

# Individual Contest

1. Let  $f(x) = ax^2 + bx + c$  ( $a \neq 0$ ) be a quadratic function, such that the equation  $f(x) = x$  has no solutions (in real numbers). Prove that the equation  $f(f(x)) = x$  has no solutions as well.

2. Given positive numbers  $a$ ,  $b$  and  $c$  such that  $a^2 = b^2 + c^2 - bc$ . Prove that

$$(a - b)(a - c) \leq 0.$$

3. Prove the equality

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} = 1 + \frac{1}{1 + \frac{1}{1 + \dots}}$$

4. The plain hexagon  $ABCDEF$  has three pairs of parallel opposite sides:  $AB \parallel DE$ ,  $BC \parallel EF$ ,  $CD \parallel FA$ . Prove that the area of the hexagon is no more than twice the area of the triangle  $ACE$ .

5. Prove that the following inequality holds for all  $x > 0$ :

$$(1 + x)^{\sqrt{3}} > 1 + \sqrt{3}x.$$

6. A bus company runs a line between two towns (there are no intermediate stops). The company would like to know the average number of passengers on the bus. For a period of one month every passenger was given a questionnaire with a single question: "How many passengers (including yourself) are on the bus?" At the end of the month all the collected answers were averaged (the arithmetic mean was calculated). Does this calculation give the correct result? Is it possible to calculate the average occupancy of the buses based on the collected data? If yes, how? How would you arrange a survey for evaluating the average occupancy of the buses?

7. In the alphabet of the Anchurian language there are only three letters:  $A$ ,  $B$  and  $C$ . Two different words have the same meaning (express the same concept) if one can be obtained from the other one by applying the following operations, in any order and any number of times:

I) if the string  $ABA$  appears anywhere in the word, it can be replaced with  $BAB$ , and vice versa. Same with  $ACA \leftrightarrow CAC$ ,  $BC \leftrightarrow CB$ .

II) a string of two identical letters can be removed or inserted — anywhere in any word.

a) Is the number of concepts that can be expressed in the language finite or infinite? If it is finite, how many different concepts can be expressed?

b) The same question if the replacement rule  $BC \leftrightarrow CB$  is canceled and the rule  $BCB \leftrightarrow CBC$  is introduced instead.

8. Let  $V$  be a linear subspace of the space of matrices of size  $n \times n$  over the real numbers, which consists of degenerate matrices only. What is the largest possible value of the dimension of  $V$ ?