ARIEL UNIVERSITY PRESS

Functional Differential Equations
Volume 27, No. 3-4, 2020, pp. 103-123
DOI: https://doi.org/10.26351/FDE/27/3-4/6
ISSN: 0793-1786 (print), 2617-8605 (online)

ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO LINEAR AUTONOMOUS NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS *

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Abstract. We investigate stability, with respect to initial data, of a linear autonomous functional differential equation of neutral type, on the basis of the well-known solution representation formula. We consider stability as a property depending on a functional space which initial functions belong to, and show that, along with the concept of asymptotic stability, a certain stronger property should be introduced, which we call strong asymptotic stability. We obtain that for initial data from the Lebesgue space L_1 the strong asymptotic stability of the equation under study is equivalent to an exponential estimate of the Cauchy function, which is the kernel of the integral operator in the solution representation formula. Moreover, we show that these properties are equivalent to the exponential stability with respect to initial data from L_p for all p from 1 to infinity inclusive. However, strong asymptotic stability with respect to initial data from L_p for p greater than one may not coincide with exponential stability.

Key Words. Neutral functional differential equation, Cauchy function, fundamental solution, Lyapunov stability, asymptotic stability, exponential stability.

AMS(MOS) subject classification. 34K25, 34K40.

1. Introduction.

The present work is devoted to some fundamental questions of stability for functional differential equations of neutral type. Stability of linear autonomous neutral equations essentially differs from stability

* The research is performed within the public contract with the Ministry of Science and High Education of the Russian Federation (contract FSNM-2020-0028) and supported by the Russian Foundation for Basic Research, grant no. 18-01-00928.

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of linear autonomous delay equations. For the latter only two qualitatively different types of stability with respect to initial data are distinguished [6, 9]: Lyapunov stability, which is uniformly continuous dependence of a solution on an initial function, and asymptotic stability, for which the change of a solution caused by a small change of initial data tends to zero as the independent variable tends to infinity. The asymptotic stability of autonomous delay equations can only be exponential, and this fact greatly simplifies the study of the asymptotic behavior of their solutions (see [6, 9, 11, 13]). In contrast, neutral linear autonomous equations can be asymptotically stable without being exponentially stable [5, 12].

In this paper we consider stability of solutions to neutral differential equations with respect to initial data as a property depending on the space from which an initial function is taken. The main subject of our research is stability with respect to initial functions from the Lebesgue spaces of integrable functions, L_p, $1 \le p \le \infty$. We obtain stability criteria in terms of the properties of the Cauchy function of the equation under study. We highlight situations when asymptotic stability is equivalent to the existence of an exponential estimate for the Cauchy function. We combine the obtained results to consruct a general description of stability conditions for linear autonomous equations of neutral type.

The paper is organized as follows. In section 2 we introduce the main object of our study and the solution representation formula. Section 3 is devoted to definitions of stability, in which a space of initial functions is taken into account. In section 4 we represent some facts that we use in our research, from paper [4] and the theory of difference equations. Section 5 contains the main results of the paper: conditions of stability, for autonomous neutral equations, with respect to initial function from the Lebesgue spaces L_p, in terms of the location of zeros of a complex variable functions with respect to the unit disk on the complex plane.

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of all positive integers and nonnegative integers respectively, and put $\mathbb{R}_+ \stackrel{\text{def}}{=} [z, r)$, $B(z,r) \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} \colon |\zeta - z| < r\}$, $B[z,r] \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} \colon |\zeta - z| \le r\}$.

2. Solution representation formula for a neutral equation. Consider a linear autonomous differential equation of neutral type with respect to the function $x \colon \mathbb{R}_+ \to \mathbb{R}$,

$$\dot{x}(t) - \sum_{k=1}^{K} a_k \dot{x}(t-kh) = \sum_{j=0}^{J} b_j x(t-jh), \quad t \in \mathbb{R}_+. \tag{1}$$

Here $h > 0$, $K \in \mathbb{N}$, $J \in \mathbb{N}_0$, $a_k, b_j \in \mathbb{R}$.

The values of the functions x and $\dot{x}$ for negative arguments are defined by initial functions φ and ψ respectively. Note that in works, where a solution of a neutral differential equation is undestood as a continuous extension of an initial function [6, 9, 11], the conditions $x(0) = \varphi(0)$ and $\psi = \dot{\varphi}$ are supposed to be satisfied. We do not suppose these conditions to be mandatory.

For further, it is convenient to represent equation (1) in operator form. Denote by S_h a *shift operator* that is acting in spaces of functions defined on $\mathbb{R}_+$: for a given $h > 0$ put

$$(S_h y)(t) \overset{\text{def}}{=} \begin{cases} y(t-h), & t-h \geq 0, \\ 0, & t-h < 0. \end{cases}$$

Define operators S and T by equalities

$$Sy \overset{\text{def}}{=} \sum_{k=1}^{K} a_k (S_h^k y), \quad Ty \overset{\text{def}}{=} \sum_{j=0}^{J} b_j (S_h^j y).$$

Consider the inhomogeneous operator equation

$$(I - S)\,\dot{x} = Tx + f, \tag{2}$$

with respect to the function $x\colon \mathbb{R}_+ \to \mathbb{R}$, where $f\colon \mathbb{R}_+ \to \mathbb{R}$.

Equation (1) can be rewritten in the form (2), by putting

$$f(t) = \sigma(t) = \sum_{k=1}^{K} a_k \psi(t - kh) + \sum_{j=1}^{J} b_j \varphi(t - jh),$$

where $\varphi(\xi) = \psi(\xi) = 0$ for $\xi > 0$.

Let the function f be integrable over every finite segment of $\mathbb{R}_+$. We define a *solution* of equation (2) as a function that is absolutely continuous in every finite interval of $\mathbb{R}_+$. According to this, a solution $x\colon \mathbb{R}_+ \to \mathbb{R}$ of (2) satisfies the equality (1) almost everywhere on $\mathbb{R}_+$.

As is known ([2], Ch. 1), equation (2) supplemented with a given initial condition $x(0)$ is uniquely solvable, and its solution is represented by the *Cauchy formula*

$$x(t) = X(t)x(0) + \int_0^t Y(t-s) f(s)\,ds, \quad t \in \mathbb{R}_+. \tag{3}$$

Here the function $X\colon \mathbb{R}_+ \to \mathbb{R}$ is called the *fundamental solution* of (2), and $Y\colon \mathbb{R}_+ \to \mathbb{R}$ the *Cauchy function* of (2). Both these functions do not depend on the initial value $x(0)$ and the external disturbance f. In what follows, we consider the functions X and Y as related to equation (1).

3. Stability with respect to the space of initial functions. Denote $\omega = \max\{Kh, Jh\}$.

Below we introduce the definitions of stability for equation (1). In contrast to ordinary differential equations, initial conditions for equation (1) are specified not at a single point, $t = 0$, but in the set $[-\omega, 0]$. The functions φ and ψ should belong to the space $L_1[-\omega, 0]$, however they can be defined in some restricted set with its own norm. This implies that the introduced above function σ is taken from the corresponding subset of $L_1[0, \omega]$. Thus, the stability properties of solutions to equation (1) are related with the choice of the set of initial functions. This fact should be explicitly reflected in the definitions of stability.

Let $\mathbb{X} \subseteq L_1[0, \omega]$ be a normed set of functions measurable in the set $[0, \omega]$. It is natural to suppose $\mathbb{X}$ to be a linear space.

By (3), a solution $x\colon \mathbb{R}_+ \to \mathbb{R}$ of equation (1) is uniquely defined by the *initial value* $x(0) = x_0 \in \mathbb{R}$ and the *initial function* $\sigma \in \mathbb{X}$. We shall call such solution *corresponding* to x_0 and σ.

Definition 1. *Equation* (1) *is called* $\mathbb{X}$-stable *(in Lyapunov sense), if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x_0 \in \mathbb{R}$ *and* $\sigma \in \mathbb{X}$ *such that* $|x_0| < \delta$ *and* $\|\sigma\|_{\mathbb{X}} < \delta$ *for the corresponding solution* $x = x(t)$ *of equation* (1) *the estimate* $\sup_{t\geq 0} |x(t)| < \varepsilon$ *is valid.*

Definition 2. *Equation* (1) *is called* asymptotically $\mathbb{X}$-stable*, if it is* $\mathbb{X}$-*stable and for all* $x_0 \in \mathbb{R}$ *and* $\sigma \in \mathbb{X}$ *the corresponding solution* $x = x(t)$ *of equation* (1) *possesses the property* $\lim_{t\to+\infty} |x(t)| = 0$.

Definition 3. *Equation* (1) *is called* exponentially $\mathbb{X}$-stable*, if there exist constants* $N, \gamma > 0$ *such that for all* $x_0 \in \mathbb{R}$ *and* $\sigma \in \mathbb{X}$ *for the corresponding solution* $x = x(t)$ *of equation* (1) *the estimate* $|x(t)| \leq Ne^{-\gamma t} (|x_0| + \|\sigma\|_{\mathbb{X}})$ *is valid.*

Remark 1. *The choice of a space* $\mathbb{X} \subseteq L_1[0, \omega]$ *is arbitrary. In most works [9, 11, 13] it is supposed that* $\mathbb{X} = C[0, \omega]$. *In papers [12, 19], where the technique of Hilbert spaces is used,* $\mathbb{X} = L_2[0, \omega]$. *In [18]* $\mathbb{X} = L_1[0, \omega]$, *in [5] the cases* $\mathbb{X} = L_p[0, \omega]$ *for all* $p \geq 1$ *are considered.*

Definitions 1–3 are formulated in traditional terms. However, note that we have in mind the Cauchy formula (3) when we relate the stability of a solution to (1) with x_0 and σ. It is natural to take the next step and use the functions X and Y explicitly to formulate the definitions of stability for equation (1).

For $\sigma \in \mathbb{X}$, denote

$$K_t\sigma = \int_0^t Y(t-s)\sigma(s)\, ds, \quad t \in \mathbb{R}_+, \tag{4}$$

and consider a family of linear functionals $\{K_t\colon \mathbb{X} \to \mathbb{R}\}_{t\in\mathbb{R}_+}$. Further, consider linear transformations $\{X_t\colon \mathbb{R} \to \mathbb{R}\}_{t\in\mathbb{R}_+}$, defined by the equalities $X_t\alpha = X(t)\alpha$, as linear functionals. Define the norms of the functionals K_t and X_t conventionally, that is $\|K_t\| = \sup_{\|\sigma\|=1} |K_t\sigma|$ and $\|X_t\| = \sup_{|\alpha|=1} |X_t\alpha|$.

Remark 2. *For $t \geq \omega$ we have $K_t\sigma = \int_0^\omega Y(t-s)\sigma(s)\,ds$.*

We see from (3) that

$$x(t) = X_t x_0 + K_t\sigma, \quad t \in \mathbb{R}_+. \tag{5}$$

Now we obtain criteria of the stability of (1) in terms of upper estimates for functions X and Y.

Theorem 1. *Equation* (1) *is $\mathbb{X}$-stable if and only if* $\sup_{t\geq 0} |X(t)| < \infty$ *and* $\sup_{t\geq 0} \|K_t\| < \infty$.

Proof. Necessity. By definition 1 the linear functionals X_t and K_t are bounded uniformly in t. Applying the solution representation (5) we obtain: there exists $N > 0$ such that for all $x_0 \in \mathbb{R}$ and $\sigma \in \mathbb{X}$ for the corresponding solution $x = x(t)$ of equation (1) the estimate $|x(t)| \leq N\left(|x_0| + \|\sigma\|_{\mathbb{X}}\right)$ is valid.

Let $\sigma = 0$. Then the boundedness of the fundamental solution is obvious from the inequality $|X(t)x_0| \leq N|x_0|$. Let now $x_0 = 0$. Then $|K_t\sigma| \leq N\|\sigma\|_{\mathbb{X}}$, that is $\|K_t\| \leq N$. Necessity is proved.

Sufficiency follows from (5) immediately. ∎

The classical definition of asymptotic stability [7, p. 68] for a solution to an ordinary differential equation suggests that every asymptotically stable solution is stable in Lyapunov sence as well. This suggestion is essential in general case. However, a solution of a *linear* equation is Lyapunov stable, if it tends to zero, so for linear equations there is no need to declare Lyapunov stability in the definition of asymptotic stability. Let us consider the question on the essentiality of the Lyapunov stability condition in the definition of asymptotic stability for equation (1).

Theorem 2. *Let $\mathbb{X}$ be a Banach space. If for all $x_0 \in \mathbb{R}$ and $\sigma \in \mathbb{X}$ the corresponding solution of* (1) *possesses the property* $\lim_{t\to+\infty} x(t) = 0$*, then equation* (1) *is $\mathbb{X}$-stable.*

Proof. It follows from (5) that the property $\lim_{t\to+\infty} x(t) = 0$ takes place for all solutions of (1) if and only if $\lim_{t\to+\infty} X(t) = 0$ and for all $\sigma \in \mathbb{X}$ we have $\lim_{t\to+\infty} K_t\sigma = 0$. In this case $\sup_{t\geq 0} |X(t)| < \infty$ and for all $\sigma \in \mathbb{X}$ we have $\sup_{t\geq 0} |K_t\sigma| < \infty$. Applying the Banach—Steinhaus theorem [15, p. 116] to the latter, we obtain that $\sup_{t\geq 0} \|K_t\| < \infty$. It remains to refer Theorem 1. ∎

THEOREM 3. *Let $\mathbb{X}$ be a Banach space. Then equation* (1) *is asymptotically $\mathbb{X}$-stable if and only if* $\lim_{t\to+\infty} X(t) = 0$ *and for all* $\sigma \in \mathbb{X}$ *we have* $\lim_{t\to+\infty} K_t\sigma = 0$.

Proof. On account of Theorem 2 and (5), equation (1) is asymptotically $\mathbb{X}$-stable if and only if $\lim_{t\to+\infty} X_t x_0 = 0$ for all $x_0 \in \mathbb{R}$ and $\lim_{t\to+\infty} K_t\sigma = 0$ for all $\sigma \in \mathbb{X}$. The former is equivalent to $\lim_{t\to+\infty} X(t) = 0$. □

Let us consider the proof of Theorem 3 in more detail. It is obvious that

$$\lim_{t\to+\infty} X(t) = 0 \Leftrightarrow \lim_{t\to+\infty} \|X_t\| = 0.$$

Thus, the condition that $\lim_{t\to+\infty} X(t)x_0 = 0$ for all $x_0 \in \mathbb{R}$ is equivalent to the uniform convergence of the family $\{X_t\}$ of linear functionals to zero. However, the condition that $\lim_{t\to+\infty} K_t\sigma = 0$ for all $\sigma \in \mathbb{X}$, which is pointwise convergence of the family $\{K_t\}$, is not equivalent to the condition of the uniform convergence of $\{K_t\}$.

Futher we investigate asymptotic stability of equation (1) having in mind both the types of convergence of functionals K_t. To separate these two cases, we introduce

DEFINITION 4. *Equation* (1) *is called* strongly asymptotically $\mathbb{X}$-stable, *if* $\lim_{t\to\infty} X(t) = 0$ *and* $\lim_{t\to\infty} \|K_t\| = 0$.

REMARK 3. *In [5] a class of neutral equations*

$$\dot{x}(t) \pm \dot{x}(t-1) = -bx(t) + cx(t-1), \quad t \in \mathbb{R}_+, \tag{6}$$

is considered. It is proved that equation (6) *is asymptotically L_1-stable if and only if* $|c| < b$. *Moreover, if* $1 < p \le \infty$, *then* (6) *proves to be strongly asymptotically L_p-stable, but for* $p = 1$ *the strong asymptotic stability is lost. This fact shows that the introduced above definitions of asymptotic stability and strong asymptotic stability are not equivalent.*

Consider the exponential $\mathbb{X}$-stability of (1).

THEOREM 4. *Equation* (1) *is exponentially* $\mathbb{X}$*-stable, if and only if there exist constants* $N, \gamma > 0$ *such that* $|X(t)| \le Ne^{-\gamma t}$ *and* $\|K_t\| \le Ne^{-\gamma t}$.

The *proof* follows from the solution representation formula (5), analogously to the proof of Theorem 1.

Theorems 1 and 4 show that Definitions 1 and 3 mean the uniform in t boundedness of the norms of the functionals $\{K_t\}$. So, there is no need to introduce the definitions of *stong* $\mathbb{X}$*-stability* and *stong exponential* $\mathbb{X}$*-stability* of (1).

4. Auxiliary results.

4.1. Some results from paper [4].

4.1.1. Relations between the functions X and Y. It is shown in [4] that

$$X(t) = Y(t) - (SY)(t), \quad t \in \mathbb{R}. \tag{7}$$

On the other hand, one can easily see that the fundamental solution of (2) satisfies the equation

$$(I - S)\dot{X} = TX.$$

By virtue of the permutability $(I - S)T = T(I - S)$, it follows that

$$\dot{X}(t) = (TY)(t) \text{ for almost all } t \in \mathbb{R}_+. \tag{8}$$

4.1.2. Jumps of the Cauchy function. In [4] it is also shown that the function Y is absolutely continuous in every interval $[nh, nh+h)$, $n \in \mathbb{N}_0$, and has finite jumps H_n at the points $t = nh$. The sequence $\{H_n\}_{n=1}^{\infty}$ is a solution of the following problem for a linear difference equation:

$$\begin{aligned} &H_n = \sum_{k=1}^{K} a_k H_{n-k}, \quad n \in \mathbb{N}; \\ &H_0 = 1; \quad H_p = 0, \quad p \in \mathbb{Z} \setminus \mathbb{N}_0. \end{aligned} \tag{9}$$

Remark 4. *The Cauchy function of equation* (2) *is the unique solution to the integral equation*

$$Y(t) = 1 + (SY)(t) + T\left(\int_0^t Y(s)\,ds\right), \quad t \in \mathbb{R}_+.$$

4.1.3. Conditions of exponential stability. Let us represent some necessary and sufficient conditions for the functions X and Y to have exponential estimates. These conditions are formulated in terms of auxiliary complex variable functions. Denote

$$P_a(z) = \sum_{k=1}^{K} a_k z^k, \quad P_b(z) = \sum_{j=0}^{J} b_j z^j.$$

Theorem 5 ([4]). *For the Cauchy function Y of equation* (1) *to have for some $N, \gamma > 0$ the exponential estimate*

$$|Y(t)| \le N e^{-\gamma t}, \quad t \in \mathbb{R}_+, \tag{10}$$

it is necessary and sufficient for all roots of equations

$$1 - P_a(z) = 0 \tag{11}$$

and

$$1 - z \exp\left(\frac{hP_b(z)}{1 - P_a(z)}\right) = 0 \tag{12}$$

to be outside the unit disk $B[0,1]$.

THEOREM 6 ([4]). *For the fundamental solution* X *of equation* (1) *to have for some* $N, \gamma > 0$ *the estimate* (10) *it is necessary and sufficient for all roots of equation* (12) *to be outside the unit disk* $B[0,1]$.

4.2. Some properties of linear autonomous difference equations. Consider the initial problem for the linear inhomogeneous difference equation of K-th order with the coefficients a_k from equation (1), with zero initial conditions:

$$\begin{aligned} y_n &= \sum_{k=1}^{K} a_k y_{n-k} + f_n, \quad n \in \mathbb{N}, \\ y_{-p} &= 0, \qquad\qquad\qquad p = 0, 1, \ldots, K-1. \end{aligned} \tag{13}$$

The following simple lemma may be deduced from the classical theory of difference equations [8]. We prove it here for the sake of clearness.

LEMMA 1. *For every sequence* $\{f_n\}_{n=1}^{\infty}$ *the solution of problem* (13) *is represented in the form*

$$y_n = \sum_{m=1}^{n} H_{n-m} f_m, \quad n \in \mathbb{N}, \tag{14}$$

where the sequence $\{H_n\}_{n \in \mathbb{N}_0}$ *is the solution of problem* (9).

Proof. It is obvious that problem (13) has a unique solution. Substituting (14) into the equation in (13) and using the properties of the sequence

$\{H_n\}$ defined by problem (9), we obtain:

$$\sum_{k=1}^{K} a_k y_{n-k} + f_n =$$

$$= a_1 \sum_{m=1}^{n-1} H_{n-m-1} f_m + a_2 \sum_{m=1}^{n-2} H_{n-m-2} f_m + \ldots + a_K \sum_{m=1}^{n-K} H_{n-m-K} f_m + f_n =$$

$$= \sum_{m=1}^{n-1} \left(a_1 H_{n-m-1} + a_2 H_{n-m-2} + \ldots + a_K H_{n-m-K}\right) f_m + f_n =$$

$$= \sum_{m=1}^{n-1} H_{n-m} f_m + H_0 f_n = \sum_{m=1}^{n} H_{n-m} f_m = y_n.$$

Equation (13) turns into identity. ∎

Note that for the equation in (13) the sequence $\{H_n\}_{n=0}^{\infty}$ is nothing but a discrete analogue of the Cauchy function [1], which is often called the *fundamental solution* of a difference equation [8]. Not only the solution of problem (13), which has zero initial conditions, but also the solution of an initial problem for the same equation with arbitrary initial conditions is expressed in terms of $\{H_n\}_{n=0}^{\infty}$ and the parameters, $\{y_{-p}\}_{p=0}^{K-1}$ and $\{f_n\}_{n=1}^{\infty}$.

The following lemma is a known transfer of the approach of prof. Azbelev, to the representation of a solution of functional differential equation, to difference equations [1].

Consider a difference Cauchy problem

$$\begin{aligned} u_n &= \sum_{k=1}^{K} a_k u_{n-k}, \quad && n \in \mathbb{N}, \\ u_{-p} &= \varphi_p, && p = 0, 1, 2, \ldots, K-1. \end{aligned} \tag{15}$$

Consider also equation (13), where

$$f_n = \begin{cases} \sum\limits_{k=n}^{K} a_k \varphi_{k-n}, & n = 1, \ldots, K; \\ 0, & n = K+1, K+2, \ldots \end{cases} \tag{16}$$

The next result is verified directly.

Lemma 2. *The solution $\{u_n\}_{n=0}^{\infty}$ of problem* (15) *coinside with the solution $\{y_n\}_{n=0}^{\infty}$ of problem* (13), *where f_n are defined by* (16).

Thus, problem (15) is reformulated in the form of problem (13) with the specific initial conditions (16).

Lemma 3. *If* $\sup_n |H_n| < \infty$, *then for the solution* $\{u_n\}$ *of problem* (15) *with arbitrary* $\{\varphi_p\}_{p=0}^{K-1}$ *we have* $\sup_n |u_n| < \infty$.

Proof. Rewrite problem (15) in the form (13), (16). It is obvious that

$$|f_n| \le \sum_{k=n}^{K} |a_k||\varphi_{k-n}| \le \max_{0\le k\le K-1} |\varphi_k| \sum_{k=0}^{K} |a_k| = A < \infty. \tag{17}$$

Taking account of $f_m = 0$ for $m \ge K+1$ and applying Lemma 1, we obtain

$$|u_n| \le \sum_{m=1}^{K} |H_{n-m}||f_m| \le A \sum_{m=1}^{K} |H_{n-m}| \le AK \cdot \sup_n |H_n|. \quad \square$$

Lemma 4. *If* $\lim_{n\to\infty} H_n = 0$, *then for the solution* $\{u_n\}$ *of problem* (15) *with arbitrary* $\{\varphi_p\}_{p=0}^{K-1}$ *we have* $\lim_{n\to\infty} u_n = 0$.

Proof. By virtue of (17) and Lemma 1 we have

$$|u_n| \le \sum_{m=1}^{K} |H_{n-m}||f_m| \le A \sum_{m=1}^{K} |H_{n-m}| \underset{n\to\infty}{\to} 0. \quad \square$$

Now we use Lemmas 3 and 4 to establish a relation between the asymptotic properties of the sequence $\{H_n\}_{n=0}^{\infty}$ and the location of roots of equation (11) with respect to the unit disk.

Lemma 5. *If* $\sup_n |H_n| < \infty$, *then equation* (11) *has no roots in* $B(0,1)$.

Proof. Assume that there is a root z_0 of (11) inside $B(0,1)$. Then $1/z_0$ is a root of the characteristic polynomial $z^K - \sum_{k=1}^{K} a_k z^{n-k}$ of the equation in (13). Put $1/z_0 = e^{\alpha+i\beta}$, where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$. Taking the initial conditions

$$v_{-p} = e^{-\alpha p}\cos\beta p, \quad w_{-p} = -e^{-\alpha p}\sin\beta p, \quad p = 0,1,2,\ldots,K-1,$$

we find the corresponding solutions of problem (15):

$$v_n = e^{\alpha n}\cos\beta n, \quad w_n = e^{\alpha n}\sin\beta n.$$

It follows from Lemma 3 that if $\sup_n |H_n| < \infty$, then $\sup_n |v_n| < \infty$ and $\sup_n |w_n| < \infty$, which is impossible, since $\alpha > 0$, and

$$v_n^2 + w_n^2 = (e^{\alpha n}\cos\beta n)^2 + (e^{\alpha n}\sin\beta n)^2 = e^{2\alpha n} \underset{n\to\infty}{\to} \infty. \quad \square$$

Lemma 6. *If* $\sup_n |H_n| < \infty$, *then there is no multiple roots of equation* (11) *on the boundary of the disk* $B[0,1]$.

Proof. Assume there is a multiple root z_0 of (11) on the boundary of $B[0,1]$. Then $1/z_0 = e^{i\beta}$, where $\beta \in \mathbb{R}$, is a multiple root of the characteristic polynomial of the equation in (13). Taking the initial conditions

$$v_{-p} = -p\cos\beta p, \quad w_{-p} = p\sin\beta p, \quad p = 0,1,2,\dots,K-1,$$

we find the following two solutions of problem (15):

$$v_n = n\cos\beta n, \quad w_n = n\sin\beta n.$$

From Lemma 3 it follows that if $\sup_n |H_n| < \infty$, then $\sup_n |v_n| < \infty$ and $\sup_n |w_n| < \infty$, which is impossible, since

$$v_n^2 + w_n^2 = (n\cos\beta n)^2 + (n\sin\beta n)^2 = n^2 \underset{n\to\infty}{\to} \infty. \quad \square$$

LEMMA 7. *If* $\lim_{n\to\infty} H_n = 0$, *then all roots of equation* (11) *are outside the disk* $B[0,1]$.

Proof. Assume there is a root z_0 of (11) in $B[0,1]$. Then $1/z_0$ is a root of the characteristic polynomial of equation in (13) and $1/z_0 = e^{\alpha+i\beta}$, where $\alpha \geq 0$. Taking the initial conditions

$$v_{-p} = e^{-\alpha p}\cos\beta p, \quad w_{-p} = -e^{-\alpha p}\sin\beta p, \quad p = 0,1,2,\dots,K-1,$$

we find the following two solutions of problem (15):

$$v_n = e^{\alpha n}\cos\beta n; \quad w_n = e^{\alpha n}\sin\beta n.$$

From Lemma 4 it follows that if $\lim_{n\to\infty} H_n = 0$, then $\lim_{n\to\infty} v_n = 0$ and $\lim_{n\to\infty} w_n = 0$, which is impossible, since

$$v_n^2 + w_n^2 = (e^{\alpha n}\cos\beta n)^2 + (e^{\alpha n}\sin\beta n)^2 = e^{2\alpha n} \geq 1. \quad \square$$

5. Criteria of L_p-stability. In this section we study $\mathbb{X}$-stability with respect to an initial function from the Lebesgue spaces $L_p[0,\omega]$ (which are Banach spaces) of integrable functions. We obtain criteria of L_p-stability of equation (1) in terms the location of zeros of the functions (11) and (12).

5.1. Lyapunov L_p-stability. Prove the following

LEMMA 8. *If* $\sup_{t\geq 0}\int_t^{t+\omega} |Y(s)|\,ds < \infty$, *then* $\sup_{t\geq 0} |X(t)| < \infty$.

Proof. From the definitions of the operators S and T, the equalities (7) and (8), and the condition of the lemma it follows that

$$\sup_{t\geq 0}\int_t^{t+\omega}|X(s)|\,ds = N_1 < \infty, \quad \sup_{t\geq 0}\int_t^{t+\omega}|\dot{X}(s)|\,ds = N_2 < \infty. \tag{18}$$

Assume that $\sup_{t\geq 0}|X(t)| = \infty$. Then there is $t_0 \in \mathbb{R}_+$ such that

$$|X(t_0)| > \frac{N_1}{\omega} + N_2.$$

Hence for all $t \in [t_0, t_0+\omega]$ we have

$$|X(t) - X(t_0)| \leq \int_{t_0}^{t}|\dot{X}(s)|\,ds \leq \int_{t_0}^{t_0+\omega}|\dot{X}(s)|\,ds \leq N_2.$$

Thus, for all $t \in [t_0, t_0+\omega]$ the estimate $|X(t)| > \frac{N_1}{\omega}$ is valid. Therefore, $\int_{t_0}^{t_0+\omega}|X(s)|\,ds > N_1$, which contradicts the first of the relations (18). ∎

Now we obtain stability criteria in terms of the Cauchy function.

THEOREM 7. *Suppose* $1 < p \leq \infty$. *Equation* (1) *is* L_p*-stable if and only if*

$$\sup_{t\geq 0}\int_t^{t+\omega}|Y(s)|^q\,ds < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For every fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_p[0,\omega] \to \mathbb{R}$ is [15]

$$\|K_t\| = \left(\int_0^{\omega}|Y(t-s)|^q\,ds\right)^{1/q} = \left(\int_t^{t+\omega}|Y(s)|^q\,ds\right)^{1/q}. \tag{19}$$

By virtue of Theorem 1 we obtain the necessity part of the theorem immediately. We obtain the sufficiency part with regard of the Hölder inequality and Lemma 8. ∎

Consider the case $p = 1$ separately.

THEOREM 8. *Equation* (1) *is* L_1*-stable if and only if*

$$\sup_{t\geq 0}|Y(t)| < \infty.$$

Proof. For every fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_1[0,\omega] \to \mathbb{R}$ is [15]

$$\|K_t\| = \operatorname*{ess\,sup}_{s\in[0,\omega]} |Y(t-s)|. \tag{20}$$

The function Y is piecewise continuous and has a finite number of jumps in each finite segment. Hence

$$\operatorname*{ess\,sup}_{s\in[0,\omega]} |Y(t-s)| = \sup_{s\in[0,\omega]} |Y(t-s)|.$$

Therefore, $\sup_{t\geq 0} \|K_t\| < \infty$ if and only if $\sup_{t\geq 0} |Y(t)| < \infty$. By virtue of (7), from $\sup_{t\geq 0} |Y(t)| < \infty$ it follows that $\sup_{t\geq 0} |X(t)| < \infty$. It remains to apply Theorem 1. ∎

From Theorems 7 and 8 we obtain

COROLLARY 1. *If the Cauchy function of* (1) *is bounded, then* (1) *is L_p-stable for all p, $1 \leq p \leq \infty$.*

Below we reduce the question on conditions of the boundedness of the function Y to the question on the location of roots of the complex variable functions (11) and (12) with respect to the unit disk.

THEOREM 9. *Suppose all roots of equation* (11) *are outside the disk $B[0,1]$. Then the Cauchy function Y of equation* (1) *is bounded if and only if equation* (12) *has no roots in $B(0,1)$ and no mutiple roots on the boundary of $B(0,1)$.*

Proof. Let $t = nh + \tau$, $\tau \in [0,h)$. Denote $Y(t) = Y(nh+\tau) = y_n(\tau)$ and consider the generating function

$$F(\tau,z) = \sum_{n=0}^{\infty} y_n(\tau) z^n$$

of the sequence $\{y_n(\tau)\}_{n=0}^{\infty}$. In [4] the explicit form of this function is found, which is

$$F(\tau,z) = \frac{\exp\{\tau \frac{P_b(z)}{1-P_a(z)}\}}{(1-P_a(z))(1 - z\exp\{h\frac{P_b(z)}{1-P_a(z)}\})},$$

and it is shown that $F(\tau,z)$ is analytic in some disk $B[0,r]$, $r<1$. Since all zeros of equation (11) are outside $B[0,1]$, there is $R>1$ such that $\min_{z\in B[0,R]} |1-P_a(z)| > 0$, and the function $1 - z\exp\{h\frac{P_b(z)}{1-P_a(z)}\}$ has a finite numbers of zeros in $B[0,R]$ by virtue of the uniqueness theorem. Denote them by $z_1, z_2, \ldots, z_m$.

By the formula for coefficients of a power series and the Cauchy theorem on residues, we have

$$(21)\quad y_n(\tau) = \frac{1}{2\pi i}\oint_{|\zeta|=r} \frac{F(\tau,\zeta)}{\zeta^{n+1}}\,d\zeta = \\ = \frac{1}{2\pi i}\oint_{|\zeta|=R} \frac{F(\tau,\zeta)}{\zeta^{n+1}}\,d\zeta + \sum_{k=1}^{m} \operatorname*{res}_{z=z_k}\left(\frac{F(\tau,z)}{z^{n+1}}\right).$$

Note that

$$\left|\oint_{|\zeta|=R} \frac{F(\tau,\zeta)}{\zeta^{n+1}}\,d\zeta\right| \le \frac{\text{const}}{R^n}.$$

From (21) it follows that $y_n(\tau)$ is bounded (in n and τ) if and only if the sum of residues is bounded at the points $z = z_k$, $k = 1, \dots, m$. This is possible only in case that $z_k \notin B(0,1)$ and if $|z_k| = 1$ then the root z_k is not multiple.

To conclude the proof it remains to note that the boundedness of $y_n(\tau)$ in n and τ is equivalent to the boundedness of $Y(t)$. ∎

Theorem 9 presents necessary and sufficient conditions of the boundedness of the Cauchy function of (1) under the assumption that all roots of (11) are outside $B[0,1]$. Below we supplement Theorem 9 by necessary conditions in terms of the location of roots of (11).

Theorem 10. *If the Cauchy function of equation* (1) *is bounded, then equation* (11) *has no roots in* $B(0,1)$.

Proof. Assume that $\sup_{t\ge 0} |Y(t)| = M < \infty$ while equation (11) has a root in $B(0,1)$. Then it follows from Lemma 5 that the sequence $\{H_n\}_{n=0}^{\infty}$ is unbounded. Therefore for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $|H_{n_k}| \ge k$, and Y has a jump H_{n_k} at the point $t_k = hn_k$. We have

$$k \le |H_{n_k}| = |Y(t_k) - \lim_{\delta\to+0} Y(t_k - \delta)| \le 2M,$$

which is impossible. ∎

Theorem 11. *If the Cauchy function of equation* (1) *is bounded, then equation* (11) *has no multiple roots on the boundary of* $B(0,1)$.

The proof of Theorem 11 is analogous to that of Theorem 10, with Lemma 6 applied instead of Lemma 5.

Remark 5. *The left-hand side of equation* (11) *is a polynomial. The problem on location of the roots of a polynomial with respect to the unit disk is well investigated (we mean theorems of the Schur—Kohn type [17]). For the sake of simplicity, when investigating the stability of equation* (1) *it is appropriate to suppose that the conditions of Theorems 10 and 11 are satisfied.*

Remark 6. *The conditions of Theorem 9 do not exhaust the class of equations of the form* (2) *with bounded Cauchy function. In paper [5] one may find examples of equation* (2)*, for which the Cauchy function is bounded while equation* (11) *has roots on the boundary of* $B[0,1]$.

Remark 7. *It is not known, whether there is an an* L_p*-stable neutral equation such that its Cauchy function is unbounded.*

Remark 8. *In paper [10] a family of equations of the form* (1) *is constructed such that they have unbounded solutions in spite of the fact that all roots of equation* (11) *are on the boundary of* $B[0,1]$. *However, this example should be considered as nonconstructive, since even the case that the initial function is not in* $L_1[0,\omega]$ *cannot be excluded.*

5.2. Asymptotic L_p-stability. For strong asymptotic stability we obtain two theorems analogous to Theorems 7 and 8.

Lemma 9. *If* $\lim_{t\to\infty}\int_t^{t+\omega}|Y(s)|\,ds = 0$*, then* $\lim_{t\to\infty}X(t) = 0$.

Proof. Suppose $\lim_{t\to\infty}\int_t^{t+\omega}|Y(s)|\,ds = 0$. Then by virtue of the definitions of the operators S and T, and the equalities (7) and (8) we have

$$\lim_{t\to\infty}\int_t^{t+\omega}|X(s)|\,ds = 0, \quad \lim_{t\to\infty}\int_t^{t+\omega}|\dot{X}(s)|\,ds = 0. \tag{22}$$

Assume that $X(t)$ does not tend to 0 as $t\to\infty$. Then there exist $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$, where $t_n \to +\infty$ as $n\to\infty$, such that $|X(t_n)| \geq \varepsilon$. Take $N\in\mathbb{N}$ such that for all integers $n \geq N$

$$\int_{t_n}^{t_n+\omega}|\dot{X}(s)|\,ds < \varepsilon/2.$$

Consider arbitrary $t\in[t_n, t_n+\omega]$. We have

$$|X(t) - X(t_n)| \leq \int_{t_n}^{t}|\dot{X}(s)|\,ds \leq \int_{t_n}^{t_n+\omega}|\dot{X}(s)|\,ds < \varepsilon/2.$$

Thus, the estimate $|X(t)| \geq \frac{\varepsilon}{2}$ is valid. Hence, $\int_{t_n}^{t_n+\omega}|X(s)|\,ds \geq \frac{\varepsilon\omega}{2} > 0$ for all $n \geq N$, which contradicts the first of the relations (22). ∎

Theorem 12. *Suppose* $1 < p \leq \infty$. *Equation* (1) *is strongly asymptotically* L_p*-stable if and only if for its Cauchy formula* Y *we have*

$$\lim_{t\to\infty}\int_t^{t+\omega}|Y(s)|^q\,ds = 0,$$

where $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. For each fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_p[0,\omega] \to \mathbb{R}$ is determined by formula (19). It follows that if $\lim_{t\to\infty} \|K_t\| = 0$, then $\lim_{t\to\infty} \int_t^{t+\omega} |Y(s)|^q\, ds = 0$, which implies the necessity. We obtain the sufficiency with regard of (19), the Hölder inequality and Lemma 9. ∎

Now consider the case $p = 1$.

THEOREM 13. *Equation* (1) *is strongly asymptotically L_1-stable if and only if for its Cauchy formula Y we have*

$$\lim_{t\to\infty} Y(t) = 0.$$

Proof. For each fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_1[0,\omega] \to \mathbb{R}$ is determined by (20). Taking account of piecewise continuity of Y we see that $\lim_{t\to\infty} \|K_t\| = 0$ if and only if $\lim_{t\to\infty} Y(t) = 0$. It remains to note that (7) implies that if $\lim_{t\to\infty} Y(t) = 0$ then $\lim_{t\to\infty} X(t) = 0$. ∎

From Theorems 12 and 13 follows

COROLLARY 2. *If* $\lim_{t\to\infty} Y(t) = 0$, *then* (2) *is L_p-stable for all p, $1 \le p \le \infty$.*

Now consider some consequences of the property $Y(t) \to 0$.

Since H_n are jumps of $Y(t)$, we have

LEMMA 10. *If $Y(t)$ has a limit as $t \to \infty$, then* $\lim_{n\to\infty} H_n = 0$.

From Lemmas 7 and 10 we obtain

THEOREM 14. *If* $\lim_{t\to\infty} Y(t) = 0$, *then all roots of equation* (11) *are outside the disk* $B[0,1]$.

Consider a function $g\colon \mathbb{C} \to \mathbb{C}$ defined by the formula

$$g(\lambda) = \lambda(1 - g_S(\lambda)) + g_T(\lambda),$$

where

$$g_S(\lambda) \stackrel{\text{def}}{=} \sum_{k=1}^{K} a_k e^{-hk\lambda}, \quad g_T(\lambda) \stackrel{\text{def}}{=} \sum_{j=0}^{J} b_j e^{-hj\lambda}.$$

We call g the *characteristic function* of equation (1). Note that it is an analytic function. Substituting $x(t) = e^{\lambda t}$ into (1), we obtain the *characteristic equation*

$$g(\lambda) = 0 \tag{23}$$

of equation (1). Eqation (23) has a countable set of roots in $\mathbb{C}$.

It is easily seen that if $\lambda_0 = \alpha + i\beta$ is a root of (23), then, taking the functions $\varphi(\xi) = e^{\alpha\xi}\cos\beta\xi$ and $\psi(\xi) = \dot\varphi(\xi)$ as initial functions for equation (1), we obtain that the function $v(t) = e^{\alpha t}\cos\beta t$ is a solution to (1). Analogously, taking initial functions $\varphi(\xi) = e^{\alpha\xi}\sin\beta\xi$ and $\psi(\xi) = \dot\varphi(\xi)$, we obtain another solution to (1), $w(t) = e^{\alpha t}\sin\beta t$.

THEOREM 15. *If the Cauchy function Y of* (1) *tends to zero as $t\to\infty$, then it has the exponential estimate* (10).

Proof. It is obvious that only the necessity needs to be proved. By virtue of Theorem 5 it is enough to obtain that equations (11) and (12) have no roots in $B[0,1]$. By Theorem 14, for equation (11) this is true.

Assume that equation (12) has a root $z_0 \in B[0,1]$. Then $1/z_0 = e^{\alpha h + i\beta h}$, where $\alpha \ge 0$. We see that

$$\exp(\alpha + i\beta) = \exp\left(-\frac{\sum_{j=0}^{J} b_j e^{-hj(\alpha+i\beta)}}{1-\sum_{k=1}^{K} a_k e^{-hk(\alpha+i\beta)}}\right).$$

It is not difficult to establish that equation (23) has a root $\lambda_1 = \alpha + i\beta_1$, with $\operatorname{Re}\lambda_1 = \alpha$. Therefore, functions $v(t) = e^{\alpha t}\cos\beta_1 t$ and $w(t) = e^{\alpha t}\sin\beta_1 t$ are solutions of equation (1). However, by Theorem 14 $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} w(t) = 0$, which is impossible, since the assumption that $\alpha \ge 0$ implies that

$$v^2(t) + w^2(t) = (e^{\alpha t}\cos\beta t)^2 + (e^{\alpha t}\sin\beta t)^2 = e^{2\alpha t} \ge 1. \quad \square$$

COROLLARY 3. *If $\lim_{t\to\infty} Y(t) = 0$, then the operator $I - S$ is invertible in each of spaces $L_p(\mathbb{R}_+)$, $1 \le p \le \infty$.*

The following statement follows immediately from Theorems 13 and 15.

COROLLARY 4. *Equation* (1) *is strongly asymptotically L_1-stable if and only if its Cauchy function Y has an exponential estimate* (10).

5.3. Exponential L_p-stability. In case $\mathbb{X} = L_p$ the definition of exponential $\mathbb{X}$-stability of (1), as those of Lyapunov and strong asymptotic stabilities, can be reformulated in terms of the Cauchy function.

LEMMA 11. *If $\int_t^{t+\omega} |Y(s)|\,ds \le Ne^{-\gamma t}$, then $|X(t)| \le Me^{-\gamma t}$.*

Proof. From (7), (8), and the definitions of the operators S and T it follows that if the conditions of the lemma are satisfied, then for all $t \in \mathbb{R}_+$ the relations

$$(24)\qquad \int_t^{t+\omega} |X(s)|\,ds \leqslant M_1 e^{-\gamma t}, \qquad \int_t^{t+\omega} \left|\dot X(s)\right| ds \leqslant M_2 e^{-\gamma t}$$

hold. Assume that there is $t_0 \in \mathbb{R}_+$ such that

$$|X(t_0)e^{\gamma t_0}| > e^{\gamma\omega}\left(\gamma M_1 + \frac{M_1}{\omega} + M_2\right).$$

Then for arbitrary $t \in [t_0, t_0+\omega]$ we have

$$\begin{aligned}\left|X(t)e^{\gamma t} - X(t_0)e^{\gamma t_0}\right| &= \left|\int_{t_0}^{t} (X(s)e^{\gamma s})'\, ds\right| \\ &\le \gamma \int_{t_0}^{t_0+\omega} |X(s)e^{\gamma s}|\, ds + \int_{t_0}^{t_0+\omega} \left|\dot X(s)e^{\gamma s}\right| ds \\ \le \gamma e^{\gamma(t_0+\omega)} \int_{t_0}^{t_0+\omega} |X(s)|\, ds &+ e^{\gamma(t_0+\omega)} \int_{t_0}^{t_0+\omega} \left|\dot X(s)\right| ds \le \gamma e^{\gamma\omega} M_1 + e^{\gamma\omega} M_2.\end{aligned}$$

Therefore for all $t \in [t_0, t_0+\omega]$ the estimate $|X(t)e^{\gamma t}| > \frac{M_1 e^{\gamma\omega}}{\omega}$ is valid. Hence

$$e^{\gamma(t_0+\omega)} \int_{t_0}^{t_0+\omega} |X(s)|\, ds \ge \int_{t_0}^{t_0+\omega} |X(s)e^{\gamma s}|\, ds > M_1 e^{\gamma\omega},$$

which contradicts the first of the relations (24). ∎

THEOREM 16. *Let $1 < p \le \infty$. Then equation* (1) *is exponentially L_p-stable if and only if there are $N, \gamma > 0$ such that*

$$\int_t^{t+\omega} |Y(s)|^q\, ds \le N e^{-\gamma t}, \quad t \in \mathbb{R}_+,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For each fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_p[0,\omega] \to \mathbb{R}_+$ if defined by formula (19). By virtue of Theorem 4, it implies the necessity part of the theorem. We obtain the sufficiency part with regard of (19), the Hölder inequality, and Lemma 11. ∎

Consider the case $p = 1$ sepapately.

THEOREM 17. *Equation* (1) *is exponentially L_1-stable if and only if the Cauchy function has exponential estimate* (10).

Proof. For each fixed $t \in \mathbb{R}_+$ the norm of the functional $K_t\colon L_1[0,\omega] \to \mathbb{R}_+$ if defined by formula (20). It follows that $\|K_t\| \le N e^{-\gamma t}$ if and only if $|Y(t)| \le N e^{-\gamma t}$. Now the exponential estimate on the fundamental solution follows from (7). ∎

Thus, the exponential L_1-stability of equation (1) is equivalent to the exponential estimate (10) of the Cauchy function. Since $L_p[0,\omega] \subset L_1[0,\omega]$, $p > 1$, the estimate (10) implies the exponential L_p-stability of equation (1) for all $p > 1$. An interesting question arises: are there exponentially L_p-stable equations of the form (1), whose Cauchy function has no exponential estimate?

In the proof of the following lemma we use the scheme of the proof of Lemma 2 from [3].

LEMMA 12. *Let λ_0 be a root of equation $1 - g_S(\lambda) = 0$. Then for all $\varepsilon > 0$ there is a root μ_0 of the same equation such that $\operatorname{Re}\lambda_0 = \operatorname{Re}\mu_0$ and there is a root of the function $g(\lambda)$ in the disk $B(\mu_0, \varepsilon)$.*

Proof. Since the function $1 - g_S(\lambda)$ is analytic, its zeros are isolated. Therefore one can choose $\varepsilon > 0$ so that the function $\eta(z) = 1 - g_S(\lambda_0 + z)$ has the only zero $z_0 = 0$ in the disk $B(0, \varepsilon)$, and the inequality $|1 - g_S(\lambda_0 + z)| \geq m > 0$ holds on the boundary $\{z : |z| = \varepsilon\}$ (which is a compact set).

Denote $\lambda_k = \lambda_0 + \frac{2\pi k i}{h}$, $k \in \mathbb{Z}$.

It is obvious that $e^{\lambda_0 h} = e^{\lambda_k h}$, hence $g_S(\lambda_0 + z) = g_S(\lambda_k + z)$. Find $k_0 \in \mathbb{Z}$ such that $|\lambda_{k_0}|$ is sufficiently large and for $|z| = \varepsilon$ the estimate

$$\left|\frac{g_T(\lambda_{k_0} + z)}{\lambda_{k_0} + z}\right| \leq \frac{\left|\sum_{j=0}^{J} b_j e^{-(\lambda_{k_0} + z)hj}\right|}{|\lambda_{k_0}| - \varepsilon} \leq \frac{\sum_{j=0}^{J} |b_j| e^{(\varepsilon - \operatorname{Re}\lambda_0)hj}}{|\lambda_{k_0}| - \varepsilon} < m$$

holds.

Denote $\mu_0 = \lambda_{k_0} = \lambda_0 + \frac{2\pi k_0 i}{h}$ (obviously, $\operatorname{Re}\lambda_0 = \operatorname{Re}\mu_0$) and consider the two functions in the disk $B[0, \varepsilon]$: $\zeta(z) = 1 - g_S(\mu_0 + z)$ and $\xi(z) = \frac{g_T(\mu_0 + z)}{\mu_0 + z}$.

With regard to the above estimates, we obtain that for $|z| = \varepsilon$ the inequalities $|\zeta(z)| \geq m > |\xi(z)|$ are satisfied.

By the Rouché theorem [14], the functions $\zeta(z)$ and $\zeta(z) + \xi(z)$ have the same number of zeros in the disk $B(0, \varepsilon)$. Since the function $\zeta(z)$ has a zero in this disk, it follows that the function

$$\zeta(z) + \xi(z) = 1 - g_S(\mu_0 + z) + \frac{g_T(\mu_0 + z)}{\mu_0 + z} = \frac{g(\mu_0 + z)}{\mu_0 + z}$$

also has a zero. This equivalent to the fact that the function $\frac{g(\lambda)}{\lambda}$ has a zero in the disk $B(\mu_0, \varepsilon)$; hence the function $g(\lambda)$ also does. ∎

THEOREM 18. *Let $1 \leq p \leq \infty$. If equation (1) is exponentially L_p-stable, then all roots of equation (11) are outside $B[0, 1]$.*

Proof. Let z_0 be a root with the least absolute value. Assume that $z_0 \in B[0,1]$, that is $z_0 = e^{-\alpha h - i\beta h}$, where $\alpha \geq 0$. Then the equation $1 - g_S(\lambda) = 0$ has a root $\lambda_0 = \alpha + i\beta$. Choose $\varepsilon < \alpha + \gamma$, where γ is the exponent in the definition of exponential L_p-stability, and find, in accordance with Lemma 12, a root μ_0 of the equation $1 - g_S(\lambda) = 0$ such that $\operatorname{Re}\mu_0 = \alpha$, and there is a root of the funtion $g(\lambda)$ in the disk $B(\mu_0, \varepsilon)$. Denote this root of $g(\lambda)$ by $\lambda_1 = \alpha_1 + i\beta_1$. By construction, $|\alpha_1 - \alpha| < \varepsilon < \alpha + \gamma$, hence $\alpha_1 + \gamma > 0$. The solution $x(t) = e^{\alpha_1 t}\cos\beta_1 t$ of equation (1) corresponds to the root λ_1. For this solution the product $|x(t)|e^{\gamma t} = e^{(\gamma+\alpha_1)t}|\cos\beta_1 t|$ is an unbounded function. Therefore equation (1) is not exponentially L_p-stable, which contradicts the condition of the lemma. Thus, equation (11) has no roots in the disk $B[0,1]$. ∎

THEOREM 19. *Let $1 \leq p \leq \infty$. Then equation* (1) *is exponentially L_p-stable if and only if for its Cauchy function the estimate* (10) *is valid.*

Proof. The sufficiency follows from Theorem 17 in view of $L_p[0,\omega] \subset L_1[0,\omega]$.

Let us prove the necessity. Suppose equation (1) is exponentially L_p-stable. Then by Theorem 19 all roots of equation (11) are outside the disk $B[0,1]$. Further, since the fundamental solution of equation (1) has an exponential estimate, it follows that by Theorem 6 all roots of equation (12) are also outside $B(0,1)$. To concude the proof it remains to apply Theorem 5. ∎

COROLLARY 5. *If equation* (1) *is exponentially L_{p_0}-stable for some $p_0 \geq 1$, then it is exponentially L_p-stable for all $p \geq 1$.*

Thus in the scale of the spaces L_p, $1 \leq p \leq \infty$, it is sufficient to establish exponential stability for a single p. Apparently, the most convenient are the spaces L_1,L_2 and L_∞.

Equally interesting is the result on the equivalence of every exponential L_p-stability to the estimate (10): by Theorem 19 the existence of this estimate can be taken as the definition of exponential stability regardless of the choice of the initial functions space L_p. A special effect is achieved by comparing this result with Theorem 15: the condition $\lim_{t\to\infty} Y(t) = 0$ can serve as a definition of exponential stability!

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