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CRITERION OF EXPONENTIAL STABILITY FOR EQUATIONS OF NEUTRAL TYPE WITH PERIODIC COEFFICIENTS *

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Abstract. For linear equations of neutral type with constant delays and periodic coefficients, exponential stability is investigated, which is defined as an exponential estimate of the Cauchy function. It is shown that exponential stability is equivalent to the location of roots of two complex variable functions on the complex plane. New exponential stability criteria for the equation under study are obtained on the basis of known exponential stability criteria for autonomous equations.

Key Words. Neutral equations, Cauchy function, exponential stability.

AMS(MOS) subject classification. 34K25, 34K40.

1. Introduction.

The purpose of the present work is to study the stability of differential-difference equations with periodic coefficients. This class of equations is quite close in properties to autonomous equations, which makes it possible to count on obtaining necessary and sufficient conditions of stability. The first such criterion was obtained by A.M. Zverkin for differential-difference equations of delayed type in 'Supplements' to the monograph [6, pp. 498–512]. It was shown there that the study of the asymp-

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totic stability of the equation

$$\dot{x}(t) + \sum_{j=0}^{J} b_j(t)x(t-jh) = 0\ , \quad t \in \mathbb{R}_+, \tag{1}$$

with continuous h-periodic coefficients b_j may be replaced by the study of the autonomous equation

$$\dot{x}(t) + \sum_{j=0}^{J} B_j x(t-jh) = 0, \quad t \in \mathbb{R}_+, \tag{2}$$

where $B_j = \frac{1}{h}\int_0^h b_j(t)\,dt$. In particular, the following result holds [6, p. 505].

THEOREM 1. *Equation* (1) *is asymptotically stable if and only if equation* (2) *is asymptotically stable.*

For equations (1) and (2) asymptotic stability coincides with exponential stability.

In paper [9] Theorem 1 is generalized to the equation

$$\dot{x}(t) - \sum_{k=1}^{K} a_k \dot{x}(t-kh) = \sum_{j=0}^{J} b_j(t)x(t-jh), \quad t \in \mathbb{R}_+, \tag{3}$$

where $h > 0$, $a_k \in \mathbb{R}$, and b_j are integrable periodic functions with period h.

THEOREM 2. *Equation* (3) *is exponentially stable if and only if the equation*

$$\dot{x}(t) - \sum_{k=1}^{K} a_k \dot{x}(t-kh) = \sum_{j=0}^{J} B_j x(t-jh), \quad t \in \mathbb{R}_+,$$

where $B_j = \frac{1}{h}\int_0^h b_j(t)\,dt$, *is exponentially stable.*

In this paper, we continue to study the stability of differential-difference equations of the form (3), but we do it without the assumption that the coefficients at the derivatives are constant.

Note that the presented method, in contrast to the methods developed in [7, sec. 12.10], does not require a priori solvability of the equation in question with respect to the derivative and the construction of the characteristic multipliers.

2. The statement of the problem. Denote by $\mathbb{N}$ the space of positive integers. Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = [0, +\infty)$.

For a measurable set E, denote by $L_1(E)$ the space of functions integrable over E , and by $L_\infty(E)$ the space of functions measurable and essentially bounded on E, with natural norms.

Denote by I the identity operator, and by Θ the zero operator, which are acting in functional spaces. Set $\Delta = \{(t,s) \in \mathbb{R}^2 \colon t \geqslant s\}$.

Consider the equation of neutral type

$$\begin{aligned} &\dot{x}(t) - \sum_{k=1}^{K} a_k(t)\dot{x}(t-kh) = \sum_{j=0}^{J} b_j(t)x(t-jh), && t \geqslant s, \\ &x(\xi) = \varphi(\xi), \quad \dot{x}(\xi) = \psi(\xi), && \xi < s, \end{aligned} \tag{4}$$

where $s \in \mathbb{R}$, $h > 0$, a_k are bounded measurable periodic functions with period h, b_j are integrable periodic functions with period h. For the statement of the problem to be correct, define the functions x and $\dot{x}$ for negative arguments by initial functions $\varphi \in L_\infty[s-\omega, s]$ and $\psi \in L_1[s-\omega, s]$ respectively, where $\omega = \max\{Kh, Jh\}$.

The background of the theory of functional differential equations shows that the choice of a space for initial functions represens a particular problem, which allows various approaches. Consider the statement accepted in the Azbelev scientific school [1].

By S_h denote the shift operator that is acting in the space of continuous (piecewise continuous, integrable) functions:

$$(S_h y)(t) = \begin{cases} y(t-h), & t-h \geqslant s, \\ 0, & t-h < s. \end{cases}$$

Set

$$(Sy)(t) = \sum_{k=1}^{K} a_k(t)(S_h^k y)(t), \quad (Ty)(t) = \sum_{j=0}^{J} b_j(t)(S_h^j y)(t).$$

Consider the equation

$$\dot{x}(t) - (S\dot{x})(t) = (Tx)(t) + f(t), \quad t \geqslant s, \tag{5}$$

where f is a locally integrable function. Note that we do not require to define a solution for $t < s$. As is known [1], equation (4) can be considered as a special case of equation (5), the initial functions φ and ψ related to the external disturbance f.

It is known [1, p. 84, Theorem 1.1] that for every given initial condition $x(s)$ there exists a unique solution of (5) in the class of locally piecewise absolutely continuous functions. It has the form

$$x(t) = X(t,s)x(s) + \int_s^t Y(t,\tau)f(\tau)\,d\tau, \tag{6}$$

where X is *the fundamental solution*, Y is *the Cauchy function*. Further, we set everywhere $X(t,s) = Y(t,s) = 0$ for $t < s$. The functions X and Y describe any solution of (5), and the study of stability is reduced to the study of asymptotic behavior of X and Y.

The main task of the present work is to investigate the stability of equation (5).

We shall say that equation (5) *is exponentially stable*, if for some $N, \gamma > 0$ for the Cauchy function the following estimate holds,

$$|Y(t,s)| \le Ne^{-\gamma(t-s)} \quad \text{for almost all } (t,s) \in \Delta. \tag{7}$$

Below we will see that the estimate (7) implies an analogous estimate for the fundamental solution $X(t,s)$, and therefore, by virtue of formula (6), an exponential estimate for every solution of equation (4).

3. The fundamental solution. In this section we describe some important properties of the fundamental solution X. From (6) it follows that for every fixed s the function $X(t,s)$ is locally absolutely continuous with respect to t, and can be defined as the solution to the corresponding to (5) homogeneous equation with the given initial condition, that is

$$\begin{aligned} &\frac{\partial X(t,s)}{\partial t} - \sum_{k=0}^{K} a_k(t) \frac{\partial X(t-kh,s)}{\partial t} = \sum_{j=0}^{J} b_j(t) X(t-jh,s), \quad t \geqslant s, \\ &X(s,s) = 1. \end{aligned} \tag{8}$$

The first property that we describe below shows that if $t - s$ increases, then the function increases not faster than the exponent. We need an auxiliary proposition.

LEMMA 1. *Let the function* $b = b(\tau)$ *is nonnegative and integrable on the segment* $[a,t]$. *Then* $\lim_{\alpha\to+\infty} \int_a^t b(\tau)e^{-\alpha(t-\tau)}\,d\tau = 0$.

Proof. Denote $E_n = \{\tau \in [a,t] \colon b(\tau) \geqslant n\}$, $n \in \mathbb{N}$. Obviously, the set E_n is Lebesgue measurable and

$$\mu E_n \leqslant \frac{1}{n} \int_{E_n} b(\tau)\,d\tau \leqslant \frac{1}{n} \int_a^t b(\tau)\,d\tau \xrightarrow[n\to\infty]{} 0.$$

Let $\varepsilon > 0$. Without loss of generality assume that $\alpha > 0$. Since Lebesgue integral is absolutely continuous, there exists N such that

$$\int_{E_N} b(\tau)e^{-\alpha(t-\tau)}\,d\tau \leqslant \int_{E_N} b(\tau)d\tau < \varepsilon/2.$$

On the other hand,

$$\int_{[a,t]\setminus E_N} b(\tau)e^{-\alpha(t-\tau)}\,d\tau \leqslant N\int_a^t e^{-\alpha(t-\tau)}\,d\tau = \frac{N}{\alpha}\left(1-e^{-\alpha(t-a)}\right) < \frac{N}{\alpha}.$$

Therewere for $\alpha > 2N/\varepsilon$ we have $\int_a^t b(\tau)e^{-\alpha(t-\tau)}\,d\tau < \varepsilon$. ∎

Let $\alpha \geqslant 0$, and $L_\infty^\alpha[s,s+l]$ be a space of measurable and essentially bounded on $[s,s+l]$ functions with the weighted norm $\|y\|_{L_\infty^\alpha} = \operatorname{vrai}\sup\limits_{t\in[s,s+l]} e^{-\alpha(t-s)}|y(t)|$. It is not difficult to verify that $L_\infty^\alpha[s,s+l]$ is a Banach space. We denote the norm of linear bounded operator that is acting in $L_\infty^\alpha[s,s+l]$ by the symbol $\|\cdot\|_\alpha$.

Note that the operators S_h and S maps the space $L_\infty^\alpha[s,s+l]$ into itself. Since

$$\|S_h y\|_{L_\infty^\alpha} = \operatorname{vrai}\sup_{t\in[s+h,s+l]} e^{-\alpha(t-s)}|y(t-h)| = \\ = \operatorname{vrai}\sup_{t\in[s,s+l-h]} e^{-\alpha(t-s+h)}|y(t)| \leqslant e^{-\alpha h}\|y\|_{L_\infty^\alpha},$$

it follows that $\|S_h\|_\alpha \leqslant e^{-\alpha h}$, therefore $\|S_h^n\|_\alpha \leqslant e^{-\alpha h n}$ for all $n\in\mathbb{N}_0$. Put $A=\sum_{k=1}^K A_k$, where $A_k = \operatorname{vrai}\sup|a_k(t)|$. Then

$$\|S\|_\alpha = \left\|\sum_{k=1}^K a_k(t)S_h^k\right\|_\alpha \leqslant \sum_{k=1}^K A_k\left\|S_h^k\right\|_\alpha \leqslant \sum_{k=1}^K A_k e^{-\alpha hk} \leqslant Ae^{-\alpha h},$$

hence $\|S^n\|_\alpha \leqslant A^n e^{-\alpha hn}$ for all $n\in\mathbb{N}_0$.

Theorem 3. *There exists $\alpha, N_1, N_2 > 0$ such that for all $(t,s)\in\Delta$ the estimates $|X(t,s)| \leqslant N_1e^{\alpha(t-s)}$ and $\int_s^t|\frac{d}{d\tau}X(\tau,s)|d\tau \leqslant N_2e^{\alpha(t-s)}$ are valid.*

Proof. Considering that the operators S and T are applied only to the first argument, the second one being fixed, rewrite (8) in the form $(I-S)\dot{X} = TX$.

Find $n\in\mathbb{N}_0$ such that $nh \leqslant t-s < (n+1)h$. Then $S^{n+1}=\Theta$, and $(I-S)^{-1} = I+S+\ldots+S^n$. Hence $\dot{X}$ may be represented in the form $\dot{X} = (I-S)^{-1}TX = (I+S+\ldots+S^n)TX$.

Integrating the last equality, we obtain

$$X(t,s) = 1 + (KX)(t,s), \tag{9}$$

where the operator K is defined by the equality

$$(Ky)(t,s) = \int_s^t ((I-S)^{-1}Ty)(\tau,s)\,d\tau.$$

Let y be an integrable in the first argument function. Make some estimates:

$$
\begin{aligned}
&\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |(S_h y)(\tau,s)|\,d\tau \\
&\qquad = \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_{s+h}^t |y(\tau-h,s)|\,d\tau \\
&\qquad = \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^{t-h} |y(\xi,s)|\,d\xi \\
&\qquad = e^{-\alpha h}\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-h-s)}\int_s^{t-h} |y(\xi,s)|\,d\xi \\
&\qquad \leqslant e^{-\alpha h}\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^{t} |y(\xi,s)|\,d\xi,
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |(Sy)(\tau,s)|\,d\tau \\
&\qquad \leqslant \operatorname{vrai}\sup_{t\in[s,s+l]} \sum_{k=1}^K e^{-\alpha(t-s)}\int_s^t |a_k(\tau)||(S_h^k y)(\tau,s)|\,d\tau \\
&\qquad \leqslant \operatorname{vrai}\sup_{t\in[s,s+l]} \sum_{k=1}^K A_k e^{-\alpha(t-s)}\int_s^t \left|(S_h^k y)(\tau,s)\right|\,d\tau \\
&\qquad \leqslant \sum_{k=1}^K A_k e^{-\alpha hk}\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |y(\xi,s)|\,d\xi \\
&\qquad \leqslant A e^{-\alpha h}\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |y(\xi,s)|\,d\xi,
\end{aligned}
$$

$$
\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |(S^n y)(\tau,s)|\,d\tau \leqslant (Ae^{-\alpha h})^n \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |y(\xi,s)|\,d\xi.
$$

Let the parameter α be subjected to the condition $\left|Ae^{-\alpha h}\right| < 1$. Then

$$
1 + Ae^{-\alpha h} + \ldots + A^n e^{-\alpha hn} \leqslant (1 - Ae^{-\alpha h})^{-1}.
$$

Therefore,

$$
\begin{aligned}
&\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t \left|((I-S)^{-1}y)(\tau,s)\right|\,d\tau \leqslant \\
&\qquad \leqslant \frac{1}{1-Ae^{-\alpha h}}\operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)}\int_s^t |y(\xi,s)|\,d\xi.
\end{aligned}
$$

It is obvious that the operator K is acting in the space $L_\infty^\alpha[s, s+l]$ and bounded. Let us estimate its norm. By the definition of the norm in the space $L_\infty^\alpha[s, s+l]$ we have:

$$\begin{aligned}
\|Ky\|_{L_\infty^\alpha[s,s+l]} &= \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)} \left|\int_s^t ((I-S)^{-1}Ty)(\tau,s)\,d\tau\right| \\
&\leqslant \frac{1}{1-Ae^{-\alpha h}} \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)} \int_s^t |(Ty)(\tau,s)|\,d\tau \\
&\leqslant \frac{1}{1-Ae^{-\alpha h}} \operatorname{vrai}\sup_{t\in[s,s+l]} e^{-\alpha(t-s)} \int_s^t \sum_{j=0}^J |b_j(\tau)||y(\tau-jh,s)|\,d\tau \\
&\leqslant \frac{1}{1-Ae^{-\alpha h}} \sum_{j=0}^J e^{-\alpha jh} \operatorname{vrai}\sup_{t\in[s,s+l]} \int_s^t |b_j(\tau)|e^{-\alpha(t-\tau)}|y(\tau-jh,s)|e^{-\alpha(\tau-s-jh)}\,d\tau \\
&\leqslant \frac{1}{1-Ae^{-\alpha h}} \sup_{t\in[s,s+l]} \left(\sum_{j=0}^J e^{-\alpha hj} \int_s^t |b_j(\tau)|\,e^{-\alpha(t-\tau)}\,d\tau\right) \|y\|_{L_\infty^\alpha[s,s+l]}\,.
\end{aligned}$$

For the norm of the operator K we obtain the estimate

$$\|K\|_\alpha \leqslant \frac{1}{1-Ae^{-\alpha h}} \left(\sum_{j=0}^J e^{-\alpha hj} \max_{t\in[s,s+l]} \int_s^t |b_j(\tau)|e^{-\alpha(t-\tau)}\,d\tau\right).$$

Applying Lemma 1, we conclude that $\|K\|_\alpha \underset{\alpha\to+\infty}{\to} 0$, hence for some $\alpha > 0$ the inequality $\|K\|_\alpha < 1$ holds. By virtue of cotraction mapping principle [8, p. 230], equation (9) has a unique solution $X \in L_\infty^\alpha[s, s+l]$ in the space $L_\infty^\alpha[s, s+l]$, that is $|X(t,s)|e^{-\alpha(t-s)} \leqslant N_1$. The estimate is valid for all $(t,s) \in \Delta$, since from (9) it follows that X is a continuous function. The first assertion of the theorem is proved.

To prove the second assertion, we use once again the inequalities obtained above while estimating $\|Ky\|_{L_\infty^\alpha[s,s+l]}$, and an estimate of the fundamental solution:

$$\begin{aligned}
&\sup_{t\in[s,s+l]} e^{-\alpha(t-s)} \int_s^t \left|\frac{d}{d\tau}X(\tau,s)\right| d\tau \\
&\qquad = \sup_{t\in[s,s+l]} e^{-\alpha(t-s)} \int_s^t |((I-S)^{-1}TX)(\tau,s)|\,d\tau \\
&\qquad \leqslant \left(\frac{1}{1-Ae^{-\alpha h}} \sum_{j=0}^J e^{-\alpha hj} \max_{t\in[s,s+l]} \int_s^t |b_j(\tau)|e^{-\alpha(t-\tau)}\,d\tau\right) \\
&\qquad\qquad \times \|X\|_{L_\infty^\alpha[s,s+l]} = N_2. \quad \square
\end{aligned}$$

Denote

$$B(r) = \{z \in \mathbb{C}\colon\ |z| < r\}, \quad P_a(\zeta, z) = \sum_{k=1}^{K} a_k(\zeta) z^k,\ \zeta \in [0, h],$$

$$P_b(\zeta, z) = \sum_{j=0}^{J} b_j(\zeta) z^j,$$

$$G(\zeta + s, z) = \exp\left\{\int_s^{\zeta+s} \frac{P_b(\tau, z)}{1 - P_a(\tau, z)}\, d\tau\right\},$$

and consider the function

$$F(\zeta + s, z) = \frac{G(\zeta + s, z)}{1 - zG(h, z)}, \quad z \in \mathbb{C}, \quad \zeta \in [0, h]. \tag{10}$$

Lemma 2. *There exists a nonempty neighbourhood of zero $B(r) \subseteq \mathbb{C}$ such that for all $z \in B(r)$ the boundary value problem*

$$(1 - P_a(\zeta + s, z))\frac{\partial F(\zeta + s, z)}{\partial \zeta} = \left(\sum_{j=0}^{J} b_j(\zeta + s) z^j\right) F(\zeta + s, z), \tag{11}$$

$$F(s, z) = zF(h + s, z) + 1 \tag{12}$$

has a unique solution, which is defined by formula (10)*.*

Proof. Since $\sum_{k=1}^{K} |a_k(\zeta)| \leqslant A$, one may choose a disk $B(r_1) \subseteq \mathbb{C}$, in which $1 - P_a(\zeta + s, z) \neq 0$. Such a disk exists, since $P_a(\zeta + s, z)$ is a polynomial in z, where $P_a(\zeta + s, 0) = 0 \neq 1$. For all $z \in B(r_1)$ the general solution of the equation (11) has the form

$$F(\zeta + s, z) = C \cdot G(\zeta + s, z), \tag{13}$$

where C is an arbitrary complex number (which is own for each fixed z). Let us show that C may be chosen so that (12) holds. Substituting (13) into (12), and taking account of the introduced above notation, we obtain:

$$C \cdot (1 - zG(h + s, z)) = 1.$$

Since a_k and b_j are periodic functions with period h,

$$G(h + s, z) = G(h, z).$$

Find a disk $B(r_2) \subseteq B(r_1)$ such that $1 - zG(h, z) \neq 0$ in it; such a disk exists by virtue of the contituity, hence the boundedness, of the function

$G(h,z)$ on every closed subset of the disk $B(r_1)$. For every $z \in B(r_2)$ the constant C is uniquely defined: $C = (1 - zG(h,z))^{-1}$. Thus, it is proved that for all $z \in B(r_2)$ the function $F(\cdot, z)$ defined by formula (10) is the unique solution of the boundary value problem (11)–(12). □

For any fixed $s \geqslant 0$ represent $t \geqslant s$ in the form $t = nh + \zeta + s$, where $n \in \mathbb{N}_0$, $\zeta \in [0,h)$, and denote $x_n(\zeta + s) = X(t,s)$. Each of the functions $x_n(\zeta + s)$ is absolutely continuous in $\zeta \in [0,h]$, and the sequence $x_n(\zeta + s)$ is uniquely defined as a solution of the following boundary value problem ($\dot{x}(\zeta + s) = \frac{\partial}{\partial \zeta} x(\zeta + s)$):

$$\begin{aligned} &\dot{x}_n(\zeta + s) - \sum_{k=0}^{K} a_k(\zeta + s)\dot{x}_{n-k}(\zeta + s) \\ &\qquad\qquad = \sum_{j=0}^{J} b_j(\zeta + s) x_{n-j}(\zeta + s), \quad n \in \mathbb{N}_0, \\ &x_n(s) = x_{n-1}(h + s), \qquad\qquad\qquad\qquad n \in \mathbb{N}, \end{aligned} \tag{14}$$

supplemented by the initial conditions $x_0(s) = 1$ and $x_n(\zeta + s) = \dot{x}_n(\zeta + s) = 0$, åñëè $n = -1, -2, \ldots$

Note that by virtue of Theorem 3 for some $N_1, N_2, R > 0$ the estimates $|x_n(\zeta + s)| \leqslant N_1 R^{-n}$ and $\int_0^h |\dot{x}_n(\zeta + s)| d\zeta \leqslant N_2 R^{-n}$ are valid.

Construct a generating function for the sequence $\{x_n(\zeta + s)\}_{n=0}^{\infty}$:

$$F_X(\zeta + s, z) = \sum_{n=0}^{\infty} x_n(\zeta + s) z^n.$$

The above estimate guarantees the convergence of this series in some nonempty neighbourhood of zero for all $\zeta + s \in [0,h)$. We should justify the possibility of the termwise differentiation of this series. To this end, prove another auxiliary statement.

Lemma 3. *Let $\alpha_n(t)$ be absolutely continuous on $[a,b]$ functions such that the functional series $\sum_{n=0}^{\infty} \alpha_n(t)$ converges at every point of the segment $[a,b]$, and the series $\sum_{n=0}^{\infty} \int_a^b |\dot{\alpha}_n(t)|\, dt$ also converges. Then the sum of the series $\sum_{n=0}^{\infty} \alpha_n(t)$ is an absolutely continuous on $[a,b]$ function, the series $\sum_{n=0}^{\infty} \dot{\alpha}_n(t)$ absolutely converges almost everywhere on $[a,b]$, and*

$$\frac{d}{dt} \sum_{n=0}^{\infty} \alpha_n(t) = \sum_{n=0}^{\infty} \dot{\alpha}_n(t).$$

Proof. By the Levi theorem [8, p. 305] we have

$$\int_a^b \sum_{n=0}^{\infty} |\dot\alpha_n(t)|\, dt = \sum_{n=0}^{\infty} \int_a^b |\dot\alpha_n(t)|\, dt < \infty,$$

hence the series $\sum_{n=0}^{\infty} \dot\alpha_n(t)$ absolutely converges almost everywhere on $[a,b]$, and its sum is in $L_1[a,b]$. Denote $\sigma(t) = \sum_{n=0}^{\infty} \alpha_n(t)$. Then

$$\sigma(t) = \lim_{m\to\infty} \sum_{n=0}^{m} \alpha_k(t) = \lim_{m\to\infty} \sum_{n=0}^{m} \left(\alpha_n(a) + \int_a^t \dot\alpha_n(s) ds \right) =$$
$$= \lim_{m\to\infty} \sum_{n=0}^{m} \alpha_k(a) + \lim_{m\to\infty} \sum_{n=0}^{m} \int_a^t \dot\alpha_n(s) ds = \sigma(a) + \lim_{m\to\infty} \int_a^t \sum_{n=0}^{m} \dot\alpha_n(s) ds.$$

It is easily seen that $\left| \sum\limits_{n=0}^{m} \dot\alpha_n(s) \right| \leqslant \sum\limits_{n=0}^{m} |\dot\alpha_n(s)| \leqslant \sum\limits_{n=0}^{\infty} |\dot\alpha_n(s)|$, hence, by the Lebesgue theorem [8, p. 302], one may pass to the limit under the integral sign,

$$\lim_{m\to\infty} \int_a^t \sum_{n=0}^{m} \dot\alpha_n(s)\, ds = \int_a^t \lim_{m\to\infty} \sum_{n=0}^{m} \dot\alpha_n(s)\, ds = \int_a^t \sum_{n=0}^{\infty} \dot\alpha_n(s)\, ds.$$

Returning to the representation of the function σ, we finally have

$$\sigma(t) = \sigma(a) + \lim_{m\to\infty} \int_a^t \sum_{n=0}^{m} \dot\alpha_n(s)\, ds = \sigma(a) + \int_a^t \sum_{n=0}^{\infty} \dot\alpha_n(s)\, ds.$$

As was shown above, $\sum_{n=0}^{\infty} \dot\alpha_n(t) \in L_1[a,b]$, hence σ is absolutely continuous, and

$$\frac{d\sigma(t)}{dt} = \sum_{n=0}^{\infty} \dot\alpha_n(t). \quad \square$$

COROLLARY 1. *The function $F_X(\zeta + s, z)$ is absolutely continuous in $\zeta \in [0,h]$, and for all $z \in B(R)$ the equality*

$$\frac{\partial}{\partial \zeta} F_X(\zeta + s, z) = \sum_{n=0}^{\infty} \dot x_n(\zeta + s) z^n$$

is valid.

Proof. The pointwise convergence of the series $\sum_{n=0}^{\infty} x_n(\zeta+s)z^n$ is noted above. From the estimate $\int_0^h |\dot{x}_n(\zeta+s)| d\zeta \leqslant N_2 R^{-n}$ it follows that for $z \in B(R)$ the series

$$\sum_{n=0}^{\infty} |z|^n \int_0^h |\dot{x}_n(\zeta+s)|\, d\zeta$$

converges. The statement of the corollary follows now from Lemma 3. □

THEOREM 4. *The generating function of the sequence* $\{x_n(\zeta+s)\}_{n=0}^{\infty}$ *is determined by the equality* (10), *that is* $F_X(\zeta+s,z) = F(\zeta+s,z)$.

Proof. Multiplying both the sides of the problem (14) by z^n, and summing in n, we obtain

$$\sum_{n=0}^{\infty} \dot{x}_n(\zeta+s)z^n - P_a(\zeta+s,z)\sum_{n=0}^{\infty} \dot{x}_n(\zeta+s)z^n = P_b(\zeta+s,z)\sum_{n=0}^{\infty} x_n(\zeta+s)z^n,$$

$$\sum_{n=0}^{\infty} x_n(s)z^n = z\sum_{n=0}^{\infty} x_n(h+s)z^n + 1.$$

Taking account of the introduced above notation and properties of the function F_X, one can give to the last two equalities the form

$$(1-P_a(\zeta+s,z))\frac{\partial F_X(\zeta+s,z)}{\partial \zeta} = P_b(\zeta,z)F_X(\zeta+s,z),$$
$$F_X(s,z) = zF_X(h+s,z)+1.$$

Therefore, for all z from some neighbourhood of zero the function $F_X(\cdot,z)$ is a solution of problem (11)–(12). Without loss of generality we may suppose that the solution of problem (11)–(12) is unique in this neighbourhood, hence it coincides with the function F. □

4. The Cauchy function. Consider the equation

$$(15)\quad Y(t,s) = 1 + \sum_{k=1}^{K} a_k(s)Y(t,s+kh) + \sum_{j=0}^{J}\int_{s+jh}^{t} b_j(\tau)Y(t,\tau)\,d\tau, \quad t \geqslant s.$$

As is shown in [1, p. 61], equation (15) may be accepted as a definition of the Cauchy function of equation (5).

LEMMA 4. *For every* $(t,s) \in \Delta$ *the equality* $Y(t+h,s+h) = Y(t,s)$ *is valid.*

Proof. Substituting the function $Y(t+h, s+h)$ into (15) and using h-periodicity of the functions a_k and b_j, we verify that this function turns (15) into identity. Since (15) has a unique solution, the lemma is proved. ∎

REMARK 1. *From Lemma 4 it follows that the behavior of the function Y is determined by the difference between arguments $t-s$, if s takes all values in the segment $[0,h]$. Therefore, in what follows, without loss of generality, we suppose that $s \in [0,h]$.*

The next statement establish a simple relation between the Cauchy function and fundamental solution of equation (5).

THEOREM 5. *For almost all $(t,s) \in \Delta$ the Cauchy function and fundamental solution are related by the formula*

$$Y(t,s) - \sum_{k=1}^{K} a_k(s) Y(t, s+kh) = X(t,s). \tag{16}$$

Proof. Suppose $hn \leqslant t-s < (n+1)h$, $n \in \mathbb{N}_0$. Then $S^{n+1} = \Theta$, the operator $I-S$ is invertible, and $(I-S)^{-1} = I + S + S^2 + \ldots + S^n$. On the segment $[s, s+(n+1)h]$ equation (5) may be rewriten in the equivalent form

$$\dot{x} = (I-S)^{-1} Tx + (I-S)^{-1} f. \tag{17}$$

One can easily see that (17) has the same fundamental solution $X(t,s)$ as equation (5) has. Since (17) is a delay equation, its solution for $x(s) = 0$ has the form

$$x(t) = \int_s^t X(t,\tau)\left(\left(I-S\right)^{-1} f\right)(\tau)\, d\tau = \sum_{i=0}^{n} \int_s^t X(t,\tau)(S^i f)(\tau)\, d\tau. \tag{18}$$

Denote

$$(S_h^* y)(t,s) = \begin{cases} y(t, s+h), & t-h \geqslant s, \\ 0, & t-h < s, \end{cases} \quad (S^* y)(t,s) = \sum_{k=1}^{K} a_k(s)(S_h^{*k}) y(t,s),$$

and note that $(S^*)^{n+1} = \Theta$, à $(I-S^*)^{-1} = I + S^* + S^{*2} + \ldots + S^{*n}$. Further, for every k we have

$$\begin{aligned} \int_s^t X(t,\tau) a_k(\tau)(S_h f)(\tau)\, d\tau &= \int_{s+h}^t X(t,\tau) a_k(\tau) f(\tau - h)\, d\tau \\ = \int_s^{t-h} X(t,\tau+h) a_k(\tau+h) f(\tau)\, d\tau &= \int_s^{t-h} X(t,\tau+h) a_k(\tau) f(\tau)\, d\tau \\ &= \int_s^t (S_h^* X)(t,\tau) f(\tau)\, d\tau, \end{aligned}$$

$$\int_s^t X(t,\tau)a_k(\tau)(S_h^k f)(\tau)\,d\tau = \int_s^t a_k(\tau)(S_h^{*k}X)(t,\tau)f(\tau)\,d\tau.$$

Therefore,

$$\int_s^t X(t,\tau)(Sf)(\tau)\,d\tau = \int_{s+h}^t X(t,\tau)\sum_{k=1}^K a_k(\tau)(S_h^k f)(\tau)\,d\tau =$$

$$= \int_s^{t-h}\sum_{k=1}^K a_k(\tau)(S_h^{*k}X)(t,\tau)f(\tau)\,d\tau = \int_s^t (S^*X)(t,\tau)f(\tau)\,d\tau,$$

$$\int_s^t X(t,\tau)(S^i f)(\tau)\,d\tau = \int_s^t (S^{*i}X)(t,\tau)f(\tau)\,d\tau,$$

$$\sum_{i=0}^n \int_s^t X(t,\tau)(S^i f)(\tau)\,d\tau = \int_s^t \sum_{i=0}^n (S^{*i}X)(t,\tau)f(\tau)d\tau$$

$$= \int_s^t ((I-S^*)^{-1}X)(t,\tau)f(\tau)\,d\tau.$$

Based on the representations (6) and (18), we conclude that for any function f from $L_1[s, s+(n+1)h]$

$$\int_s^t ((I-S^*)^{-1}X)(t,\tau)f(\tau)\,d\tau = \int_s^t Y(t,\tau)f(\tau)\,d\tau,$$

hence $(I-S^*)^{-1}X = Y$ or $X(t,s) = Y(t,s) - (S^*Y)(t,s)$. By virtue of the definition of the operator S^*, we obtain the representation (16). ∎

Combining Lemma 4 and Theorem 5, we obtain

Corollary 2. *For all $(t,s)\in\Delta$ the equality $X(t+h,s+h) = X(t,s)$ holds.*

From Theorem 5 and Lemma 4 another formula follows, which relates the Cauchy function with fundamental solution. It was applied in the investigation of stability for autonomous equations ([2], [4]) and for equations of the form (3) [9].

Corollary 3. *The Cauchy function with fundamental solution of equation (5) are related by the formula*

$$Y(t,s) - \sum_{k=1}^K a_k(s)Y(t-kh,s) = X(t,s),\ (t,s)\in\Delta. \tag{19}$$

Note another important property of the Cauchy funcion.

Corollary 4. *There exists $\alpha > 0$, $M > 0$ such that for almost all $(t, s) \in \Delta$ the estimate $|Y(t, s)| \leqslant M e^{\alpha(t-s)}$ is valid.*

Proof. Find $n \in \mathbb{N}_0$ such that $nh \leqslant t - s < (n+1)h$. We have $(S^*)^{n+1} = \Theta$, and $(I - S^*)^{-1} = I + S^* + \ldots + S^{*n}$. Hence, using Theorem 5, the Cauchy function may be represented in the form $Y = (I - S^*)^{-1} X = (I + S^* + \ldots + S^{*n})X$.

Based on Theorem 3, it is easy for every $i = 0, \ldots, n$ to establish the estimate $|(S^{*i}X)(t, s)| \leqslant N e^{\alpha(t-s)} A^i e^{-\alpha h i}$ for almost all $(t, s) \in \Delta$, from which it follows that $|Y(t, s)| \leqslant N e^{\alpha(t-s)} \sum_{i=0}^{n} A^i e^{-\alpha h i}$. Note that in the proof of Theorem 3 the exponent α was chosen so that the inequality $|A e^{-\alpha h}| < 1$ holds. Thus, $|Y(t, s)| \leqslant \frac{N e^{\alpha(t-s)}}{1 - A e^{-\alpha h}} = M e^{\alpha(t-s)}$ for almost all $(t, s) \in \Delta$, which was to be proved. ∎

Note that if a_k are continuous, then the estimate (19) is true for all $(t, s) \in \Delta$.

For any fixed s represent $t \geqslant s$ in the form $t = nh + \zeta + s$, where $n \in \mathbb{N}_0$ and $\zeta \in [0, h)$, denote $y_n(\zeta + s) = Y(t, s)$, and construct a generating function for the sequence $\{y_n(\zeta + s)\}_{n=0}^{\infty}$:

$$F_Y(\zeta + s, z) = \sum_{n=0}^{\infty} y_n(\zeta + s) z^n. \tag{20}$$

Corollary 4 guarantees the convergence of this series in some neighbourhood of zero for all $\zeta + s \in [0, h)$. The equality (19) may be rewritten in terms of the sequence $\{x_n(\zeta + s)\}_{n=0}^{\infty}$, $\{y_n(\zeta + s)\}_{n=0}^{\infty}$ in the form

$$y_n(\zeta + s) - \sum_{k=0}^{K} a_k(s) y_{n-k}(\zeta + s) = x_n(\zeta + s).$$

Multiplying both the sides of this equality by z^n, and summing it in n, we obtain

$$\sum_{n=0}^{\infty} y_n(\zeta + s) z^n - P_a(s, z) \sum_{n=0}^{\infty} y_n(\zeta + s) z^n = \sum_{n=0}^{\infty} x_n(\zeta + s) z^n.$$

In view of the introduced above notation this means that

$$(1 - P_a(s, z)) F_Y(\zeta + s, z) = F_X(\zeta + s, z).$$

Since for the function F_X the explicit form is found (formula (10)), it is finally established

THEOREM 6. *The generating function of the sequence* $\{y_n(\zeta+s)\}_{n=0}^{\infty}$ *has the form*

$$F_Y(\zeta+s,z) = \frac{1}{1-P_a(s,z)} \cdot \frac{G(\zeta+s,z)}{1-zG(h,z)}. \tag{21}$$

Denote by $z_0(s)$ a point which is the nearest to zero, and where the analyticity of the function $F_Y(\zeta+s,\cdot)$ is broken (from (21) it follows that z_0 does not depend on ζ, but depends on s). By virtue of the periodicity of coefficients a_k it may be supposed that $s \in [0,h]$. It is obvious that $|z_0(s)| > 0$, hence the function $F_Y(\zeta+s,\cdot)$ is analytic in the disk $B(|z_0(s)|)$ for all $\zeta \in [0,h]$.

THEOREM 7. *For all elements of the sequence* $\{y_n(\zeta+s)\}_{n=0}^{\infty}$ *the estimate* $|y_n(\zeta+s)| \leqslant Ne^{-\gamma n}$ *with positive and independent of* ζ *and* s *constants* N,γ *is valid, if and only if for some* $R>1$ *the inequality* $|z_0(s)| \geqslant R$ *holds for almost all* $s \in [0,h]$.

Proof. Necessity. Suppose $|y_n(\zeta+s)| \leqslant Ne^{-\gamma n}$. Then $|y_n(\zeta+s)z^n| \leqslant Ne^{-\gamma n}|z|^n$, hence the series (20) converges in the disk $B(e^{\gamma})$, where $\gamma > 0$. Therefore $|z_0(s)| > e^{\gamma} > 1$ for almost all s.

Sufficienty. Suppose $|z_0(s)| \geqslant R > 1$. By virtue of the Cauchy inequality for coefficients of a power series we obtain the estimate $|y_n(\zeta+s)| \leqslant NR^{-n}$, where $N = \max_{|z|=R;\ \zeta,s\in[0,h]} |F_Y(\zeta+s,z)|$. Putting $\gamma = \ln R$, we arrive at the desired inequality. ∎

THEOREM 8. *The Cauchy funcion of equation* (5) *has exponential estimate* (7)*, if and only if the following conditions hold:*

(a) there exists $r>1$ *such that for almost all* $s \in [0,h]$ *all roots of the equations* $P_a(s,z)=1$ *are outside the disk* $B(r)$*;*

(b) all roots of the equation $zG(h,z)=1$ *are outside the disk* $|z| \leqslant 1$.

Proof. From formula (21) it follows that the analyticity domain of the function F_Y is determined by the zeros of its denominator. If condition (a) holds, then the function G is analytic in the disk $B(r)$, therefore the equation $zG(h,z)=1$ can only have a finite number of zeros in this disk. Let z^* be the least of them in absolute value. It is easy to see that for the number $R = \min\{r, |z^*|\}$ the conditions of Theorem 7 hold. To conclude the proof it remains to remind the definition of the functions y_n. ∎

COROLLARY 5. *If the Cauchy function of equation* (5) *has the estimate* (7)*, then an analogous estimate holds for the fundamental solution, that is* $|X(t,s)| \leqslant Me^{-\gamma(t-s)}$, $(t,s) \in \Delta$.

Proof. The corollary follows from the relation (16). ∎

Note that the converse is not true: an exponential estimate for the fundamental solution *does not imply* a similar estimate for the Cauchy function. A corresponding example is given in paper [4].

From Theorem 8 it follows that the absence of roots of the equation $1 = P_a(s, z)$ in the disk $\{z \in \mathbb{C} : |z| \leqslant 1\}$ for almost all s is a necessary condition for the existence of the estimate (7). It is shown in [3] that one may give a number of equivalent reformulations of this condition, in terms of the invertibility of the operator $I - S$ and exponenetial stability criteria for difference equations.

REMARK 2. *All relations for the Cauchy function that must be satisfied pointwise are valid for for almost all $(t, s) \in \Delta$. This is due to the fact that a_k are measurable and (essentially) bounded functions. But if a_k are continuous, then all the relations hold for all $(t, s) \in \Delta$.*

5. Examples. Of all classes of functional differential equations, autonomous equations are best studied. In particular, for the equation of the form

$$\dot{x}(t) - a\dot{x}(t-1) = -bx(t) + cx(t-1), \quad t \in \mathbb{R}_+, \tag{22}$$

where $a, b, c \in \mathbb{R}$, in paper [5] a criterion of exponential stability is found in the form of a domain in the three-dimensional space of parameters. Let us represent it here. In the space $Ouvw$, define a surface

$$\Gamma = \{(u, v, w) : u = \cos\theta + \sin\theta/\theta,\ w = -\theta\sin\theta + v\cos\theta,\ \theta \in \mathbb{R}\}.$$

In the surface Γ limit the range of the parameter θ: set $\theta \in (\theta_1, \pi)$ for $v \geqslant 0$, where θ_1 is the least positive root of the equation $\cos y + \frac{vsiny}{y} = -1$, and $\theta \in (0, \theta_2)$ for $v \in (-2, 0)$, where θ_2 is the least positive root of the equation $\cos y + \frac{vsiny}{y} = 1$. The surface Γ together with the planes $v = w$, $u = \pm 1$ bound a domain D (the boundaries not in D) in the space $Ouvw$, which is shown in fig. 1.

THEOREM 9. *The Cauchy function of equation* (22) *has an exponential estimate, if and only if the point with coordinates* (a, b, c) *is in* D.

Now we will show, how one can obtain new stability conditions for equations with periodic coefficients on the basis of Theorems 8 and 9.

EXAMPLE 1. *Consider the equation*

$$\dot{x}(t) - a\dot{x}(t-1) = -b(t)x(t) + c(t)x(t-1), \quad t \in \mathbb{R}_+, \tag{23}$$

where $a \in \mathbb{R}$, b and c are integrable over $[0, 1]$ *functions, with* $b(t+1) = b(t)$, $c(t+1) = c(t)$.

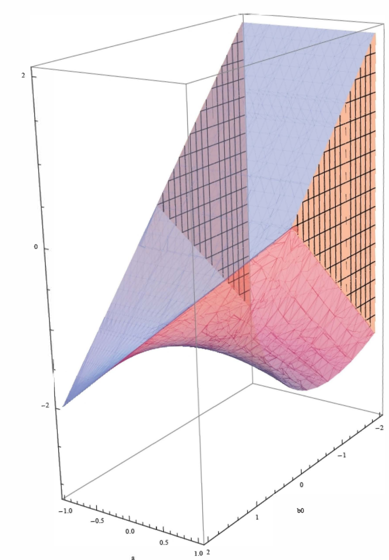

FIG. 1. *The domain D*

PROPOSITION 1. *Equation* (23) *is exponentially stable, if and only if the point with coordinates* $\left(a, \int_0^1 b(t)\,dt, \int_0^1 c(t)\,dt\right)$ *is in* D.

Proof. Put the following equation

$$\dot{x}(t) - a\dot{x}(t-1) = -\left(\int_0^1 b(t)\,dt\right) x(t) + \left(\int_0^1 c(t)\,dt\right) x(t-1), \quad t \in \mathbb{R}_+, \tag{24}$$

in correspondence to equation (23). Since

$$\int_0^1 \frac{-b(t) + c(t)z}{1 - az} dt = \frac{-\int_0^1 b(t)dt + z\int_0^1 c(t)dt}{1 - az},$$

it follows that the functions $G(1, z)$ are coincide for equations (23) and (24), and the condition $|a| < 1$ is common for them. Therefore, by virtue of Theorem 8 equation (24) is exponentially stable, if and only if the autonomous equation (23) is exponentially stable. Applying Theorem 9 we obtain the desired criterion. ∎

EXAMPLE 2. *Consider the equation*

$$\dot{x}(t) - a(t)\dot{x}(t-1) = -b(t)x(t), \quad t \in \mathbb{R}_+. \tag{25}$$

Let the coefficient a be defined by the equality: $a(t) = 0$, if $t \in e$, and $a(t) = \alpha$, if $t \notin e$, where $\alpha \in \mathbb{R}$, e is an arbitrary measurable subset of the segment $[0, 1]$, and $a(t + 1) = a(t)$; further, let b be an arbitrary integrable over $[0, 1]$ function, and $b(t + 1) = b(t)$.

PROPOSITION 2. *Equation (25) is exponentially stable, if and only if the point with coordinates $\left(\alpha, \int_0^1 b(t)\,dt, \alpha \int_e b(t)\,dt\right)$ is in D.*

Proof. Put the following equation,

$$\dot{x}(t) - \alpha\dot{x}(t-1) = -\left(\int_0^1 b(t)\,dt\right) x(t) + \alpha\left(\int_e b(t)\,dt\right) x(t-1), \quad t \in \mathbb{R}_+, \tag{26}$$

in correspondence to equation (25). Since

$$\int_0^1 \frac{b(t)}{1 - a(t)z} dt = \int_e b(t)dt + \frac{1}{1 - \alpha z} \int_{[0,1]\setminus e} b(t)\,dt = \frac{1}{1 - \alpha z}\left(\int_0^1 b(t)\,dt - \alpha z \int_e b(t)\,dt\right),$$

it follows that the functions $G(1, z)$ are coincide for equations (25) and (26), and the condition $|\alpha| < 1$ is common for them. Therefore, by virtue of Theorem 8, equation (25) is exponentially stable, if and only if the autonomous equation (26) is exponentially stable. Since (26) is a special case of (22), Theorem 9 can be applied to (26). This implies the desired criterion. ∎

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