2} \nabla \psi \nabla (\psi - W_\varepsilon H - u_\varepsilon) dx +
\]

\[
+ \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, \psi)(\psi - H - u_\varepsilon) ds -
\]

\[
- \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-2} |\nabla (W_\varepsilon H)\nabla (\psi - W_\varepsilon H - u_\varepsilon) | dx.
\]

It is easy to see that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla \psi |^{p-2} \nabla \psi \nabla (\psi - W_\varepsilon H - u_\varepsilon) dx = \int_{\Omega} |\nabla \psi |^{p-2} \nabla \psi \nabla (\psi - u) dx.
\]

It follows from (16) that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^{p-2} |\nabla (W_\varepsilon H)\nabla (\psi - W_\varepsilon H - u_\varepsilon) | dx =
\]

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla W_\varepsilon |^{p-2} |\nabla W_\varepsilon \nabla (|H|^{p-2} H (\psi - W_\varepsilon H - u_\varepsilon)) | dx.
\]
By the definition of $W_{\varepsilon}$, we have
\[ \mathcal{L}_{\varepsilon} \equiv \int \left| \nabla W_{\varepsilon} \right|^{p-2} \nabla W_{\varepsilon} \nabla \left( |H|^{p-2} H (\psi - W_{\varepsilon} H - u_{\varepsilon}) \right) dx = \]
\[ = \sum_{j \in T_{\varepsilon}} \int |\nabla w_{\varepsilon}^{j}|^{p-2} \partial_{\nu} w_{\varepsilon}^{j} |H|^{p-2} H (\psi - u_{\varepsilon}) ds + \]
\[ + \sum_{j \in T_{\varepsilon} \setminus \partial T_{\varepsilon}} \int |\nabla w_{\varepsilon}^{j}|^{p-2} \partial_{\nu} w_{\varepsilon}^{j} |H|^{p-2} (\psi - H - u_{\varepsilon}) ds, \]
where $\partial_{\nu} g$ is the normal derivative of $g$.

It is easy to see that
\[ \partial_{\nu} w_{\varepsilon}^{j} \bigg|_{\partial T_{\varepsilon}^{j}} = \frac{d}{dr} w_{\varepsilon}^{j} \bigg|_{r = \varepsilon/4} = \frac{(n-p)2^{2n-2} C_{0}^{n-p} \varepsilon^{1-p}}{(p-1)(1-a_{\varepsilon}^{p-n} \varepsilon^{p+n} 2^{2n-2p})}, \]
\[ \partial_{\nu} w_{\varepsilon}^{j} \bigg|_{\partial G_{\varepsilon}^{j}} = - \frac{d}{dr} w_{\varepsilon}^{j} \bigg|_{r = \alpha_{\varepsilon}} = \frac{(n-p)\varepsilon^{n-p}}{(p-1)C_{0}(1-a_{\varepsilon}^{p-n} \varepsilon^{p+n} 2^{2n-2p})}. \]

By using this relation and (17)-(21), inequality (15) can be rewritten as
\[ \int_{\Omega_{\varepsilon}} |\nabla \psi|^{p-2} \nabla \psi \nabla (\psi - u) dx + Z_{\varepsilon} + A_{\varepsilon} \varepsilon \sum_{j \in T_{\varepsilon} \setminus \partial T_{\varepsilon}} \int |H|^{p-2} H (\psi - u_{\varepsilon}) ds - \]
\[ - \varepsilon^{-1} \int_{S_{\varepsilon}} \left( \left( \frac{n-p}{p-1} \right)^{p-1} C^{1-p}_{0} |H|^{p-2} \sigma (x, \psi - H) + g \right) (\psi - H - u_{\varepsilon}) ds - \]
\[ - Q_{\varepsilon} \geq \int_{\Omega_{\varepsilon}} f (\psi - W_{\varepsilon} H - u_{\varepsilon}) dx, \]
where $Z_{\varepsilon} \to 0$ as $\varepsilon \to 0$, and
\[ A_{\varepsilon} = \left( \frac{n-p}{p-1} \right)^{p-1} \frac{2^{2n-2} C_{0}^{n-p}}{(1-a_{\varepsilon}^{p-n} \varepsilon^{p+n} 2^{2n-2p})^{p-1}}, \]
\[ Q_{\varepsilon} = \frac{1 - (1-a_{\varepsilon}^{p-n} \varepsilon^{p+n} 2^{2n-2p})^{p-1}}{(1-a_{\varepsilon}^{p-n} \varepsilon^{p+n} 2^{2n-2p})^{p-1}} \left( \frac{n-p}{p-1} \right)^{p-1} \varepsilon^{-1} \int_{S_{\varepsilon}} |H|^{p-2} H (\psi - H - u_{\varepsilon}) ds. \]

It follows from (6) that
\[ \lim_{\varepsilon \to 0} Q_{\varepsilon} = 0. \]

Since $H$ solves equation (10), we find from (22) as $\varepsilon \to 0$ that $u$ satisfies
\[ \int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \nabla (\psi - u) dx + \lim_{\varepsilon \to 0} A_{\varepsilon} \varepsilon \sum_{j \in T_{\varepsilon} \setminus \partial T_{\varepsilon}} \int |H|^{p-2} H (\psi - u_{\varepsilon}) ds \geq \]
\[ \int_{\Omega} f (\psi - u) dx. \]
To find the limit on the left-hand side of (23), we use the lemma proved in [11].

**Lemma 3.** Let \( h_\varepsilon \in H^1_0(\Omega) \) and \( h_\varepsilon \rightharpoonup h \) in \( H^1_0(\Omega) \) as \( \varepsilon \to 0 \). Then

\[
2^{2n-2}\varepsilon \sum_{j \in \Gamma, \partial T_j} h_\varepsilon ds \to \omega_n \int h dx, \quad \varepsilon \to 0.
\]

Lemma 3, combined with (23), implies that

\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \nabla (\psi - u) dx + A \int_{\Omega} |H|^{p-2} H(\psi - u) dx \geq \int_{\Omega} f(\psi - u) dx,
\]

where \( A = \left( \frac{n-p}{p-1} \right)^{p-1} C_0^{n-p} \omega_n \).

Setting in (24) \( \psi = u + \lambda v, \lambda > 0, v \in W^{1,p}_0(\Omega) \), and passing to the limit as \( \lambda \to 0 \), we obtain

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_{\Omega} |H(x,u)|^{p-2} H(x,u) v dx \geq \int_{\Omega} f v dx.
\]

Replacing \( v \) by \(-v\) in (25) yields an integral identity for \( u \):

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_{\Omega} |H(x,u)|^{p-2} H(x,u) v dx = \int_{\Omega} f v dx.
\]

Therefore, \( u \) is a weak solution of problem (9).

\[\square\]

2. **The case** \( \alpha \in (1, n/(n-p)), \gamma = \alpha(n-1) - n \).

**Theorem 3.** Let \( n \geq 3, 2 \leq p < n, \alpha \in (1, n/(n-p)), \gamma = \alpha(n-1) - n \) and \( u_\varepsilon \) be a weak solution of problem (1). Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the problem

\[
\left\{ \begin{array}{l}
-\Delta_p u + A \sigma(x,u) = f(x) + A g(x), \quad \Omega; \\
\quad u = 0, \quad \partial \Omega;
\end{array} \right.
\]

where \( A = C_0^{n-1} |\partial G_0| \). Then \( z_\varepsilon \rightharpoonup u \) and \( z_\varepsilon^j \rightharpoonup \partial_{x_j} u \) in \( L_p(\Omega) \) as \( \varepsilon \to 0 \).

**Proof.** Integral identity (4) and the monotonicity of \( |\lambda|^{p-2} \lambda \) and \( \sigma(x,u) \) with respect to \( u \) imply that
\[
\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla (v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, v)(v - u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} g(v - u_\varepsilon) ds,
\]

where \( v \) is an arbitrary function from \( W^{1,p}(\Omega_\varepsilon, \partial \Omega) \).

Let us pass to the limit in (27) as \( \varepsilon \to 0 \).

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla (v - u_\varepsilon) dx = \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx,
\]

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx = \int_{\Omega} f(v - u) dx.
\]

Let’s prove that for \( \alpha \in (1, \frac{n}{n-p}) \), \( \gamma = \alpha(n-1) - n \) we have following:

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, v)(v - u_\varepsilon) ds = A \int_{\Omega} \sigma(x, v)(v - u) dx.
\]

We define a function \( M_\varepsilon(x) \) as \( Y_\varepsilon \)-periodic solution of the boundary problem:

\[
\begin{align*}
\Delta_p M_\varepsilon &= \mu_\varepsilon, \quad x \in Y_\varepsilon = \varepsilon Y \setminus a_\varepsilon G_0; \\
\partial_{\nu_\varepsilon} M_\varepsilon &= -1, \quad x \in \partial(a_\varepsilon G_0) = S^0_\varepsilon,
\end{align*}
\]

where \( \mu_\varepsilon = \frac{c^n_{n-1} \varepsilon^{\alpha(n-1)-n} |\partial G_0|}{1 - (a_\varepsilon^{-1})^{n-1} |\partial G_0|} \).

It is assumed that:

\[
\int_{Y_\varepsilon} M_\varepsilon(x) dx = 0.
\]

The function \( M_\varepsilon(x) \) satisfies the integral identity:

\[
-\int_{Y_\varepsilon} |\nabla M_\varepsilon|^{p-2} \nabla M_\varepsilon \varphi dx + \int_{S^0_\varepsilon} \varphi ds = \mu_\varepsilon \int_{Y_\varepsilon} \varphi dx.
\]

Applying the methods of [9], we obtain the following result.
LEMMA 4. Let \( u \in W^{1,p}(Y_\varepsilon) \) and \( < u >_{Y_\varepsilon} = 0 \), \( 2 \leq p < n \), \( n \geq 3 \). Then
\[
\|u\|_{L_p(Y_\varepsilon)} \leq K_1\varepsilon \|\nabla u\|_{L_p(Y_\varepsilon)},
\]
where constant \( K_1 \) independent of \( \varepsilon \).

Setting in (33) \( \varphi = M_\varepsilon \) and applying Lemma 2, Lemma 4 and also using the definition of the function \( M_\varepsilon(x) \), we obtain:
\[
\|\nabla M_\varepsilon\|_{L_p(Y_\varepsilon)}^p \leq \left( \int_{S_\varepsilon} M_\varepsilon ds \right) + \mu_\varepsilon \left( \int_{Y_\varepsilon^j} M_\varepsilon dx \right) \leq \left( \int_{S_\varepsilon^j} 1 ds \right)^{p-1} \|M_\varepsilon\|_{L_p(S_\varepsilon^j)} \leq C_1 a_\varepsilon \|M_\varepsilon\|_{L_p(S_\varepsilon^j)} \leq C_2 a_\varepsilon (n-1) \frac{p-1}{p} \|M_\varepsilon\|_{L_p(Y_\varepsilon)} + a_\varepsilon \frac{p-1}{p} \|\nabla M_\varepsilon\|_{L_p(Y_\varepsilon)} \leq C_3 (a_\varepsilon (n-1) \frac{p-1}{p} + a_\varepsilon \frac{p-1}{p}) \|\nabla M_\varepsilon\|_{L_p(Y_\varepsilon)} \leq C_4 a_\varepsilon \frac{p-1}{p} \|\nabla M_\varepsilon\|_{L_p(Y_\varepsilon)} \Rightarrow
\]
\[
\|\nabla M_\varepsilon\|_{L_p(Y_\varepsilon)} \leq K a_\varepsilon^\frac{p}{p-1} \Rightarrow \|\nabla M_\varepsilon\|_{L_p(\Omega_\varepsilon)} \leq C(a_\varepsilon^{-(n-1)} \frac{p}{p-1}).
\]

Let \( M_\varepsilon^j(x) \) be an restriction of \( M_\varepsilon(x) \) on \( Y_\varepsilon^j \).

Knowing the definition of the function \( M_\varepsilon^j(x) \), we can make following transformation
\[
\varepsilon^{-\gamma} \int_{S_\varepsilon^j} \sigma(x,v)(v-u_\varepsilon) ds =
\]
\[
= -\varepsilon^{-\gamma} \sum_{j \in \Omega_\varepsilon} \int_{Y_\varepsilon^j} dv(|\nabla M_\varepsilon^j|^{p-2} \nabla M_\varepsilon^j \sigma(x,v)(v-u_\varepsilon)) =
\]
\[
= \varepsilon^{-\gamma} \sum_{j \in \Omega_\varepsilon} \int_{Y_\varepsilon^j} |\nabla M_\varepsilon^j|^{p-2} \nabla M_\varepsilon^j \nabla (\sigma(x,v)(v-u_\varepsilon)) dx - \varepsilon^{-\gamma} \sum_{j \in \Omega_\varepsilon} \int_{Y_\varepsilon^j} \Delta_p M_\varepsilon^j (v-u_\varepsilon) dx =
\]
\[
= \varepsilon^{-\gamma} \sum_{j \in \Omega_\varepsilon} \int_{Y_\varepsilon^j} |\nabla M_\varepsilon^j|^{p-2} \nabla M_\varepsilon^j \nabla (\sigma(x,v)(v-u_\varepsilon)) dx - \varepsilon^{-\gamma} \sum_{j \in \Omega_\varepsilon} \mu_\varepsilon \int_{Y_\varepsilon^j} (v-u_\varepsilon) dx.
\]

Using (36), we get:
\[
\varepsilon^{-\gamma} \int_{\Omega_\varepsilon} |\nabla M_\varepsilon|^{p-1} |\nabla (\sigma(x,v)(v-u_\varepsilon))| dx \leq C \varepsilon^{-\gamma} \left( \int_{\Omega_\varepsilon} |\nabla M_\varepsilon|^p \right)^{\frac{p-1}{p}} \leq C \varepsilon^{-\alpha(n-1)+n\varepsilon^{(\alpha-1)(p-1)p}} = C \varepsilon^{\frac{1}{p}(n-\alpha(n-1))}.
\]
Taking in account that $\alpha \in (1, \frac{n}{n-p})$, we deduce from (38):

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \sum_{j \in T_{\varepsilon}} \int_{Y_{\varepsilon j}^j} |\nabla M_{\varepsilon}^j|^{p-2} \nabla M_{\varepsilon}^j \nabla (\sigma(x, v)(v - u_\varepsilon)) dx = 0.
\]

Then, using (39) we obtain that the limit of the right-hand side of (36) is equal to:

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \sum_{j \in Y_{\varepsilon}} \mu_{\varepsilon} \int_{Y_{\varepsilon j}^j} (v - u_\varepsilon) dx =
\]

\[
= C_{n-1}^{\varepsilon} |\partial G_0| \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} \sigma(x, v)(v - u_\varepsilon) dx =
\]

\[
= C_{n-1}^{\varepsilon} |\partial G_0| \int_{\Omega} \sigma(x, v)(v - u) dx.
\]

In a similar manner, we find that:

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_{\varepsilon}} g(v - u_\varepsilon) ds = A \int_{\Omega} g(v - u) dx.
\]

Taking in account (40), (41), and passing to the limit as $\varepsilon \to 0$ in (27), we obtain:

\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A \int_{\Omega} \sigma(x, v)(v - u) dx \geq
\]

\[
\geq \int_{\Omega} f(v - u) dx + A \int_{\Omega} g(v - u) dx.
\]

Setting in (42) $\psi = u + \lambda v$, $\lambda > 0$, $v \in W^{1,p}_0(\Omega)$, and passing to the limit as $\lambda \to 0$, we get

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_{\Omega} \sigma(x, u)v dx \geq \int_{\Omega} f v dx + A \int_{\Omega} g v dx.
\]

Replacing $v$ by $-v$ in (43) yields an integral identity for $u$:

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_{\Omega} \sigma(x, u)v dx = \int_{\Omega} f v dx + A \int_{\Omega} g v dx.
\]

Therefore, $u$ is a weak solution of problem (26).
3. The case: $\alpha > \frac{n}{n-p}$, $\gamma$ is an arbitrary. Let $\psi_\varepsilon \in C_0^\infty(\Omega)$: $\psi_\varepsilon^j(\rho) = 0, \rho \geq 2C\varepsilon^\alpha, \psi_\varepsilon^j = 1, \rho \leq C\varepsilon^\alpha$, such as $|\nabla \psi_\varepsilon^j| \leq C_1\varepsilon^{-\alpha}$, where $\rho = |x - P_\varepsilon^j|$. Then, we construct a function: $\psi_\varepsilon(x) = \sum_{j=1}^{N_\varepsilon} \psi_\varepsilon^j(x)$. It is easy to see that $\psi_\varepsilon \to 0$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$.

Let us study the integral identity (4) of the problem (1), assuming that $v = \varphi(1 - \psi_\varepsilon)$.

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (\varphi(1 - \psi_\varepsilon)) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\sigma(x, u_\varepsilon) - g(x))\varphi(1 - \psi_\varepsilon) ds = \int_{\Omega_\varepsilon} f\varphi(1 - \psi_\varepsilon) dx.$$  

It is easy to see that $1 - \psi_\varepsilon = 0, x \in S_\varepsilon$. Therefore, we get:

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (\varphi(1 - \psi_\varepsilon)) dx = \int_{\Omega_\varepsilon} f\varphi(1 - \psi_\varepsilon) dx.$$  

Let us pass to the limit as $\varepsilon \to 0$.

We transform left-hand side of the equation, obtaining the result:

(44) $$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi(1 - \psi_\varepsilon) dx + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (1 - \psi_\varepsilon) \varphi dx.$$  

For the first summand from (44) we have following:

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi(1 - \psi_\varepsilon) dx =$$

$$= \lim_{\varepsilon \to 0} \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi dx - \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi \psi_\varepsilon dx \right) =$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi dx$$

because

$$\left| \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi \psi_\varepsilon dx \right| \leq \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega_\varepsilon} |\nabla \psi_\varepsilon|^p dx \right)^{\frac{1}{p}} \leq$$

$$\leq K\varepsilon^{\frac{n(n-p)-n}{p}} \to 0.$$  

Then we obtain:

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi dx = \int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi dx.$$
For the second summand from (44) we have following:

\[
\left| \int_{\Omega} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (1 - \psi_\varepsilon) \varphi dx \right| = \int_{\bigcup_{j=1}^{N} T_j^{c,\varepsilon}} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla (1 - \psi_\varepsilon) \varphi dx \leq K \|\nabla \psi_\varepsilon\|_{L^p(\Omega)} \|\nabla u_\varepsilon\|_{L^p(\Omega)}^{p-1} \leq K \varepsilon^{\frac{n(n-p)-n}{p}} \to 0.
\]

It is easy to see that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} f \varphi (1 - \psi_\varepsilon) dx = \int_{\Omega} f \varphi dx.
\]

Therefore, passing to the limit in (43) as \( \varepsilon \to 0 \) we obtain integral identity for the function \( u \)

(45) \[
\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \int_{\Omega} f \varphi dx.
\]

So the following theorem is proved.

**Theorem 4.** Let \( n \geq 3, 2 \leq p < n, \alpha > n/(n-p) \), \( \gamma \) is an arbitrary and \( u_\varepsilon \) be a weak solution of problem (1). Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the problem

(46) \[
\begin{cases}
-\Delta_p u = f(x), & \Omega; \\
u(x) = 0, & \partial \Omega;
\end{cases}
\]

Then \( z_\varepsilon \rightharpoonup u \) and \( z^j_\varepsilon \rightharpoonup \partial_{x_j} u \) in \( L^p(\Omega) \) as \( \varepsilon \to 0 \).

4. **The case:** \( \alpha \geq 1, \gamma < \alpha(n-1) - n \). Let’s study integral identity (4) of the problem (1).

(47) \[
\int_{\Omega} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, u_\varepsilon) \varphi ds = \int_{\Omega} f \varphi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} g \varphi ds.
\]

We pass to the limit in (47) as \( \varepsilon \to 0 \).

\[
\lim_{\varepsilon \to 0} \int_{\Omega} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi dx = \int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi dx,
\]
\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f \varphi \, dx = \int_{\Omega} f \varphi \, dx.
\]

Then using (6) we obtain:
\[
\left| \varepsilon^{-\gamma} \int_{S_\varepsilon} d\varphi \, ds \right| \leq K \varepsilon^{-\gamma} |S_\varepsilon| = K \varepsilon^{-\gamma} \varepsilon^{(n-1)-n} \to 0.
\]

In the similar manner we have that
\[
\left| \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, u_\varepsilon) \varphi \, ds \right| \leq \varepsilon^{-\gamma} \int_{S_\varepsilon} |\sigma(x, u_\varepsilon)||\varphi| \leq K \varepsilon^{-\gamma} \|u_\varepsilon\|_{L_p(S_\varepsilon)}^{\frac{p-1}{p}} |S_\varepsilon|^\frac{1}{p} \leq K \varepsilon^{\frac{(n-1)+n-\gamma}{p}} \to 0.
\]

Therefore, we get integral identity:
\[(48) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx.\]

The following theorem is proved.

**Theorem 5.** Let \( n \geq 3, 2 \leq p < n, \alpha \geq 1, \gamma < \alpha(n-1) - n \) and \( u_\varepsilon \) be a weak solution of problem (1). Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the problem
\[
\begin{cases}
-\Delta_p u = f(x), & \Omega;

u(x) = 0, & \partial\Omega;
\end{cases}
\]

Then \( z_\varepsilon \rightharpoonup u \) and \( z_\varepsilon^j \rightharpoonup \partial_{x_j} u \) in \( L_p(\Omega) \) as \( \varepsilon \to 0 \).

**5. The case:** \( 1 \leq \alpha < \frac{n}{n-p}, \gamma > \alpha(n-1) - n \). Suppose in addition that \( g(x) \equiv 0 \). Let’s evaluate \( \|u_\varepsilon\|_{L_p(\Omega_\varepsilon)}^p \). Using (6) and Lemma 1 we obtain:
\[
\|u_\varepsilon\|_{L_p(\Omega_\varepsilon)}^p \leq K(\alpha^{1-n} \varepsilon^n \|u_\varepsilon\|_{L_p(S_\varepsilon)} + \alpha^{p-n} \varepsilon^n \|\nabla u_\varepsilon\|_{L_p(\Omega_\varepsilon)}) =
K(\varepsilon^{\alpha(1-n)+n} \|u_\varepsilon\|_{L_p(S_\varepsilon)} + \varepsilon^{\alpha(p-n)+n} \|\nabla u_\varepsilon\|_{L_p(\Omega_\varepsilon)}) \leq
K(\varepsilon^{\alpha(1-n)+n+\gamma} + \varepsilon^{\alpha(p-n)+n}).
\]

In this case we have \( 1 \leq \alpha < \frac{n}{n-p}, \gamma > \alpha(n-1) - n \), then \( \alpha(1-n)+n+\gamma > 0, \alpha(p-n) + n > 0 \), therefore:
\[(50) \quad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L_p(\Omega_\varepsilon)}^p = 0.\]

So we obtain the following theorem.
Theorem 6. Let \( n \geq 3, 2 \leq p < n, 1 \leq \alpha < \frac{n}{n-p}, \gamma > \alpha(n-1)-n \) and \( u_\varepsilon \) be a weak solution of problem (1). Then \( \|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \to 0 \) as \( \varepsilon \to 0 \).

6. The case: \( \alpha = \frac{n}{n-p}, \gamma > \alpha(n-1)-n = \frac{n}{n-p}(p-1) \).

Theorem 7. Let \( n \geq 3, 2 \leq p < n, \alpha = \frac{n}{n-p}, \gamma > \alpha(n-1)-n = \frac{n}{n-p}(p-1) \) and \( u_\varepsilon \) be a weak solution of problem (1) with \( g(x) \equiv 0 \). Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the problem

\[
\begin{cases}
-\Delta_p u + A|u|^{p-2}u = f(x), & \Omega; \\
u = 0, & \partial \Omega;
\end{cases}
\]

where \( A = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n, \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

Then \( z_\varepsilon \to u \) and \( z'_\varepsilon \to \partial \varepsilon_j u \) in \( L^p(\Omega) \) as \( \varepsilon \to 0 \).

Proof. Integral identity (4) and the monotonicity of \( |\lambda|^{p-2} \lambda \) and \( \sigma(x,u) \) with respect to \( u \) imply that

\[
\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla(v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,v)(v - u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx.
\]

where \( v \) is an arbitrary function from \( W^{1,p}(\Omega_\varepsilon, \partial \Omega) \).

Let \( v = \varphi(1 - W_\varepsilon) \), where \( W_\varepsilon \) is a function defined in (13), that satisfies the following: \( (1 - W_\varepsilon)|_{S_\varepsilon} = 0, (1 - W_\varepsilon)|_{T_{\varepsilon/4}} = 1 \), be a test function in (52). Then

\[
\int_{\Omega_\varepsilon} |\nabla(\varphi - W_\varepsilon \varphi)|^{p-2} \nabla(\varphi - W_\varepsilon \varphi) \nabla(\varphi - W_\varepsilon \varphi - u_\varepsilon) dx - \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,0) u_\varepsilon ds \geq \int_{\Omega_\varepsilon} f(\varphi - W_\varepsilon \varphi - u_\varepsilon) dx.
\]

We transform integral inequality applying that \( \sigma(x,0) = 0 \):

\[
\int_{\Omega_\varepsilon} |\nabla(\varphi - W_\varepsilon \varphi)|^{p-2} \nabla(\varphi - W_\varepsilon \varphi) \nabla(\varphi - W_\varepsilon \varphi - u_\varepsilon) dx \geq \int_{\Omega_\varepsilon} f(\varphi - W_\varepsilon H - u_\varepsilon) dx.
\]

Let us pass to the limit in (53) as \( \varepsilon \to 0 \). Defining
\[ \mathcal{P}_\varepsilon \equiv \int_{\Omega_\varepsilon} \left( |\nabla (\varphi - W\varepsilon \varphi)|^{p-2} - |\nabla \varphi|^{p-2} \right) \nabla \varphi \nabla (\varphi - W\varepsilon \varphi - u_\varepsilon) \, dx. \]

In the similar manner as in (16), (17), in the first case, we prove that \( \mathcal{P}_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

We denote
\[ \mathcal{F}_\varepsilon \equiv \int_{\Omega_\varepsilon} \left( |\nabla (\psi - W\varepsilon H)|^{p-2} - |\nabla (W\varepsilon H)|^{p-2} \right) \nabla (W\varepsilon H) \nabla (\psi - W\varepsilon H - u_\varepsilon) \, dx. \]

In the similar way we obtain that \( \mathcal{F}_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

Therefore, the limit of the left-hand side of (53) as \( \varepsilon \to 0 \) coincides with the limit of the expression
\[ \int_{\Omega_\varepsilon} |\nabla \varphi|^{p-2} \nabla \varphi \nabla (\varphi - W\varepsilon \varphi - u_\varepsilon) \, dx - \int_{\Omega_\varepsilon} |\nabla (W\varepsilon \varphi)|^{p-2} \nabla (W\varepsilon \varphi) \nabla (\varphi - W\varepsilon \varphi - u_\varepsilon) \, dx. \]

It is easy to see that
\[ \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla \varphi|^{p-2} \nabla \varphi \nabla (\varphi - W\varepsilon H \varphi - u_\varepsilon) \, dx = \int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \nabla (\varphi - u) \, dx. \]

It follows from (16) that
\[ \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla (W\varepsilon \varphi)|^{p-2} \nabla (W\varepsilon \varphi) \nabla (\varphi - W\varepsilon \varphi - u_\varepsilon) \, dx = \]
\[ = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla W\varepsilon|^{p-2} \nabla W\varepsilon \nabla \left( |\varphi|^{p-2} \varphi (\varphi - W\varepsilon \varphi - u_\varepsilon) \right) \, dx. \]

By the definition of \( W\varepsilon \) we have
\[ \mathcal{L}_\varepsilon \equiv \int_{\Omega_\varepsilon} |\nabla W\varepsilon|^{p-2} \nabla W\varepsilon \nabla \left( |\varphi|^{p-2} \varphi (\varphi - W\varepsilon \varphi - u_\varepsilon) \right) \, dx = \]
\[ = \sum_{j \in Y^* \rho T^*_j} \int |\nabla w^j_\varepsilon|^{p-2} \partial^j \varphi |\varphi|^{p-2} \varphi (\varphi - u_\varepsilon) \, ds - \]

[Note: The text is a segment of a mathematical document discussing homogenization limits for boundary value problems. The mathematical expressions involve integrals, partial derivatives, and boundary conditions, typical of advanced mathematical physics or engineering contexts.]
\[- \sum_{j \in \mathcal{T}_e \cap \partial \Omega^j_e} \int |\nabla w^j_\varepsilon|^p \partial_{\varepsilon} w^j_\varepsilon |\varphi|^{p-2} \varphi u_\varepsilon ds,\]

where \( \partial_{\varepsilon} g \) is the normal derivative of \( g \).

It is easy to see that

\[
\begin{align*}
\partial_{\varepsilon} w^j_\varepsilon \bigg|_{\partial T^j_\varepsilon} &= \frac{d}{dr} w^j_\varepsilon \bigg|_{r=\varepsilon/4} = -(n-p) \frac{2^{2n-2-p} C_0^\frac{n-p}{p-1} \varepsilon^{-\frac{1}{p-1}}}{(p-1)(1-\frac{n-p}{p-1} \varepsilon^{-\frac{p-n}{p-1}} 2^{2n-2p})}, \\
\partial_{\varepsilon} w^j_\varepsilon \bigg|_{\partial \Omega^j_\varepsilon} &= -\frac{d}{dr} w^j_\varepsilon \bigg|_{r=a_\varepsilon} = \frac{(n-p) \varepsilon^{-\frac{n}{p}}}{(p-1) C_0 (1-\frac{n-p}{p-1} \varepsilon^{-\frac{p-n}{p-1}} 2^{2n-2p})}.
\end{align*}
\]

Then we get:

\[
\begin{align*}
\sum_{j \in \mathcal{T}_e \cap \partial \Omega^j_\varepsilon} \int |\nabla w^j_\varepsilon|^p \partial_{\varepsilon} w^j_\varepsilon |\varphi|^{p-2} \varphi u_\varepsilon ds &\leq \\
\leq K \varepsilon^{-\alpha(p-1)}|S_\varepsilon|^\frac{1}{p} \|u_\varepsilon\|_{L_p(S_\varepsilon)} &\leq K \varepsilon^{-\alpha(p-1)} \to 0.
\end{align*}
\]

Therefore, inequality (53) can be rewritten as

\[
\int_{\Omega_\varepsilon} |\nabla \varphi|^{p-2} \nabla \varphi \nabla (\varphi - u_\varepsilon) dx + Z_\varepsilon + A_\varepsilon \sum_{j \in \mathcal{T}_e \cap \partial \Omega^j_\varepsilon} \int_{\partial T^j_\varepsilon} |\varphi|^{p-2} \varphi (\varphi - u_\varepsilon) ds \geq \\
\geq \int_{\Omega_\varepsilon} f(\psi - W_\varepsilon H - u_\varepsilon) dx,
\]

where \( Z_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) and

\[
A_\varepsilon = \left( \frac{n-p}{p-1} \right)^{p-1} \frac{2^{2n-2-p} C_0^{n-p}}{(1-\frac{n-p}{p-1} \varepsilon^{-\frac{p-n}{p-1}} 2^{2n-2p})^{p-1}}.
\]

Lemma 3 implies that

\[
\lim_{\varepsilon \to 0} A_\varepsilon \varepsilon \sum_{j \in \mathcal{T}_e \cap \partial \Omega^j_\varepsilon} \int_{\partial T^j_\varepsilon} |\varphi|^{p-2} \varphi (\varphi - u_\varepsilon) ds = A \int_{\Omega} |\varphi|^{p-2} \varphi (\varphi - u) dx.
\]

Let us pass to the limit in (54) by using obtained:

\[
(55) \int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \nabla (\varphi - u) dx + A \int_{\Omega} |\varphi|^{p-2} \varphi (\varphi - u) dx \geq \int_{\Omega} f(\varphi - u) dx,
\]
where $A = \left( \frac{n-p}{p-1} \right)^p C_0^{n-p} \omega_n$.

Setting in (55) $\varphi = u + \lambda v$, $\lambda > 0$, $v \in W_0^{1,p}(\Omega)$, and passing to the limit as $\lambda \to 0$, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_\Omega |u|^{p-2} uv dx \geq \int_\Omega fv dx. \quad (56)$$

Replacing $v$ by $-v$ in (56) yields an integral identity for $u$:

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + A \int_\Omega |u|^{p-2} uv dx = \int_\Omega fv dx.$$

Therefore, $u$ is a weak solution of problem (51). \qed

REFERENCES


SYSTEM OF DIFFERENCE EQUATIONS FOR DEFINING THE MARKOV-GEOMETRIC DISTRIBUTION OF ORDER K

E. SHMERLING *

Abstract. Simple formulas for calculating the values of cumulative distribution functions and moments about the origin of Markov-geometric random variables of order $k$ are derived from a system of first-order difference equations for the probabilities of non-occurrence of a success run of length $k$ in the first $n$ trials. Results of experimental calculations that verified the obtained formulas are presented.

AMS(MOS) subject classification. 2000 Mathematics Subject Classification: 60J10; 97K60

Key Words. Markov chain; Geometric distribution of order $k$; Difference equations

1. Introduction. The distribution of the number of trials until the first occurrence of consecutive $k$ successes in Bernoulli trials with success probability $p$ is called a geometric distribution of order $k$. This definition is due to Philippou, Georgiou and Philippou (see [4]). The distribution has drawn interest since De Moivre (1667-1754) first addressed this topic in his works (see [5]) Since Philippou et al. called the distribution a geometric distribution of order $k$ a lot of important results in studying the properties of this distribution and its application in different areas have been obtained by various researchers. The contribution of Johnson, Kotz and Kemp (see [3]) and Balakrishnan and Koutras [1] seems to be very important.

Markov-geometric distribution of order $k$ designated $MG_k$ is an extension of the geometric distribution of order $k$. Since $MG_k$ was first introduced many authors investigated the distribution theory based on a Markov chain and many studies on application of Markov-geometric random variables in

* Department of Computer Science and Mathematics, Ariel University Center of Samaria, Ariel 44837, Israel

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solving problems in reliability theory and many other fields have been ac-
quired (see [1]).

Let \( X_1, X_2, \ldots \) be a time homogeneous two-state Markov chain with trans-
sition probability matrix

\[
\begin{bmatrix}
  p_{00} & p_{01} \\
  p_{10} & p_{11}
\end{bmatrix},
\]

that is,

\[
P_{ij} = P(X_t = j | X_{t-1} = i), \quad t \geq 2, \quad 0 \leq i, j \leq 1
\]

and initial probabilities \( p_j = P(X_1 = j), \quad j = 0, 1 \). The distribution of waiting
time for the first occurrence of a success run of length \( k \) in this case will be
called Markov-geometric distribution of order \( k \).

An equivalent reformulation of this as a two-coin tossing game is the
following: consider two biased coins \( C_1 \) and \( C_2 \) with

\[
P(\text{coin } C_1 \text{ shows head}) = p_{01} = 1 - p_{00} = 1 - P(\text{ coin } C_1 \text{ shows tail})
\]

\[
P(\text{coin } C_2 \text{ shows head}) = p_{11} = 1 - p_{10} = 1 - P(\text{coin } C_2 \text{ shows tail}).
\]

For the first trial, toss another coin with \( p_0 = 1 - p_1 \) as the probability of
tail. In the subsequent tosses, choose \( C_1 \) if tail was observed in the previous
trial and choose \( C_2 \) otherwise. Obviously, the waiting time for the first run
of length \( k \) of heads follows a Markov-geometric distribution of order \( k \).

An explicit expression for the probability generating function has been
derived. It has the form

\[
G(z) = \frac{p_1 + (p_0p_{01} - p_1p_{00})z(p_{11}z)^k}{p_{11} + (p_{01}p_{10} - p_0p_{11})z - p_0p_{10}zA(z)},
\]

where

\[
A(z) = \frac{1 - (p_{11}z)^k}{1 - p_{11}z}.
\]

We can see from formula (2) that \( G(z) \) is a rational function that can be
rewritten in the form \( G(z) = C + U(z)/V(z) \), where \( C \) is a constant, \( U \) is
a polynomial of order \( k \), \( V \) is a polynomial of order \( k + 1 \) and \( V \) and \( U \)
don’t have common zeros. Let us state the following well known property of
random variables with such probability generating functions (see [2]).

Let \( z_1, \ldots, z_{k+1} \) be the distinct zeros of \( V \) (real or complex). Then all the
probabilities \( p_i = P(MG_k = i), \quad i = 1, 2, \ldots \) can be expressed as

\[
p_i = \frac{p_1}{z_1^{i+1}} + \frac{p_2}{z_2^{i+1}} + \ldots + \frac{p_{k+1}}{z_{k+1}^{i+1}}, \quad \text{where}
\]

\[
p_j = \frac{U(s_j)}{V(s_j)}, \quad 1 \leq j \leq k + 1.
\]
Utilizing formula (3) which defines the probability mass function (pmf) of $MG_k$ we can derive the formula for calculating the values of cumulative distribution function (cdf) of $MG_k$ and the formulas for calculating its moments about the origin. Obviously, these formulas are complicated and unsuitable for practical use. However it’s more convenient to calculate the values of cdf and the moments of higher order of $MG_k$ by these formulas than by direct differentiation of (2).

In the presented article we derive simple formulas for calculating the values of cdf of $MG_k$. These formulas are obtained from a system of first order difference equations described in section 2. Simple formulas for calculating the moments of higher order of $MG_k$ are also derived and presented in this section.

Matlab functions for calculating the values of cdf and moments about the origin of $MG_k$ via the presented formulas have been developed (they are presented in Appendices B,C,F,G). Results of experimental calculations carried out utilizing these functions, which proved the validity of the derived formulas are presented in section 3.

**2. Derivation of simple formulas for defining the cdf of $MG_k$ and its moments about the origin.** Let $NR(i)$, $i=1,2,...$ designate the following vector of order $k$:

$$NR(i) = \begin{bmatrix} NR(1,i) \\ \vdots \\ NR(k,i) \end{bmatrix},$$

where $NR(1,i)$ is the probability that no run of length $k$ occurs in first $i$ trials and the $i-th$ trial is a failure, $NR(j,i)$, $1 \leq k$ is the probability that no run of length $k$ occurs during the first $i$ trials, trial with index $i-j+1$ is a failure, trials with indexes greater than $i-j+1$ and less equal to $i$ are successful.

Obviously, the sum $\text{sum}(NR(i)) = \sum_{j=1}^{k} NR(j,i)$ is equal to the probability that no run of length $k$ occurs during the first $i$ trials, and the complement of this sum is the $i-th$ value of cdf of $MG_k$:

$$\text{cdf}(i) = P(MG_k \leq k) = 1 - \text{sum}(NR(i)).$$

Luckily, the sequence $NR(i)$, $i = 1,2,...$ coincides with the solution of a system of difference equations. The following recursive equalities hold for any $i \geq 1$
\[ NR(1, i + 1) = NR(1, i) \cdot p_{00} + NR(2, i) \cdot p_{10} + NR(3, i) \cdot p_{10} + \ldots + NR(k, i) \cdot p_{10} \]

\[ NR(2, i + 1) = NR(1, i) \cdot p_{11} \]

\[ NR(3, i + 1) = NR(2, i) \cdot p_{11} \]

\[ \vdots \]

\[ NR(k, i + 1) = NR(k - 1, i) \cdot p_{11}. \]

In matrix form the system of equalities (4) can be written as

\[ NR(i + 1) = TRM \cdot NR(i), \]

where the transition probability matrix TRM has the form:

\[
TRM = \begin{bmatrix}
p_{00} & p_{10} & p_{10} & \cdots & p_{10} & p_{10} \\
p_{01} & 0 & 0 & \cdots & 0 & 0 \\
0 & p_{11} & 0 & \cdots & 0 & 0 \\
0 & 0 & p_{11} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_{11} & 0.
\end{bmatrix}
\]

Thus, we have the following formula for calculating \( NR(i) \)

\[ NR(i) = TRM^{i-1} \cdot NR(1). \]

Obviously, the initial value \( NR(1) \) has the form

\[
NR(1) = \begin{bmatrix}
p_0 \\
p_1 \\
\vdots \\
0
\end{bmatrix}.
\]

Finally, we have obtained the following simple formula for calculating the values of cdf:

\[ cdf(i) = 1 - \text{sum}(TRM^{i-1} \cdot NR(1)). \]

The values of the pmf of \( MG_k \) can be calculated in the following way:

\[ pmf(i) = p_{11} \cdot NR(k, i - 1), \quad i > 1, \quad pmf(1) \equiv 0. \]
Let $P_{11}$ designate the vector of order $k$: $P_{11} = [0, \ldots, 0, p_{11}]$. The equality can be rewritten in the following way:

$$pmf(i) = P_{11} \cdot NR(i - 1).$$

In view of (5) we obtain the following formula

$$(6) \quad pmf(i) = P_{11} \cdot TRM^{i-2} \cdot NR(1), \ i > 1.$$

This formula enables one to derive a method of calculating the moments about the origin of $MG_k$ alternative to direct differentiation of (2).

The formula for calculating the mean of $MG_k$ derived by differentiation of (2) is rather simple and has the form

$$M_1 = \left( p_{01} + p_{10} \right) - p_1 p_{11}^{k-1} + (p_1 - p_{01}) p_{11}^k,$$

but formulas for calculating the moments about the origin of order higher than 1 derived by differentiation of (2) are very complicated and absolutely unsuitable for practical use.

Luckily, utilizing (6) enables one to derive simple formulas for calculating moments about the origin.

Let’s first derive the formula for the mean of $MG_k$.

$$M_1 = \sum_{i=2}^{\infty} i \cdot pmf(i) = P_{11} \cdot \left( \sum_{i=2}^{\infty} i \cdot TRM^i \right) \cdot TRM^{-2} \cdot NR(1).$$

We have the following well-known formula for calculating the sum of an infinite series $\sum_{i=2}^{\infty} i \cdot q^i$, where $q$ is some positive number less than 1:

$$\sum_{i=2}^{\infty} i \cdot q^i = \frac{q}{(1-q)^2} - q.$$

Substituting $TRM$ instead of $q$ and the identity matrix of order $k$ eye($k$) instead of 1, we obtain the following formula

$$\sum_{i=2}^{\infty} i \cdot TRM^i = TRM \cdot (eye(k) - TRM)^{-2} - TRM.$$

Thus the formula for calculating $M_1$ via $TRM$ takes the form

$$M_1 = P_{11} \cdot (TRM \cdot (eye(k) - TRM)^{-2} - TRM) \cdot TRM^{-2} \cdot NR(1).$$
Now let’s derive the formula for the second moment about the origin of $MG_k$. We have \[ \sum_{i=2}^{\infty} i^2 \cdot q^i = (q^2 + q) \cdot (1 - q)^{-3} - q, \] from which follows the corresponding formula for the sum of matrix series and the formula for calculating $M_2$:

\[
M_2 = P_{11} \cdot ((TRM^2 + TRM) \cdot (\text{eye}(k) - TRM)^{-3} - TRM) \cdot TRM^{-2} \cdot NR(1).
\]

Analogously, simple formulas for calculating moments of $MG_k$ of order greater than 2 can be derived.

3. Results of experimental calculations. Matlab functions for calculating specific values of $cdf$ and $pmf$ of $MG_k$ based on the presented formulas designated $cdf\_mgk$ and $cdf\_pmf$ were developed and presented in Appendices B and D. Functions $pmf\_mgk\_tab$ and $cdf\_mgk\_tab$ that build tables of $pmf$ and $cdf$ distributions, respectively, and also plot the graphs of these functions, are also presented in Appendices C and E.

Functions $mean\_mgk$ for calculating the mean of $MG_k$ and $smao\_mgk$ for calculating the second moment about the origin via the formulas described in section 2 are presented in Appendices F and G.

In experimental calculations we assumed $k = 4$, $p_{11} = 0.5$, $p_{11} = 0.7$, $p_{01} = 0.6$ The results of experimental calculations are presented in tables 1 and 2.

<table>
<thead>
<tr>
<th>$MG_4$</th>
<th>$PMF_MG_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-8</td>
<td>0.1715 0.1029 0.0720 0.0689 0.0686</td>
</tr>
<tr>
<td>9-16</td>
<td>0.0580 0.0506 0.0454 0.0406 0.0359 0.0318 0.0283 0.0251</td>
</tr>
<tr>
<td>17-24</td>
<td>0.0223 0.0198 0.0176 0.0156 0.0139 0.0123 0.0109 0.0097</td>
</tr>
<tr>
<td>25-32</td>
<td>0.0086 0.0076 0.0068 0.0060 0.0053 0.0047 0.0042 0.0037</td>
</tr>
<tr>
<td>33-40</td>
<td>0.0033 0.0029 0.0026 0.0023 0.0020 0.0018 0.0016 0.0014</td>
</tr>
<tr>
<td>41-48</td>
<td>0.0013 0.0011 0.0010 0.0009 0.0008 0.0007 0.0006 0.0005</td>
</tr>
<tr>
<td>49-56</td>
<td>0.0005 0.0004 0.0003 0.0003 0.0002 0.0002 0.0002</td>
</tr>
</tbody>
</table>
Table 2: Cdf distribution

<table>
<thead>
<tr>
<th>$MG_4$</th>
<th>$CDF_{MG_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-8</td>
<td>0 0 0 0.1715 0.2744 0.3464 0.41537 0.4840</td>
</tr>
<tr>
<td>9-16</td>
<td>0.0580 0.0506 0.0454 0.0406 0.0359 0.0318 0.0283 0.0251</td>
</tr>
<tr>
<td>17-24</td>
<td>0.54202 0.5926 0.6380 0.6786 0.7145 0.7464 0.7748 0.7999</td>
</tr>
<tr>
<td>25-32</td>
<td>0.8223 0.8421 0.8598 0.8755 0.8894 0.9017 0.9127 0.9225</td>
</tr>
<tr>
<td>33-40</td>
<td>0.9311 0.9388 0.9457 0.9517 0.9571 0.9619 0.9662 0.9699</td>
</tr>
<tr>
<td>41-48</td>
<td>0.9733 0.9763 0.9789 0.9813 0.9834 0.9852 0.9869 0.9883</td>
</tr>
<tr>
<td>49-56</td>
<td>0.9896 0.9908 0.9918 0.9927 0.9935 0.9942 0.9949 0.9954</td>
</tr>
</tbody>
</table>
The mean and second moment about the origin were calculated. All the values obtained in experimental calculations utilizing the described above functions were verified by comparing them to the calculated values obtained via alternative methods.

**Remark** The developed Matlab functions are compact (the compiled code is short), simple and readable, which makes them convenient for potential users.

**REFERENCES**


**A. TRM and NR(1) initialization function.**

```matlab
function [trm, nr] = init_trm_nr1(k, p1, p11, p01)

% Input Parameters
% k - the order of geometric distribution
% p1 - success probability in the first trial
% p11 - success probability after successful trial
% p01 - success probability after failed trial
% n - number of trials
p10=1-p11; p0=1-p1; p00=1-p01; nr(:,1)=[p0; p1; zeros(k-2,1)];
trm=zeros(k); trm(1,1)=p00; trm(1,2:end)=p10; trm(2,1)=p01; for i=3:k
    trm(i,i-1)=p11;
end;
```

**B. Function for calculating a single cdf value.**

```matlab
function [cdf] = cdf_mgk(k, p1, p11, p01, n)
% Output : cdf - probability that a run of length k
% occurs during first n trials
[trm, nr1]=init_trm_nr1(k, p1, p11, p01);
cdf=1-sum(trm.^((n-1)*nr1(:,1));
```
C. Function for building cdf table and plotting its graph.

function [cdf_tab] = cdf_mgk_tab(k,p1,p11,p01,eps)
% All CDF values less than 1-eps are calculated
n=1; tmp=0; while (tmp<1-eps)
    tmp= cdf_mgk(k,p1,p11,p01,n);
    cdf_tab(n)=tmp;
    n=n+1;
end; cdf_tab=cdf_tab(1:end-1); plot(cdf_tab);

D. Function for calculating a single pmf value.

function p=pmf_mgk(k,p1,p11,p01,n)
% Output : p - probability that the first run of length k
% occurs at n-th trial
[trm,nr1]=init_trm_nr1(k,p1,p11,p01);
vec_p11=[zeros(1,k-1) p11];
if n>1 p=vec_p11*trm^(n-2)*nr1(:,1); else
    p=0;
end;

E. Function for building pmf distribution table and plotting its graph.

function pmf_tab =pmf_mgk_tab(k,p1,p11,p01,eps)
    n=k;
    tmp=1;
    while (tmp>eps)
        tmp= pmf_mgk(k,p1,p11,p01,n);
        pmf_tab(n)=tmp;
        n=n+1;
    end; pmf_tab=pmf_tab(1:end-1);
    plot(pmf_tab);

F. Function for calculating the mean.

function mn = mean_mgk(k,p1,p11,p01)
% Output : mn - mean of Markov-geometric random variable of order k
[trm,nr1]=init_trm_nr1(k,p1,p11,p01);
vec_p11=[zeros(1,k-1) p11];
mn=vec_p11*(trm*(eye(k)-trm)^(-2)-trm)*trm^(-2)*nr1(:,1);

G. Function for calculating second moment about the origin.

function ms = smao_mgk(k,p1,p11,p01)
% Output: ms - the second moment about the origin of
% Markov-geometric random variable of order k
[trm,nr1]=init_trm_nr1(k,p1,p11,p01);
vec_p11=zeros(1,k-1); p11];
ms=vec_p11*((trm^2+trm)*(eye(k)-trm)^(-3)-trm)*trm^(-2)*nr1(:,1);
ON THE SMOOTH PARAMETER-DEPENDENCE OF THE RESOLVENT FUNCTION OF ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED STATE-DEPENDENT DELAY

B. SLEZÁK

Abstract. The continuity, Lipschitz continuity and the differentiability of resolvent function of Abstract Functional Differential Equations with unbounded state-dependent delay are investigated with respect to the parameters and initial functions, without compatibility or monotonicity condition. A nonlinear variation-of-constant formula is proved. As special cases new results are obtained for Abstract Functional Differential Equations and for Abstract Ordinary differential equations, giving stronger results than the classical ones.

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Key Words. Upper semicontinuity of the resolvent function, Lipschitz continuity of the resolvent function, differentiability of resolvent function, nonlinear variation-of-constant formula.

1. Introduction. Let $Y$ be Banach space, $\Theta$ and $\Omega$ be metric spaces and let $r \geq 0$ be fixed, let $f : U \subset \mathbb{R} \times \mathcal{C}([-r,0],Y) \times Y \times \Theta \to Y$ and $\tau : W \subset \mathbb{R} \times \mathcal{C}([-r,0],Y) \times \Omega \to \mathbb{R}^+_0$ be functions, defined on the open sets $U$ and $W$, respectively. Retarded functional differential equation

\begin{align}
    x'(t) &= f(t, x_t, x(t - \tau(t, x_t, \theta), \omega), t \geq \sigma \\
    x(t) &= \phi(t - \sigma) \text{ if } t \in [\sigma - r, \sigma],
\end{align}

will be considered with initial condition, where $x_t \in \mathcal{C}([-r,0],Y)$ denotes the so called segment function

\begin{align}
    x_t : [-r,0] \to Y, s \mapsto x(t + s).
\end{align}

* University of Pannonia, Veszpré, Hungary, email: slezakb@chello.hu
We use the notations of the monograph [4]. The norm $\|\cdot\|_C$ will denote the maximum-norm on $C([-r,0],Y)$, that is $\|\phi\|_C := \max \{\|\phi(t)\| \mid t \in [-r,0]\}$. $W^{1,\infty}_\lambda$ denotes the subspace of $C([-r,0],Y)$, where $\lambda \in [-r,0]$ and

$$\phi \in W^{1,\infty}_\lambda \iff \phi \text{ is lipschitzian on } [\lambda,0],$$

equipped with the norm $\|\cdot\|_{W^{1,\infty}_\lambda}$, where

$$\phi \in W^{1,\infty}_\lambda \implies \|\phi\|_{W^{1,\infty}_\lambda} := \max \{\|\phi\|_C, \min\{L \mid L \text{ is a Lipschitz-constant for } \phi \text{ on } [\lambda,0]\}\}.$$

The image of the solutions can be in an infinite dimensional Banach space $Y$ and the functions $f$ and $\tau$ sometimes can be not continuous.

The function $x$ is said to be a solution of (1.1) on $[\sigma-r,b]$, satisfying the initial condition, if $x_\sigma = \phi$, $(t,x_t) \in \text{dom}(f)$ and $x$ satisfies equation (1.1), whenever $t \in [\sigma,b]$. If $x$ is not differentiable on the interval $[\sigma-r,b]$ but the right-hand derivative of $x$ exists at the point $a$ then $x'(a)$ denotes the right-hand derivative of $x$ at $a$. Sometimes the solutions will be considered on a compact interval $[\sigma-r,b]$; in this case $x'(b)$ denotes the left hand derivative of $x$ at $b$. In the special case when $\tau$ is the constant zero function we get the classical functional differential equations, if furthermore $r = 0$ then we get the ordinary differential equations.

**Definition 1.** Let us denote by $\Phi(\sigma,\phi,\theta,\omega)$ the set of the noncontinuable solutions of the initial value problem (1.1) and let $\Phi^{[a,b]}$ denote the (usually multivalued) function, having $(\sigma,\phi,\theta,\omega)$ in its domain if and only if each element $x$ of $\Phi(\sigma,\phi,\theta,\omega)$ is defined on the interval $[a,b]$, and let $\Phi^{[a,b]}(\sigma,\phi,\theta,\omega)$ the set of the restrictions of these solutions to the interval $[a,b]$. The function $\Phi$ will be named the resolvent function of (1.1).

In this article the properties of the resolvent function will be investigated, rather than the pointwise continuity or pointwise differentiability. On this way stronger results are obtained. The results are usually generalizations of the corresponding results of [14].

In Section 2 the continuity of the resolvent function $\Phi$ is investigating. It will be proved that - under quite weak assumptions - $\Phi^{[a,b]}$ is upper semi-continuous.

Some results are given with respect to the openness of the domain of $\Phi^{[a,b]}$. The question of the openness of $\text{Dom}(\Phi^{[a,b]})$ is closely relative to the question that whether every initial value problem has a solution. In the abstract case it is possible that an initial value problem has not a solution (example is given for instance in [3]). This is closely relative also to the question that whether the noncontinuable solutions reach to the boundary, in the sense of Definition 2 below.
Definition 2. Let $\sigma < b \leq \infty$, let $W \subset \mathbb{R} \times C([-r,0], Y)$ be a non empty set, let $I$ be an interval and let $x : I \rightarrow Y$ be a function, $(\sigma, x_\sigma) \in W$. We say that $x$ reaches to the boundary of $W = \text{Dom}(\tau)$ on the right if for each compact subset $C \subset W$ there exists an element $t_C \in I$ such that $(t, x_t) \in W \setminus C$ for all $t \in I$, $t \geq t_C$.

Let $x$ be a solution of the initial value problem
\begin{equation}
(1.3) \quad x'(t) = f(t, x_t, x(t - \tau(t, x_t))), \quad x_\sigma = \phi
\end{equation}
on the interval $I$.

We say that $x$ reaches to the boundary of the domain of $f$ on the right, if for each compact subset $C \subset \text{Dom}(f)$ there exists an element $t_C \in I$ such that $(t, x_t, x(t - \tau(t, x_t))) \in \text{Dom}(f) \setminus C$ for all $t \in I$, $t \geq t_C$.

We say that $x$ reaches to the boundary on the right if either $x$ reaches to the boundary of the domain of $\tau$ on the right, or $x$ reaches to the boundary of the domain of $f$ on the right.

The question that when the noncontinuable solutions reach to the boundary is detailed in [12].

In the literature the following condition is usually used for the function $\tau$:
\begin{equation}
(1.4) \quad \tau(t, \psi, \theta) \leq r.
\end{equation}

Lemma 2.1 [12] gives an explanation for this fact, under the supposition that the functions $f$ and $\tau$ are continuous. We invoke it below in Lemma 1.

Lemma 1. Let the functions $f$ and $\tau$ be continuous.

(I) Suppose that the initial value problem (1.1) has a solution, whenever $(\sigma, \phi, \theta) \in \text{Dom}(\tau)$ and $(\sigma, \phi, -\tau(\sigma, \phi, \theta), \omega) \in \text{Dom}(f)$. Then the following (i) - (iv) statements hold:

(i) $\tau \leq r$.

(ii) Let $\Lambda$ stand for the function defined by the following formula:
\begin{equation}
(1.5) \quad \Lambda : \text{Dom}(\tau) \rightarrow Y, \quad (t, \psi, \theta) \mapsto \psi(-\tau(t, \psi, \theta)),
\end{equation}

and let $\tilde{f}$ defined on the following way:
\[
\tilde{f}(t, \psi, \theta, \omega) : = f(t, \psi, \Lambda(t, \psi, \theta), \omega),
\]

\[
(t, \psi, \theta) \in \text{Dom}(\tau), (t, \psi, \Lambda(t, \psi, \theta)) \in \text{Dom}(f).
\]

The domain of $\tilde{f}$ is open and $\tilde{f}$ is continuous. The function $x$ is a solution of (1.1) if and only if it is a solution of the initial value problem
\begin{equation}
(1.6) \quad x'(t) = \tilde{f}(t, x_t, \theta, \omega), \text{ if } t \in [\sigma, b], \quad x(t) = \phi(t - \sigma), \text{ if } t \in [\sigma - r, \sigma].
\end{equation}
(iii) If \( x \) is a noncontinuable solution of (1.1) then \( x \) reaches to the boundary of \( \text{Dom} \left( \hat{f} (\cdot, \cdot, \theta, \omega) \right) \) on the right.

(iv) Let \( G \subset \mathbb{R} \times \mathcal{C}([-r,0], Y) \times \Theta \times \Omega \) be a compact set such that if \( (\sigma, \phi, \theta, \omega) \in G \) then \( (\sigma, \phi, \theta) \in \text{Dom} (\tau) \) and \( (\sigma, \phi, \phi(-\tau(t, \phi, \theta)), \omega) \in \text{Dom} (f) \). Denote \( \Phi (\sigma, \phi, \theta, \omega) \) the set of the noncontinuable solutions \( x \) of (1.1), satisfying the initial condition \( x_\sigma = \phi \).

For every neighborhood \( V \subset \mathbb{R} \times \mathcal{C}([-r,0], Y) \times \Theta \times \Omega \) of the compact set \( \text{pr}_{\mathbb{R} \times \mathcal{C}([-r,0], Y)} (G) \) there is a neighborhood \( E \) of \( G \), moreover there are positive numbers \( \alpha \) and \( K \) such that

\[
\left\{ (\sigma, \phi, \theta, \omega) \in E, \ x \in \Phi (\sigma, \phi, \theta, \omega) \right\} \implies \left\{ [\sigma - r, \sigma + \alpha] \subset \text{Dom} (x), \ \{(t, x_t) \mid t \in [\sigma, \sigma + \alpha]\} \subset V, \right.
\]

and \( x \) is lipschitzian on the interval \([\sigma, \sigma + \alpha]\) with the constant \( K \).

(II) Suppose furthermore that \( Y = \mathbb{R}^n \). The initial value problem (1.1) has a solution, whenever \( (\sigma, \phi, \theta) \in \text{Dom} (\tau) \) and \( (\sigma, \phi, \phi(-\tau(\sigma, \phi, \theta)), \omega) \in \text{Dom} (f) \), if and only if \( \tau \leq r \).

Lemma 1 shows that if \( \tau \leq r \) then the initial value problem (1.1) is equivalent to the initial value problem (1.6), where there is not a state-dependent delay. However, in lots of problems condition (1.4) is too strong. In this paper it is sometimes omitted, sometimes it is replaced by property

\[
(1.7) \ \ \sigma - r \leq t - \tau (t, \psi, \theta), \ \text{whenever} \ (t, \psi, \theta) \in \text{Dom} (\tau) \text{ and } t \in [\sigma, b],
\]

where \( \sigma \) is fixed.

**Remark 1.** It is evident that the initial value problem (1.1) has a solution \( x \) on the interval \([\sigma - r, b]\) then \( \sigma - r \leq t - \tau (t, x_t, \theta) \), whenever \( t \in [\sigma, b] \). In some statements this assumption (or the supposition \( \sigma - r < t - \tau (t, x_t, \theta) \)) would be enough, weaker than condition (1.7). For instance if the function \( \tau \) is continuous and locally lipschitzian in its second and third variables, the function \( f \) is continuous and locally lipschitzian in its second, third and fourth variables, and \( \sigma - r < t - \tau (t, x_t, \theta) \), then \( (\phi, \theta, \omega) \) is an interior point of the domain of the function \( \Phi^{[\sigma - r,b]} (\sigma, \cdot, \cdot, \cdot) \). But the solutions are usually not known, hence this weaker supposition does not seem to be useful. That is why we avoid it for the simplicity.

Lemma 2 below gives the possibility that if we investigate the smoothness properties of the resolvent function on a compact interval, then we can suppose that \( \tau \leq r \), without loss of generality.

**Lemma 2.** (Lemma 2.2 [12]) Let \( r \leq \tilde{r} \) and let \( \tilde{\tau} \) stand for the function,
defined by the following formula:

\[
\tilde{\tau} : \left\{ \left( t, \tilde{\psi} \right) \mid \tilde{\psi} \in C \left( [\tilde{r}, 0], Y \right), \left( t, \tilde{\psi}_{|[-r, 0]} \right) \in \text{Dom} (\tau) \right\} \to \mathbb{R},
\]

\[
\left( t, \tilde{\psi} \right) \longmapsto \tau \left( t, \tilde{\psi}_{|[-r, 0]} \right),
\]

and let \( \tilde{f} \) stand for the function, defined on the following way:

\[
\tilde{f} : \left\{ \left( t, \tilde{\psi}, y \right) \mid \tilde{\psi} \in C \left( [\tilde{r}, 0], Y \right), \left( t, \tilde{\psi}_{|[-r, 0]}, y \right) \in \text{Dom} (f) \right\} \to Y,
\]

\[
\left( t, \tilde{\psi}, y \right) \longmapsto f \left( t, \tilde{\psi}_{|[-r, 0]}, y \right).
\]

(i) \( \tilde{\tau} \) and \( \tilde{f} \) are continuous or lipschitzian or (\( n \) times, continuously) differentiable functions in their some variables if \( \tau \) and \( f \) have these properties, respectively. \( \text{Dom} (\tilde{\tau}) \) and \( \text{Dom} (\tilde{f}) \) are open if \( \text{Dom} (\tau) \) and \( \text{Dom} (f) \) are open.

(ii) Let the function \( x \) be a solution of the initial value problem

\[
(1.8) \quad x' (t) = f \left( t, x_t, x \left( t - \tau (t, x_t) \right) \right), \quad x_\sigma = \phi.
\]

If \( \tilde{\phi} \in C \left( [\tilde{r}, 0], Y \right) \) and \( \tilde{\phi}_{|[-r, 0]} = \phi \) then the function

\[
\tilde{x} (t) = \left\{ \begin{array}{ll}
\tilde{\phi} (t - \sigma) & \text{if } t \in [\sigma - \tilde{r}, \sigma - r] \\
x (t) & \text{if } t \geq \sigma - r \text{ and } t \in \text{Dom} (x)
\end{array} \right.
\]

is a solution of the initial value problem

\[
(1.9) \quad \tilde{x}' (t) = \tilde{f} \left( t, \tilde{x}_t, \tilde{x} \left( t - \tilde{\tau} (t, \tilde{x}_t) \right) \right), \quad \tilde{x}_\sigma = \tilde{\phi},
\]

where \( \tilde{x}_t : [-\tilde{r}, 0] \to Y, \ s \longmapsto \tilde{x} (t + s) \).

(iii) If the function \( \tilde{x} \) is a solution of the initial value problem (1.9), then \( x = \tilde{x}_{|[\text{Dom} (\tilde{x}) \cap [\sigma - r, \infty]}} \) is a solution of the initial value problem (1.8), where \( \phi := \tilde{\phi}_{|[-r, 0]} \), if and only if for each point \( t \geq \sigma, t \in \text{Dom} (x) \) the inequality

\[
\tau (t, x_t) \leq t - \sigma + r
\]

holds.

The only problem is that the existence of the solution \( \tilde{x} \) of the consequence-initial value problem (1.9) on the interval \( [\sigma, b] \) does not imply the existence of the solution of (1.8) on this interval. The following consequence of Lemma 2, which we need in the proof of Theorem 2, gives a relation between the domain of \( x \) and \( \tilde{x} \).

**Corollary 1.** If the function \( \tau \) is continuous then at least on of the following statements holds

(i) \( \text{Dom} (x) = \text{Dom} (\tilde{x}) \cap [\sigma - r, \infty] \),

(ii) \( \text{Dom} (x) \) is a compact interval.
Proof. By item (ii) of Lemma 2 \( \text{Dom}(x) \subset \text{Dom}(\tilde{x}) \). If (i) does not hold then by item (iii) of Lemma 2 there exists a point \( t_0 \in \text{Dom}(\tilde{x}) \cap [\sigma - r, \infty] \) such that

\[
t_0 - \tilde{\tau}(t_0, \tilde{x}_{t_0}) < \sigma - r,
\]

therefore

\[
\beta := \sup \{ \gamma \mid \forall t \in [\sigma, \gamma] : t - \tilde{\tau}(t, \tilde{x}_t) \geq \sigma - r \} < t_0,
\]

and

\[
\beta - \tilde{\tau}(\beta, \tilde{x}_\beta) = \sigma - r, \text{ Dom}(x) = [\sigma - r, \beta].
\]

It follows that investigating the continuity of the resolvent function \( \Phi \) one can suppose without loss of generality that \( \tau \leq r \), because the function \( \Lambda \), defined by (1.5), is continuous, if \( \tau \) is continuous. Nevertheless, usually one can not investigate the functional differential equations with state-dependent delay as simple problems type of (1.6), because the function \( \Lambda \) does not inherit the ”good” properties of the function \( \tau \), so we have to be careful using equivalences, formulated in Lemma 2 and Lemma 1. For example, if the initial function \( \phi \) is not lipschitzian then the initial value problem (1.1) can have more than one noncontinuable solution, even if the functions \( f \) and \( \tau \) are lipschitzian (see in [5], [16]).

Remark 2. The situation is similar also if we need the openness of \( \text{Dom}(\Phi^{[a,b]}) \), for instance if \( a > \sigma - r \) and the question is the Fréchet-differentiability of \( \Phi^{[a,b]} \), then one has to suppose that condition (1.7) yields. Nevertheless, in this paper condition (1.7) is sometimes replaced by the supposition that \( \Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0) \) is compact, or \( \Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0) \) has one element. This is not a very strong assumption, for instance if \( Y = \mathbb{R}^n \) then by Theorem 4.2 [12] it holds, whenever \((\sigma_0, \phi_0, \theta_0, \omega_0) \in \text{Dom}(\Phi^{[\sigma_0-r,b]}) \).

The openness of the \( \text{Dom}(\Phi^{[a,b]}) \) does not follow from it, but \( \Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0) \) compact implies that the set

\[
H := \{(t, x_1, (t - \tau(t, x_1, \theta_0)), \omega_0) \mid t \in [\sigma_0, b], x \in \Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0)\}
\]

is also compact (Theorem 4.2 [12]) and the set \( H \) has a neighbourhood \( V \subset \text{Dom}(f) \). Similarly, the set

\[
G := \{(t, x_1, \theta_0) \mid t \in [\sigma_0, b], x \in \Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0)\}
\]

is compact and it has a neighbourhood \( W \subset \text{Dom}(\tau) \). As \( \tau \) is continuous, one can suppose that there is a number \( \tilde{\tau} \) such that

\[
|\tau(t, \psi, \theta)| \leq \tilde{\tau}
\]
also yields, whenever \((t, \psi, \theta) \in W\). By Theorem 2 in Section 2 if \(\delta\) is small enough then the restricted solution \(u_{|\text{Dom}(u)\cap\sigma-r,b}\) of the initial value problem (1.1) is equal to the restricted solution \(u_{|\text{Dom}(u)\cap\sigma-r,b}\) of the initial value problem (1.1), replacing \(f\) by its restriction \(f|_V\) and \(\tau\) by \(\tau|_W\). Using the notations of Lemma 2, we obtain that the restrictions \(u_{|\text{Dom}(u)\cap\sigma-r,b}\) are equal to the restrictions \(\tilde{u}_{|\text{Dom}(u)\cap\sigma-r,b}\) of the solutions \(\tilde{u}\) of the consequence-

equation
\[
\tilde{u}'(t) = \tilde{f}(t, \tilde{\alpha}, \tilde{\beta} (t - \tilde{\tau} (t, \tilde{\alpha}, \beta)), \omega), t \geq \sigma,
\]
\[
\tilde{u}_{\sigma} = \tilde{\phi}.
\]

For problem (1.10) condition (1.7) holds. Therefore one can obtain information about properties of the solutions of problem (1.1) from the solutions of problem (1.10).

In Section 2 the upper semicontinuity of the resolvent function \(\Phi\) is investigated, under the suppositions that the functions \(f\) and \(\tau\) are continuous. The results are stronger than the classical ones, even in the case of the ordinary differential equations.

In Section 3 the Lipschitz-continuity of the resolvent function \(\Phi\) is investigated under the suppositions that the function \(f\) is locally lipschitzian in its second, third and fourth variables and the function \(\tau\) is locally lipschitzian in its second and third variables in the classical sense of Definition 4, which follows from the continuous differentiability. In the literature usually the following different definitions are used:

**Definition 3.** \(f(t,\nu,\omega,\omega)\) is locally Lipschitz-continuous in \(\nu, \omega\) and \(\omega\) if for every \(\alpha > 0\), \(M_1 \subset \Omega_1\), \(M_2 \subset \Omega_2\), \(M_3 \subset \Omega_3\) where \(M_1\) and \(M_2\) are compact subsets of \(\mathbb{R}^n\) and \(M_3\) is a closed, bounded subset of \(\Omega\), there is a constant \(L_1 = L_1(\alpha, M_1, M_2, M_3)\) such that

\[
\|f(t,\nu,\omega,\omega) - f(t,\nu,\omega,\omega)\| \leq L_1(\|\nu - \nu\| + \|\omega - \omega\| + \|\omega - \omega\|_\Omega),
\]

for \(t \in [0,\alpha]\), \(\nu, \nu \in M_1, \omega, \omega \in M_2,\) and \(\omega, \omega \in M_3,\)

\(\tau(t,\psi,\theta)\) is locally Lipschitz-continuous in \(\psi\) and \(\theta\) if for every \(\alpha > 0\), \(M_1 \subset \Omega_4, M_5 \subset \Omega_5\), where \(M_4\) is a compact subsets of \(C\), and \(M_5\) is a closed, bounded subset of \(\Theta\), there is a constant \(L_2 = L_2(\alpha, M_4, M_5)\) such that

\[
|\tau(t,\psi,\theta) - \tau(t,\psi,\theta)| \leq L_2(\|\psi - \psi\| + \|\theta - \theta\|_\Theta),
\]

for \(t \in [0,\alpha], \psi, \psi \in M_4\) and \(\theta, \theta \in M_5.\)

These local Lipschitz-continuities do not follow from the continuous differentiability if the parameter-spaces are not finite dimensional (for instance in the control-theory). In this and some other senses our suppositions are
weaker, and the obtained consequences sometimes are stronger than the known ones. For instance not only pointwise Lipschitz-continuity of the resolvent function is proved, not supposing that the solutions are unique. The statements are generalizations of the corresponding statements of [14].

The main aim of this paper is to investigate the smooth parameter-dependence of the resolvent function of Abstract Functional Differential Equations with unbounded state-dependent delay with respect to the parameters and initial functions, without the following compatibility condition (1.11)

\[ \phi \text{ is continuously differentiable and } \phi'(0) = f(\sigma, \phi, \phi(\sigma - \tau(\sigma, \phi, \theta)), \omega) \]

or

\[ \phi \text{ is lipschitzian and the Brocate-Colonius-type monotonicity condition} \]

(1.12)

\[
\text{ess inf} \left\{ \frac{d}{dt} (t - \tau(t, x_t)) : \text{a.e. } t \in [\sigma, b] \right\} > 0
\]

holds.

Condition (1.11) is not convenient, because the set of the initial functions, satisfying this condition, usually has not an algebraic structure.

In the recent paper of F. Hartung [9] pointwise continuous differentiability of the characteristic function with respect to the initial parameters (including the initial time-parameter \( \sigma \)) is proved in finite dimensional case, without parameters \( \theta \) and \( \omega \), under the following assumptions:

the function \( \tau \) is continuously differentiable and it is lipschitzian in its second variable in the sense of Definition 3, moreover \( \tau \leq r \)

\( f \) is continuous and continuously differentiable in its second and third variables and is lipschitzian in its second and third variables in the sense of Definition 3,

condition (1.12) is satisfied,

the initial functions are lipschitzian,

but not supposing the compatibility condition.

Therefore the set of the initial functions, where the characteristic function can be differentiable, has a vector space structure, which is a significant difference with respect to the situation was before. However, condition (1.12) is an implicit one and it is difficult to control it.

In Section 4 the continuous differentiability of the resolvent function \( \Phi^{[\sigma-r,b]}(\sigma, \cdot, \cdot, \cdot) \) is investigated. Our aim is, roughly speaking, to proof that the resolvent function of (1.1) is continuously differentiable at the points of the vector space \( C^1([-r,0], Y) \), under weaker suppositions, not using for example either (1.12) or (1.11) conditions and the assumptions, formulated
in Definition 3. In fact, we will prove only the tangential condition

\[
\|\Phi^{[\sigma-r,b]}(\sigma, \psi, \vartheta, \varpi) - \Phi^{[\sigma-r,b]}(\sigma, \phi, \theta, \omega) - D\Phi^{[\sigma-r,b]}(\sigma, \cdot, \cdot, \cdot)(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)\| \\
\leq \varepsilon \|((\psi, \vartheta, \varpi) - (\phi, \theta, \omega))\|
\]

with respect to two different norms on the domain of \(D\Phi^{[\sigma-r,b]}(\sigma, \cdot, \cdot, \cdot)\), and replacing the condition with the supposition that \(\Phi^{[\sigma_0-r,b]}(\sigma_0, \phi_0, \theta_0, \omega_0)\) has one element, corresponding to Remark 2.

In Subsection 4.1 the continuous differentiability of the function \(\Phi^{[\sigma-r,b]}(\sigma, \cdot, \cdot, \cdot)\) is proved, showing also that this differentiability is uniform in the initial parameter \(\sigma\), but supposing that the compatibility condition holds.

In Subsection 4.2 a nonlinear variation-of-constant formula is given for our problem (1.1).

Using this formula, in Subsection 4.3 we show that the statements of Subsection 4.1 are also valid without condition (1.11), but supposing that the initial functions are continuously differentiable on the interval \([\lambda, 0]\). So one can obtained stronger results also for the "classical" case of functional differential equations, when \(\tau = 0\), even in the finite dimensional case.

**Remark 3.** The results of this article seem to be usable in the investigation of the smooth-dependence of the resolvent function with respect to the initial time-parameter \(\sigma\). But this question is over the frame of this paper. The author would like to investigate it in a forthcoming article.

**2. The continuity of \(\Phi^{[a,b]}\).** In this section we investigate the openness of the domain of the resolvent function and the upper-semicontinuity of the resolvent function. Usually we suppose that the functions \(f\) and \(\tau\) are continuous, but sometimes weaker conditions will be assumed. If \(f\) and \(\tau\) are continuous and the fixed solution \(\Phi(\sigma, \phi)\) is unique (or even more generally, \(\Phi^{[a,b]}(\sigma, \phi)\) is compact), then the resolvent function is upper-semicontinuous at the point \((\sigma, \phi)\); this fact does not depend on the openness of the domain of the resolvent function. \(\text{Dom} \left(\Phi^{[a,b]}\right)\) can be not open, if \(Y\) is infinite dimensional (even in the case of the ordinary differential equations), because it is possible that an initial value problem has not a solution. It is also possible (even in the finite dimensional case) that the initial value problem (1.3) has not a solution. The very simple Example 1 [14] shows a situation when a solutions \(x\) is defined on the interval \([0, r]\) but there is not a solution \(u\), satisfying the initial condition \(u^r_0 = x^r_0\). \(\text{Dom} \left(\Phi\right)\) can be not open if the condition \(\tau \leq r\) does not hold. However \(\text{Dom} \left(\Phi^{[a,b]}\right)\) can be open in this case. It will be shown that \((\sigma, \phi) \in \text{Dom} \left(\Phi^{[a,b]}\right)\) is an interior point if
\( \Phi^{[a,b]}(\sigma, \phi) \) is compact and there is a neighbourhood \( V \) of \( (\sigma, \phi) \) such that if \( (\rho, \psi) \in V \) then the noncontinuable solutions \( \Phi(\rho, \psi) \) reach to the boundary on the right. This condition is weaker than the condition \( r \leq s \). It means that the condition of the openness of \( \text{Dom}(\Phi^{[a,b]}) \) can be replaced by the condition that the noncontinuable solutions reach to the boundary on the right.

The following Lemma is a variant of Lemma 7. [11], where it was proved under the assumption that the function \( f \) is continuous. The proof is a simple adaptation of the proof given in [11].

**Lemma 3.** Let \( Y \) be a Banach space, \( 0 \leq r, \) let \( \alpha \) and \( \delta \) be positive numbers, let \( C := [\sigma_0 - 3\alpha, \sigma_0 + 3\alpha] \times \{ \phi \mid \|\phi - \phi_0\|_{C} \leq 3\delta \} \) a subset of \( \mathbb{R} \times C([-r, 0], Y) \) and let \( f : C \to Y \) be a function, bounded on \( C, \) say \( \|f(\sigma, \phi)\| \leq K, \) whenever \( (\sigma, \phi) \in C. \) Let \( \alpha \) and \( \delta \) be chosen so that \( \alpha < \frac{\delta}{K}, \) and \( \|\phi_0(s_1) - \phi_0(s_2)\| \leq \delta, \) whenever \( s_1, s_2 \in [-r, 0], \|s_1 - s_2\| \leq \alpha. \) Denote \( \Phi(\sigma, \phi) \) the set of the noncontinuable solutions \( x \) of the initial value problem

\[
(x'(t) = f(t, x_t), \quad x_0 = \phi.
\]

Let \( (\sigma, \phi) \in C \) be fixed. Suppose furthermore that \( |\sigma - \sigma_0| \leq \alpha, \|\phi - \phi_0\|_{C} \leq \delta \) and \( x \in \Phi(\sigma, \phi). \)

(i) If \( t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(x) \) then \( (t, x_t) \in C. \)

(ii) If the solution \( x \) reaches to the boundary on the right then \( [\sigma - r, \sigma + \alpha] \subseteq \text{Dom}(x). \)

**Proof.** Denote

\[
\gamma := \sup \left\{ \rho \in [\sigma, \sigma + \alpha] \mid [\sigma, \rho] \subseteq \text{Dom}(x) \text{ and } \forall t \in [\sigma, \rho] : \|x_t - \phi_0\| \leq 3\delta \right\}.
\]

(i) The closure of the set \( \Gamma_x := \{(t, x_t) \mid t \in [\sigma, \gamma]\} \) is included in \( C. \) We show that if \( \gamma \in \text{Dom}(x) \) then \( \gamma = \sigma + \alpha. \) Suppose that \( \gamma < \sigma + \alpha \) and \( \gamma \in \text{Dom}(x). \)

If \( -r \leq s \leq \sigma - \gamma \) then \( -r \leq \gamma + s - \sigma \leq 0 \) and the estimation

\[
||x_\gamma - \phi_0(s)|| = ||\phi(\gamma + s - \sigma) - \phi_0(s)||
\]

\[
\leq ||\phi(\gamma + s - \sigma) - \phi_0(\gamma + s - \sigma)|| + ||\phi_0(\gamma + s - \sigma) - \phi_0(s)|| \leq 2\delta
\]
yields, because \( ||\phi - \phi_0|| \leq \delta \) and \( |\gamma + s - \sigma - s| = |\gamma - \sigma| \leq \alpha. \)

By the Mean Value Theorem \( x \) is lipschitzian on \( [\sigma, \sigma + \gamma] \) with the constant \( K. \)
If $\sigma - \gamma \leq s \leq 0$, $-r \leq s$ then (using the equality $x(\sigma) = \phi(0)$) we get that

$$
\|x(\gamma + s) - \phi_0(s)\| \\
\leq \|x(\gamma + s) - x(\sigma)\| + \|\phi(0) - \phi_0(0)\| + \|\phi_0(0) - \phi_0(s)\| \\
\leq K(\gamma + s - \sigma + \delta + \delta < 3\delta),
$$

because $\gamma - \sigma < \alpha$ and

$$
|\gamma + s - \sigma| = \gamma + s - \sigma < \alpha + s \leq \alpha < \frac{\delta}{K}.
$$

Consequently $\|x_\gamma - \phi_0\| < 3\delta$, which is a contradiction.

It means that $\Gamma_\alpha := \{(t, x_t) \mid t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(x)\} \subset C$.

(ii) As $\Gamma_\alpha \subset C \subset \text{Dom}(f)$ and $C$ is closed, so the closer of $\Gamma_\alpha$ is included in $C$. As $x$ is lipschitzian on $[\sigma, \sigma + \alpha] \cap \text{Dom}(x)$ with the constant $K$ therefore $\text{cl}(\Gamma_\alpha) \subset C$ is compact. Indeed, if $\{(t_n, x_{t_n}) \mid n \in \mathbb{N}\} \subset \Gamma_\alpha$ then one can suppose that $t_n \to t \in [\sigma, \sigma + \alpha]$. As the function $\phi$ is uniformly continuous, $x : [\sigma, \sigma + \alpha] \to Y$ is lipschitzian with the constant $K$, so $x : [\sigma - r, \sigma + \alpha] \to Y$ is uniformly continuous. Hence

$$
\lim_{t_n, t_m \to t} \|x_{t_n} - x_{t_m}\|_C = \lim_{t_n, t_m \to t} \max_{s \in [-r, 0]} \|x(t_n + s) - x(t_m + s)\| = 0,
$$

therefore $\{t_n, (x_{t_n}) \mid n \in \mathbb{N}\}$ is convergent.

This contradicts to the supposition that $x$ reaches to the boundary on the right.

In the following Lemma 4 a generalization of Theorem 12 [11] is given. The proof is a simple adaptation of the proof of Theorem 12 [11].

**Lemma 4.** Let $Y$ be a Banach space, let $0 \leq r$, let $P$ be a topological space, let $U \subset \mathbb{R} \times C([-r, 0], Y) \times P$ be an open set, $f : U \to Y$ is a function. Suppose that $f$ is either continuous or it is bounded. Denote $\Phi(\sigma, \phi, p)$ the set of the noncontinuable solutions of the initial value problem

$$
x'(t) = f(t, x_t, p), \quad x_\sigma = \phi.
$$

Let $G \subset U$ be a compact subset.

(i) For every neighborhood $V \subset \text{pr}_{\mathbb{R} \times C([-r, 0], Y)}(U) \subset \mathbb{R} \times C([-r, 0], Y)$ of the compact set $\text{pr}_{\mathbb{R} \times C([-r, 0], Y)}(G)$ there is a neighborhood $E \subset U$ of $G$, moreover there are positive numbers $\alpha$ and $K$ such that

$$
((\sigma, \phi, p) \in E, \quad x \in \Phi(\sigma, \phi, p)) \implies \{(t, x_t) \mid t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(x)\} \subset V,
$$

(ii) Let $\sigma_0 \in \mathbb{R}$ and $\alpha$, such that $\alpha > 0$. Then $\text{pr}_{\mathbb{R} \times C([0, \sigma_0], Y)}(\Phi(\sigma, \phi, p))$ is a closed subset of $\mathbb{R} \times C([0, \sigma_0], Y) \times P$.
and $x$ is Lipschitzian on the interval $[\sigma, \sigma + \alpha] \cap \text{Dom}(x)$ with the constant $K$.

(ii) The number $\alpha$ and the neighborhood $E \subset U$ of $G$ can be chosen so that
if $(\sigma, \phi, p) \in E$, $x \in \Phi(\sigma, \phi, p)$ and $x$ reaches to the boundary on the right
then $[\sigma - r, \sigma + \alpha] \subset \text{Dom}(x)$.

Proof. Let $V \subset \mathbb{R} \times C([-r, 0], Y)$ be a fixed neighborhood of the set
$\text{pr}_{\mathbb{R} \times C([-r, 0], Y)}(G)$. We can suppose that $G$ has a neighbourhood $W$ such
that $f$ is bounded on $W$, say $\|f(\sigma, \phi, p)\| \leq K$ if $(\sigma, \phi, p) \in W$, because
if $f$ is continuous then it follows from the compactness of $G$. Every point
$(\sigma_i, \phi_i, p_i) \in G$ has a closed neighborhood

$$[\sigma_i - 3\alpha_i, \sigma_i + 3\alpha_i] \times \{\phi : \|\phi - \phi_i\| \leq 3\delta_i\} \times P_i \subset W,$$

such that $\alpha_i < \frac{\delta_i}{K}$, and $\|\phi_i(s_1) - \phi_i(s_2)\| \leq \delta_i$, whenever $|s_1 - s_2| \leq \alpha_i$.

(i) By Lemma 3 (i) if $H_i := [\sigma_i - \alpha_i, \sigma_i + \alpha_i] \times \{\phi : \|\phi - \phi_i\| \leq \delta_i\} \times P_i$
then

$$(\sigma, \phi, p) \in H_i, \quad x \in \Phi(\sigma, \phi, p) \implies \Gamma_x := \{(t, x_t) \mid t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(x)\} \subset V.$$

As $G \subset \bigcup_{(\sigma_i, \phi_i, p_i) \in G} H_i$, $G$ compact implies there are finite many $H_1, \ldots, H_n$
neighborhoods such that $G \subset E := \bigcup_{i=1}^n H_j$ and $E$ is a neighborhood of $G$.
Consequently,

$$(\sigma, \phi, p) \in E, \quad x \in \Phi(\sigma, \phi, p), \quad t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(x) \implies (t, x_t) \in V,$$

where $\alpha := \min \{\alpha_j \mid j = 1, \ldots, n\}$. Moreover, by the Mean Value Theorem $x$
is Lipschitzian on the interval $[\sigma, \sigma + \alpha] \cap \text{Dom}(x)$ with the constant $K$.

(ii) By Lemma 3 (ii) if $(\sigma, \phi, p) \in H_i$, and $x \in \Phi(\sigma, \phi, p)$ then
$[\sigma - r, \sigma + \alpha_i] \subset \text{Dom}(x)$. It implies that

$$(\sigma, \phi, p) \in E, \quad x \in \Phi(\sigma, \phi, p) \implies [\sigma - r, \sigma + \alpha] \subset \text{Dom}(x).$$
is compact. Let $\sigma_0 - r < a < b$.

(i) For each positive number $\varepsilon$ there is a neighbourhood $W$ of $(\sigma_0, \phi_0, p_0)$ such that

$$
(\sigma, \phi, p) \in W, \ u \in \Phi (\sigma, \phi, p) \implies \exists x \in \Phi (\sigma_0, \phi_0, p_0) : \left\| (x - u)_{|[a,b] \cap \text{Dom}(u)} \right\| \leq \varepsilon.
$$

The function $\Phi_{[a,b]}$ is upper-semicontinuous at $(\sigma_0, \phi_0, p_0)$.

(ii) The neighbourhood $W$ can be chosen so that if $(\sigma, \phi, p) \in W$, $u \in \Phi (\sigma, \phi, p)$ and $u$ reaches to the boundary on the right then $[a, b] \subset \text{Dom}(u)$.

Proof. It is easily seen that the set

$$
\Gamma := \{(t, x_t) \mid t \in [\sigma_0 - r, b], \ x \in \Phi (\sigma_0, \phi_0, p_0)\} \subset \mathbb{R} \times C([-r, 0], Y)
$$

is compact. Let $\varepsilon > 0$ be fixed and let $V \subset \mathbb{R} \times C([-r, 0], Y)$ be the $\varepsilon$-neighbourhood of the compact set $\Gamma$, that is

$$
V := \{(s, \psi) \in \mathbb{R} \times C([-r, 0], Y) \mid \exists (t, x_t) \in \Gamma : \| (t, x_t) - (s, \psi) \| \leq \varepsilon\}.
$$

The numbers $\varepsilon$ and $K$ can be given so that if $(s, \psi) \in V$ then $\| f (s, \psi) \| \leq K$.

(i) By Lemma 4 there is a neighborhood $E$ of $G := \Gamma \times \{p_0\}$, moreover there are positive numbers $\alpha$ and $K$ such that if $(\sigma, \phi, p) \in E$, $u \in \Phi (\sigma, \phi, p)$ then

$$
(2.4) \ \{(t, u_t) | t \in [\sigma, \sigma + \alpha] \cap \text{Dom}(u)\} \subset V.
$$

We can suppose that $E = H \times S$, where $H$ is a neighbourhood of $\Gamma$ and $S$ is a neighbourhood of $p_0$. Let us fix $(\sigma, \phi, p) \in E$ and $u \in \Phi (\sigma, \phi, p)$.

Let $\gamma := \sup \{ \rho \mid \text{graph} \left( u_{|[a,\rho]} \right) \subset V \}$. Evidently (by Lemma 4), $\gamma > a$. If $\gamma$ is the endpoint of $\text{Dom}(u)$ then $\text{graph}(u)$ is included in the closed set $V$. If $\gamma$ is an inner point of $\text{Dom}(u)$ then we have to prove that $\gamma > b$. Suppose that it is not true, that is $\gamma \leq b$. But $[a, \gamma - \frac{a}{2} + \alpha] = [a, \gamma + \frac{a}{2}]$ and $\text{graph} \left( u_{|[a,\gamma-\frac{a}{2}]} \right) \subset V$ implies that $\text{graph} \left( u_{|[a,\gamma+\frac{a}{2}]} \cap \text{Dom}(u) \right) \subset V$. This contradicts to the definition of $\gamma$.

Denote $\beta$ the right endpoint of the interval $[a, b] \cap \text{Dom}(u)$. For every natural number $n$ the distance of the compact sets $\text{graph} \left( u_{|[a,\beta - \frac{1}{n}]} \right)$ and

$$
\Gamma_{[a,\beta - \frac{1}{n}]} := \{(t, x_t) \mid t \in [a, \beta - \frac{1}{n}], \ x \in \Phi (\sigma_0, \phi_0, p_0)\} \subset \mathbb{R} \times C([-r, 0], Y)
$$

is not greater than $\varepsilon$, hence there is an element $x_n \in \Phi (\sigma_0, \phi_0, p_0)$ such that $\left\| (x_n)_{|[a,\beta - \frac{1}{n}]} - u_{|[a,\beta - \frac{1}{n}]} \right\| \leq \varepsilon$. $\Phi_{[a,b]} (\sigma_0, \phi_0, p_0)$ compact implies that we can
suppose that the sequence $x_n$ tends to an element $x \in \Phi(\sigma_0, \phi_0, p_0)$. It is clear that $\|x|_{[a,b] \cap \text{Dom}(u)} - u|_{[a,b] \cap \text{Dom}(u)}\| \leq \varepsilon$.

It means that the function $\Phi_{[a,b]}$ is upper-semicontinuous at $(\sigma_0, \phi_0, p_0)$.

(ii) The proof is similar to the proof of Lemma 3 (ii). 

As a corollary of Theorem 1 we can establish an analogous statement for the state-dependent case:

**Theorem 2.** Let $Y$ be Banach space, let $\Theta$ and $\Omega$ be topological spaces, let $r \geq 0$ be fixed. Let $f : U \subset \mathbb{R} \times C([-r, 0], Y) \times Y \times \Theta \rightarrow Y$ and $\tau : W \subset \mathbb{R} \times C([-r, 0], Y) \times \Omega \rightarrow \mathbb{R}_0^+$ be continuous functions.

Suppose that every noncontinuable solution $u \in \Phi(\sigma_0, \phi_0, \theta_0, \omega_0)$ of the problem

$$
(2.5) \quad x'(t) = f(t, x_t, x(t - \tau(t, x_t), \theta_0), \omega_0), \quad x_{\sigma_0} = \phi_0
$$

is defined on the compact interval $[\sigma_0 - r, b]$ and the set $\Phi|_{[\sigma_0 - r, b]}(\sigma_0, \phi_0, \theta_0, \omega_0)$ $$
\subset C([\sigma_0 - r, b], Y) \text{ is compact. Let } \sigma_0 - r < a < b.
$$

(i) For each positive number $\varepsilon$ there is a neighbourhood $V$ of $(\sigma_0, \phi_0, \theta_0, \omega_0)$ such that

$$(\sigma, \phi, \theta, \omega) \in V, \ u \in \Phi(\sigma, \phi, \theta, \omega) \quad \Rightarrow \quad \exists x \in \Phi(\sigma_0, \phi_0, \theta_0, \omega_0) : \| (x - u) |_{[a,b] \cap \text{Dom}(u)} \| \leq \varepsilon.
$$

The function $\Phi_{[a,b]}$ is upper-semicontinuous at $(\sigma_0, \phi_0, \theta_0, \omega_0)$.

(ii) The neighbourhood $V$ can be chosen so that if $(\sigma, \phi, \theta, \omega) \in V, \ u \in \Phi(\sigma, \phi, \theta, \omega)$ and $u$ reaches to the boundary on the right then $[a, b] \subset \text{Dom}(u)$.

**Proof.** (i) By Lemma 2.1 (I) (ii) [12] and Lemma 2.2 [12] it is enough to prove the statement for the initial value problem type of

$$
(2.5) \quad x'(t) = f(t, x_t, p), \quad x_{\sigma} = \phi.
$$

(ii) By Proposition 3.1 [12] $\text{Dom}(u)$ is not compact. By Corollary 1 $\text{Dom}(u) = \text{Dom}(\tilde{u}) \cap [\sigma - r, \infty[$. Therefore the statement follows from Theorem 1. 

**Remark 4.** In Theorem 5.2 [12] it is proved that under the suppositions that every initial value problem has a solution and $\Phi_{[a,b]}$ is upper-semicontinuous at $(\sigma_0, \phi_0, \theta_0, \omega_0)$, the point $(\sigma_0, \phi_0, \theta_0, \omega_0)$ is the inner point of the domain of the multivalued function $\Phi_{[a,b]}$. Theorem 2 is a generalization of that statement, saying that the upper-semicontinuity is not a supposition but it is a consequence.
Suppose that

\[ x'(t) = f(t, t, x(t), \omega(t)), \quad t \in [\sigma, b], \]
\[ x(t) = \phi_0(t - \sigma_0) \quad \text{if} \ t \in [\sigma_0 - r, \sigma_0] \]

has a unique solution on \([\sigma_0 - r, b]\) (that is the set \(\Phi^{[\sigma_0 - r, b]}(\sigma_0, \phi_0, \theta_0, \omega_0)\) has exactly one element) then \(\Phi^{[a, b]}\) is upper-semicontinuous at \((\sigma_0, \phi_0, \theta_0, \omega_0)\).

**Remark 5.** In the special case of the ordinary differential equations, if \(f \in \mathbb{R} \times Y \times P \to Y\) is a continuous function and it is locally lipschitzian in its second variable then the complete solutions exist, that is \(\Phi^{[a, b]}(\sigma, \phi, \theta, \omega)\) has one element, hence it is compact. Therefore we get the classical theorem on the continuity of \(\Phi^{[a, b]}\), given for instance in [3] (Theorem 2.10.1).

**Lemma 5.** Let \(A\) and \(B\) be topological spaces, \(A\) be compact, \(g : A \to B\) be an upper-semicontinuous multivalued function. If for every point \(a \in A\) the set \(g(a) \subset B\) is compact then \(g(A) \subset B\) is compact.

**Proof.** Let \(\{V_i \mid i \in I\}\) be an open covering of \(g(A)\), that is \(g(A) \subset \bigcup_{i \in I} V_i\). For each element \(a \in A\) there exists finite sets \(V_{j_1}^a, \ldots, V_{j_n}^a \in \{V_i \mid i \in I\}\) such that \(g(a) \subset \bigcup_{k=1}^n V_{j_k}^a\). For the brevity denote \(H_a := \bigcup_{k=1}^n V_{j_k}^a\). As \(g\) is upper-semicontinuous, for each \(a \in A\) there is an open neighbourhood \(W_a\) of \(a\) such that \(g(W_a) \subset H_a\). It follows from the compactness of \(A\) that there are finite elements \(a_1, \ldots, a_p \in A\) such that \(A = \bigcup_{i=1}^p W_{a_i}\). It implies that \(g(A) \subset \bigcup_{i=1}^p g(W_{a_i}) \subset \bigcup_{i=1}^p H_{a_i} = \bigcup_{i=1}^p \bigcup_{k=1}^n V_{j_k}^{a_i}\)\.

**Theorem 3.** Let \(Y\) be Banach space, let \(\Theta\) and \(\Omega\) be topological spaces, let \(r \geq 0\) be fixed. Let \(f\) and \(\tau : \mathbb{R} \times \mathbb{C}([-r, 0], Y) \times \Omega \to \mathbb{R}^+\) be continuous functions.

(i) (Weierstrass-property) Let \([a, b] \subset [\sigma_0 - r, \infty]\) and let \(\emptyset \neq H \subset \text{Dom}(\Phi^{[a, b]})\) be such that \(\Phi^{[a, b]}(\sigma, \phi, \theta, \omega) \subset \mathbb{C}([a, b], Y)\) is compact, whenever \((\sigma, \phi, \theta, \omega) \in H\).

If \(H\) is compact then the set \(\Phi^{[a, b]}(H) \subset \mathbb{C}([a, b], Y)\) is compact.

(ii) (Bolzano-property) Suppose that \(V\) is an open subset of the domain of \(\Phi^{[a, b]}\), and \(\emptyset \neq H \subset V\).

If \(\Phi^{[a, b]}(\sigma, \phi, \theta, \omega) \subset \mathbb{C}([a, b], Y)\) is connected, whenever \((\sigma, \phi, \theta, \omega) \in H\), and \(H\) is connected then \(\Phi^{[a, b]}(H) \subset \mathbb{C}([a, b], Y)\) is connected.

(iii) If \(\text{Dom}(\Phi^{[a, b]})\) is open, \(\Phi^{[a, b]}(\sigma, \phi, \theta, \omega) \subset \mathbb{C}([a, b], Y)\) is a continuum, whenever \((\sigma, \phi, \theta, \omega) \in H\), and \(H\) is a continuum then \(\Phi^{[a, b]}(H) \subset \mathbb{C}([a, b], Y)\) is a continuum.

**Proof.** By Theorem 2 the restricted function \(\Phi^{[a, b]}_H\) is upper-semicontinuous on \(H\).
(i) $H \subset \mathcal{C}([a, b], Y)$ compact implies $H$ is compact in the inherited topology of $H$. By Lemma 5 $\Phi^{[a,b]}(H)$ is compact.

(ii) If $H$ is a connected subset of the open set $\text{Dom} (\Phi^{[a,b]})$ then it is connected in the inherited topology of $\text{Dom} (\Phi^{[a,b]})$. By Lemma 2.1 [13] $\Phi^{[a,b]}(H)$ is connected.

(iii) By definition $\Phi^{[a,b]}(H)$ is a continuum if and only if it is compact and connected. Hence the statement follows immediately from (i) and (ii). □

REMARK 6. If $Y$ is finite dimensional and every initial value problem has a solution then the set $\Phi^{[a,b]}(\sigma, \phi, \theta, \omega) \subset \mathcal{C}([a, b], Y)$ is always compact and connected, moreover $\text{Dom} (\Phi^{[a,b]})$ is open (see in [12] and [13]). So we have got above a generalizations of those theorems.

The question arises that under what assumptions $\text{Dom} (\Phi^{[a,b]})$ is open.

Theorem 4. Let $Y$ be Banach space, let $\Theta$ and $\Omega$ be topological spaces, let $r \geq 0$ be fixed. Let $f$ and $\tau : W \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \Omega \rightarrow \mathbb{R}^+_0$ be continuous functions.

(i) If $\Phi^{[a,b]}(\sigma, \phi, \theta, \omega) \subset \mathcal{C}([a, b], Y)$ is compact and every noncontinuable solution reaches to the boundary, then $(\sigma, \phi, \theta, \omega)$ is an inner point of $\text{Dom}(\Phi^{[a,b]})$.

(ii) If $x = \Phi(\sigma_0, \phi_0, \theta_0, \omega_0)$ is a complete solution, $[a, b] \subset \text{Dom}(x)$ and

$$t \in \text{Dom}(\Phi(\sigma_0, \phi_0, \theta_0, \omega_0)) \implies \sigma_0 - r < t - \tau(t, x_t, \theta_0),$$

and the neighbourhood $V$ of $(\sigma_0, \phi_0, \theta_0, \omega_0)$ is chosen so that if $(\sigma, \phi, \theta, \omega) \in V$ and $u \in \Phi(\sigma, \phi, \theta, \omega)$ then $u$ reaches to the boundary on the right, then the neighbourhood $V$ can be chosen so that $V \subset \text{Dom}(\Phi^{[a,b]})$.

Proof. (i) This statement follows immediately from Theorem 2.

(ii) By Theorem 2 there is a neighbourhood $V$ of $(\sigma_0, \phi_0, \theta_0, \omega_0)$ such that if $(\sigma, \phi, \theta, \omega) \in V$ and $u \in \Phi(\sigma, \phi, \theta, \omega)$ then $\sigma - r < t - \tau(t, x_t, \theta)$ also holds. By Theorem 3.4 [12] $u$ reaches to the boundary. Hence by Theorem 2 $V$ can be chosen so that $V \subset \text{Dom}(\Phi^{[a,b]})$. □

3. The Lipschitz continuity of $\Phi^{[a,b]}$. Suppose that $Y$ is a Banach space, $\Theta$ and $\Omega$ are metric spaces, $r \geq 0$. Let $f : U \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times Y \times \Omega \rightarrow Y$ and $\tau : W \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \Theta \rightarrow \mathbb{R}^+_0$ be functions.

In this section we suppose that the function $\tau$ is continuous and locally lipschitzian in its second and third variables, and the function $f$ is continuous and locally lipschitzian in its second, third and fourth variables in the sense of Definition 4 below.

In the following definition the continuity of the functions $f$ and $\tau$ and the openness of their domain are not supposed.
Definition 4. We say that the function $\tau$ is locally lipschitzian in its second and third variables if for every point $\kappa := (t, \phi, \theta) \in \text{Dom}(\tau)$ there are positive numbers $\rho_\kappa$ and $(L_\tau)_\kappa$ such that if $(s, \psi, \vartheta) \in \text{Dom}(\tau)$, $(s, \varphi, \xi) \in \text{Dom}(\tau)$,

$$\max \{ |s - t|, \| \phi - \psi \|_C, \| \phi - \varphi \|_C, d(\theta, \vartheta), d(\theta, \xi) \} \leq \rho_\kappa,$$

then

$$|\tau(s, \varphi, \xi) - \tau(s, \psi, \vartheta)| \leq (L_\tau)_\kappa (\| \phi - \psi \|_C + d(\xi, \vartheta)).$$

The function $f$ is locally lipschitzian in its second, third and fourth variables if for every point $\gamma := (t, \phi, y, \omega) \in \text{Dom}(f)$ there are positive numbers $\rho_\gamma$ and $(L_f)_\gamma$ such that if $(s, \psi, z, \varepsilon) \in \text{Dom}(f)$, $(s, \varphi, v, \eta) \in \text{Dom}(f)$,

$$\max \{ |s - t|, \| \phi - \psi \|_C, \| \phi - \varphi \|_C, \| z - y \|, \| v - y \|, d(\omega, \varepsilon), d(\omega, \eta) \} \leq \rho_\lambda,$$

then

$$\| f(s, \varphi, v, \eta) - f(s, \psi, z, \varepsilon) \| \leq (L_f)_\gamma (\| \phi - \psi \|_C + \| v - z \| + d(\eta, \varepsilon)).$$

Notation 1. Let $\sigma \in \text{pr}_R(\text{Dom}(\tau))$ and $[\sigma, b]$ be fixed. Denote $\lambda(\sigma)$ the number

$$\lambda(\sigma) := \inf \{ t - \tau(t, \psi, \vartheta) - \sigma \mid (t, \psi, \vartheta) \in \text{Dom}(\tau), \ t \in [\sigma, b] \}.$$

The following theorem is a sharper variant of Theorem 3 [14].

Theorem 5. If $\tau$ is locally lipschitzian in its second and third variables, $f$ is locally lipschitzian in its second, third and fourth variables, moreover if $x \in \Phi(\sigma, \psi, \vartheta, \omega)$ and $\phi$ is lipschitzian on the interval

$$[\inf \{ t - \tau(t, x_t, \theta) \mid t \geq \sigma, \ t \in \text{Dom}(x) \} - \sigma, 0],$$

then $x$ is the unique element of $\Phi(\sigma, \phi, \theta, \omega)$.

Proof. One can adopt directly the proof of Theorem 3 [14] without any difficulties, so we omit this proof. □

If $\Phi(\sigma, \phi, \theta, \omega)$ has one element then this fact will be signed by equation

$$x = \Phi(\sigma, \phi, \theta, \omega).$$

In Proposition 1 [14] it was proved that if $\sigma$ is fixed, $f$ and $\tau$ are continuous functions, $f$ is locally lipschitzian in its second, third and fourth variables, the function $\tau$ is locally lipschitzian in its second and third variables then $\Phi(\sigma, \phi, \theta, \omega)$ is locally lipschitzian in some sense. In Theorem 6 below a stronger variant of Proposition 1 [14] is given, showing that there is a uniform Lipschitz-constant, not depending on the parameters $\sigma, \phi, \theta$ and $\omega$. The proof is a slight modification of the proof of Proposition 1 [14].

We do not suppose in Theorem 6 that $\Phi(\sigma, \psi, \vartheta, \varepsilon)$ has one element.
Lemma 6. Let $M$ be metric space with the distance $d$, $C \subset M$ be compact. If $x \in M$ and $\rho > 0$ then denote

$$B(x, \rho) := \{u \in M \mid d(u, x) < \rho\}.$$ 

If $\{B(x_i, \rho_i) \mid i = 1, \ldots, k\}$ is a covering of $C$ then there is a number $n \in \mathbb{N}$ such that $C \subset \bigcup_{i=1}^{k} B(x_i, \rho_i - \frac{1}{n})$.

Proof. Suppose that the statement is not true, that is for every number $n \in \mathbb{N}$ there is an element $c_n \in C$ such that $c_n \notin \bigcup_{i=1}^{k} B(x_i, \rho_i - \frac{1}{n})$. We can suppose that $c_n \rightarrow c \in C$. Then $c \notin \bigcup_{i=1}^{k} B(x_i, \rho_i - \frac{1}{n})$ because for every fixed $i$

$$c \in \cap_{n \in \mathbb{N}} \left(\overline{B(x_i, \rho_i - \frac{1}{n})}\right) = \overline{\bigcup_{n \in \mathbb{N}} B(x_i, \rho_i - \frac{1}{n})},$$

where $\overline{B(x_i, \rho_i - \frac{1}{n})}$ denotes the complement of $B(x_i, \rho_i - \frac{1}{n})$. It is a contradiction.

Corollary 3. Let $M$ be metric space with the distance $d$, $C \subset M$ be compact. If $\{B(x_i, \rho_i) \mid i = 1, \ldots, k\}$ is a covering of $C$ then there is a number $\delta > 0$ such that if $u, x \in M$, $c \in C$, $d(c, x), d(u, x) \leq \delta$

holds then there is an element $x_i$ such that

$$d(u, x_i), d(x, x_i) < \rho_i.$$ 

Proof. By Lemma 6 $C \subset \bigcup_{i=1}^{k} B(x_i, \rho_i - \frac{1}{n})$ for some $n \in \mathbb{N}$. Let $0 < \delta < \frac{1}{3n}$. Then there is an element $x_i$ such that

$$d(x, x_i) \leq d(x, c) + d(c, x_i) < \frac{1}{3n} + \rho_i - \frac{1}{n} = \rho_i - \frac{2}{3n},$$

$$d(u, x_i) \leq d(u, x) + d(x, x_i) < \frac{1}{3n} + \rho_i - \frac{1}{2n} < \rho_i.$$

Theorem 6. Let $Y$ be Banach space, $\Theta$ and $\Omega$ be metric spaces, $r \geq 0$, $f : U \subset \mathbb{R} \times C([-r, 0], Y) \times Y \times \Omega \rightarrow Y$ and $\tau : W \subset \mathbb{R} \times C([-r, 0], Y) \times \Theta \rightarrow \mathbb{R}_0^+$ be continuous functions, $f$ be locally lipschitzian in its second, third and fourth variables, the function $\tau$ be locally lipschitzian in its second and third variables. Suppose that $(\sigma_0, \phi_0, \theta_0, \omega_0) \in \text{Dom} \left(\Phi^{(\sigma_0 - r, b)} \right)$ and $\Phi^{(\sigma_0 - r, b)}(\sigma_0, \phi_0, \theta_0, \omega_0)$ has one element. For every fixed number $m > 0$ there
exist positive numbers $\delta$ and $L$, depending only on $f$, on $\tau$ and on the number $m$, such that if

\[
\max \{ |\sigma - \sigma_0|, \|\phi - \phi_0\|_C, d(\theta_0, \theta), d(\omega_0, \omega), \|\psi - \phi_0\|_C, d(\theta_0, \vartheta), d(\omega_0, \varpi) \} \leq \delta,
\]

\[
\sigma_0 \leq \sigma,
\]

$\phi$ is Lipschitzian on the interval $[\lambda(\sigma), 0]$ with the constant $m$,

\[
(\phi, \theta, \omega), (\psi, \vartheta, \varpi) \in \text{Dom} \left( \Phi(\sigma, \cdot, \cdot, \cdot) \right), \ u \in \Phi(\sigma, \psi, \vartheta, \varpi), \ x = \Phi[^{\sigma - r,b}](\sigma, \phi, \theta, \omega),
\]

then items (i) and (ii) below hold.

(i) For every element $t \in [\sigma - r, b] \cap \text{Dom}(u)$ the inequality

\[
(3.1) \quad \|(x - u)(t)\| \leq L \left( \|\phi - \psi\|_C + d(\theta, \vartheta) + d(\omega, \varpi) \right)
\]

holds, moreover $x - u$ is Lipschitzian on the interval $[\sigma, b] \cap \text{Dom}(u)$ with the constant $L \left( \|\phi - \psi\|_C + d(\theta, \vartheta) + d(\omega, \varpi) \right)$.

(ii) If furthermore $\phi - \psi$ is Lipschitzian with the Lipschitz-constant $L_{\phi - \psi}$ on the interval $[\alpha, 0] \subset [-r, 0]$, then $x - u$ is Lipschitzian on the interval $[\alpha + \sigma, b] \cap \text{Dom}(u)$ with the constant

\[
\max \{ L \left( \|\phi - \psi\|_C + d(\theta, \vartheta) + d(\omega, \varpi) \right), L_{\phi - \psi} \}.
\]

Proof. By Lemma 2 we can suppose that if $(t, \psi, \theta) \in \text{Dom}(\tau)$ then $\tau(t, \psi, \theta) \leq r$. By Theorem 5 $\Phi(\sigma, \phi, \theta, \omega)$ has one element, so the equation

\[
x = \Phi[^{\sigma - r,b}](\sigma, \phi, \theta, \omega),
\]

(i) For the brevity denote $y = \Phi[^{\sigma_0 - r,b}](\sigma_0, \phi_0, \theta_0, \omega_0)$. The sets

\[
C_\tau := \{(s, y_s, \theta_0) \mid s \in [\sigma_0, b] \},
\]

and

\[
C_f := \{(s, y_s, \Lambda(s, y_s, \theta_0), \omega_0) \mid s \in [\sigma_0, b] \}
\]

are compact, where $\Lambda(s, y_s, \theta_0) := y(s - \tau(s, y_s, \theta_0))$. There are positive numbers $K$ and $\rho$ such that $f$ is bounded on the $\rho$-neighbourhood $V$ of $C_f$ with a constant $K$. Moreover, the set $V$ and the $\rho$-neighbourhood $W$ of $C_\tau$ can be given so that if

\[
(3.2) \quad \|(x - y)|[^{\sigma, b}]\|_C, \|(y - u)|[^{\sigma, b} \cap \text{Dom}(u)\| \leq \rho
\]
then for every point \( t \in [\sigma, b] \cap \text{Dom} (u) \)

\[
\tag{3.3} \| \tau (t, x_t, \theta) - \tau (t, u_t, \vartheta) \| \leq L_\tau (\| x_t - u_t \|_C + d(\theta, \vartheta)),
\]

and

\[
\tag{3.4} \| f (t, x_t, x (t - \tau (t, x_t, \theta)), \omega) - f (t, u_t, u (t - \tau (t, u_t, \vartheta)), \varpi) \| 
\leq L_f (\| x_t - u_t \|_C + \| \tau (t, x_t, \theta) - \tau (t, u_t, \vartheta) \| + d(\omega, \varpi))
\]
yields for some \( L_\tau \) and \( L_f \). As by Theorem 2 the function \( \Phi^{[\sigma_0 - r, b]} \) is upper-semicontinuous at \((\sigma_0, \phi_0, \theta_0, \omega_0)\), the number \( \delta \) can be chosen so that

\[
\{(s, x_s, \Lambda (s, x_s, \theta), \omega) \mid x = \Phi^{[\sigma - r, b]} (\sigma, \phi, \theta, \omega), s \in [\sigma, b] \} \subset V,
\]

\[
\{(s, u_s, \Lambda (s, u_s, \theta), \varpi) \mid u \in \Phi^{[\sigma - r, b]} (\sigma, \psi, \theta, \varpi), s \in [\sigma, b] \} \subset V,
\]

and if \( s \in [\sigma, b] \) then

\[
(s, x_s, \theta), (s, u_s, \vartheta) \in W
\]

also yields. Therefore \( x \) is lipschitzian on \([\lambda(\sigma), b]\) with the constant \( L_x := \max\{m, K\} \) and \( u \) is lipschitzian on \([\sigma, b] \cap \text{Dom} (u) \) with the constant \( K \). Furthermore the number \( \delta \) can be chosen so that if \( s \in [\sigma, b] \cap \text{Dom} (u) \) then

\[
\| x_s (\tau (s, x_s, \theta)) - x_s (\tau (s, u_s, \vartheta)) \| \leq L_x L_\tau (\| x_s - u_s \|_C + d(\theta, \vartheta)).
\]

Hence

\[
\| x_s (\tau (s, x_s, \theta)) - u_s (\tau (s, u_s, \vartheta)) \| \leq L_x L_\tau (\| x_s - u_s \|_C + d(\theta, \vartheta)) + \| x_s - u_s \|_C
\]

\[
\leq (L_x L_\tau + 1) (\| x_s - u_s \|_C + d(\theta, \vartheta)).
\]

Using inequality (3.5),

\[
\| f (s, x_s, \Lambda (s, x_s, \theta), \omega) - f (s, u_s, \Lambda (s, u_s, \theta), \varpi) \|
\leq L_f (\| x_s - u_s \|_C + \| x - \tau (s, x_s, \theta) \| - u (s - \tau (s, u_s, \vartheta)) \| + d(\omega, \varpi))
\leq L_f (\| x_s - u_s \|_C + (L_x L_\tau + 1) (\| x_s - u_s \|_C + d(\theta, \vartheta)) + d(\omega, \varpi))
\leq k (\| x_s - u_s \|_C + d(\theta, \vartheta) + d(\omega, \varpi)),
\]

where \( k := L_f (L_x L_\tau + 2) \).

If \( t \in [\sigma, b] \cap \text{Dom} (u) \) then

\[
\| (x - u) (t) \|
\leq \| \phi - \psi \|_C + \int_\sigma^t \| f (s, x_s, \Lambda (s, x_s, \theta), \omega) - f (s, u_s, \Lambda (s, u_s, \theta), \varpi) \| \, ds
\]

\[
\leq \| \phi - \psi \|_C + \int_\sigma^t k (\| x_s - u_s \|_C + d(\theta, \vartheta) + d(\omega, \varpi)) \, ds
\]

\[
\leq \| \phi - \psi \|_C + k (b - \sigma) (d(\theta, \vartheta) + d(\omega, \varpi)) + k \int_\sigma^t \| x_s - u_s \|_C \, ds
\]
If conditions of Theorem 6 are satisfied, then there exist numbers δ and \( L \) such that for every element \( s \in [\sigma, b] \)

\[
\|\Lambda(s, x, \theta) - \Lambda(s, u, \vartheta)\| \leq L \left( \|\phi - \psi\|_C \right) + \|\theta - \vartheta\| + \|\omega - \varpi\|
\]

Proof. By (3.5) and (3.1) in Theorem 6

\[
\|\Lambda(s, x, \theta) - \Lambda(s, u, \vartheta)\| \\
\leq (L_L T + 1) \|x - u\|_C + d(\theta, \vartheta) \\
\leq (L_L T + 1) \left( L \left( \|\phi - \psi\|_C \right) + d(\theta, \vartheta) + d(\omega, \varpi) \right) + d(\theta, \vartheta)
\]

This proves the statement.
4. Differentiability. Remind that if $\sigma$ is fixed then

$$\lambda (\sigma) := \inf \{ t - \tau (t, \psi, \vartheta) - \sigma \mid t \in [\sigma, b], \ (t, \psi, \vartheta) \in \text{Dom} (\tau) \} ,$$

and

$$\Lambda (t, \cdot, \cdot) : \{(\phi, \theta) \mid (t, \phi, \theta) \in \text{Dom} (\tau) \} \rightarrow Y, \ (\phi, \theta) \longmapsto \phi (-\tau (t, \phi, \theta)) .$$

**Notation 2.** In this section the following notations will be used:

(i) if

$$g \in \mathbb{R} \rightarrow Y$$

is a function, lipschitzian on the interval $[\alpha, \beta] \subset \text{Dom} (g)$, then

$$L_{[\alpha, \beta]} (g) := \min \{ L \mid L \text{ is a Lipschitz-constant for } g \text{ on } [\alpha, \beta] \} .$$

(ii) $W_{\lambda (\sigma)}^{1, \infty}$ is a Banach space equipped with the norm $\| \cdot \|_{W_{\lambda (\sigma)}^{1, \infty}}$, where

$$W_{\lambda (\sigma)}^{1, \infty} \subset C ([\lambda (\sigma), 0] , Y) ,$$

$$\psi \in W_{\lambda (\sigma)}^{1, \infty} \iff \psi \text{ is lipschitzian on the interval } [\lambda (\sigma), 0] ,$$

$$\| \cdot \|_{W_{\lambda (\sigma)}^{1, \infty}} : W_{\lambda (\sigma)}^{1, \infty} \rightarrow \mathbb{R}, \ \| \psi \|_{W_{\lambda (\sigma)}^{1, \infty}} := \max \{ \| \psi \|_C , L_{[\lambda (\sigma), 0]} (\psi) \} .$$

**Remark 7.** In this section we do not suppose that $\text{Dom} (\Phi^{[\sigma-r,b]})$ is open. That is why we do not say sometimes that the resolvent function is Fréchet-differentiable. However, if $Y = \mathbb{R}^n$ and $\lambda (\sigma) \in [-r, 0]$ then $\text{Dom} (\Phi^{[\sigma-r,b]} (\sigma, \cdot, \cdot, \cdot))$ is open by Theorem 5.3 [12]. Moreover, by Theorem 5 if $\lambda (\sigma) \in [-r, 0]$ then $\text{Dom} (\Phi^{[\sigma-r,b]} (\lambda (\sigma), \cdot, \cdot, \cdot, \cdot, \cdot))$ is open in the space $W_{\lambda (\sigma)}^{1, \infty}$. If the condition $\lambda (\sigma) \in [-r, 0]$ does not hold, see Remark 2 in the introduction.

4.1. The uniform continuity of the function $D_{2,3,4} \Phi^{[\sigma-r,b]}$ under compatibility condition. In this subsection we use the following conditions:

**Condition 1.** The function $f$ is continuous and continuously differentiable in its second, third and fourth variables, $\tau$ is continuous and continuously differentiable in its second and third variables, $(\sigma_0, \phi_0, \theta_0, \omega_0) \in \text{Dom} (\Phi^{[\sigma_0-r,b]})$ and $\Phi^{[\sigma_0-r,b]} (\sigma_0, \phi_0, \theta_0, \omega_0)$ has one element,

$$\max \{ |\sigma_0 - \sigma| , \| \phi_0 - \phi \|_C , \| \phi_0 - \psi \|_C , \| \theta - \theta_0 \| , \| \xi - \theta_0 \| , \| \omega - \omega_0 \| , \| \varphi - \omega_0 \| \} \leq \delta ,$$

$$\sigma \leq \sigma_0 ,$$

$$(\sigma, \phi, \theta, \omega) \in \text{Dom} (\Phi^{[\sigma-r,b]}), \ (\sigma, \psi, \xi, \varphi) \in \text{Dom} (\Phi^{[\sigma-r,b]}),$$

and

$$\sum_{i=1}^4 \| \partial_{\sigma} \Phi^{[\sigma-r,b]} (\sigma_0, \phi_0, \theta_0, \omega_0) \partial_{\varphi} \Phi^{[\sigma-r,b]} (\sigma_0, \phi_0, \theta_0, \omega_0) \| < \infty .$$

**Condition 2.** The function $f$ is continuous and continuously differentiable in its second and fourth variables, $\tau$ is continuous and continuously differentiable in its second vari-
\( m \geq 0 \) is fixed, \( \phi \) is Lipschitzian on \([\lambda(\sigma),0]\) with the constant \( m \),
\[
u \in \Phi^{[\sigma\,-\,\tau],b}(\sigma,\psi,\varpi), \quad x = \Phi^{[\sigma\,-\,\tau],b}(\sigma,\phi,\theta,\omega),
\]

**Condition 2.** \( x \) is continuously differentiable on the interval \([\lambda(\sigma) + \sigma, b]\).

**Remark 8.** (i) If \( \lambda(\sigma) < 0 \) and \( x \) is continuously differentiable on the interval \([\lambda(\sigma) + \sigma, b]\) then the compatibility condition
\[
\phi'(0) = f(\sigma,\phi,\sigma - \tau(\sigma,\phi,\theta),\omega)
\]
holds for the initial function \( \phi \).

(ii) In Condition 1 we do not suppose that \( \Phi(\sigma,\psi,\vartheta,\varpi) \) has one element.

By Lemma 2 we can suppose without loss of generality that \( \tau \leq r \).

First we reformulate some statements given in [14] Sharper forms in the sense that the corresponding estimations hold uniformly.

In the following lemma some frequently used estimations are formulated.

**Lemma 7.** Let \( \Phi \) stand for the resolvent function of the equation
\[
x'(t) = f(t,x,t - \tau(t,x,t),\omega).
\]

Suppose that Condition 1 holds.

(i) The numbers \( \delta > 0 \) and \( k > 0 \) can be chosen so that if \( t \in [\sigma, b] \) then the inequalities, formulated in items (a) and (b) below, yield.

\[
\begin{align*}
\|D_{2,3}\tau(t,u_t,\vartheta)\| & \leq k, \\
|\tau(t,x_t,\theta) - \tau(t,u_t,\vartheta)| & \leq k(\|\psi - \phi\|_C + \|\theta - \vartheta\| + \|\varpi - \omega\|), \\
\|D_2 f(t,u_t,\Lambda(t,u_t,\vartheta),\varpi)\| & \leq k, \\
\|D_3 f(t,u_t,\Lambda(t,u_t,\vartheta),\varpi)\| \|D_{2,3}\tau(t,u_t,\vartheta)\| & \leq k, \\
\|D_4 f(t,u_t,\Lambda(t,u_t,\vartheta),\varpi)\| & \leq k.
\end{align*}
\]

If furthermore Condition 2 is satisfied then
\[
\|x'_t(-\tau(t,x_t,\theta))\| \leq k.
\]

(ii) If \( \varphi \) is Lipschitzian with the number \( L_\varphi \) on the interval \([\lambda(\sigma),0]\) then
\[
\|\varphi(-\tau(t,x_t,\theta)) - \varphi(-\tau(t,u_t,\vartheta))\| \leq L_\varphi k(\|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|).
\]

(ii) For every positive number \( \varepsilon \) the number \( \delta > 0 \) can be chosen so that if \( t \in [\sigma, b] \) then
\[
\|D_{2,3}\tau(t,x_t,\theta) - D_{2,3}\tau(t,u_t,\vartheta)\| \leq \varepsilon.
\]
(iii) If furthermore Condition 2 is satisfied and \( u \) is continuously differentiable on \( [\lambda(\sigma) + \sigma, b] \) and \( \phi_0 \) is continuously differentiable on the interval \( [\kappa, 0] \), where
\[
\kappa = \inf \{ \lambda(\sigma) \mid |\sigma - \sigma_0| \leq \delta \},
\]
then for every fixed positive numbers \( \varepsilon \) and \( m \) the number \( \delta > 0 \) can be chosen so that if
\[
\max \{ \|\phi_0 - \phi\|_{W_1^1}, \|\phi_0 - \psi\|_{W_1^1} \} \leq \delta,
\]
then for every point \( t \in [\sigma, b] \)
\[
\|x'_t(\tau(t, x_t, \theta)) - u'_t(\tau(t, u_t, \vartheta))\| \leq \varepsilon.
\]

**Proof. (i) (a)** By Theorem 6 one can suppose that if \( t \in [\sigma, b] \) then
\[
|\tau(t, x_t, \theta) - \tau(t, u_t, \vartheta)| \leq k (\|\psi - \phi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|).
\]
For the brevity denote
\[
y := \Phi^{[\sigma_0 - r, b]}(\sigma_0, \phi_0, \theta_0, \omega_0).
\]
\(D_{2,3}\tau\) is continuous, therefore it is bounded on the compact set
\[
H_y := \{(t, v_t, \theta_0) \mid t \in [\sigma_0, b], \ v \in \Phi^{[\sigma_0 - r, b]}(\sigma_0, \phi_0, \theta_0, \omega_0)\}.
\]
As \(\Phi^{[\sigma_0 - r, b]}\) is upper semicontinuous at \((\sigma_0, \phi_0, \theta_0, \omega_0)\), this implies that the numbers \( \delta > 0 \) and \( k > 0 \) can be chosen so that if \( t \in [\sigma, b] \) then
\[
\|D_{2,3}\tau(t, u_t, \vartheta)\| \leq k;
\]
Similarly, as the functions \(\Lambda, D_2f, D_3f\) and \(D_4f\) are continuous, one can suppose that
\[
\|D_2f(t, u_t, \Lambda(t, u_t, \vartheta), \varpi)\| \leq k, \quad \|D_3f(t, u_t, \Lambda(t, u_t, \vartheta), \varpi)\| \leq k, \quad \|D_4f(t, u_t, \Lambda(t, u_t, \vartheta), \varpi)\| \leq k.
\]
By Theorem 2 one can suppose that if \( t \in [\sigma, b] \) then
\[
\|x'(t)\| = \|f(t, x_t, \Lambda(t, x_t, \theta), \omega)\| \leq k,
\]
hence if \( x \) is differentiable on \([\lambda(\sigma) + \sigma, b]\) and \( m \leq k \) then
\[
\|x'_t(\tau(t, x_t, \theta))\| \leq k.
also yields.

(b) If \( \varphi \) is lipschitzian with the number \( L_\varphi \) on the interval \([\lambda (\sigma), 0]\) then by item (a)

\[
\| \varphi (-\tau (t, x_t, \theta)) - \varphi (-\tau (t, u_t, \vartheta)) \| \\
\leq L_\varphi | \tau (t, x_t, \theta) - \tau (t, u_t, \vartheta) | \leq L_\varphi k (\| \varphi - \psi \|_C + \| \theta - \vartheta \| + \| \omega - \varpi \|).
\]

(ii) As \( D_{2,3} \tau \) is uniformly continuous on the compact set \( H_y \), there is a \( \rho \)-neighbourhood \( H_\rho \) of \( H_y \) such that

\[
\| D_{2,3} \tau (t, x_t, \theta) - D_{2,3} \tau (t, u_t, \vartheta) \| \leq \varepsilon,
\]

whenever

\[
(t, x_t, \theta), (t, u_t, \vartheta) \in H_\rho.
\]

As \( \Phi^{[\sigma_0-\tau, \theta]} \) is upper semicontinuous at \((\sigma_0, \phi_0, \theta_0, \omega_0)\), one can suppose that this last condition holds.

(iii) By Theorem 5 \( x \) and \( u \) are unique solutions. As \( \phi'_0 \) is uniformly continuous on \([\kappa, 0]\), there exists a positive number \( \mu \) such that

\[
\text{if } s, q \in [\kappa, 0] \text{ and } |s - q| \leq \mu \text{ then } \| \phi'_0 (s) - \phi'_0 (q) \| \leq \varepsilon.
\]

If furthermore

\[
\| \phi_0 - \phi \|_{W^{1,\infty}_{\lambda(\sigma)}} \leq \| \phi_0 - \psi \|_{W^{1,\infty}_{\lambda(\sigma)}} \leq \delta \leq \frac{\varepsilon}{3},
\]

then

\[
(4.1) \quad \| \phi'_0 (s) - \phi'_0 (q) \| \leq 2 \| \phi_0 - \phi \|_{W^{1,\infty}_{\lambda(\sigma)}} + \| \phi'_0 (s) - \phi'_0 (q) \| \leq \varepsilon,
\]

\[
\| \psi'_0 (s) - \psi'_0 (q) \| \leq \varepsilon.
\]

\[
\| x'(t - \tau (t, x_t, \theta)) - u'(t - \tau (t, u_t, \vartheta)) \| \\
\leq \| x'_t (t - \tau (t, x_t, \theta)) - x'_t (t - \tau (t, u_t, \vartheta)) \| \\
+ \| x'_t (t - \tau (t, u_t, \vartheta)) - u'_t (t - \tau (t, u_t, \vartheta)) \|.
\]

By Theorem 6 there is a number \( L \) such that the function \( x - u \) is lipschitzian on \([\lambda (\sigma) + \sigma, b]\) with the constant

\[
\max \{ L (\| \phi - \psi \|_C + \| \theta - \vartheta \| + \| \omega - \varpi \|) , L_{\phi - \psi} \},
\]

if \( \phi - \psi \) is lipschitzian on \([\lambda (\sigma), 0]\) with the constant \( L_{\phi - \psi} \). Therefore

\[
\| x'_t (t - \tau (t, u_t, \vartheta)) - u'_t (t - \tau (t, u_t, \vartheta)) \| \\
\leq \max \{ L (\| \phi - \psi \|_C + \| \theta - \vartheta \| + \| \omega - \varpi \|) , \| \phi - \psi \|_{W^{1,\infty}_{\lambda(\sigma)}} \} \leq \varepsilon,
\]
if $\delta$ is small enough. Moreover, if $s, q \in [\sigma, b]$ then it follows from the compactness of $H_y$, the continuity of $f$ and from the continuity of $\Phi^{[\sigma_0, b]}$ at $(\sigma_0, \phi_0, \theta_0, \omega_0)$ that

$$\|x'(s) - x'(q)\| = \|f(s, x_s, x_s(-\tau(s, x_s, \theta))) - f(q, x_q, x_q(-\tau(q, x_q, \theta)))\| \leq \varepsilon,$$

if $\delta$ is small enough. Consequently, $\delta$ can be chosen so that if

$$t - \tau(t, x_t, \theta) \in [\sigma, b], \ t - \tau(t, u_t, \vartheta) \in [\sigma, b]$$

then

$$\|x'_t(-\tau(t, x_t, \theta)) - x'_t(-\tau(t, u_t, \vartheta))\| \leq \varepsilon.$$  \hspace{1cm} (4.2)

If

$$t - \tau(t, x_t, \theta) \in [\sigma, b], \ t - \tau(t, u_t, \vartheta) \in [\lambda(\sigma) + \sigma, \sigma]$$

then

$$|t - \tau(t, u_t, \vartheta) - \sigma| \leq |\tau(t, x_t, \theta) - \tau(t, u_t, \vartheta)| \leq k(\|\psi - \phi\|_C + \|\theta - \vartheta\| + \|\varpi - \omega\|),$$

and by (4.2)

$$\|x'(t - \tau(t, x_t, \theta)) - \phi'(0)\| \leq \varepsilon,$$

therefore by (4.1)

$$\|x'(t - \tau(t, x_t, \theta)) - x'(t - \tau(t, u_t, \vartheta))\| \leq \|x'(t - \tau(t, x_t, \theta)) - \phi'(0)\| + \|\phi'(0) - \phi'(t - \tau(t, u_t, \vartheta) - \sigma)\| \leq 2\varepsilon,$$

if $\delta$ is small enough.

Similarly, if

$$t - \tau(t, x_t, \theta) \in [\lambda(\sigma) + \sigma, \sigma], \ t - \tau(t, u_t, \vartheta) \in [\lambda(\sigma) + \sigma, \sigma]$$

then if

$$|\tau(t, x_t, \theta) - \tau(t, u_t, \vartheta)| \leq k(\|\psi - \phi\|_C + \|\theta - \vartheta\| + \|\varpi - \omega\|)$$

is small enough then

$$\|x'(t - \tau(t, x_t, \theta)) - x'(t - \tau(t, u_t, \vartheta))\| = \|\phi'(t - \tau(t, x_t, \theta) - \sigma) - \phi'(t - \tau(t, u_t, \vartheta) - \sigma)\| \leq \varepsilon.$$

This proves the statement. \qed
By Lemma 3 [14] if \( \tau \) is continuous and continuously differentiable in its second and third variables, \((t, \phi, \theta) \in \text{Dom}(\tau), \tau \leq r \) and the function \( \phi \) is continuously differentiable at the point \(-\tau(t, \phi, \theta)\) then the function \( \Lambda(t, \cdot, \cdot) \) is differentiable at \((\phi, \theta)\) and

\[
D_{2,3} \Lambda(t, \phi, \theta) = \varphi(-\tau(t, \phi, \theta)) - [D_{2,3} \tau(t, \phi, \theta)(\varphi, \xi)] \phi'(-\tau(t, \phi, \theta)).
\]

The following Lemma 8 is a sharper variant of Lemma 4 [14].

**Lemma 8.** Let \( \Phi \) stand for the resolvent function of the equation

\[
x'(t) = f(t, x(t), x(t - \tau(t, x(t), \theta)), \omega).
\]

Suppose that Conditions 1 and 2 hold, \( \tau \leq r \) and let \( \varepsilon \) be an arbitrary fixed positive number and \( \psi \in W^{1,\infty}_\lambda(\sigma) \).

(i) The positive number \( \delta \) can be fixed so that for every point \( s \in [\sigma, b] \)

\[
\|\Lambda(s, x_s, \theta) - \Lambda(s, u_s, \theta) - D_{2,3} \Lambda(s, x_s, \theta)(x_s - u_s, \theta - \vartheta)\| 
\leq \varepsilon \left( \|\phi - \psi\|_{W^{1,\infty}_\lambda(\sigma)} + \|\theta - \vartheta\| + \|\omega - \varpi\| \right),
\]

and there is a number \( k \) such that for every point \((\varphi, \vartheta) \in C([-r, 0], Y) \times \Theta \)

\[
\|D_{2,3} \Lambda(s, x_s, \theta)(\varphi, \vartheta)\| \leq k (\|\varphi\|_C + \|\vartheta\|).
\]

(ii) If furthermore \( \phi_0 \in W^{1,\infty}_\lambda(\sigma) \) then the number \( \delta \) can be chosen so that

\[
\|\Lambda(s, x_s, \theta) - \Lambda(s, u_s, \theta) - D_{2,3} \Lambda(s, x_s, \theta)(x_s - u_s, \theta - \vartheta)\| 
\leq \varepsilon (\|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|)
\]

yields, whenever

\[
\|\phi_0 - \phi\|_{W^{1,\infty}_\lambda(\sigma)}, \|\phi_0 - \psi\|_{W^{1,\infty}_\lambda(\sigma)} \leq \delta.
\]

(iii) If \( \lambda(\sigma) = 0 \) then the number \( \delta \) can be chosen so that

\[
\|\Lambda(s, x_s, \theta) - \Lambda(s, u_s, \theta) - D_{2,3} \Lambda(s, x_s, \theta)(x_s - u_s, \theta - \vartheta)\| 
\leq \varepsilon (\|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|).
\]

**Proof.** Modifying the proof of Lemma 4 [14], first we show that for each point \( t \in [\sigma, b] \) there is an interval neighbourhood (with respect to the inherited
that inequalities hold, whenever \( s, t, x \) is continuously differentiable in its second and third variable at the fixed topology of \([\sigma, b] \)

Under our suppositions the function

\[
h : \text{Dom} \, (\tau) \to Y, \quad (t, \varphi, \xi) \mapsto x (t - \tau (t, \varphi, \xi))
\]
is continuously differentiable in its second and third variable at the fixed point \((t, x_t, \theta)\), and

\[
D_{2,3} h (t, x_t, \theta) (x_t - u_t, \theta - \vartheta) = D_{2,3} \Lambda (t, x_t, \theta) (x_t - u_t, \theta - \vartheta) - (x_t - u_t) (-\tau (t, x_t, \theta)).
\]

Let \( \varepsilon > 0 \) and \( \mu > 0 \) be fixed, and let the positive number \( \delta_t \) and the interval \( I_t \) be chosen so that properties (I)-(II) below hold:

(I) if \( \|x_s - u_s\|_C + \|\theta - \vartheta\| \leq \delta_t \) and \( s \in I_t \) then

\[
\|x (s - \tau (s, x_s, \theta)) - x (s - \tau (s, u_s, \vartheta)) - D_{2,3} h (s, x_s, \theta) (x_s - u_s, \theta - \vartheta)\| \\
\leq \mu (\|x_s - u_s\|_C + \|\theta - \vartheta\|),
\]

(II) if \( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \leq \delta_t \) then for some constant \( k \)

\[
|\tau (s, x_s, \theta) - \tau (s, u_s, \vartheta)| \leq k (\|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|)
\]

holds, whenever \( s \in I_t \cap \text{Dom} \,(u) \).

By our supposition on \( h \) (I) can be satisfied. By Lemma 7 condition (II) is satisfied if \( \delta_t \) is small enough.

If \([a, b] \subset \text{Dom} \,(x - u)\) then in this proof \( L_{[a,b]} (x - u) \) denotes the minimal Lipschitz-constant of \( x - u \) on the interval \([a, b]\).

We show that if (I)-(II) hold and \( \mu \) and \( \delta_t \) are small enough then inequality (4.4) also holds. Indeed, using the estimations of Lemma 7 we get that inequalities

\[
\|\Lambda (s, x_s, \theta) - \Lambda (s, u_s, \vartheta) - D_{2,3} \Lambda (s, x_s, \theta) (x_s - u_s, \theta - \vartheta)\| \\
= \|x_s (-\tau (s, x_s, \theta)) - u_s (-\tau (s, u_s, \vartheta)) - (x_s - u_s) (-\tau (s, x_s, \theta))\| \\
\leq \|u_s (-\tau (s, x_s, \theta)) - u_s (-\tau (s, u_s, \vartheta)) - x_s (-\tau (s, x_s, \theta))\| \\
+ \|x_s (-\tau (s, x_s, \theta)) - x_s (-\tau (s, x_s, \theta))\| + \|x_s (-\tau (s, x_s, \theta)) - x_s (-\tau (s, u_s, \vartheta))\|
\]

\[
\leq L_{[\lambda(\sigma) + \sigma, b]} (x - u) |\tau (s, x_s, \theta) - \tau (s, u_s, \vartheta)| + \mu (\|x_s - u_s\|_C + \|\theta - \vartheta\|) \\
\leq L_{[\lambda(\sigma) + \sigma, b]} (x - u) L_\tau (\|x_s - u_s\|_C + \|\theta - \vartheta\|) + \mu (\|x_s - u_s\|_C + \|\theta - \vartheta\|) \\
\leq L_{[\lambda(\sigma) + \sigma, b]} (x - u) k (\|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\|)
\]
yields for some constant \( k \).

By Theorem 6 there is a number \( L \) such that if \( \delta_t \) is small enough then

\[
L_{[\sigma, b]} (x - u) \leq L \left( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \right).
\]

Consequently, using the inequalities

\[
L_{([\lambda(\sigma), 0])} (x - u) + L \left( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \right)
\]

\[
\leq L \|\phi - \psi\|_{W^1_{\lambda(\sigma)}} + L \left( \|\theta - \vartheta\| + \|\omega - \varpi\| \right),
\]

\[
\|\phi - \psi\|_C \leq \|\phi - \psi\|_{W^1_{\lambda(\sigma)}},
\]

and the estimations (4.7), we get that

\[
(4.8) \quad \left\| \Lambda (s, x_s, \theta) - \Lambda (s, u_s, \vartheta) - D_{2,3} \Lambda (s, x_s, \theta) (x_s - u_s, \theta - \vartheta) \right\|
\]

\[
\leq L \left( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \right) \cdot k \left( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \right).
\]

(i) Estimation (4.8) implies that \( \delta_t \) can be chosen so that

\[
\|\Lambda (s, x_s, \theta) - \Lambda (s, u_s, \vartheta) - D_{2,3} \Lambda (s, x_s, \theta) (x_s - u_s, \theta - \vartheta) \| \leq \varepsilon \left( \|\phi - \psi\|_{W^1_{\lambda(\sigma)}} + \|\theta - \vartheta\| + \|\omega - \varpi\| \right).
\]

Moreover, by Lemma 7 we can suppose that

\[
\left\| D_{2,3} \Lambda (t, x_t, \theta) (\varphi, \vartheta) \right\|
\]

\[
\leq \|\varphi (-\tau (t, x_t, \theta))\| + \|\left[ D_{2,3} \tau (t, x_t, \theta) (\varphi, \vartheta) \right] x_t' (-\tau (t, x_t, \theta))\|
\]

\[
\leq \|\varphi\|_C + \|x_t\|_{W^1_{\lambda(\sigma)}} \|D_{2,3} \tau (t, x_t, \theta) (\varphi, \vartheta)\|
\]

\[
\leq k \left( \|\varphi\|_C + \|\vartheta\| \right).
\]

(ii) The triangle inequality and inequalities (4.8) imply that \( \delta_t \) can be chosen so that

\[
\|\Lambda (s, x_s, \theta) - \Lambda (s, u_s, \vartheta) - D_{2,3} \Lambda (s, x_s, \theta) (x_s - u_s, \theta - \vartheta) \| \leq \varepsilon \left( \|\phi - \psi\|_C + \|\theta - \vartheta\| + \|\omega - \varpi\| \right).
\]

As \([\sigma, b]\) is compact, this proves the statement.

(iii) If \( \lambda (\sigma) = 0 \) then \( L_{[\lambda(\sigma), 0]} (x - u) = 0 \), therefore by (4.7) the statement is verified. \( \blacksquare \)
The function $\Lambda$ is not continuously differentiable in its second and third variables. However, its restriction has continuity property, formulated in the following lemmas, playing important role in the proof of the continuity of $D_{2,3,4}\Phi^{[\sigma-r,0]}$.

**Lemma 9.** Let $\tau$ be continuous and continuously differentiable in its second and third variables, and $\tau \leq \rho \leq r$. The function $\Lambda$ is continuously differentiable in the following sense: for every point $(t, \alpha, \theta) \in \text{Dom}(\tau)$ and for every positive number $\varepsilon$ there is a positive number $\delta$ such that if $(s, \beta, \vartheta) \in \text{Dom}(\tau)$, $\alpha$ and $\beta$ are continuously differentiable on the interval $[-\rho, 0] \subset [-r, 0]$, \[
\max\left\{|t-s|, \|\theta - \vartheta\|, \|\alpha - \beta\|_{W^{1,\infty}_\rho}\right\} \leq \delta,
\]
then for every element $(\varphi, \xi) \in W^{1,\infty}_\rho \times \Theta$
\[
||[D_{2,3,4}\Lambda(t, \alpha, \theta) - D_{2,3}\Lambda(s, \beta, \vartheta)](\varphi, \xi)|| \leq \varepsilon \left(||\varphi||_{W^{1,\infty}_\rho} + ||\xi||\right).
\] (4.9)

**Proof.** By our suppositions the numbers $\delta$ and $k$ can be chosen so that
\[
\|D_{2,3}\tau(t, \alpha, \theta)\| \leq k, \delta \leq \frac{\varepsilon}{2k},
\]
\[
\|D_{2,3}\tau(t, \alpha, \theta) - D_{2,3}\tau(s, \beta, \vartheta)\| \leq \frac{\varepsilon}{k},
\]
\[
\|\beta'(-\tau(s, \beta, \vartheta))\| \leq \|\alpha\|_{W^{1,\infty}_\rho} + \delta \leq k,
\]
moreover
\[
|\tau(t, \alpha, \theta) - \tau(s, \beta, \vartheta)| \leq \varepsilon,
\]
\[
\mu_{\alpha'}(\tau(t, \alpha, \theta) - \tau(s, \beta, \vartheta)) \leq \frac{\varepsilon}{2k},
\]
where $\mu_{\alpha'}$ denotes the uniform continuity modulus of $\alpha'$. Consequently
\[
\|\alpha'(-\tau(s, \alpha, \theta)) - \beta'(-\tau(s, \beta, \vartheta))\| \leq \|\alpha'(-\tau(t, \alpha, \theta)) - \alpha'(-\tau(s, \beta, \vartheta))\| + \|\alpha - \beta\|_{W^{1,\infty}_\rho} \leq \frac{\varepsilon}{k}.
\]
If $\delta$ is small enough then
\[
\|\phi(-\tau(t, \alpha, \theta)) - \phi(-\tau(s, \beta, \vartheta))\| \leq \|\phi\|_{W^{1,\infty}_\rho} |\tau(t, \alpha, \theta) - \tau(s, \beta, \vartheta)| \leq \varepsilon \|\phi\|_{W^{1,\infty}_\rho}.\]
Therefore, using the estimations above, we get that
\[
\| D_{2,3} \Lambda \left( t, \alpha, \theta \right) - D_{2,3} \Lambda \left( s, \beta, \vartheta \right) \left( \phi, \xi \right) \|
\leq \| \varphi \left( -\tau \left( t, \alpha, \theta \right) \right) - \varphi \left( -\tau \left( s, \beta, \vartheta \right) \right) \| + \\
\| D_{2,3} \tau \left( t, \alpha, \theta \right) \left( \varphi, \xi \right) \alpha' \left( -\tau \left( t, \alpha, \theta \right) \right) - D_{2,3} \tau \left( s, \beta, \vartheta \right) \left( \varphi, \xi \right) \beta' \left( -\tau \left( s, \beta, \vartheta \right) \right) \|
\leq \varepsilon \| \varphi \|_{W_{\lambda(\sigma)}^{1,\infty}} + \| \left( D_{2,3} \tau \left( t, \alpha, \theta \right) - D_{2,3} \tau \left( s, \beta, \vartheta \right) \right) \left( \varphi, \xi \right) \left( \beta' \left( -\tau \left( s, \beta, \vartheta \right) \right) \right) \| + \\
\| D_{2,3} \tau \left( t, \alpha, \theta \right) \left( \varphi, \xi \right) \alpha' \left( -\tau \left( t, \alpha, \theta \right) \right) - \beta' \left( -\tau \left( s, \beta, \vartheta \right) \right) \|
\leq \varepsilon \| \varphi \|_{W_{\lambda(\sigma)}^{1,\infty}} + 2 \varepsilon \left( \| \varphi \|_{C} + \| \xi \| \right).
\]

This proves the statement. 

**Lemma 10.** Suppose that Conditions 1 and 2 hold. For every positive number \( \varepsilon \) there is a positive number \( \delta \) such that if \( \psi \) is continuously differentiable on the interval \( [\lambda(\sigma), 0] \), the compatibility condition

\[
\psi'(0) = f \left( \sigma, \psi, \psi \left( -\tau \left( \sigma, \psi, \vartheta \right) \right) \right), \quad \varphi \in W_{\lambda(\sigma)}^{1,\infty},
\]

\[
\| \phi_0 - \phi \|_{W_{\lambda(\sigma)}^{1,\infty}}, \| \phi_0 - \psi \|_{W_{\lambda(\sigma)}^{1,\infty}} \leq \delta.
\]

then for every point \( t \in [\sigma, b] \) and for every element \( (\varphi, \xi) \in W_{\lambda(\sigma)}^{1,\infty} \times \Theta \)

\[
\| D_{2,3} \Lambda \left( t, x_t, \xi \right) - D_{2,3} \Lambda \left( t, u_t, \xi \right) \left( \varphi, \xi \right) \|
\leq \varepsilon \left( \| \varphi \|_{W_{\lambda(\sigma)}^{1,\infty}} + \| \xi \| \right).
\]

**Proof.** Let \( \varepsilon > 0 \) be fixed. By (4.3)

\[
\| (D_{2,3} \Lambda \left( t, x_t, \theta \right) - D_{2,3} \Lambda \left( t, u_t, \vartheta \right) \left( \varphi, \xi \right) \|
\leq \| \varphi \left( -\tau \left( t, x_t, \theta \right) \right) - \varphi \left( -\tau \left( t, u_t, \vartheta \right) \right) \| + \\
\| D_{2,3} \tau \left( t, x_t, \theta \right) \left( \varphi, \xi \right) x_t' \left( -\tau \left( t, x_t, \theta \right) \right) - D_{2,3} \tau \left( t, u_t, \vartheta \right) \left( \varphi, \xi \right) u_t' \left( -\tau \left( t, u_t, \vartheta \right) \right) \|
\]

Furthermore

\[
\| D_{2,3} \tau \left( t, x_t, \theta \right) \left( \varphi, \xi \right) x_t' \left( -\tau \left( t, x_t, \theta \right) \right) - D_{2,3} \tau \left( t, u_t, \vartheta \right) \left( \varphi, \xi \right) u_t' \left( -\tau \left( t, u_t, \vartheta \right) \right) \|
\leq \| (D_{2,3} \tau \left( t, x_t, \theta \right) - D_{2,3} \tau \left( t, u_t, \vartheta \right) \left( \varphi, \xi \right) x_t' \left( -\tau \left( t, x_t, \theta \right) \right) \| + \\
\| (D_{2,3} \tau \left( t, u_t, \vartheta \right) \left( \varphi, \xi \right) u_t' \left( -\tau \left( t, u_t, \vartheta \right) \right) \|.
\]
By Lemma 7 one can suppose that
\[ \| D_{2,3} \tau (t, x_t, \theta) - D_{2,3} \tau (t, u_t, \vartheta) \| \leq \varepsilon, \]
therefore one can suppose that there is a number \( k \) such that
(4.12)\[ \| [D_{2,3} \tau (t, x_t, \theta) - D_{2,3} \tau (t, u_t, \vartheta)] (\varphi, \xi) x'_t (-\tau (t, x_t, \theta)) \| \leq \varepsilon, \]
By Lemma 7 if \( \delta \) is small enough then
(4.13)\[ \| D_{2,3} \tau (t, u_t, \vartheta) \| \leq k, \]
and
(4.14)\[ \| x'_t (-\tau (t, x_t, \theta)) - u'_t (-\tau (t, u_t, \vartheta)) \| \leq \varepsilon. \]
By (4.11), (4.12), (4.13) and (4.14)
\[ \| D_{2,3} \Lambda (t, x_t, \theta) - D_{2,3} \Lambda (t, u_t, \vartheta) (\varphi, \xi) \| \leq \varepsilon \left( \| \varphi \|_{W_1^1} + \| \xi \| \right), \]
if \( \delta \) is small enough. □

**Definition 5.** Suppose that Conditions 1 and 2 are satisfied and \( \tau \leq r \).
For every point \( t \in [\sigma, b] \) let \( A (t, \sigma, \phi, \theta, \omega) \) stand for the function
(4.15)\[ A (t, \sigma, \phi, \theta, \omega) : C([-r, 0], Y) \times \Theta \times \Omega \to Y, \]
\[ (\varphi, \xi, \eta) \mapsto D_2 f (t, x_t, \Lambda (t, x_t, \theta), \omega) (\varphi) + [D_3 f (t, x_t, \Lambda (t, x_t, \theta), \omega) \circ D_{2,3} \Lambda (t, x_t, \theta)] (\varphi, \xi) + D_4 f (t, x_t, \Lambda (t, x_t, \theta), \omega) (\eta). \]

The state-independent parametric linear functional differential equation
(4.16)\[ y' (t) = A (t, \sigma, \phi, \theta, \omega) (y_t, \xi, \eta), \quad t \geq \sigma \]
is named the variational equation of the equation
(4.17)\[ x' (t) = f (t, x_t, x (-\tau (t, x_t, \theta)), \omega), \quad t \geq \sigma. \]
In the case of the state-independent functional differential equations (that is when \( \tau = 0 \))
\[ A (t, \sigma, \phi, \omega) : C([-r, 0], Y) \times \Omega \to Y, \]
\[ (\varphi, \eta) \mapsto D_2 f (t, x_t, \omega) (\varphi) + D_3 f (t, x_t, \omega) (\eta), \]
and 
\[ y'(t) = A(t, \sigma, \phi, \omega)(y_t, \eta) \]
is named the variational equation of the equation 
\[ x'(t) = f(t, x_t, \omega). \]

**Notation 3.** Let \( \Delta \) stand for the resolvent function of the variational equation (4.16), that is let \( \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta) \) denote the set of the noncontinuable solutions \( v \) of the equation (4.16), satisfying the initial condition \( v_\sigma = \varphi \).

**Remark 9.** By Theorem 5 \( x \) is the unique element of \( \Phi(\sigma, \phi, \theta, \omega) \).

By Theorem 7, [14] the solution \( \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta) \) exists, it is unique and \( \Delta(\sigma, \phi, \theta, \omega) \) is continuous and linear.

In the following Theorem 7 a modification of Theorem 8 [14] is given, saying that if \( \phi \) is continuously differentiable and the compatibility condition is satisfied then the resolvent function \( \Phi \) is differentiable in its variables \((\phi, \theta, \omega)\), uniformly with respect to the parameters. The proof is the suitable modification of the proof of Theorem 8 [14].

**Theorem 7.** Suppose that Conditions 1 and 2 are satisfied, moreover \( \psi \in W^{1,\infty}_{\lambda(\sigma)} \).

For every positive number \( \varepsilon \) the number \( \delta > 0 \) can be given so that

(i) 
\[ \| u - x - \Delta^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega) [(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)] \| \leq \varepsilon \left( \| \psi - \phi \|_{W^{1,\infty}_{\lambda(\sigma)}} + \| \vartheta - \theta \| + \| \varpi - \omega \| \right) \]

(ii) if \( \phi_0 \in W^{1,\infty}_{\lambda(\sigma)} \),
\[ \| \phi_0 - \phi \|_{W^{1,\infty}_{\lambda(\sigma)}}, \| \phi_0 - \psi \|_{W^{1,\infty}_{\lambda(\sigma)}} \leq \delta \]
then
\[ \| u - x - \Delta^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega) [(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)] \| \leq \varepsilon (\| \psi - \phi \|_{C} + \| \vartheta - \theta \| + \| \varpi - \omega \|) \]
holds.

**Proof.** We can suppose that \( \tau \leq r \). Evidently, \( x \) is lipschitzian on \([\lambda(\sigma), b]\) with some constant \( L_x \). It follows from our suppositions that we can suppose that \( f \) is uniformly differentiable in its second, third and fourth variables.
at the points of the compact set \( \{ (t, x_t, \Lambda(t, x_t, \theta), \omega) \mid t \in [\sigma, b] \} \) and \( \tau \) is uniformly differentiable in its second and third variables at the points of the compact set \( \{ (t, x_t, \theta) \mid t \in [\sigma, b] \} \).

Let \( \varepsilon > 0 \) be fixed.

First we show that if \( \delta \) is small enough then

\[
\begin{align*}
\text{(4.18)} & \quad \left| f(t, x_t, \Lambda(t, x_t, \theta), \omega) - f(t, u_t, \Lambda(t, u_t, \theta), \omega) \right| \\
& \leq c \left( \| \phi - \psi \|_C + \| \theta - \vartheta \| + \| \omega - \bar{\omega} \| \right).
\end{align*}
\]

Indeed, if \( s \in [\sigma, b] \) then

\[
\begin{align*}
\text{(4.19)} & \quad \left| f(s, x_s, \Lambda(s, x_s, \theta), \omega) - f(s, u_s, \Lambda(s, u_s, \theta), \omega) \right| \\
& \leq \left| D_3 f(s, x_s, \Lambda(s, x_s, \theta), \omega) (\Lambda(s, x_s, \theta) - \Lambda(s, u_s, \theta)) \right| \\
& \quad + \left| D_2 f(s, x_s, \Lambda(s, x_s, \theta), \omega) (D_3 \Lambda(s, x_s, \theta) (x_s - u_s, \theta - \vartheta)) \right| \\
\text{(4.20)} & \quad + \left| D_4 f(s, x_s, \Lambda(s, x_s, \theta), \omega) (\omega - \bar{\omega}) \right|.
\end{align*}
\]

The estimation of the first part (4.19): by Corollary 4 for every positive number \( \mu \) the number \( \delta \) can be chosen so that

\[
\begin{align*}
\text{(4.22)} & \quad \left| f(s, x_s, \Lambda(s, x_s, \theta), \omega) - f(s, x_s, \Lambda(s, u_s, \theta), \omega) \right| \\
& \leq \mu \left| \Lambda(s, x_s, \theta) - \Lambda(s, u_s, \theta) \right| \leq \mu L \left( \| \phi - \psi \|_C + \| \theta - \vartheta \| + \| \omega - \bar{\omega} \| \right).
\end{align*}
\]

The estimation of the part (4.20): by Lemma 8 if

\[
\sup_{s \in [\sigma, b]} \| D_3 f(s, x_s, \Lambda(s, x_s, \theta), \omega) \| \leq k
\]

then

\[
\begin{align*}
\text{(4.23)} & \quad \left| D_3 f(s, x_s, \Lambda(s, x_s, \theta), \omega) \right| \\
& \leq \varepsilon k \left\{ \begin{array}{ll}
(\| \phi - \psi \|_{W_{\Lambda(s)}^{1,\infty}} + \| \theta - \vartheta \|), & \text{if } \| \phi - \psi \|_C \leq \delta \\
(\| \phi - \psi \|_C + \| \theta - \vartheta \|), & \text{if } \| \phi - \psi \|_{W_{\Lambda(s)}^{1,\infty}} \leq \delta
\end{array} \right.
\end{align*}
\]
The estimation of (4.21): the numbers \( \delta \) and \( \mu \) can be given so that

\[
\begin{align*}
&\left| f(s, x_s, \Lambda (s, u_s, \theta), \omega) - f(s, u_s, \Lambda (s, u_s, \theta), \varpi) \right| \\
&\quad - D_2 f(s, x_s, \Lambda (s, x_s, \theta), \omega) (x_s - u_s) \\
&\quad - D_4 f(s, x_s, \Lambda (s, x_s, \theta), \omega) (\omega - \varpi)
\end{align*}
\]

\[
\leq \mu (\| x_s - u_s \|_C + \| \omega - \varpi \|) \leq \varepsilon (\| \phi - \psi \|_C + \| \omega - \varpi \|)
\]

yields. Therefore if \( t \in [\sigma, b] \) then

\[
\begin{align*}
&\left| (u - x - \Delta^{[\sigma-r,b]} (\sigma, \phi, \theta, \omega) [(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)]) (t) \right| \\
&\quad \leq \int_{\sigma}^{t} \left| f(s, u_s, \Lambda (s, u_s, \theta), \omega) - f(s, x_s, \Lambda (s, x_s, \theta), \omega) \\
&\quad - A(s, \sigma, \phi, \theta, \omega) (u_s - x_s, \theta - \theta, \varpi - \varpi) \right| \, ds
\end{align*}
\]

\[
+ \int_{\sigma}^{t} \| A(s, \sigma, \phi, \theta, \omega) (u_s - x_s) \\
&\quad - \left[ \Delta^{[\sigma-r,b]} (\sigma, \phi, \theta, \omega) (\psi - \phi, \vartheta - \theta, \varpi - \varpi) \right] (s) \| \, ds
\]

\[
\leq \varepsilon (b - \sigma) (\| \psi - \phi \|_C + \| \vartheta - \theta \| + \| \varpi - \varpi \|)
\]

\[
+ K \int_{\sigma}^{t} \| (u_s - x_s) - \| \Delta^{[\sigma-r,b]} \phi (\psi, \vartheta, \varpi) \| (\psi - \phi, \vartheta - \theta, \varpi - \varpi) \| (s) \| \, ds,
\]

where \( K := \max_{s \in [\sigma, b]} \| A(s, \sigma, \phi, \theta, \omega) \| \). Therefore, by the Bellmann-Gronwall inequality,

\[
\begin{align*}
&\left| (u - x - \Delta^{[\sigma-r,b]} (\sigma, \phi, \theta, \omega) [(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)]) (t) \right| \\
&\quad \leq \varepsilon (b - \sigma) (\| \psi - \phi \|_C + \| \vartheta - \theta \| + \| \varpi - \varpi \|) e^{K(b-\sigma)}.
\end{align*}
\]

If \( t \in [\sigma - r, \sigma] \) then

\[
\left| (u - x - \Delta^{[\sigma-r,b]} (\sigma, \phi, \theta, \omega) [(\psi, \vartheta, \varpi) - (\phi, \theta, \omega)]) (t) \right| = 0.
\]

The statement is proved.

**Lemma 11.** If Conditions 1 and 2 are satisfied then the numbers \( \delta \) and \( K \) can be chosen so that

\[
\| A(s, \sigma, \phi, \theta, \omega) (\varphi, \xi, \eta) \| \leq K (\| \varphi \|_C + \| \xi \| + \| \eta \|),
\]

whenever \( s \in [\sigma, b] \) and \( (\varphi, \xi, \eta) \in C([-r, 0], Y) \times \Theta \times \Omega \).

**Proof.** By Lemma 7 the number \( k \) can be chosen so that the inequality

\[
\| D_3 f(s, x_s, \Lambda (s, x_s, \theta), \omega) \| \| D_{2,3} \tau (s, x_s, \theta) \| \| x' (s - \tau (s, x_s, \theta)) \| \leq k
\]

also holds.
Hence one can suppose that
\[ \|A(s, \sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)\| \]
\[ \leq \|D_2 f(s, x_s, x(s - \tau(s, x_s, \theta)), \omega)\| \|\varphi\|_C \]
\[ + \|D_3 f(s, x_s, x(s - \tau(s, x_s, \theta)), \omega)\| \|\varphi\|_{-\tau(s, x_s, \theta)} \| \xi\| \]
\[ + \|D_3 f(s, x_s, x(s - \tau(s, x_s, \theta)), \omega)\| \cdot \]
\[ \|D_2 \tau(s, x_s, \theta)\| (\|\varphi\|_C + \|\xi\|) \|x_s'(t - \tau(s, x_s, \theta))\| \]
\[ + \|D_4 f(s, x_s, x(s - \tau(s, x_s, \theta)), \omega)\| \|\eta\| \]
\[ \leq k (\|\varphi\|_C + \|\xi\| + \|\eta\|), \]
which proves the statement.

**Lemma 12.** Suppose that Conditions 1 and 2 yield. Denote \( \Delta(\sigma, \phi, \theta, \omega) \) the resolvent function of the variational equation (4.16). The function \( \Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega) \) is continuous and linear. Moreover, there is a number \( k \) such that if \( \delta \) is small enough then
\[ \|\Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| \leq k (\|\varphi\|_C + \|\xi\| + \|\eta\|). \]

**Proof.** The linearity and the continuity of \( \Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega) \) was proved in Theorem 7 (ii) [14].

If \( t \in [\sigma - r, \sigma] \) then
\[ \|\Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| \leq \|\varphi(t - \sigma)\| \leq \|\varphi\|_C. \]

By Lemma 11 if \( y = \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta) \) and \( t \in [\sigma, b] \) then
\[ \|y(t)\| = \|\Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| \]
\[ \leq \|\varphi\|_C + \int_{\sigma}^{t} \|A(s, \sigma, \phi, \theta, \omega)(y_s, \xi, \eta)\| \] \[ \leq \|\varphi\|_C + K \int_{\sigma}^{t} \|y_s\|_C + \|\xi\| + \|\eta\| \] \[ \leq \|\varphi\|_C + K (b - \sigma) (\|\xi\| + \|\eta\|) + K \int_{\sigma}^{t} \|y_s\|_C \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] \[ \] holds, where \( K \) does not depend on the point \((\varphi, \xi, \eta)\). By the Bellmann-Gronwall inequality
\[ \|\Delta^{[\sigma - r, b]}(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| \leq [\|\varphi\|_C + K (b - \sigma) (\|\xi\| + \|\eta\|)] e^{K(b - \sigma)}, \]
which proves the statement. \( \blacksquare \)
The following Lemma 13 is a sharper variant of Lemma 6 [14], where the parameter $\sigma$ was fixed. The proof is a suitable modification of the proof given there.

**Lemma 13.** If Conditions 1 and 2 are satisfied then the numbers $\delta$ and $K$ can be chosen so that

(i) $\Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)$ is Lipschitzian on the interval $[\sigma, b]$ with the constant $K(\|\varphi\|_C + \|\xi\| + \|\eta\|)$;

(ii) if moreover $\varphi \in W^{1, \infty}_{\lambda(\sigma)}$, then $\Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)$ is Lipschitzian on the interval $[\lambda(\sigma) + \sigma, b]$ with the constant $K\left(\|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}} + \|\xi\| + \|\eta\|\right)$.

**Proof.** Let denote $y := \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)$.

(i) If $t \in [\sigma, b]$ then by Lemma 11 and Lemma 12

$$\|(\Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta))'(t)\| = \|A(t, \sigma, \phi, \theta, \omega)(y_t, \xi, \eta)\| \leq K(\|y_t\|_C + \|\xi\| + \|\eta\|)$$

$$\leq K(k(\|\varphi\|_C + \|\xi\| + \|\eta\|) + \|\xi\| + \|\eta\|),$$

hence the statement follows from the Mean Value Theorem.

(ii) If $\sigma + \lambda(\sigma) \leq t \leq p \leq \sigma$ then

$$\|\Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(p) - \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| = \|\varphi(p - \sigma) - \varphi(t - \sigma)\| \leq \|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}}(p - t).$$

If $\sigma + \lambda(\sigma) \leq t \leq \sigma \leq p$ then by estimation (4.25) in Lemma 12 and by Lemma 11 the numbers $K$ and $L$ can be chosen so that

$$\|\Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(p) - \Delta(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta)(t)\| \leq \|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}}(\sigma - t) + K\int_\sigma^p \|y_s\| ds + K(p - \sigma)(\|\xi\| + \|\eta\|)$$

$$\leq \|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}}(\sigma - t) + K\int_\sigma^p k(\|\varphi\|_C + \|\xi\| + \|\eta\|) ds$$

$$+ K(p - \sigma)(\|\xi\| + \|\eta\|)$$

$$\leq \|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}}(\sigma - t) + L(p - \sigma)(\|\varphi\|_C + \|\xi\| + \|\eta\|)$$

$$\leq (L + 1)(\|\varphi\|_{W^{1, \infty}_{\lambda(\sigma)}} + \|\xi\| + \|\eta\|)(p - t).$$

The statement follows from this and from item (i).
In Theorem 9 [14] it is proved that if \( \Delta^{[\sigma-r,b]} (\sigma, \phi, \theta, \omega) \) is restricted to the space \( W^{1,\infty}_{\lambda(\sigma)} \times \Theta \times \Omega \) then \( \Delta^{[\sigma-r,b]} (\sigma, \cdots) \) is continuous at the point \((\phi, \theta, \omega)\), if \((\sigma, \phi, \theta, \omega) \in \text{Dom} \left( \Delta^{[\sigma-r,b]} \right)\). Modifying its proof, in Theorem 8 below we show that this continuity is uniform in \((\sigma, \phi, \theta, \omega)\).

**Lemma 14.** If Conditions 1 and 2 are satisfied and \( \tau \leq r \) then for every positive number \( \varepsilon \) the number \( \delta \) can be given so that if \( |\sigma - \rho| \leq \delta, \rho \leq \sigma, \Phi^{[\sigma-r,b]} (\rho, \psi, \vartheta, \varpi) \) is continuously differentiable on \([\lambda(\rho) + \rho, b] \) then for every point \((\varphi, \xi, \eta) \in W^{1,\infty}_{\min \{\lambda(\rho), \lambda(\sigma)\}} \times \Theta \times \Omega \) and for every point \( t \in [\sigma, b] \)

\[
\| \left( A(t, \sigma, \phi, \theta, \omega) (\varphi, \xi, \eta) - A(t, \rho, \psi, \vartheta, \varpi) \right) (\varphi, \xi, \eta) \| \\
\leq \varepsilon \left( \| \varphi \|_{W^{1,\infty}_{\min \{\lambda(\rho), \lambda(\sigma)\}}} + \| \xi \| + \| \eta \| \right).
\]

**Proof.** Denote \( u := \Phi^{[\sigma-r,b]} (\rho, \psi, \vartheta, \varpi) \). If \( t \in [\sigma, b] \) then

\[
\| \langle A(t, \sigma, \phi, \theta, \omega) - A(t, \rho, \psi, \vartheta, \varpi) \rangle \varphi, \xi, \eta \| \\
\leq \| D_2 f (t, x, \lambda(t, x, \theta), \omega) - D_2 f (t, u, \lambda(t, u, \vartheta), \varpi) \| \| \varphi \|_C \\
+ \| D_3 f (t, x, \lambda(t, x, \theta), \omega) - D_3 f (t, u, \lambda(t, u, \vartheta), \varpi) \| \| \varphi \|_C \\
+ \| D_4 f (t, x, \lambda(t, x, \theta), \omega) - D_4 f (t, u, \lambda(t, u, \vartheta), \varpi) \| \| \varphi \|_C.
\]

For the brevity denote

\[
\lambda(\rho, \sigma) := \min \{\lambda(\rho), \lambda(\sigma)\}.
\]

If \( \delta \) is small enough then by Lemma 7 and by Lemma 10 we can suppose that

\[
D_3 f (t, x, \lambda(t, x, \theta), \omega) \leq k,
\]

\[
\left\| D_2 \Lambda (t, x, \theta) - D_2 \Lambda (t, u, \vartheta) \right\| (\varphi, \xi) \| \leq \frac{\varepsilon}{k} \left( \| \varphi \|_{W^{1,\infty}_{\min \{\lambda(\sigma), \lambda(\rho)\}}} + \| \xi \| \right),
\]

moreover

\[
\left\| \langle D_3 f (t, x, \lambda(t, x, \theta), \omega) - D_3 f (t, u, \lambda(t, u, \vartheta), \varpi) \rangle \varphi, \xi \| \right\| \leq \frac{\varepsilon}{k} \left( \| \varphi \|_{W^{1,\infty}_{\min \{\lambda(\sigma), \lambda(\rho)\}}} + \| \xi \| \right)
\]

and

\[
\| D_4 f (t, x, \lambda(t, x, \theta), \omega) - D_4 f (t, u, \lambda(t, u, \vartheta), \varpi) \| \| \eta \| \leq \varepsilon \| \eta \|.
\]
If Conditions 1 and 2 are satisfied, moreover

\[ \| (A(t, \sigma, \phi, \theta, \omega) - A(t, \rho, \psi, \vartheta, \varpi)) (\varphi, \xi, \eta) \| \]

\[ \leq \varepsilon \left( \| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \| \right). \]

The supposition yields:

Therefore, if \( t \in [\sigma, \rho] \) then

\[ \| v(t) - w(t) \| \leq \max \{ A, B, C, D \}, \]

Theorem 8. If Conditions 1 and 2 are satisfied, moreover \((\rho, \psi, \vartheta, \varpi) \in \text{Dom}(\Delta^{[\rho-r,b]})\), and if \( \phi_0 \in W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))} \) then the number \( \delta \) can be given so that for every number \( \varepsilon > 0 \)

\[ \rho \leq \sigma, \sigma - \rho \leq \delta, \| \psi - \phi_0 \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}}, \| \phi - \phi_0 \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} \leq \delta \]

imply

\[ \left\| \left[ \Delta^{[\sigma-r,b]}(\sigma, \phi, \theta, \omega) - \Delta^{[\sigma-r,b]}(\rho, \psi, \vartheta, \varpi) \right] \right\|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} \leq \varepsilon, \]

that is for every point \((\varphi, \xi, \eta) \in W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))} \times \Theta \times \Omega\) the following inequality yields:

\[ \left\| \left( \Delta^{[\sigma-r,b]}(\rho, \psi, \vartheta, \varpi) - \Delta^{[\sigma-r,b]}(\sigma, \phi, \theta, \omega) \right) (\varphi, \xi, \eta) \right\| \]

\[ \leq \varepsilon \left( \| \phi \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} + \| \xi \| + \| \eta \| \right). \]

Proof. The supposition \( \| \phi - \psi \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} \leq \delta \leq 1 \) implies that

\[ \| \psi \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} \leq \| \phi \|_{W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))}} + \delta \leq m + 1. \]

Hence Condition 1 holds, replacing \( m \) by \( m + 1 \) if \( \delta \leq 1 \).

Let us fix the positive number \( \varepsilon \).

Let the element \((\varphi, \xi, \eta) \in W^{1,\infty}_{\min(\lambda(\rho), \lambda(\sigma))} \times \Theta \times \Omega\) be fixed and let \( v = \Delta^{[\sigma-r,b]}(\sigma, \phi, \theta, \omega)(\varphi, \xi, \eta), w = \Delta^{[\sigma-r,b]}(\rho, \psi, \vartheta, \varpi)(\varphi, \xi, \eta) \). For each \( t \in [\sigma - r, b] \)

\[ v(t) - w(t) \]

\[ = \begin{cases} 
\varphi(t - \sigma) - \varphi(t - \rho), & \text{if } \sigma - r \leq t \leq \rho \\
\varphi(t - \sigma) - \varphi(0) - \int_{\rho}^{t} A(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta) ds, & \text{if } t \in [\rho, \sigma] \\
f_{\sigma}^{t} A(s, \sigma, \phi, \theta, \omega)(v_s, \xi, \eta) ds - \int_{\rho}^{t} A(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta) ds, & \text{if } t \in [\sigma, b] 
\end{cases} \]

Therefore, if \( t \in [\sigma - r, b] \) then

\[ \| v(t) - w(t) \| \leq \max \{ A, B, C, D \}, \]
where

\[ A = \| \varphi(t - \sigma) - \varphi(t - \rho) \| \leq L \varphi (\sigma - \rho) \leq \| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} |\sigma - \rho|, \]

\[ B = \left\| \varphi(t - \sigma) - \varphi(0) - \int_{\rho}^{t} \mathcal{A}(s, \rho, \psi, \vartheta, \varpi) \, ds \right\|, \quad t \in [\rho, \sigma] \]

\[ C = \int_{\rho}^{\sigma} \| \mathcal{A}(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta) \| \, ds, \]

\[ D = \int_{\sigma}^{t} \| (\mathcal{A}(s, \sigma, \phi, \theta, \omega)(v_s, \xi, \eta) - \mathcal{A}(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta)) \| \, ds, \quad t \in [\sigma, b]. \]

By Lemma 11 and Lemma 12 there is a number \( K \) such that

\[ B \leq L \varphi (\sigma - \rho) + \int_{\rho}^{t} \| \mathcal{A}(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta) \| \, ds \]

\[ \leq L \varphi (\sigma - \rho) + (\sigma - \rho) K (\| w_s \|_{\infty} + \| \xi \| + \| \eta \|) \]

\[ \leq L \varphi (\sigma - \rho) + (\sigma - \rho) K (K (\| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \|) + \| \xi \| + \| \eta \|). \]

Hence the number \( k \) can be given so that

\[ B \leq k (\sigma - \rho) (\| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \|). \]

By Lemma 11 and Lemma 12 there are numbers \( K \) and \( k \) such that

\[ C = \int_{\rho}^{\sigma} \| \mathcal{A}(s, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta) \| \, ds \leq (\sigma - \rho) K (\| w_s \|_{\infty} + \| \xi \| + \| \eta \|) \]

\[ \leq (\sigma - \rho) K (K (\| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \|) + \| \xi \| + \| \eta \|) \]

\[ \leq k (\sigma - \rho) (\| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \|), \]

By Lemma 14 and Lemma 11

\[ \| (\mathcal{A}(t, \sigma, \phi, \theta, \omega)(\varphi, \xi, \eta) - \mathcal{A}(t, \rho, \psi, \vartheta, \varpi)(\varphi, \xi, \eta)) \| \]

\[ \leq \varepsilon \left( \| \varphi \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \| \right). \]

\[ \| (\mathcal{A}(t, \sigma, \phi, \theta, \omega)(v_s, \xi, \eta) - \mathcal{A}(t, \rho, \psi, \vartheta, \varpi)(w_s, \xi, \eta)) \| \]

\[ \leq \| (\mathcal{A}(t, \sigma, \phi, \theta, \omega) - \mathcal{A}(t, \rho, \psi, \vartheta, \varpi))(v_s, \xi, \eta))\| + \| \mathcal{A}(t, \rho, \psi, \vartheta, \varpi)(v_s - w_s, 0, 0)\| \]

\[ \leq \varepsilon \left( \| v_s \|_{W^{1,\infty}_{\min(\lambda(\sigma), \lambda(\rho))}} + \| \xi \| + \| \eta \| \right) + K \| v_s - w_s \|_{\infty}.
By Lemma 13 there are numbers $K$ and $k$ such that

$$D \leq \int_{\sigma}^{t} \varepsilon \left( K \left( \| \varphi \|_{W_{\min(\lambda(\sigma), \lambda(\rho))}^{1,\infty}} + \| \xi \| + \| \eta \| \right) + \int_{\sigma}^{t} K \| v_s - w_s \|_C \right) ds$$

$$\leq \varepsilon (b - \sigma) K \left( \| \varphi \|_{W_{\min(\lambda(\sigma), \lambda(\rho))}^{1,\infty}} + \| \xi \| + \| \eta \| \right) + \int_{\sigma}^{t} K \| v_s - w_s \|_C \right) ds.$$

Consequently, if

$$D \leq \max \{ A, B, C \}$$

then the statement is proved. If

$$\max \{ A, B, C, D \} = D$$

then

$$\| v(t) - w(t) \| \leq \varepsilon (b - \sigma) k \left( \| \varphi \|_{W_{\min(\lambda(\sigma), \lambda(\rho))}^{1,\infty}} + \| \xi \| + \| \eta \| \right) + \int_{\sigma}^{t} \| v_s - w_s \|_C ,$$

and the Bellmann-Gronwall inequality implies the statement.


(4.26) \[ y'(t) = f(t, y_t) + g(t, y_t) . \]

The following variation-of-constant formula is not a generalization of Shanholt’s one, because the perturbed term here does not depend on the segment function. However, Theorem 9 below is not weaker, because in formula (4.26) the perturbation is state independent, moreover our formula is proved for the abstract case.

LEMMA 15. Let $Y$ be a Banach spaces. Let $M \subset \mathbb{R} \times Y$ be open set,

$$G : M \rightarrow Y$$

be a function, continuous and continuously differentiable in its second variable. Let $\Gamma$ stand for the characteristic function of the equation

(4.27) \[ u'(t) = G(t, u(t)) . \]

(i) If

$$x : [\sigma - r, b] \rightarrow Y ,$$

$$\phi \in C([-r, 0], Y)$$
are functions such that
\[ t \in [\sigma - r, \sigma] \implies x(t) = \phi(t - \sigma) \]
and
\[ t \in [\sigma, b] \implies (\sigma, t, x(t)) \in \text{Dom}(\Gamma) \]
then the function
\[ c : [\sigma - r, b] \to Y, \ t \mapsto c(t) = \begin{cases} \Gamma(\sigma, t, x(t)), & \text{if } t \in [\sigma, b] \\
\phi(t - \sigma), & \text{if } t \in [\sigma - r, \sigma] \end{cases} \]
is continuously differentiable at the point \( t \in [\sigma - r, b] \) if and only if \( x \) also satisfies this condition.

(ii) Let \( V \) stand for the function
\[
V : \begin{cases} (t, \psi) \in [\sigma, b] \times \mathcal{C}([-r, 0], Y) | s \in [-r, 0], \ t + s \geq \sigma \\ \implies (t + s, \sigma, \psi(s)) \in \text{Dom}(\Gamma) \} \to \mathcal{C}([-r, 0], Y), \\
V(t, \psi)(s) = \begin{cases} \Gamma(t + s, \sigma, \psi(s)), & \text{if } t + s \geq \sigma \\
\psi(s), & \text{if } t + s < \sigma \end{cases} \end{cases}
\]
If \( (\sigma, \psi) \in \text{Dom}(V) \) then \( V(\sigma, \psi) = \psi \).
The function \( V \) is continuous and its domain is open.
Moreover \( V \) continuously differentiable in its second variable, that is
\[
\forall (t, \phi) \in \text{Dom}(V) \ \forall \epsilon > 0 \ \exists \delta > 0 : \\
(q, \psi) \in \text{Dom}(V), |t - q|, \|\phi - \psi\|_\mathcal{C} \leq \delta \\
\implies \|D_2 V(t, \phi) - D_2 V(q, \psi)\| \leq \epsilon.
\]

Proof. (i) The statement follows from the fact that \( \Gamma \) is continuously differentiable, and
\[
x(t) = \Gamma(t, \sigma, c(t)), \text{ if } t \in [\sigma, b] \\
x_\sigma = c_\sigma = \phi.
\]

(ii) \( \text{Dom}(V) \neq \emptyset \), because \( (\sigma, \phi) \in \text{Dom}(V) \). The equality \( V(\sigma, \psi) = \psi \) is evident.
Let the element \( (t, \phi) \in \text{Dom}(V) \) be fixed. The set
\[
H := \{(t + s, \phi(s)) | s \in [-r, 0], \ t + s \geq \sigma \} \subset \mathbb{R} \times Y
\]
is contained in the open set \( \text{Dom}(\Gamma(\cdot, \sigma, \cdot)) \) and it is compact. It follows that \( \text{Dom}(V) \) is open. Moreover, \( \Gamma(\cdot, \sigma, \cdot) \) is uniformly continuous at the points
of $H$. It implies that the function $V$ is continuous, because if $q + s < \sigma \leq t + s$ then

$$0 \leq t + s - \sigma \leq t + s - (q + s) = t - q,$$

moreover if $(t, \phi), (q, \psi) \in \text{Dom}(V)$ then

$$\|V(t, \phi) - V(q, \psi)\| = \max_{s \in [-r, 0]} \|V(t, \phi)(s) - V(q, \psi)(s)\|$$

$$= \max_{s \in [-r, 0]} \begin{cases} \|\Gamma(t + s, \sigma, \phi(s)) - \Gamma(q + s, \sigma, \psi(s))\|, & \text{if } q + s, t + s \geq \sigma \\ \|\phi(s) - \psi(s)\|, & \text{if } q + s, t + s < \sigma. \end{cases}$$

The function $V$ is differentiable in its second variable, namely if $(t, \phi) \in \text{Dom}(V)$ then $D_2V(t, \phi) = B$, where

$$B \in \mathcal{L}(\mathcal{C}([-r, 0], Y), \mathcal{C}([t, 0], Y)), \xi \mapsto B(\xi),$$

$$B(\xi) : [-r, 0] \to Y, s \mapsto \begin{cases} D_3\Gamma(t + s, \sigma, \phi(s))(\xi(s)), & \text{if } t + s \geq \sigma \\ \xi(s), & \text{if } t + s < \sigma \end{cases}.$$ 

Indeed, $B$ is evidently linear. It is also continuous, because $D_3\Gamma(\cdot, \sigma, \cdot)$ is continuous at the points of the compact set $H$ and

$$\|B(\xi)\| = \max_{s \in [-r, 0]} \begin{cases} \|D_3\Gamma(t + s, \sigma, \phi(s))(\xi(s))\|, & \text{if } t + s \geq \sigma \\ \|\xi(s)\|, & \text{if } t + s < \sigma \end{cases},$$

where

$$K := \max_{(t+s,\phi(s))\in H} \|D_3\Gamma(t + s, \sigma, \phi(s))\|.$$ 

Moreover, for every positive number $\varepsilon$ there is a number $\delta > 0$ such that if $\max \{|t - q|, \|\phi - \psi\|_C\} \leq \delta$ then

$$\forall s \in [-r, 0] : \|D_3\Gamma(t + s, \sigma, \phi(s)) - D_3\Gamma(q + s, \sigma, \psi(s))\| \leq \varepsilon.$$

By the Mean Value Theorem it implies that the function

$$\Gamma(t + s, \sigma, \cdot) - D_3\Gamma(t + s, \sigma, \phi(s))$$

is lipschitzian with the constant $\varepsilon$ on the $\delta$-neighbourhood of $\phi(s)$, therefore if $\sigma \leq t + s$ then

$$\|(V(t, \psi) - V(t, \phi) - B(\psi - \phi))(s)\| \leq \varepsilon \|\phi - \psi\|_C.$$
If \( t + s < \sigma \) then
\[
\| (V(t,\psi) - V(t,\phi) - B(\psi - \phi))(s) \| = 0 \leq \varepsilon \| \phi - \psi \|_C.
\]

Hence
\[
D_2V(t,\phi) = B.
\]

Let \( \varepsilon > 0 \) be fixed and \( \delta > 0 \) be chosen so that inequality (4.30) yields. If \( \sigma \leq t + s \) and \( \sigma \leq q + s \) and \( \xi \in C([-r,0],Y) \) then
\[
\| (D_2V(t,\phi) - D_2V(q,\psi))(\xi)(s) \| = \| (D_3\Gamma(t+s,\sigma,\phi(s)) - D_3\Gamma(q+s,\sigma,\psi(s)))(\xi)(s) \| \leq \varepsilon \| \xi \|_C.
\]

If \( q + s < \sigma \) and \( t + s < \sigma \) and \( \xi \in C([-r,0],Y) \) then
\[
\| (D_2V(t,\phi) - D_2V(q,\psi))(\xi)(s) \| = 0.
\]

\[\text{Theorem 9. (A nonlinear variation-of-constant formula for abstract functional differential equations with state-dependent delay).}\]

Let \( Y \) be a Banach space, \( r \geq 0 \). Let \( U \subset \mathbb{R} \times C([-r,0],Y) \times Y, W \subset \mathbb{R} \times C([-r,0],Y) \) and \( M \subset \mathbb{R} \times Y \) be open sets,
\[
F : U \to Y, \quad \tau : W \to \mathbb{R}^+_0, \quad \tau \leq r, \quad G : M \to Y
\]

be functions, \( G \) be continuous and continuously differentiable in its second variable. Let \( \Gamma \) stand for the characteristic function of the equation
\[
(4.31) \quad u'(t) = G(t,u(t)).
\]

Let \( V \) stand for the function
\[
V : \left\{ (t,\psi) \in [\sigma,b] \times C([-r,0],Y) \mid s \in [-r,0], t + s \geq \sigma, (t+s,\sigma,\psi(s)) \in \text{Dom}(\Gamma) \right\} \to C([-r,0],Y),
\]
\[
V(t,\psi)(s) = \begin{cases} 
\Gamma(t+s,\sigma,\psi(s)), & \text{if } t + s \geq \sigma \\
\psi(s), & \text{if } t + s < \sigma
\end{cases}
\]
Let $\tilde{\tau}$ stand for the function
$$\tilde{\tau} := \tau \circ (pr_{\mathbb{R}}, V), \quad (t, \psi) \mapsto \tau (t, V(\psi)),$$

$\tilde{F}$ stand for the function
$$(4.32) \quad \tilde{F} : (t, \psi) \mapsto D_{3} \Gamma (\sigma, t, \psi (0)) (F (t, V(\psi)), V(\psi)(-\tilde{\tau} (t, \psi))) .$$

(i) The function
$$[\sigma, b] \ni t \mapsto x(t) = \Gamma (t, \sigma, c(t)),$$
$$[\sigma - r, \sigma] \ni t \mapsto \phi (t - \sigma)$$
is the solution of the state-dependent equation
$$(4.33) \quad x'(t) = F (t, x(t), x(t - \tau(t, x(t)))) + G (t, x(t)), \quad t \in [\sigma, b],$$
satisfying the initial condition $x_{\sigma} = \phi$, if and only if the function
$$c : [\sigma - r, b] \to Y$$
is a solution of the equation
$$(4.34) \quad c' (t) = \tilde{F} (t, c(t)), \quad t \in [\sigma, b],$$
satisfying the initial condition $c_{\sigma} = \phi$.

(ii) If $F$ and $\tau$ are continuous then the function $\tilde{F}$ is continuous.
If $F$ is lipschitzian in its second and third variables and $\tau$ is lipschitzian in its second variable then the function $\tilde{F}$ is lipschitzian in its second variable.

(iii) If the function $F$ is continuously differentiable in its second and third variables and $\tau$ is continuously differentiable in its second variable then the functions $\tilde{F}$ and $\tilde{\tau}$ also satisfy the corresponding conditions, respectively.

Proof. (i) Let $x$ be the solution of the initial value problem on the interval $[\sigma - r, b]$.
$$x'(t) = F (t, x(t), x(t - \tau(t, x(t)))) + G (t, x(t)), \quad x_{\sigma} = \phi$$

Suppose that there exists a function $c : [\sigma - r, b] \to Y$ such that
$$x(t) := \begin{cases} \Gamma (t, \sigma, c(t)), & t \in [\sigma, b] \\ \phi (t - \sigma) = c(t), & t \in [\sigma - r, \sigma] \end{cases} .$$

The function $c$ exists if and only if
$$(4.35) \quad (\sigma, t, x(t)) \in \text{Dom} (\Gamma)$$
\[ c : [\sigma - r, b] \rightarrow Y, \]
\[ c(t) : = \begin{cases} 
\Gamma (\sigma, t, x(t)) , & t \in [\sigma, b] \\
\phi (t - \sigma) , & t \in [\sigma - r, \sigma] 
\end{cases} . \]

\[ s \in [-r, 0] , \ t + s \in [\sigma, b] \implies 
\left[ (t + s, \sigma, c_t(s)) \in \text{Dom}(\Gamma) \iff (\sigma, t + s, \Gamma (t + s, \sigma, c_t(s))) \in \text{Dom}(\Gamma) \right], \]

hence by \((4.35)\) \((t, c_t) \in \text{Dom}(V), \) whenever \(t \in [\sigma, b].\)

\[ x(t) := \begin{cases} 
\Gamma (t, \sigma, c(t)) , & t \in [\sigma, b] \\
\phi (t - \sigma) , & t \in [\sigma - r, \sigma] 
\end{cases} \]

therefore if \(t \geq \sigma\) then

\[ x_t = V (t, c_t) \text{ and } (t, V (t, c_t)) \in \text{Dom}(\tau) , \]

\[ D_1 \Gamma (t, \sigma, c(t)) = G (t, \Gamma (t, \sigma, c(t))) . \]

As \(\Gamma\) is the characteristic function of \((4.31)\), it is continuously differentiable and \(D_3 \Gamma (t, \sigma, y)\) is invertible, namely

\[ [D_3 \Gamma (t, \sigma, y)]^{-1} = D_3 \Gamma (\sigma, t, y) , \ [D_3 \Gamma (\sigma, \sigma, y)]^{-1} = \text{id}_Y . \]

Therefore \(x\) is the solution of the initial value problem

\[ x'(t) = F (t, x_t, x (t - \tau (t, x_t))) + G (t, x (t)) , \ x_\sigma = \phi \]

if and only if \(c_\sigma = \phi\) and for \(t \geq \sigma\)

\[ x'(t) = F (t, V (t, c_t), V (t, c_t) (-\tilde{\tau} (t, c_t))) + G (t, \Gamma (t, \sigma, c(t))) \]

\[ = D_1 \Gamma (t, \sigma, c(t)) + D_3 \Gamma (t, \sigma, c(t)) (c'(t)) . \]

Hence \((4.37)\) yields if and only if

\[ c'(t) = D_3 \Gamma (\sigma, t, c_t (0)) F (t, V (t, c_t), V (t, c_t) (-\tilde{\tau} (t, c_t))) \]

\[ = \tilde{F} (t, c_t) , \ t \in [\sigma, b], \ c_\sigma = \phi . \]

Suppose now that \(c\) is the solution of \((4.38)\). Then

\[ V (t, c_t) : [-r, 0] \rightarrow Y, \ s \mapsto \begin{cases} 
\Gamma (t + s, \sigma, c(t + s)) , & t + s \in [\sigma, b] \\
\phi (t + s - \sigma) , & t + s \in [\sigma - r, \sigma] 
\end{cases} \]
is defined, if \( t \in [\sigma, b] \), therefore \( x(t) = \Gamma(t, \sigma, c(t)) \) also exists. This proves the statement.

(ii) By Lemma 15 this statement is evident.

(iii) The functions \( V \) and \( \bar{\tau} \) are continuously differentiable in their second variable by Lemma 15 and by the chain-rule.

The following special case will be important for us.

**Theorem 10.** Let \( Y \) be a Banach spaces, \( r \geq 0 \). Let \( \sigma \in \mathbb{R} \) be fixed, \( J \subset \mathbb{R} \) be open interval, \([\sigma - r, b] \subset J \) and \( g : J \to Y \) be continuous function, \( U \subset \mathbb{R} \times C([-r, 0], Y) \times Y \) be open set, \( F : U \to Y \), \( \tau : W \to \mathbb{R}_+^+ \) be functions, \( \tau \leq r \). Let \( \bar{F} \) stand for the function

\[
\bar{F} : (t, \psi, y) \mapsto F \left( t, V(t, \psi) , y + \left( \int_\sigma^t I \cdot g \right) (t - \tau(t, V(t, \psi))) \right),
\]

where

\[
I : \mathbb{R} \to \mathbb{R}, \quad q \mapsto \begin{cases} 0 & \text{if } q < \sigma \\ 1 & \text{if } q \geq \sigma \end{cases},
\]

and

\[
V : [\sigma, b] \times C([-r, 0], Y) \to C([-r, 0], Y), \quad V(t, \psi) := \psi + \left( \int_\sigma^t I \cdot g \right)_t.
\]

(i) The function

\[
t \mapsto X(t, \sigma, \phi) = \begin{cases} c(t) + \int_\sigma^t g, & t \geq \sigma, \\ \phi(t - \sigma), & t \in [\sigma - r, \sigma] \end{cases}
\]

is the solution of the state-dependent equation

\[
x'(t) = F(t, x(t), x(t - \tau(t, x(t)))) + g(t), \quad t \in [\sigma, b],
\]

satisfying the initial condition \( x_\sigma = \phi \), if and only if the function

\[
c : [\sigma - r, b] \to Y
\]

is a solution of the equation

\[
c'(t) = \bar{F}(t, c_t, c(t - \bar{\tau}(t, c_t))) , \quad t \in [\sigma, b],
\]

satisfying the initial condition \( c_\sigma = \phi \), where \( \bar{\tau} \) stands for the function

\[
\bar{\tau}(t, \psi) \mapsto \tau \left( t, \psi + \left( \int_\sigma^t I \cdot g \right)_t \right).
\]
If $F$ and $\tau$ are continuous then the functions $\tilde{F}$ and $\tilde{\tau}$ are continuous.

(ii) If the function $F$ is lipschitzian in its second and third variables and $\tau$ is lipschitzian in its second variables then the functions $\tilde{F}$ and $\tilde{\tau}$ also satisfy the corresponding conditions, respectively.

(iii) If the function $F$ is continuously differentiable in its second and third variables and $\tau$ is continuously differentiable in its second variable then the functions $\tilde{F}$ and $\tilde{\tau}$ also satisfy the corresponding conditions, respectively.

Proof. We use the notations of Theorem 9.

Let $G$ stand for the function

\begin{equation}
G : J \times Y \to Y, \ (t,y) \mapsto g(t).
\end{equation}

Then the function $\Gamma$ is the characteristic function of the equation

$$u'(t) = G(t, u(t)),$$

and if $t \in [\sigma, b]$ and $\psi \in C([-r, 0], Y)$ then

$$V(t, \psi) : [-r, 0] \to Y, \ s \to \begin{cases} \omega + \int_{\sigma}^{t+s} g = \Gamma(t + s, \sigma, \psi(s)), & \text{if } \sigma \leq t + s, \\ \psi, & \text{if } t + s < \sigma. \end{cases}$$

Therefore

$$V(t, \psi)(-\tilde{\tau}(t, \psi)) = \left(\omega + \left(\int_{\sigma}^{t} I \cdot g\right)ight)(-\tilde{\tau}(t, \psi))$$

$$= \psi(-\tilde{\tau}(t, \psi)) + \left(\int_{\sigma}^{t} I \cdot g\right)(-\tau(t, V(t, \psi))).$$

Hence the statement follows from Theorem 9, because $D_3\Gamma(t, \sigma, y) = id_Y$. \qed

4.3. Differentiability with respect to the initial function without compatibility condition. Now we can prove the main statement of this section. In the following Theorem 11 the consequence part of Theorem 7 is obtained without assuming the compatibility condition.

Theorem 11. Let $\Phi$ stand for the resolvent function of the equation

$$x'(t) = f(t, x, x(t - \tau(t, x_t, \theta)) , \omega).$$

Suppose that Condition 1 is satisfied. The function $\Phi$ is differentiable in its second, third and fourth variables, uniformly with respect to the parameters $(\sigma, \phi, \theta, \omega)$ in the following sense:

for every positive number $\varepsilon$ the number $\delta > 0$ can be given so that if $\phi$ is
Smooth Parameter Dependence \[ \Delta^{[\sigma - r, b]} (\sigma, \phi, \theta, \omega) : C \left([-r, 0]\right) \times \Theta \times \Omega \to C \left([\sigma - r, b], Y\right) \]
such that

(I) (i) \[
\|u - x - \Delta^{[\sigma - r, b]} (\sigma, \phi, \theta, \omega) \left[[\psi, \vartheta, \varpi] - (\phi, \theta, \omega)\right]\| 
\leq \varepsilon \left(\|\psi - \phi\|_{W^{1, \infty}} + \|\vartheta - \theta\| + \|\varpi - \omega\|\right),
\]

(ii) if moreover \(\|\phi - \phi_0\|_{W^{1, \infty}_{\lambda(\sigma)}} \leq \delta\) and \(\|\psi - \psi_0\|_{W^{1, \infty}_{\lambda(\rho)}} \leq \delta\) then

\[
\|u - x - \Delta^{[\sigma - r, b]} (\sigma, \phi, \theta, \omega) \left[[\psi, \vartheta, \varpi] - (\phi, \theta, \omega)\right]\| 
\leq \varepsilon \left(\|\psi - \phi\|_{C} + \|\vartheta - \theta\| + \|\varpi - \omega\|\right)
\]
holds.

(II) The derivative function of \(\Phi\) is continuous in the following sense:
if moreover \(\rho \leq \sigma\), then for every positive number \(\varepsilon\) the number \(\delta\) can be chosen so that if \(\phi_0\) and \(\psi\) are continuously differentiable on the interval \([\min \{\lambda(\sigma), \lambda(\rho)\}, 0]\), and

(4.41) \[
\|\phi_0 - \psi\|_{W^{1, \infty}_{\min\{\lambda(\sigma), \lambda(\rho)\}}} \leq \delta
\]
holds then

(4.42) \[
\left\|\left[\Delta^{[\sigma - r, b]} (\rho, \psi, \vartheta, \varpi) - \Delta^{[\sigma - r, b]} (\sigma, \phi_0, \theta, \omega)\right]\|_{W^{1, \infty}_{\min\{\lambda(\sigma), \lambda(\rho)\}} \times \Theta \times \Omega}\right\| \leq \varepsilon,
\]
that is if \((\varphi, \xi, \eta) \in W^{1, \infty}_{\min\{\lambda(\sigma), \lambda(\rho)\}} \times \Theta \times \Omega\) then

(4.43) \[
\left\|\left(\Delta^{[\sigma - r, b]} (\rho, \psi, \vartheta, \varpi) - \Delta^{[\sigma - r, b]} (\sigma, \phi_0, \theta, \omega)\right) (\varphi, \xi, \eta)\right\| 
\leq \varepsilon \left(\|\varphi\|_{W^{1, \infty}_{\min\{\lambda(\sigma), \lambda(\rho)\}}} + \|\xi\| + \|\eta\|\right).
\]
Proof. We use the notations of Theorems 9 and 10. One can suppose that \(\tau \leq r\).

Let \(g: \mathbb{R} \to \mathbb{R}\) be the constant function
\[
g(t) = \eta := f (\sigma, \phi, \phi \left(-\tau (\sigma, \phi, \theta)\right), \omega) - \phi' (0).
\]
Let $F$ stand for the function

$$F : \text{Dom}(f) \to Y, \quad F(t, \psi, y, \omega) := f(t, \psi, y, \omega) - \eta.$$  

Then

$$\hat{F}(\sigma, \phi, \phi(-\tau(\sigma, \phi, \theta))\omega) = F(\sigma, \phi, \phi(-\tau(\sigma, \phi, \theta))\omega) = \phi'(0).$$

Let $\Phi$ stand for the resolvent function of the equation

$$(4.44) \quad x'(t) = f(t, x_t, x(t - \tau(t, x_t, \theta), \omega),$$

and $\mp$ stand for the resolvent function of the equation

$$c'(t) = \hat{F}(t, c_t, c(t - \tilde{\tau}(t, c_t, \theta)), \omega), \quad t \in [\sigma, b], \quad c_{\sigma} = \phi.$$

By Theorem 10

$$(4.45) \quad \Phi^{[\sigma, b]}(\sigma, \phi, \theta, \omega) = \mp^{[\sigma, b]}(\sigma, \phi, \theta, \omega) + (id_{[\sigma, b]} - \sigma)\eta,$$

$$\Phi^{[\sigma-r, c]}(\sigma, \phi, \theta, \omega) = \mp^{[\sigma-r, c]}(\sigma, \phi, \theta, \omega) = \phi \circ (id_{[\sigma-r]} - \sigma).$$

By Theorem 7 the function $\mp^{[\sigma-r, b]}(\sigma, \cdot, \cdot, \cdot)$ is differentiable at $(\phi, \theta, \omega)$, namely $\Delta^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega)$ is the derivative of $\mp^{[\sigma-r, b]}(\sigma, \cdot, \cdot, \cdot)$ at $(\phi, \theta, \omega)$ in the following sense:

(i) if $\max\left\{\|\phi - \psi\|_{C}, \|\theta - \vartheta\|, \|\omega - \varpi\|\right\} \leq \delta$ then

$$\|u - x - \Delta^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega)(\psi - \phi, \theta - \vartheta, \omega - \varpi)\| \leq \varepsilon\left(\|\psi - \phi\|_{W^{1,\infty}(\lambda(\sigma), \lambda(\rho))} + \|\theta - \vartheta\| + \|\omega - \varpi\|\right)$$

(ii) if moreover $\max\left\{\|\phi - \psi\|_{W^{1,\infty}(\lambda(\sigma), \lambda(\rho))}, \|\theta - \vartheta\|, \|\omega - \varpi\|\right\} \leq \delta$ then

$$\|u - x - \Delta^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega)(\psi - \phi, \theta - \vartheta, \omega - \varpi)\| \leq \varepsilon(\|\psi - \phi\| + \|\theta - \vartheta\| + \|\omega - \varpi\|_{C})$$

holds.

By (4.45)

$$(4.46) \quad D_{2,3,4}\Phi^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega) = D_{2,3,4}\mp^{[\sigma-r, b]}(\sigma, \phi, \theta, \omega).$$

This proves items (i) and (ii).

(II) Let $\Pi$ be the subspace of the vector-space $C([-r, 0], Y)$, containing the continuously differentiable functions on the interval $[\min\{\lambda(\sigma), \lambda(\rho)\}, 0],\ldots$
and equipped with the norm $\|\cdot\|_{W^{1,\infty}_{\min(\lambda(\sigma),\lambda(\rho))}}$. Then $\Pi$ is a Banach-space. Let stand $\tilde{f}$ for the function

$$
\tilde{f} : \begin{cases}
(t, \psi, \vartheta, \varpi, \beta) \mid \beta \in \Pi, \ (t, \beta, \vartheta) \in \text{Dom}(\tau),
(t, \psi, \vartheta, \varpi, \beta) \in \text{Dom}(f) \end{cases} \rightarrow Y,
$$

$$(t, \psi, \vartheta, \varpi, \beta) \rightarrow f(t, \psi, \vartheta, \varpi) - f(\tau(\sigma, \beta, \vartheta)), \varpi) + \beta'(0).$$

By Lemma 9 if $\sigma$ is fixed then the function

$$
\Pi \times \Theta \rightarrow Y, \ (\beta, \vartheta) \mapsto \beta(-\tau(\sigma, \beta, \vartheta))
$$

is continuously differentiable in its second and third variables. Evidently, the function

$$
\Pi \rightarrow Y, \ \beta \mapsto \beta'(0)
$$

is continuous and linear. Therefore $\tilde{f}$ is continuous and continuously differentiable in its second, third, fourth and fifth variables. By item (I) the resolvent function $\tilde{\Phi}$ of the equation

$$
u'(t) = \tilde{f} (t, v, \tau(t, \psi, \vartheta)), \zeta, \beta)
$$

is differentiable in its second, third, fourth and fifth variables and by Theorem 8, replacing $\alpha$ by $\phi_0$ and $\beta$ by $\psi$, for every point $(\varphi, \xi, \eta) \in W^{1,\infty}_{\min(\lambda(\sigma),\lambda(\rho))} \times \Theta \times \Omega$ the inequality

$$
\left\| \left( \tilde{\Delta}^{[\sigma-r,b]}(\rho, \gamma, \psi, \vartheta, \varpi) - \tilde{\Delta}^{[\sigma-r,b]}(\sigma, \phi_0, \theta, \omega, \phi_0) \right)(\varphi, \xi, \eta) \right\|
\leq \varepsilon \left( \|\varphi\|_{W^{1,\infty}_{\min(\lambda(\sigma),\lambda(\rho))}} + \|\xi\| + \|\eta\| \right),
$$

yields, whenever $\|\phi_0 - \psi\|_{W^{1,\infty}_{\min(\lambda(\sigma),\lambda(\rho))}}$ is small enough, where

$\tilde{\Delta}^{[\sigma-r,b]}(\rho, \gamma, \psi, \vartheta, \varpi)$ and $\tilde{\Delta}^{[\sigma-r,b]}(\sigma, \phi_0, \theta, \omega, \phi_0)$ denote the derivative

$D_{2,3,4,5}^{[\sigma-r,b]}(\rho, \gamma, \psi, \vartheta, \varpi)$ and $D_{2,3,4,5}^{[\sigma-r,b]}(\sigma, \phi_0, \theta, \omega, \phi_0)$, respectively.

By equality (4.46) the statement is valid. $\blacksquare$

REFERENCES


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+972 54424173; elitsyn@gmail.com.