ARIEL UNIVERSITY PRESS

Functional Differential Equations
Volume 29, No. 3-4, 2022, pp. 197-203
DOI: https://doi.org/10.26351/FDE/29/3-4/6
ISSN: 0793-1786 (print), 2617-8605 (online)

ON EXACT CONSTRUCTION OF THE CAUCHYY FUNCTION FOR A DIFFERENTIAL EQUATION WITH LINEAR DELAY

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Abstract. Within the framework of a constructive approach to the study of functional differential equations, a way for the exact construction of the Cauchy function for differential equation with linear delay is proposed. For purposes of illustration, an example of the exact solution of one control problem is given.

Key Words. differential equation with delay; Cauchy function; constructive approach; reliable computing; control problem.

AMS(MOS) subject classification. 34K06, 34H05, 65L10

1. Introduction.

Our study is based on the main results of the theory of functional differential equations [1] and is a development of earlier studies in this area (see, for instance, [2, 3]). We will consider the differential equation with linear delay

$$\begin{aligned} &\dot{x}(t) + p(t)x[h(t)] = f(t), \quad t \in [0,T], \\ &x(\xi) = 0,\, \xi < 0, \end{aligned} \tag{1}$$

where p, f are polynomials with rational coefficients, h is a linear increasing function with rational coefficients, $h(t) < t$. As is known, for $x(0) = 0$ the general solution x of (1) has the presentation

$$x(t) = \int_0^t C(t,s)\, f(s)\, ds, \quad t \in [0,T], \tag{2}$$

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where $C(\cdot,\cdot)$ is the desired Cauchy function. In Section 1 one way of the exact construction of this function is proposed. An example of construction the Cauchy function is given in Section 2. In Section 3 an example of solving one control problem based on the proposed way for constructing the Cauchy function is described.

2. Method of construction. Define the points t_q, $q = 0, \dots, m+1$, by the equalities

$$t_0 = 0,\ t_q = h^{-1}(t_{q-1}), q = 1, \dots, m,\ t_{m+1} = T, \tag{3}$$

where constant m is such that both conditions $t_m < T$ and $h^{-1}(t_m) \geq T$ are fulfilled. Denote by $\mathfrak{I}_q$ the intervals

$$\mathfrak{I}_0 = (-\infty, 0)\,,\ \mathfrak{I}_q = [t_{q-1}, t_q)\,, q = 1, \dots, m,\ \mathfrak{I}_{m+1} = [t_m, T]\,. \tag{4}$$

By construction we have

$$x(t) \in \begin{cases} \mathfrak{I}_0, & t < 0, \\ \mathfrak{I}_{q-1}, & t \in \mathfrak{I}_q,\ q = 1, \dots, m+1. \end{cases} \tag{5}$$

Under the above assumptions the solution x of the Cauchy problem

$$\begin{array}{l} \dot{x}(t) \,+\, p(t)x[h(t)] \,=\, f(t), \quad t \in [0,T], \\ x(\xi) \,=\, 0,\, \xi < 0, \\ x(0) = 0 \end{array} \tag{6}$$

can be found in the form

$$x(t) \,=\, \sum_{q=1}^{m+1} \chi_{\mathfrak{I}_q}(t)\, x_q(t), \quad t \in [0,T], \tag{7}$$

where $\chi_{\mathfrak{I}_q}(\cdot)$ is the characteristic function of $\mathfrak{I}_q$, x_q is defined as follows:

$$
(8)\qquad
\begin{aligned}
x_1(t) &= \int_0^t f(s)\,ds, \quad t \in \mathfrak{I}_1, && q = 1,\\
x_q(t) &= x_{q-1}(t_{q-1}) + \int_{t_{q-1}}^t x_{q-1}[h(s)]\,ds +\\
&\quad + \int_{t_{q-1}}^t f(s)\,ds, \quad t \in \mathfrak{I}_q, && q = 2, \dots, m+1.
\end{aligned}
$$

We will construct the Cauchy function $C(\cdot,\cdot)$ in the form

$$
(9)\qquad C(t,s) = \sum_{q=1}^{m+1} \chi_{\mathfrak{I}_q}(t)\, C_q(t,s), \quad 0 \le s \le t \le T.
$$

Below the way of construction of this function based on equalities (2), (7), (8) will be described. Define the functions ${}^{\nu}\omega$ as follows:

$$
(10)\qquad {}^{1}\omega(t) = t,\ {}^{\nu}\omega(t) = {}^{\nu-1}\omega[h(t)], \nu = 2, \dots, m+1, \quad t \in [0,T],
$$

and let ${}^{\nu}\omega(t) = c_1^{\nu} t + c_0^{\nu}$.

Step 1. $t \in \mathfrak{I}_1$**.** In this case $h(t) = 0$ and we get

$$
(11)\qquad x_1(t) = \int_0^t C_1(t,s)\, f(s)\,ds, \quad C_1(t,s) = \kappa_1^1(t,s)\, p_1^1(t,s),
$$

where $\kappa_1^1(t,s)$ is the characteristic function of the set $\{(t,s) : 0 \le s \le {}^{1}\omega(t)\}$, $p_1^1(t,s) = 1$.

Step 2. $t \in \mathfrak{I}_2$**.** Since $h(t) \in \mathfrak{I}_1$, we obtain

$$
\begin{aligned}
x_2(t) &= \int_0^{t_1} C_1(t_1,s)\, f(s)\,ds + \int_{t_1}^t f(s)\,ds +\\
&\quad + \int_{t_1}^t -p(s) \int_0^{h(s)} C_1(h(s),\tau)\, f(\tau)\,d\tau\,ds,
\end{aligned}
$$

whence the equality

$$C_2(t,s) \;=\; \delta_1(s)\, r_2^1(t) \;+\; \kappa_2^1(t,s)\, p_2^1(t,s) \;+\; \kappa_2^2(t,s)\, p_2^2(t,s), \tag{12}$$

follows, where $\kappa_2^1(t,s)$ is the characteristic function of the set $\{(t,s) : t_1 \le s \le {}^1\omega(t)\}$, $\kappa_2^2(t,s)$ is the characteristic function of the set $\{(t,s) : 0 \le s \le {}^2\omega(t)\}$, $\delta_1(s)$ is the characteristic function of the set $\{s : 0 \le s \le t_1\}$,

$$p_2^2(t,s) \;=\; \int\limits_{t_1+\frac{s}{c_1^2}}^{t} -p(\tau)\, d\tau, \quad p_2^1(t,s) = 1, \quad r_2^1(t) = 1.$$

Step q. $q = 3, \ldots, m+1$**,** $t \in \Im_q$**,** $h(t) \in \Im_{q-1}$**.** We have

$$\begin{aligned} x_q(t) \;&=\; \int\limits_0^{t_{q-1}} C_{q-1}(t_{q-1}, s)\, f(s)\, ds \;+ \int\limits_{t_{q-1}}^{t} f(s)\, ds \,+ \\ &+ \int\limits_{t_{q-1}}^{t} -p(s) \int\limits_0^{h(s)} C_{q-1}(h(s), \tau)\, f(\tau)\, d\tau\, ds. \end{aligned}$$

After some transformations we get

$$C_q(t,s) \;= \sum_{i=1}^{q-1} \delta_i(s)\, C_i(t_i, s)\, r_q^i(t) \;+ \sum_{j=1}^{q} \kappa_q^j(t,s)\, p_q^j(t,s), \tag{13}$$

where

$$r_q^{q-1}(t) = 1, \quad r_q^{q-i}(t) \;=\; \int\limits_{t_{q-1}}^{t} -p(s)\, r_{q-1}^i[h(s)]\, ds, \;\; i = 1, \ldots, q-2,$$

$\kappa_q^j(t,s)$ is the characteristic function of the set $\{(t,s) : {}^j\omega(t_{q-1}) \le s \le {}^j\omega(t)\}$, $\delta_i(s)$ is the characteristic function of the set $\{s : 0 \le s \le t_i\}$,

$$p_q^1(t,s) \;=\; 1, \;\; p_q^j(t,s) \;=\; \int\limits_{t_{q-1}+\frac{s-{}^j\omega(t_{q-1})}{c_1^j}}^{t} -p(\tau)\, p_{q-1}^{j-1}(h(\tau), s)\, d\tau,$$

$$j = 2, \ldots, q.$$

3. Example. Computer aided implementation of the above method for constructing the Cauchy function was created using the programming language Python (the module SymPy). It was constructed the Cauchy function for the equation (1), where

$$p(t) = -\tfrac{3}{2}t^2 + 5t, \quad h(t) = \tfrac{2}{3}t - \tfrac{2}{9}, \quad t \in [0, \tfrac{65}{24}].$$

In the case $m = 3$ and the intervals $\mathfrak{I}_q$ are defined as follows:

$$\mathfrak{I}_1 = \left[0, \tfrac{1}{3}\right), \quad \mathfrak{I}_2 = \left[\tfrac{1}{3}, \tfrac{5}{6}\right), \quad \mathfrak{I}_3 = \left[\tfrac{5}{6}, \tfrac{19}{12}\right), \quad \mathfrak{I}_4 = \left[\tfrac{19}{12}, \tfrac{65}{24}\right].$$

The graph of this Cauchy function is demonstrated in Fig.1.

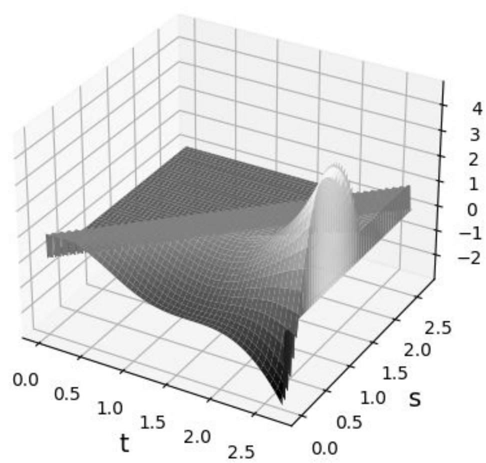

FIG. 1. *Cauchy function*

4. Control problem. As an example of the application of the constructed Cauchy function, consider the following control problem

$$(14) \qquad \begin{aligned} \dot{x}(t) &= -p(t)x[h(t)] + b(t)u(t), \quad t \in [0, T], \\ x(\xi) &= 0, \ \xi < 0, \\ x(0) &= 0, \quad x(t_q) = \alpha_q, \ q = 1, \ldots, m+1, \end{aligned}$$

where p, h are from (1), t_q are from (3), b is a polynomial with rational coefficients, α_q are rational numbers. We have

$$x(t_q) \;=\; \int\limits_0^T M_q(s)\, u(s)\, ds, \quad M_q(s) \;=\; \chi_{[0,t_q]}(s)\, C_q(t_q, s)\, b(s). \tag{15}$$

Let $\mathbf{M}(\cdot) = \operatorname{col}\{M_1(\cdot), \ldots, M_n(\cdot)\}$ and

$$\mathbf{W} \;=\; \int\limits_0^T \mathbf{M}(s)\, \mathbf{M}^\top(s)\, ds, \tag{16}$$

$(\cdot)^\top$ stands for transposition. As it is known [1], the fulfillment of the condition $\det \mathbf{W} \neq 0$ is the necessary and sufficient condition for the solvability of (14) for any $\alpha = \operatorname{col}\{\alpha_1, \ldots, \alpha_m\}$. If $\det \mathbf{W} \neq 0$, then the control $\bar{u}$ which has the minimal norm $\|\cdot\|_{\mathbb{L}_2}$ can be defined as

$$\bar{u}(s) \;=\; M^\top(s)\, d, \quad d \;=\; \mathbf{W}^{-1}\alpha. \tag{17}$$

Here $\mathbb{L}_2$ is the space of square-summable functions $u : [0, T] \to \mathbb{R}$ with the norm $\|u\|_{\mathbb{L}_2} = \sqrt{\int\limits_0^T u^2(s)\, ds}$. The problem (14) was successfully solved for the following values of parameters

$$t \in [0, \tfrac{65}{24}], \quad p(t) = -\tfrac{3}{2}t^2 \,+\, 5t, \quad h(t) = \tfrac{2}{3}t - \tfrac{2}{9}, \quad b(t) = t^2 \,-\, 1,$$

$$t_1 = \tfrac{1}{3},\ \alpha_1 = \tfrac{3}{2}, \quad t_2 = \tfrac{5}{6},\ \alpha_3 = 4, \quad t_3 = \tfrac{19}{12},\ \alpha_3 = \tfrac{5}{2}, \quad t_4 = \tfrac{65}{24},\ \alpha_4 = 5.$$

The solution found in this case turned out to be as follows:

$$\begin{array}{ll} \tilde{u}(t) \;=\; \sum\limits_{q=1}^4 \tilde{u}_q(t),\ \tilde{u}_q(t) \;=\; \sum\limits_{i=1}^q \chi_i(t)\varphi_q^i(t), & \\ & t \in [0, \tfrac{65}{24}], \\ \tilde{x}(t) \;=\; \sum\limits_{q=1}^4 \chi_q(t)\tilde{x}_q(t), & \end{array}$$

where $\chi_i(\cdot)$ is the characteristic function of the interval $[t_{i-1}, t_i]$, $i = 1, \ldots, 4$,

$$\varphi_1^1(t) = \frac{1409076710526215387572911596691365835396356432905344522886827733305}{3583650954919162728535497163902449777467179227283277104752016672 1}t^2 + \ldots,$$

$$\varphi_2^1(t) = -\frac{9137510194900729188465868360009123825336087087081059321848379738 90}{3583650954919162728535497163902449777467179227283277104752016672 1}t^5 + \ldots,$$

$$\varphi_2^2(t) = \frac{2707410428118734574360257291854555207506988766542536095362482885 60}{3583650954919162728535497163902449777467179227283277104752016672 1}t^2 + \ldots,$$

$$\varphi_3^1(t) = \frac{2250236487243718813500768760521448053629595983351533398899790815 25}{1886132081536401436071314296790763040772199593306987949869482459}t^8 + \ldots,$$

$$\varphi_3^2(t) = -\frac{3951032515325048122470485615181692055755669216309962620839138880 0}{1886132081536401436071314296790763040772199593306987949869482459}t^5 + \ldots,$$

$$\varphi_3^3(t) = \frac{1170676300837051295546810552646427275779457545573322258026411520 0}{1886132081536401436071314296790763040772199593306987949869482459}t^2 + \ldots,$$

$$\varphi_4^1(t) = \frac{1016438667340554466671819073798288715816977608336762381668264412 00}{1886132081536401436071314296790763040772199593306987949869482459}t^{11} + \ldots,$$

$$\varphi_4^2(t) = -\frac{7931970700782866741898664316182347546079752102856612397716070400}{1886132081536401436071314296790763040772199593306987949869482459}t^8 + \ldots,$$

$$\varphi_4^3(t) = \frac{1392719135391230374434882074720631667898776775261517677513932800}{1886132081536401436071314296790763040772199593306987949869482459}t^5 + \ldots,$$

$$\varphi_4^4(t) = -\frac{412657521597401592425150244361668642340378303781190422967091200}{1886132081536401436071314296790763040772199593306987949869482459}t^2 + \ldots,$$

$$\tilde{x}_1(t) = \\ = \frac{726027619528967476194156481284491939869269720240544558334474580 0}{1886132081536401436071314296790763040772199593306987949869482459}t^{14} + \ldots,$$

$$\tilde{x}_2(t) = \\ = \frac{43888208048042973984292502354641754240507671489333540257792 00}{1886132081536401436071314296790763040772199593306987949869482459}t^{17} + \ldots,$$

$$\tilde{x}_3(t) = \\ = \frac{541218611781278513143433313258442989138194222637124891443 20}{1527766986044485163217764580400518063025481670578660239394280791 79}t^{20} + \ldots,$$

$$\tilde{x}_4(t) = \\ = \frac{2467429769857243035642994477753413546863352805199764652912803840}{1775664698434231853061213501676945571491994784828364388732967984475766955 93}t^{23} + \ldots.$$

We give here incomplete expressions for functions x_q, φ_q^i due to limited space.

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