COUNTING LOCAL SYSTEMS WITH WILD MONODROMY

YUVAL Z. FLICKER

ABSTRACT. Let X_1 be a smooth projective absolutely irreducible curve over $\mathbb{F}_q$ of genus g, S_1^D, S_1^w and S_1^t disjoint reduced divisors consisting of $N_1^D \geq 2$, $N_1^w \geq 0$, $N_1^t \geq 0$ closed points of X_1, $S_1 = S_1^D \sqcup S_1^w \sqcup S_1^t$; suppressing the index 1 indicates an extension of scalars from $\mathbb{F}_q$ to an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, replacing it by m indicates an extension of scalars to $\mathbb{F}_{q^m}(\subset \mathbb{F})$. We count the number of equivalence classes of $\overline{\mathbb{Q}}_\ell$-smooth sheaves of rank $n \geq 2$ on $X - S$, fixed by the Frobenius, with principal unipotent monodromy at each $s \in S^D$, monodromy with a fixed Artin conductor $c_s > n$ at each $s \in S^w$, and conductor $1 \leq c_s \leq n$ at each $s \in S^t$. We use the Galois-Automorphic dictionary to reduce the count to that of automorphic representations on $\mathrm{GL}(n)$; the inner-forms correspondence to reduce further to a question on division algebras where the trace formula simplifies – thus the assumption that N_1^D is ≥ 2 (and even, or n odd) amounts to using a trace formula reminiscent of the "simple" one; and new pseudo-coefficients for generic irreducible admissible representations with a fixed conductor, of independent interest. If there is $s \in S_1^w$, with $c_s > n$, the elliptic regular terms in the trace formula are 0, in contrast to the tame case considered at [DF13]. En hors-d'œuvre, we also compute the number when $n = 1$ by elementary means. This suggests how to reduce the exceptional case S_1^w empty and $c_v \leq n$ for all v, to [DF13].

CONTENTS

1. INTRODUCTION

Consider a smooth projective geometrically irreducible curve X_1 of genus g over a finite field $\mathbb{F}_q$. Let $F_1 = k(X_1)$ denote its field of rational functions, $|X_1|$ its set of closed points,

---

2010 *Mathematics Subject Classification.* 11S37, 11R39, 11F70, 11F72, 22E35, 11R58, 11G20, 14H30.

Key words and phrases. $\overline{\mathbb{Q}}_\ell$-smooth sheaves, Galois representations, counting, Artin conductor, pseudo-coefficients, automorphic representations, generic representations, $\mathrm{GL}(n)$, trace formula.

Correspondence with P. Deligne greatly improved my understanding of the subject matter of this paper. I thank D. Kazhdan for useful comments. Some of this work was done at Humboldt Universität, Berlin, visiting E. Große-Klönne, and at MPIM, Bonn.

that we identify with the set of places of F_1. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$; it is the union, thus $\varinjlim_m \mathbb{F}_{q^m}$, of the degree m extensions $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ in $\mathbb{F}$. Put $F_m = F_1 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ for the field of rational functions on $X_m = X_1 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$. Then $F_1 \otimes_{\mathbb{F}_q} \mathbb{F}$ is the field $k(X)$ of rational functions on $X = X_1 \otimes_{\mathbb{F}_q} \mathbb{F}$, the base change of X_1 to $\mathbb{F}$. If F_1^{ur} is the maximal unramified extension of F_1 in a fixed algebraic closure $\overline{F}_1$ of F_1 containing $\mathbb{F}$, then $\mathrm{Gal}(F_1^{\mathrm{ur}}/F_1)$ is an extension of $\mathrm{Gal}(k(X)/k(X_1)) \simeq \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}}$ by $\pi_1(X, o) \simeq \mathrm{Gal}(F_1^{\mathrm{ur}}/k(X))$, the fundamental group based at the geometric point $o : \mathrm{Spec}\,\mathbb{F} \to X$.

Fix a finite set C of rational positive integers $c > 0$. Let S_1^D, S_1^{wt} be disjoint reduced effective divisors on X_1, thus disjoint finite subsets of $|X_1|$. Choose a surjection $\iota : S_1^{wt} \twoheadrightarrow C$ onto C, $\iota(s) = c_s \in C$. Put $S_1^c = \{s \in S_1^{wt};\ c_s = c\}$ for the fiber of ι at c, $S_1^w = \sqcup_{c>n} S_1^c$, $S_1^t = \sqcup_{1 \le c \le n} S_1^c$. Then $S_1^{wt} = S_1^w \sqcup S_1^t$. Put $S_1 = S_1^D \sqcup S_1^{wt}$. Write S_m^* for $S_1^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ and S^* for $S_1^* \otimes_{\mathbb{F}_q} \mathbb{F}$, where $*$ is c, w, t, wt, D, etc. Write $N_m^* = |S_m^*|$ for the cardinality of S_m^*, and $N^* = |S^*|$ for that of S^*. The superscripts w and t in S^w and S^t hint (only) at "wild" and "tame".

Fix a prime ℓ not dividing q. The pullback by the Frobenius endomorphism Fr of X defines a permutation Fr^* of the set of isomorphism classes of rank n irreducible $\overline{\mathbb{Q}}_\ell$-smooth sheaves on $X - S$. These are the "local systems" of the title, see [DF13, section 1.1] for elaboration. It maps to itself the subset $E_n(X - S, S^D, S^{wt}, C, \iota)$ of those classes for which the local monodromy at each $s \in S^D$ is principal unipotent (thus with a single Jordan block), such that the Artin conductor at each $s \in S^w$ is c_s, $c_s > n$, where c_s depends only on the Fr^*-orbit of s, and such that the Artin conductor at each $s \in S^t$ is c_s, $1 \le c_s \le n$.

Let $T = T_\ell(X_1, S_1^D, S_1^{wt}, C, \iota, n, m) = T_\ell(X_1, S_1^D, S_1^w, S_1^t, n, m, (c_s;\ s \in S_1^w \sqcup S_1^t))$ be the number of fixed points of $\mathrm{Fr}^{*\,m}$ acting on this subset. We omit m from the notation of T if $m = 1$, and usually also the fixed X_1, n. We compute T assuming $n > 2$ *is odd or* N_1^D *is even* (thus $(-1)^{N_1^D(n-1)} = 1$), and $N_1^D \ge 2$, $N_1^w + N_1^t \ge 1$, so $c_s > n$ for all $s \in S_1^w$ and $1 \le c_s \le n$ for all $s \in S_1^t$. The case of $n \ge 2$, odd $N_1^D \ge 2$, should be reducible to that as in [F2], using the trace formula, on working with a division algebra D whose localization D_s stays a division algebra for all $s \in S_1^D - \{w\}$ for some place $w \in S_1^D$, D is split outside $S_1^D - \{w\}$, and using a pseudo-coefficient of a Steinberg representation up to an $\mathbb{F}_q$-twist at w, inspired by [DF13, section 5], which uses masses of categories techniques.

As in [DF13] we use the Galois-automorphic dictionary to reduce the computation of T to counting automorphic representations of $\mathrm{GL}(n)$, and the assumption $N_1^D \ge 2$ to transfer the computation to the multiplicative group of a division algebra, where the trace formula in this compact quotient case is simpler, reminiscent of the "simple trace formula", and so easier to use, as in [F17]. In fact [DF13] and [F17] correspond to the case $N_1^w = 0 = N_1^t$.

Our main technical result, the computation of the number of the automorphic representations of $\mathbf{a}^{\mathbb{Z}} \backslash \mathrm{GL}(n, \mathbb{A})$, $\mathbb{A} = \mathbb{A}_{F_1}$, up to twisting by a character of the quotient $\mathbb{Z}$ of $\mathrm{GL}(n, \mathbb{A})$ via $\deg \circ \det$, unramified outside S_1, Steinberg up to an unramified twist at S_1^D, of fixed conductor $c_v > n$ at S_1^w and $1 \le c_v \le n$ at S_1^t (the sum of $N_1^w = |S_1^w|$ and $N_1^t = |S_1^t|$ is > 0), stated as Theorem 5.1, is independent of the Galois-automorphic translation. So are the rank one and pseudo-coefficients results. Here $\mathbf{a}$ is a fixed idèle of degree 1.

The new idea is to use a novel "generic pseudo-coefficient" that detects generic admissible local representations with a given conductor. It detects also components of non-generic representations that occur in the discrete-series non-cuspidal spectrum for $\mathrm{GL}(n)$, such as the one-dimensional representations. Since $N_1^D \geq 2$, by our choice of test functions at $v \in S_1^D$ these discrete-series non-cuspidal representations do not appear in the trace formula that we consider, except for one-dimensional representations.

Our result looks rather different than that of [DF13] when $N_1^w > 0$ so there is a place v with conductor $c(\pi_v) > n$, as the use of the generic pseudo-coefficient kills all elliptic orbital integrals. Only the contribution from the identity element remains in the geometric side of the trace formula, as in Kottwitz' reduction of the computation of the Tamagawa number of a reductive group over a number field to that of its quasisplit inner form, a case reduced by King Lai to Langlands' computation in the split case. However, when $1 \leq c_v \leq n$ for all v, the number we compute reduces to that computed in [DF13], where $S_1^{wt} = S_1^w \sqcup S_1^t$ is empty.

As noted above, we deal here only with the case where N_1^D is even or $n > 2$ is odd. The case $N_1^D \geq 3$ is odd should reduce to this case as in [F2] or [DF13, section 5]. Put $q_s = q^{\deg(s)}$.

The zeta function of X_1 over $\mathbb{F}_q$ is $\zeta(X_1, t) = \prod_i \det(1 - t \cdot \mathrm{Frob}; H^i(X))^{(-1)^{i+1}} = \frac{\mathbf{h}_1(t)}{(1-t)(1-qt)}$. Here $\mathbf{h}_1(t) = \sum_{0 \leq i \leq 2g} a_i t^i = \det(1 - t \cdot \mathrm{Frob}; H^1(X)) = \prod_\alpha (1 - \alpha t)$, $a_i \in \mathbb{Z}$, is a polynomial of degree $2g$ over $\mathbb{Z}$ satisfying the functional equation $\mathbf{h}_1(1/qt) = q^{-g} t^{-2g} \mathbf{h}_1(t)$, or $a_{2g-i} = q^{g-i} a_i$, where $g = \mathrm{genus}(X_1)$. The Picard number of X_1 over $\mathbb{F}_q$ is $h_1 = \mathbf{h}_1(1)$. Note that $a_0 = \mathbf{h}_1(0) = \det(1; H^1(X))$ is 1. The complex absolute values of the $2g$ algebraic numbers α are $q^{1/2}$. These α can be labeled so that $\alpha_{g+i} = \overline{\alpha}_i = q\alpha_i^{-1}$. Assume $(\ell, q) = 1$.

Theorem 1.1. *If $N_1^D \geq 2$, $N_1^w \geq 1$ so $S_1^w = \{s;\, c_s > n\} \neq \emptyset$, $S_1^t = \{s;\, 1 \leq c_s \leq n\}$, and $m = 1$, the number $T_\ell(X_1, S_1^D, S_1^{wt}; n, C, \iota)$ is $\frac{h_1 I^D}{q^n - 1}$ times $\prod_{c \in C} \prod_{s \in S_1^c} I_s = \prod_{s \in S_1^w \sqcup S_1^t} I_s$, where*

$$I^D = \prod_{1 \leq j < n} \mathbf{h}_1(q^j) \frac{\prod_{s \in S_1^D} (q_s^j - 1)}{(q^j - 1)^2},$$

$I_s = I_s(c_s, q_s)$ is $\sum_{0 \leq k < c_s} (-1)^k \binom{n}{k} q_s^{n(c_s - k - 1)} (q_s^n - 1) + (-1)^{c_s} \binom{n}{c_s}$ if $s \in S_1^t$, where $1 \leq c_s \leq n$; $I_s = I_s(c_s, q_s)$ is $q_s^{n(c_s - n - 1)} (q_s^n - 1)^{n+1}$ when $s \in S_1^w$, where $c_s > n$.
Put

$$\delta(n) = \delta(n, S_1^w, S_1^t;\, C,\, \iota) \quad \text{for} \quad (-1)^{\sum_{s \in S_1^t} c_s} \prod_{s \in S_1^{wt}} \binom{n}{c_s} = (-1)^{\sum_c c|S_1^c|} \prod_c \binom{n}{c}^{|S_1^c|}.$$

It is zero if $c_s > n$, namely $S_1^w \neq \emptyset$. When $N_1^w = 0$, thus S_1^w is empty, $N_1^t \geq 1$, thus all $c_s \neq 0$ are $\leq n$, the number $T_\ell(X, S_1^D, S_1^w, S_1^t; n, C, \iota)$ is

$$\frac{h_1 I^D}{q^n - 1} \left(\prod_{s \in S_1^t} I_s - \delta(n) \right) + h_1(\delta(n) - \delta(n-1)) + \delta(n)(-1)^{N_1^D(n-1)} T(S_1^D).$$

We are assuming that $N_1^D \geq 2$ is even, or $n \geq 2$ is odd, so the factor $(-1)^{N_1^D(n-1)}$ is 1.

By $T(S_1^D)$ we mean the $T(X_1, S_1, n)$ of [DF13, Theorem 2.3] (with S_1 replaced by our S_1^D), or T of [F17, Theorem 1.1] with N equal our $N_1^D = |S_1^D|$. This is the case where our S_1^w and S_1^t are empty. It is an integral multiple of h_1 ([DF13, Theorem 6.18]). I^D is an integer as $N_1^D = |S_1^D|$ is ≥ 2. $I_s/(q^n - 1)$ is an integer for $s \in S_1^w$, where $c_s > n$, and $N_1^w \geq 1$.

If $n = 1$ then $I_s = q_s^{c_s-2}(q_s - 1)^2$ if $c_s \geq 2$, $I_s = q_s - 2$ and $\sum_{s\in S_1^t} c_s = |S_1^t|$ if $c_s = 1$, as in Theorem 2.1 below. Theorem 2.1 computes the number of characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times$ with specified ramification $c_s \geq 1$, $s \in S_1$. The form Theorem 2.1 takes suggests the form Theorem 1.1 has in the exceptional case of $c_s \leq n$ for all S_1. We give three (short) proofs of Theorem 2.1: computing the cardinality of the characters χ with conductor $c(\chi_s) \leq c_s$ and using the inclusion-exclusion principle to count those with $c(\chi_s) = c_s$, analogous approach based on the trace formula for GL(1), and the trace formula at test functions embodying this principle locally. The third approach is here developed for all $n \geq 2$ using the trace formula.

The variation with m is discussed in section 7. In particular, our very explicit formula, as in [DF13, Corollary 6.8], implies

Corollary 1.2. *When $(X_1,\, S_1^D)/\mathbb{F}_q$ is replaced by $(X_m,\, S_m^D)/\mathbb{F}_{q^m}$, the factor I^D, as a function of $m \in \mathbb{Z}_{\geq 1}$, has the form $\sum_k n_k\gamma_k^m$ where $n_k \in \mathbb{Z}$ and each γ_k is a q-Weil number, the product of a root of unity and a monomial in q and the eigenvalues of the action of* Frob *on $H^1(X)$.*

A new approach of Deligne to show this property of the tame factor I^D is in section 7.

In the case of tame ramification an analogous claim is discussed in [F1] in rank two, and by Hongjie Yu [Yu] in general rank, relating the number of points on a coarse moduli space of certain Higgs bundles with a truncated trace formula, using a suitable pseudo coefficient in the tame case. We believe this technique applies in our wild case with our pseudo coefficient, and hope to use it in a subsequent work to remove the restriction $|N_1^D| \geq 2$, and show that Corollary 1.2 applies to T_ℓ of the Theorem, not only its tame part I^D.

Corollary 1.3. *The number T_ℓ computed in the theorem is independent of the prime ℓ. It depends on q, on X_1 (via $\mathbf{h}_1$), on S_1^w, S_1^t, S_1^D, $q_s = q^{d_s}$, c_s, but not on the relative position of S_1^D, S_1^w, S_1^t in X_1, that is, if S_1^D, S_1^w, S_1^t are replaced by $\widehat{S}_1^D$, $\widehat{S}_1^w$, $\widehat{S}_1^t$ with $(q_s, c_s;\, s \in S_1^*) = (q_s, c_s;\, s \in \widehat{S}_1^*)$, $* = w,\, t,\, D$, the result does not change.*

As a function of $q \to \infty$, the leading term of $\mathbf{h}_1(q^j)$ in q is q^{2gj+g}, of $h_1 = \mathbf{h}_1(1)$ is q^g. So

Corollary 1.4. *The leading term in $q \to \infty$ of T is q to the power*

$$(g-1)n^2 + \frac{n(n-1)}{2}|S^D| + n\sum_{c\in C} c|S^c|, \qquad |S^D| = \sum_{s\in S_1^D} \deg(s),$$

$\sum_{c\in C} c|S^c| = \sum_{c\in C} c\sum_{s\in S_1^c} \deg(s) = \sum_{s\in S_1^w\sqcup S_1^t} \deg(s)c_s$.

Indeed, $(g-1)n^2$ is g plus

$$\sum_{1\leq j<n} ((2j+1)g - 2j) - n = \sum_{1\leq j<n} (2j+1)(g-1) - 1 = (g-1)(n^2-1) - 1.$$

The case considered here, where the conductor might be large, includes the case where there are σ_s with positive Swan conductor, or depth – representation theoretically, so we use the word wild. The tame case requires other techniques, see [F24], [FÖ23], [F1].

In [F24] we compute the cardinality of a similar set of equivalence classes, invariant under the Frobenius action, of $\overline{\mathbb{Q}}_\ell$-smooth sheaves with fixed tame monodromy of principal series (PS) or discrete series (DS) type, under compatibility and general position assumptions, introduced by Deligne [D15]. Here we only put constraints on the monodromy, such as to have conductor c_s. In [F24] we expect a dominant term q^d, where d is half the dimension of a related complex space of moduli, which is complex symplectic. It is not so here. In [FÖ23] we permit in addition one point of principal unipotent monodromy. Both [F24], [FÖ23] are in rank two and use the explicit trace formula as in [F15]. In [F1] we use the much simpler, compact quotient trace formula to study the case where in addition to PS and DS types tame ramification, at least at two points there is principal unipotent monodromy. This most likely can be done in any rank, using the anisotropic trace formula, as here.

Example 1.1. To appreciate the general results here we examine now a special case that can be compared with geometric, direct computations. Thus we consider here the projective line $X = \mathbb{P}^1$ over $\mathbb{F}_q$, the set $S = \{0, 1, \infty\}$ of three rational points on X, thus $q_s = q$ for $s \in S$, and an irreducible two ($n = 2$) dimensional ℓ-adic representation σ of the fundamental group $\pi_1(X - S, o)$. Denote by σ_s the restriction of σ to the decomposition group D_s.

In the notation of Theorem 1.1, suppose $S^D = \{0, 1\}$, $S^w = \emptyset$, $S^t = \{\infty\}$, $c_s = 1$ at $s = \infty$, $n = 2$. Then $h_1 = 1$, $I_D = 1$, $I_\infty = \sum_{k=0}(q^2 - 1) - 2 = q^2 - 3$, $\delta(2) = (-1)^1 \cdot \binom{2}{1} = -2$, $\prod_{s \in S_1^t} I_s - \delta(2) = q^2 - 1$, $\delta(1) = -1$, $\delta(2) - \delta(1) = -1$, $T(S_1^D) = 0$. Hence Theorem 1.1 implies that $T = \frac{q^2-1}{q^2-1} - 1$ is 0, so there are no irreducible rank 2 local systems with principal unipotent monodromy (twisted, that is: multiplied by a quadratic character α) at 0, 1, and Artin conductor $c_\infty = 1$ at $s = \infty$, as is explained geometrically below.

If however $c_\infty = 2$ ($S^D = \{0, 1\}$, $S^w = \emptyset$, $S^t = \{\infty\}$, $n = 2$) then again $h_1 = 1$, $I_D = 1$, and I_∞ is $q^2(q^2 - 1)$ (term of $k = 0$), plus $-2(q^2 - 1)$ (term of $k = 1$), plus 1, thus $I_\infty = (q^2 - 2)(q^2 - 1) + 1$; $\delta(2) = 1$, $\delta(1) = 0$; $I_\infty - \delta(2) = (q^2 - 2)(q^2 - 1)$, so that $T = \frac{(q^2-2)(q^2-1)}{q^2-1} + 1$ equals $q^2 - 1$.

If c_∞ is $c > n$, then $I_\infty = q^{n(c-n-1)}(q^n - 1)^{n+1}$, $I_\infty/(q^n - 1) = q^{n(c-n-1)}(q^n - 1)^n$ and so when $n = 2$ ($T(S_1^D) = 0$) we have $T = q^{2(c-3)}(q^2 - 1)^2$. So $T = (q^2 - 1)^2$ when $c_\infty = 3$.

Note that for all $s \in |X| - S$, σ_s is unramified: the inertia subgroup I_s acts trivially on the space V of σ, i.e., $V = V^{I_s}$, where V^{I_s} abbreviates $V^{\sigma_s(I_s)}$. In this case the Artin conductor $a(\sigma_s) = \dim(V/V^{I_s}) + \mathrm{Sw}(\sigma_s)$ is zero. If $a(\sigma_s) = 1$ then $\dim V^{I_s} = 1$, and σ_s may (1) have unipotent monodromy, or be (2) semisimple regular: a direct sum of two distinct tame characters $(\varepsilon_{1s}, \varepsilon_{2s})$, thus of $\mathcal{O}_s^\times/(1 + \varpi_s \mathcal{O}_s)$, of principle series PS type (the Frobenius does not permute them), and one of them, say ε_{2s}, be 1, to have conductor 1. The case where σ_2 is discrete series DS type: $(\eta_s, \overline{\eta}_s)$, η_s a character of $\mathcal{O}_{E_s}^\times/(1 + \varpi_s \mathcal{O}_{\mathbb{E}_s})$, that the Frobenius permutes, has conductor 2.

Now a general question is to find rigid cases, where local monodromy determines global monodromy.

For example, there is no irreducible rank 2 local system on $X = \mathbb{P}^1 - S$, $S = \{0, 1, \infty\}$ where the three points are of degree one, with Artin conductor c_s equals 1 at 0, 1, ∞. A geometric way to see this is as follows. This is a tame case, so it suffices to check over $\mathbb{C}$, where – as we are in rank 2 – this follows from the theory of hypergeometric functions. At each $s = 0,\, 1,\, \infty$ we have two eigenvalues α_s, β_s, one of which, say β_s, is 1 as $\dim V^{I_s}$ is 1. As $\prod_{s\in\{0,1,\infty\}} \det_s = 1$, we have $\alpha_0\alpha_1\alpha_\infty = 1$, a reducible case.

Our way to see this uses our counting: By [DF13, Corollary 6.19(ii), p. 968, l. 3], and [F17, p. 142, l. 13-14], there are no irreducible two-dimensional representations σ of $\pi_1(X - S, o)$ with nontrivial unipotent monodromy at each $s \in S$, $o : \operatorname{Spec}\overline{\mathbb{F}}_q \to X - S$ a geometric point. [In fact, these references show that when $n = 2$, $X = \mathbb{P}^1$ so $g = 0$, $S \subset |X|$ is a set of closed points s with degrees $d_s \geq 1$ whose sum is 2 or 3, $|S| \geq 2$, thus the degrees $(d_s,\, s \in S)$ are $(1, 1)$, $(1, 2)$ or $(1, 1, 1)$, there are no such σ. By [F15, Theorem 3.1], there are no irreducible two-dimensional representations σ of $\pi_1(X, o)$, $X = \mathbb{P}^1$, thus $S = \emptyset$. By [F15, Theorem 4.1], there are no irreducible two-dimensional representations σ of $\pi_1(X - S, o)$, $X = \mathbb{P}^1$, with unipotent monodromy at $S = \{s\}$, if s is a closed point of degree $d \leq 3$, but there are q such σ when $d = 4$. This reproves [DF13, Proposition 7.1].]

Thus there are no cuspidal automorphic representations of $\mathbf{a}^{\mathbb{Z}}\backslash \operatorname{GL}(2, \mathbb{A})$ with special components at all $s \in S$, unramified elsewhere; $\mathbb{A}$ is the ring of adèles of the function field $F = k(X)$ ($\mathbb{F}_q(t)$ in our case of $X = \mathbb{P}^1$), $\mathbf{a}$ is a fixed idèle of degree 1.

According to [F24, Example 2, p. 780, l. 10], the number of σ with $\sigma_s = \varepsilon_{1s} \oplus \varepsilon_{2s}$, $s \in \{0,\, 1,\, \infty\}$, $\varepsilon_{1s} \neq \varepsilon_{2s}$, is 1, but none if ε_{2s} is 1.

If σ_∞ is principal unipotent, and σ_0, σ_1 are determined by $\varepsilon_{1s} \neq \varepsilon_{2s}$, thus $|S^D| = 1$, $|S^{(1,1)}| = 2$ in the notation of [FÖ23], then $X(q) = (q + 1)^2$, $X(-1) = 0$, $X(1) = 4$ in the notation of [FÖ23, Theorm 2.1], so $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_4$ there is

$$\frac{(q+1)^2}{(q+1)(q-1)} - \frac{4}{2(q-1)} = \frac{q+1-2}{q-1} = 1.$$

As $h_1 = 1$, there is a unique two dimensional irreducible representation of $\pi_1(\mathbb{P}^1 - \{0,\, 1, \infty\})$ with local components σ_0, σ_1, but none for which $\varepsilon_{2s} = 1$ and $\varepsilon_{1,0} \cdot \varepsilon_{1,1} = 1$.

As in the second paragraph in this Example, there are no irreducible σ with $|S^D| = 2$ and $c_\infty = 1$ at a third place $\infty \in \mathbb{P}^1 - S^D$.

Let us return to the case of the third paragraph in this example. That is, let us count geometrically the $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$-invariant isomorphism classes of irreducible ℓ-adic rank two local systems on $X = \mathbb{P}^1 - S$, $S = \{0,\, 1,\, \infty\}$, over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, with principal unipotent local monodromy (twisted by a quadratic character α_s) at $s \in S^D = \{0,\, 1\}$, and Artin conductor two at ∞. There are two cases.

Tame cases: At ∞ the local representation is ("regular":) the sum of two *distinct* characters α, α^{-1} (as the central character is 1) of $\widehat{\mathbb{Z}}(1)$ (as it is "tame"). Thus $\alpha^2 \neq 1$.

Or the local representation is $\alpha\cdot$(principal unipotent), where the square of α is one.

And we have the condition of stability of $\{\alpha,\, \alpha^{-1}\}$ under $x \mapsto x^q$.

When q is odd, $\alpha = \alpha^q$, $\alpha \neq 1$ and not of order two on $\mathbb{F}_q^\times$, up to $\alpha \mapsto \alpha^{-1}$, gives $\frac{(q-1)-2}{2} = \frac{q-3}{2}$ pairs $\{\alpha,\, \alpha^{-1}\}$; plus one $\alpha\cdot$(principal unipotent) ($\alpha \neq 1 = \alpha^2$ on $\widehat{\mathbb{Z}}(1)$; $\alpha \neq 1$ as there are no local systems with principal unipotent monodromy at three places by [DF13, Corollary 6.19(ii), p. 968, ℓ. 3]).

When $\alpha \neq \alpha^q$ then $\alpha^q = \alpha^{-1}$, so $\alpha^{q+1} = 1$, $\alpha \neq 1$, not of order two, up to $\alpha \sim \alpha^{-1} = \alpha^q$, we have $\frac{(q+1)-2}{2} = \frac{q-1}{2}$ possibilities, giving a total of $q - 1$ local representations.

When q is even a similar computation gives $\frac{q-2}{2} + \frac{q}{2} = q - 1$.

Wild cases: σ_∞ equals 1 plus a wild one dimensional character, with Swan conductor $\mathrm{Sw}(\sigma_\infty) = 1$. These latter are characters of $\mathbb{F}_q$, nontrivial. This gives $q-1$ possible local data at ∞. For q prime, the $q-1$ local data at ∞ are conjugate under $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, hence must give the same contribution. Expecting this to hold for all q, it is to be expected from our count of q^2-1 (for $c_\infty = 2$), that it is $q^2-1 = (q-1)+(q-1)q$ (the first $q-1$ is for the tame case, the second $q-1$ is the expected number of wild cases, each coming with multiplicity q).

In the case of the 4th paragraph in this Example: $c = c_\infty = 3 > n = 2$, Deligne checked that our result: $T = (q^2-1)^2$, is the same as what follows from his geometric expectations.

To drop the requirement that the central character is 1, we now consider SL(2), rather than GL(2), and note that there is a cuspidal automorphic representations of $\mathrm{SL}(2,\mathbb{A})$ with special components at all $s \in S$, unramified elsewhere, namely irreducible two-dimensional representations σ of $\pi_1(X-S,o)$ into the Langlands dual group ${}^L\mathrm{SL}(2) = \mathrm{PGL}(2,\mathbb{C})$ of SL(2), with nontrivial unipotent monodromy at each $s \in S$; or of $\mathrm{GL}(2,\mathbb{A})$, with a component special tensored with $\varepsilon_v \circ \det$, ε_v a character of order two of $F_v^\times$.

This is suggested by the existence of three nontrivial unipotent matrices: A_0, A_1, A_∞ in $\mathrm{SL}(2,\mathbb{C})$ whose product is $-I$. We take them to fix different lines, spanned by the columns $\ell_0 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\ell_1 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. We fix $A_0 = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$, $A_1 = \left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$. Then $A_\infty = -(A_0A_1)^{-1} = \left(\begin{smallmatrix}-1&2\\c&-1-2c\end{smallmatrix}\right)$. Now A_∞ is unipotent iff $c=-2$. For this c, A_∞ fixes the line spanned by $\ell_\infty = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$.

The same calculation shows there are no three nontrivial unipotent matrices: A_0', A_1', A_∞' in $\mathrm{SL}(2,\mathbb{C})$, fixing different lines, whose product is I. This is compatible with [DF13, Corollary 6.19(ii), p. 968, l. 3], and [F17, p. 142, l. 13-14], mentioned above.

We get a rank two local system on $\mathbb{P}^1 - S$ with nontrivial unipotent monodromy at 0 and 1, and monodromy with a single Jordan block and eigenvalue -1 at ∞.

This local system can be obtained from the universal Tate module of the Legendre family of elliptic curves on $\mathbb{P}^1 - \{0,\,1,\,\infty\}$ defined by $y^2 = x(x-1)(x-\lambda)$, $\lambda \in \mathbb{P}^1 - \{0,\,1,\,\infty\}$.

Question 1.5. (1) *Obtain the computation using the mass of categories technique of* [DF13, section 4]*, rather than using the trace formula as in* [F17].
(2) *In the case* $n=2$ *use the explicit trace formula of* [F14] *to compute* T*, especially with* N_1^D *equals* 0 *and* 1*, as in* [F15].
(3) *Extend the results to the case of* $n \geq 2$*, odd* $N_1^D \geq 2$*, as in* [F2] *using the trace formula, or* [DF13, section 5] *using masses of categories techniques, on working with a division algebra* D *whose localization* D_s *stays a division algebra for all* $s \in S_1^D - \{w\}$ *for some place* $w \in S_1^D$*,* D *is split outside* $S_1^D - \{w\}$*, and using a pseudo-coefficient of a Steinberg representation up to an* $\mathbb{F}_q$*-twist at* w*.*
(4) *Compute the trace at the generic pseudo-coefficient of the discrete series non generic* (*Speh*) *representations.*
(5) *Extend Corollary* 1.2 *to establish geometricity of the cardinality* T_ℓ *of Theorem* 1.1*, not only its tame part* I^D*, also without assuming* $|N_1^D| \geq 2$*, on using the coarse moduli space used by Hongjie Yu* [Yu] *in the analogous tame case but using instead our pseudo coefficient in the wild case.*

In the representation theoretic sections 2, 4, 5, we consider only $m = 1$, so erase the subscript 1 from X_1, S_1, etc.

The proof of the theorem is based on translating Galois to automorphic representations. The translation preserves local conductors, thus if $\sigma \to \pi$, then $\mathrm{Artin}(\sigma_v) = c(\pi_v)$. Here $\mathrm{Artin}(\sigma_v)$ is the Artin conductor of the Galois representation σ_v. The conductor of the *generic* irreducible admissible representation π_v of $\mathrm{GL}(n, F_v)$ is denoted by $c(\pi_v)$. It is defined by [JPS81], see also [J12], [M13], and section 4 below. It is an integer ≥ 0, $= 0$ iff π_v is unramified. Here we used the representation theoretic notation v instead of the geometric s, for a place in $S_1^w \sqcup S_1^t$. There are related notions as follows.

The restriction σ_v of the Galois representation σ to the decomposition group at the place $v \in |X|$ is called *unramified* if its restriction to the inertia subgroup I_v $(= \pi_1(X^*_{(v)}, \overline{\eta})$ in the notation of [D15, section 2]) acts trivially. It is *tamely ramified* if its restriction to the *wild ramification subgroup* – the maximal pro-p subgroup – P_v of I_v is trivial. Otherwise σ_v is called *wild.* The quotient of I_v by P_v is $\widehat{\mathbb{Z}}(1)$, by which we mean $\widehat{\mathbb{Z}}^{p'}(1)(k) = \varprojlim \mu_m(k)$, m ranges over the positive integers prime to p, $\mu_m(k)$ indicated the group of mth roots of 1 in the algebraic closed field k, and k is taken to be the algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Then $\widehat{\mathbb{Z}}(1)$ is $\simeq \varprojlim \mathbb{F}_{q^n}^\times$, projective limit with respect to the norm maps between the finite fields $\mathbb{F}_{q^n}$. The Galois representation σ is wild at v if the Swan conductor of σ_v is positive. The Swan conductor $\mathrm{Sw}(\sigma_v)$ of a Galois representation σ_v measures the wildness of σ_v. It is an integer ≥ 0. It is related to the Artin conductor by $\mathrm{Artin}(\sigma_v) = \dim(V/V^{\sigma_v(I_v)}) + \mathrm{Sw}(\sigma_v)$, where V is the space of σ_v, I_v is the inertia subgroup, and $V^{\sigma_v(I_v)}$ the subspace fixed by the inertia subgroup. The Artin conductor is additive, see, e.g., [GR10, p. 436, l. 3].

The *depth*, $d(\pi_v)$, of an irreducible admissible representation π_v of $\mathrm{GL}(n, F_v)$, defined by A. Moy and G. Prasad [MP94], [MP96], is defined as follows. In this paragraph (only) let G be the group of F_v-points of a connected reductive algebraic group defined over F_v. Let $\mathcal{B} = \mathcal{B}(G)$ be the Bruhat-Tits building attached to G. For any point $x \in \mathcal{B}$ let G_x be the parahoric subgroup of G associated to x. Moy and Prasad defined a decreasing filtration $\{G_{x,r}\}$ of G_x indexed by the non-negative real number r (see [MP96, section 3]). Put $G_{x,r^+} = \cup_{s>r} G_{x,s}$. The depth $d(\pi_v)$ of an irreducible admissible representation (π_v, V) of G is the smallest non-negative number r such that the space $V^{G_{x,r+}}$ is nontrivial for some $x \in \mathcal{B}$. According to [MP96, Theorem 5.2], if π_v is a subquotient of the induced $I_P^G(\pi_{1v} \times \cdots \times \pi_{kv})$ from $\pi_{1v} \times \cdots \times \pi_{kv}$ on the Levi factor $M(F_v)$ of a parabolic $P(F_v)$, then $d(\pi_v) = d(\pi_{1v} \times \cdots \times \pi_{kv})$.

When π_v is cuspidal, σ_v is irreducible, $c(\pi_v) = \mathrm{Artin}(\sigma_v)$, and $\mathrm{Sw}(\sigma_v) = \mathrm{Artin}(\sigma_v) - n$ is $c(\pi_v) - n = nd(\pi_v) \geq 0$. When E_v/F_v is an unramified field extension of degree n, and σ_v is induced, from a character not fixed by $\mathrm{Gal}(E_v/F_v)$ on the Weil group $W(E_v/E_v) = E_v^\times$, to the relative Weil group $W(E_v/F_v)$ (an extension of $\mathrm{Gal}(E_v/F_v) = \mathbb{Z}/n$ by $W(E_v/E_v)$), then σ_v is tame irreducible, with Swan conductor 0.

When π_v is essentially square integrable (e.s.i.), the Galois representation, or the $\overline{\mathbb{Q}}_\ell$-smooth sheaf, σ_v, associated with π_v by the local Langlands correspondence, is indecomposable. Conversely, an indecomposable σ_v corresponds to an e.s.i. π_v. In this case it is known (see Lansky and Raghuram [LR03]) that $\mathrm{Sw}(\sigma_v) = nd(\pi_v)$, and $d(\pi_v) = \max(\frac{c(\pi_v)-n}{n}, 0)$.

When π_v is generic it is induced $I(\pi_{1v} \times \cdots \times \pi_{kv})$ (parabolically, normalized induction) from e.s.i. π_{jv}, and it corresponds to $\sigma_{1v} \oplus \cdots \oplus \sigma_{kv}$, where the σ_{jv} are indecomposable, of dimension n_j if the π_{jv} are representations of $\mathrm{GL}(n_j, F_v)$. Then

$$c(\pi_v) = c(I(\pi_{1v} \times \cdots \times \pi_{kv})) = \mathrm{Artin}(\sigma_{1v} \oplus \cdots \oplus \sigma_{kv}) = \sum_{1 \le j \le k} \mathrm{Artin}(\sigma_{jv}) = \sum_{1 \le j \le k} c(\pi_{jv}).$$

The 3rd = is the additivity of the Artin conductor.

2. Rank one appetizer

Let us consider first the case of $n = 1$, thus $\mathrm{GL}(1) = \mathbb{G}_m$. It is a good indicator for the general case $n \geq 2$. Thus we wish to count the number of characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times$ with specified conductors c_v. As above, F is the function field of a smooth projective absolutely irreducible curve X over $\mathbb{F}_q$, $\mathbb{A}$ the ring of adèles of F, $\mathbb{A}^\times$ the group of idèles, $\mathbf{a}$ is an idèle of degree 1, and $\mathcal{O}_{\mathbb{A}}^\times = \prod_v \mathcal{O}_v^\times$ where $\mathcal{O}_v^\times$ is the group of units in the ring of integers $\mathcal{O}_v$ of the completion F_v of F at the place v of F, or closed point of X, in $|X|$. We shall say the character χ_v is *unramified* if its value on $\mathcal{O}_v^\times$ is 1, and that it has *conductor* $c_v = c(\chi_v) > 0$ if it is ramified and $c_v \geq 1$ is the smallest integer such that χ_v is 1 on $1 + \varpi_v^{c_v} \mathcal{O}_v$ where ϖ_v is a generator of the maximal ideal $\mathfrak{m}_v$ in $\mathcal{O}_v$. Noting that $F^\times \cap \mathbf{a}^{\mathbb{Z}} \mathcal{O}_{\mathbb{A}}^\times$ equals $\mathbb{F}_q^\times$, denoting the cardinality $|(\mathrm{Pic}^0 X)(\mathbb{F}_q)|$ of the Picard group by h_1, using the exact sequence

$$1 \to \mathbb{F}_q^\times \to \mathcal{O}_{\mathbb{A}}^\times \to \mathbb{A}^\times/F^\times \mathbf{a}^{\mathbb{Z}} \to (\mathrm{Pic}^0 X)(\mathbb{F}_q) \to 1,$$

we see that the volume $|\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times|$ is $h_1 |\mathcal{O}_{\mathbb{A}}^\times|/(q-1)$, or the cardinality $|\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} \mathcal{O}_{\mathbb{A}}^\times F^\times|$ is h_1. Hence the number of nowhere ramified characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times$ is h_1.

Theorem 2.1. *Let S be a nonempty reduced divisor on the curve X over $\mathbb{F}_q$, namely a finite subset of the set $|X|$ of closed points of $X/\mathbb{F}_q$, or places of F. Fix positive integers c_v, $v \in S$. Put $I_v = q_v^{c_v - 2}(q_v - 1)^2$ if $v \in S^w = \{v \in S;\, c_v > 1\}$, $I_v = q_v - 2$ if $v \in S^t = \{v \in S;\, c_v = 1\}$. The number of characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times$ unramified outside S with conductor c_v, $v \in S$, is $\frac{h_1}{q-1} \prod_{v \in S} I_v$ if S^w is not empty. It is $\frac{h_1}{q-1}[\prod_{v \in S} I_v - (-1)^{|S|}] + (-1)^{|S|} h_1$ when S^w is empty.*

When S is empty we saw above that the number of everywhere unramified characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times$ is $h_1 = |\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} F^\times \mathcal{O}_{\mathbb{A}}^\times|$. The denominator $q - 1$ appears only when $S \neq \emptyset$.

The local factors I_v here are the same as in Theorem 1.1 at $n = 1$.

Proof. Consider first the number of characters with ramification (= conductor) $\leq c_v$ at each $v \in S$. This number is equal to the cardinality of the group

$$\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} \cdot \prod_{v \notin S} \mathcal{O}_v^\times \cdot \prod_{v \in S} (1 + \mathfrak{m}_v^{c_v})^\times \cdot F^\times.$$

Here $(1 + \mathfrak{m}_v^{c_v})^\times$ is $\mathcal{O}_v^\times$ if $c_v = 0$. This is an extension of $\mathrm{Pic}^0(X)(\mathbb{F}_q) = \mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}} \cdot \prod_{v \in |X|} \mathcal{O}_v^\times \cdot F^\times$, of cardinality h_1, by $(\prod_{v \in S} \mathcal{O}_v^\times/(1 + \mathfrak{m}_v^{c_v})^\times)/\mathbb{F}_q^\times$; here $\mathbb{F}_q^\times$ acts freely when $S \neq \emptyset$. This shows there are $\frac{h_1}{q-1} \prod_{v \in S} q_v^{c_v - 1}(q_v - 1)$ characters with conductor $\leq c_v$ for all $v \in S$ when S is not empty (there are h_1 characters if S is empty).

To find the number of characters with exact conductor c_v at $v \in S$ we use the inclusion-exclusion principle to remove the characters where for some v the conductor is $\leq c_v - 1$, then add back those removed twice, etc. Thus we take the alternating sum of the number of characters, some conductors $\leq c_v$, some $\leq c_v - 1$, where each of these numbers is given by the same formula, with some c_v replaced by $c_v - 1$, *except* when all the c_v are 1, and all are replaced by $c_v - 1$, in which case the number of unramified characters is h_1, and not the mass $h_1/(q-1)$.

This is why the case "at least one c_v is ≥ 2" is simpler: we get $\frac{h_1}{q-1}$ times

$$\sum_{T\subset S}(-1)^{|T|}\prod_{v\in S-T} q_v^{c_v-1}(q_v-1)\prod_{v\in T}(\delta(c_v\geq 2)q_v^{c_v-2}(q_v-1)+\delta(c_v=1)),$$

(here $\delta(Y)$ is 1 if Y holds, 0 if not) which is the product (*):
$\prod_{v\in S}$[(factor for c_v) - (factor for $c_v - 1$, $= 1$ if $c_v = 1$)]. Each factor in this product is I_v, $= q_v^{c_v-1}(q_v-1) - q_v^{c_v-2}(q_v-1) = q_v^{c_v-2}(q_v-1)^2$ if $c_v \geq 2$, $q_v - 2$ if $c_v = 1$.

When $c_v = 1$ for all $v \in S$, the term in the sum over T corresponding to $T = S$ is h_1 and not $h_1/(q-1)$ as appears in the product (*). Removing it from this product, and adding it back, we get the formula of the theorem.

Let us obtain the theorem using the trace formula $\sum_\chi \operatorname{tr}\chi(f) = |G(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}G(F)|\sum_\gamma f(\gamma)$ for $G = \mathrm{GL}(1)$. The sum on the left is over the characters χ of $G(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}G(F) = \mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times$, that on the right over $\gamma \in \mathbf{a}^{\mathbb{Z}}F^\times$, and the test function is smooth and compactly supported. We take $f = \otimes_{v\in|X|}f_v$ with $f_v = f_v^0 = f_v^{(0)}$, the characteristic function of $\mathcal{O}_v^\times$ for all $v \notin S$, and for $v \in S$ we take $f_v = f_v^{(c_v)}$, the characteristic function $\mathrm{char}(1+\mathfrak{m}_v^{c_v})$ of $1+\mathfrak{m}_v^{c_v}$, divided by the volume of $1+\mathfrak{m}_v^{c_v}$, so f_v is $q_v^{c_v-1}(q_v-1) = |\mathcal{O}_v^\times/(1+\mathfrak{m}_v^{c_v})|$ times $\mathrm{char}(1+\mathfrak{m}_v^{c_v})$. The volume in front of the right side is the quotient of $h_1 = |\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times\mathcal{O}_{\mathbb{A}}^\times|$ by $q-1$, the cardinality of $\mathbb{F}_q^\times = \mathcal{O}_{\mathbb{A}}^\times \cap \mathbf{a}^{\mathbb{Z}}F^\times$, since the sequence $1 \to \mathbb{F}_q^\times \to \mathcal{O}_{\mathbb{A}}^\times \to \mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times \to \mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times\mathcal{O}_{\mathbb{A}}^\times \to 1$ is exact. Recall that $c(\chi_v)$ is the *conductor* of a character χ_v, the smallest integer $c \geq 0$ such that χ_v is 1 on $(1+\mathfrak{m}_v^c)^\times$, $= \mathcal{O}_v^\times$ when $c = 0$. Then $\operatorname{tr}\chi_v(f_v^{(c_v)})$ is 1 if $c(\chi_v) \leq c_v$, 0 otherwise.

For our f, the left side of the trace formula is then the number of χ with $c(\chi_v) \leq c_v$ for all $v \in S$. The sum $\sum_\gamma f(\gamma)$ on the right is $q - 1$ if S is empty, then the right side is h_1. It is $f(e) = \prod_v(q_v^{c_v-1}(q_v-1)\delta(c_v\geq 1) + \delta(c_v = 0))$ if S is non-empty, as then $\sum_\gamma f(\gamma)$ reduces to $\gamma = e$. To count the χ with $c(\chi_v) = c_v$ for all $v \in S$ we subtract the number of χ with $c(\chi_v) \leq c_v$ for $v \in S - \{v_0\}$, and $c(\chi_v) \leq c_v - 1$ for $v = v_0$. We add back the number $A(S,T)$ of χ with $c(\chi_v) \leq c_v$ for $v \in S - T$, and $c(\chi_v) \leq c_v - 1$ for $v \in T$, $T \subset S$, $|T| = 2$. Continuing we obtain $\sum_{T\subset S}(-1)^{|T|}A(S,T)$, except that when all $c_v = 1$, $A(S,S)$ is h_1 and not $\frac{h_1}{q-1}$. When at least one c_v is ≥ 2 we get $\frac{h_1}{q-1}$ times

$$\sum_{T\subset S}(-1)^{|T|}\prod_{v\in S-T} q_v^{c_v-1}(q_v-1)\prod_{v\in T}(\delta(c_v\geq 2)q_v^{c_v-2}(q_v-1)+\delta(c_v=1)),$$

$= \prod_{v\in S}$[(factor for c_v) - (factor for $c_v - 1$, $= 1$ if $c_v = 1$)]$= \prod_{v\in S} I_v$. However when $c_v = 1$ for all $v \in S$ we get $\frac{h_1}{q-1}[\prod_{v\in S} I_v - (-1)^{|S|}] + (-1)^{|S|}h_1$.

And here is another variant of the proof, based on the trace formula. We use the test function $\widetilde{f}^{(c_v;\, v\in S)} = f^{S,0}\cdot\prod_{v\in S}\widetilde{f}_v^{(c_v)}$, where $f^{S,0} = \prod_{v\notin S} f_v^0$, $\widetilde{f}_v^{(c_v)} = f_v^{(c_v)} - f_v^{(c_v-1)}$. Here $f_v^{(c)}$

is the constant measure supported on $(1+\varpi_v^c\mathcal{O}_v)^\times$ of volume 1, namely the characteristic function of this set divided by its volume. The value of the convolution operator $\chi_v^{(j)}(f_v^{(c)})$ is 1 if $j \leq c$, and 0 if $j > c$. Hence $\chi_v^{(j)}(f_v^{(c)} - f_v^{(c-1)})$ is 1 if $j = c$, and 0 if $j \neq c$.

We are to count the number of characters χ of $\mathbb{A}^\times/F^\times\mathbf{a}^{\mathbb{Z}}$ with conductors $c(\chi_v)$ equal 0 if $v \notin S$ and c_v if $v \in S$. This is $\sum_\chi \chi(\widetilde{f}^{(c_v;\, v\in S)})$, χ ranges over the characters of $\mathbb{A}^\times/\mathbf{a}^{\mathbb{Z}}F^\times$. By the trace formula for GL(1), this is $\frac{h_1}{q-1}$ times $\sum_{\gamma\in F^\times} \widetilde{f}^{(c_v;\, v\in S)}(\gamma)$. If there is $c_v \geq 2$, the sum reduces to $\gamma = 1$. Indeed $F^\times \cap \operatorname{supp} f^{(c_v;\, v\in S)}$ is the identity element if there is $c_v \geq 1$. The value $\widetilde{f}^{(c_v)}(e) = f^{(c_v)}(e) - f^{(c_v-1)}(e)$ is $q_v^{c_v}(1-q_v^{-1}) - q_v^{c_v-1}(1-q_v^{-1}) = q_v^{c_v}(1-q_v^{-1})^2$ if $c_v > 1$, and $q_v(1-q_v^{-1}) - 1 = q_v - 2$ if $c_v = 1$.

But if $c_v = 1$ for all $v \in S$, the sum over $\gamma \in F^\times$ reduces only to $\mathbb{F}_q^\times = F^\times \cap \mathcal{O}_{\mathbb{A}}$. So we express the test function $\widetilde{f}^{(c_v;\, v\in S)}$, when $c_v = 1$ for all $v \in S$, $|S| \geq 1$, as the sum of $\phi_1 = f^{S,0} \cdot [\prod_{v\in S}(f_v^{(1)} - f_v^{(0)}) - (-1)^{|S|}]$, for which the sum over $\gamma \in F^\times$ on the right side of the trace formula reduces to $\gamma = e$, and $\phi_2 = (-1)^{|S|}f^0$, for which that sum ranges over $\mathbb{F}_q^\times$. So applying the trace formula separately for ϕ_1 and ϕ_2 we get that $\sum_\chi \chi(\widetilde{f}^{(c_v;\, v\in S)}) = \sum_\chi \chi(\phi_1) + \sum_\chi \chi(\phi_2)$ is $\frac{h_1}{q-1}[\prod_S((q_v-1)-1) - (-1)^{|S|}] + h_1(-1)^{|S|}$, as asserted. □

The third variant of the proof is that generalized to the higher rank case.

3. Translations

Assume $|S^D| \geq 2$ and $S^w \sqcup S^t$ is not empty. As in [DF13, 1.11], from the Galois-automorphic correspondence we have

(i) There is a bijective correspondence between

(A1) isomorphism classes of rank n irreducible $\overline{\mathbb{Q}}_\ell$-smooth sheaves on $X - S$, fixed by the Frobenius, with principal unipotent monodromy at each $s \in S^D$, such that the Artin conductor of the representation V_1 of the Weil group $W(X_1 - S_1, o)$ at each $s_1 \in S_1^w$ is $c_{s_1} > n$, and $c_{s_1} \leq n$ at each $s_1 \in S_1^t$, and

(B1) classes modulo $\mathbb{F}_q$-twisting (see [DF13, 1.10]) of cuspidal (irreducible) automorphic representations π of $\mathrm{GL}(n,\mathbb{A})$ over $\overline{\mathbb{Q}}_\ell$, unramified outside S_1, such that for all $s_1 \in S_1^D$ the representation π_{s_1} is of the form $\mathrm{St} \otimes \chi \circ \det$ for an unramified character $\chi : F_{1,s_1}^\times \to \overline{\mathbb{Q}}_\ell^\times$, for all $s_1 \in S_1^w$ the representation π_{s_1} has conductor $c_{s_1} > n$, and $c_{s_1} \leq n$ for all $s_1 \in S_1^t$.

(ii) The $\mathbb{F}_q$-twists of any π as in (B1) are all distinct.

Note that under the local Langlands correspondence ε-factors are matching, hence the Artin conductor of an n-dimensional irreducible Galois representation σ_v is equal to the conductor of the corresponding generic cuspidal admissible representation $\pi_v = \pi_v(\sigma_v)$ of $\mathrm{GL}(n, F_v)$, as defined by [JPS81, Théorème, section 5], see section 4 below.

Let D be a division algebra over F_1 of rank n, unramified (split) outside S_1^D and such that for each $s_1 \in S_1^D$ the completion D_{s_1} is a division algebra over F_{1,s_1}. Such a division algebra exists iff $N_1^D = |S_1^D| \geq 2$ is even, or n is odd. In fact if $|S_1^D|$ is even, take D to be defined by equal number of invariants $\frac{1}{n}$ and $-\frac{1}{n}$ at the $s_1 \in S_1^D$, and 0 at at all $s_1 \in |X_1| - S_1^D$. If $N_1^D = |S_1^D| \geq 3$ is odd, and n is odd, at three places in S_1^D the invariants are $\frac{n-1}{2n}$, $\frac{n-1}{2n}$, $\frac{1}{n}$, the others in S_1^D are in pairs $\frac{1}{n}$ and $-\frac{1}{n}$, and 0 at all $s_1 \in |X_1| - S_1^D$. If $N_1^D = |S_1^D| \geq 3$ is odd, as explained in [F2] or [DF13, section 5], one may choose a place w in S_1^D, consider the division algebra attached in this paragraph to $S_1^D - \{w\}$, and use at w a pseudo-coefficient of the Steinberg representation up to $\mathbb{F}_q$-twists to reduce the case of an odd $N_1^D = |S_1^D| \geq 3$ to that of even $|S_1^D - \{w\}| \geq 2$.

Denote by $D^\times$ the algebraic group over F_1 such that for any commutative F_1-algebra R the group $D^\times(R)$ of R-points of $D^\times$ is the multiplicative group $(D \otimes_F R)^\times$ of $D \otimes_F R$. The group of F_1-points of $D^\times$ is the multiplicative group of D.

The reduced norm defines a homomorphism denoted det of algebraic groups from $D^\times$ to the multiplicative group $\mathbb{G}_m$; $\mathbb{F}_q$-twists of automorphic representations of $D^\times(\mathbb{A})$ are defined as for $\mathrm{GL}(n)$: if $\chi : \mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$ is a character of the ring $\mathbb{Z}$ of rational integers, it defines a character $\chi \circ \deg \circ \det : D^\times(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell^\times$ of $D^\times(\mathbb{A})$, and a twist $\pi\chi = \pi \otimes \chi$ of a representation (we mean by this an irreducible one) π of $D^\times(\mathbb{A})$ or $\mathrm{GL}(n, \mathbb{A})$. As in [DF13, 1.13], we shall use the well-known

Fact. Suppose $n \geq 2$. There is a bijective correspondence, compatible with $\mathbb{F}_q$-twists, between:

(A2) cuspidal automorphic representations π of $\mathrm{GL}(n, \mathbb{A})$ whose local components at each $s_1 \in S_1^D$ are of the form $\mathrm{St} \otimes \chi \circ \det$ where χ is an unramified character $F_{1,s_1}^\times \to \overline{\mathbb{Q}}_\ell^\times$, and

(B2) automorphic representations π' of $D^\times(\mathbb{A})$ of $\dim > 1$ whose local components at each $s_1 \in S_1^D$ is one-dimensional of the form $\chi \circ \det$ for χ an unramified character of $F_{1,s_1}^\times$.

The representation π occurs with multiplicity one in the cuspidal spectrum of $\mathrm{GL}(n, \mathbb{A})$, and π' in that of $D^\times(\mathbb{A})$, so we need not distinguish between π (or π') and its equivalence class. If π corresponds to π', at each $s_1 \notin S_1^D$ we have $D_{s_1}^\times \simeq \mathrm{GL}(n, F_{1,s_1})$ and $\pi_{s_1} \simeq \pi'_{s_1}$.

The number of classes (A1) or (B1) is then equal to the number of classes modulo $\mathbb{F}_q$-twisting of automorphic representations π' of $D^\times(\mathbb{A})$ of dimension > 1 with local components $\chi \circ \det$, χ unramified character of $F_{1,s_1}^\times$, at each $s_1 \in S_1^D$; with local components at each $s_1 \in S_1^w$ of the fixed conductor $c_{s_1} > n$; at each $s_1 \in S_1^t$ of the conductor $c_{s_1} \leq n$; and unramified outside S_1. For such π', the $\mathbb{F}_q$-twists $\pi'\chi$ are all distinct.

So the number $T(X_1, S_1^D, S_1^w, S_1^t, n; (c_{s_1}; s_1 \in S_1^w \sqcup S_1^t))$ we wish to compute is also the number, up to $\mathbb{F}_q$-twists, of the automorphic representations π of $D^\times(\mathbb{A})$ such that (i) π is unramified outside S_1; (ii) for v in S_1^D the local component π_v is of the form $\chi \circ \det$ for χ an unramified character of $F_v^\times$; (iii) the component π_v ($v \in S_1^w$) has the chosen conductor $c_v > n$; (iv) the component π_v ($v \in S_1^t$) has the conductor $c_v \leq n$; (v) $\dim \pi > 1$.

Note that as we intend to use next the trace formula on $D^\times(\mathbb{A})$, we switched from the arithmetic-geometric notation s_1 for closed points of X_1 to that of places v on F_1.

From now on we shall denote F_1 by F, until we study the variation to F_m.

We shall count these representations up to $\mathbb{F}_q$-twists. Let $\mathbf{a} \in \mathbb{A}^\times$ be an idèle of F of positive degree $\deg(\mathbf{a}) > 0$. View $\mathbb{A}^\times$ as the center of $D^\times(\mathbb{A})$. Then $\deg(\det(\mathbf{a})) = \deg(\mathbf{a}^n) = n\deg(\mathbf{a})$. If χ is a character of the quotient $\mathbb{Z}$ of $D^\times(\mathbb{A}) \overset{\det}{\to} \mathbb{A}^\times \overset{\deg}{\to} \mathbb{Z}$ then the central character $\omega_{\pi\chi}$ of $\pi\chi$ is $\omega_{\pi\chi}(\mathbf{a}) = \omega_\pi(\mathbf{a})\chi(n\deg(\mathbf{a}))$ at $\mathbf{a}$, so any π has an $\mathbb{F}_q$-twist π' with $\omega_{\pi'}(\mathbf{a}) = 1$. The π' is not unique, we may twist it by a character of $\mathbb{Z}/n\deg(\mathbf{a})$. Hence $T(X_1, S_1^D, S_1^w, S_1^t, n; (c_{s_1}; s_1 \in S_1^w \sqcup S_1^t))$ is the number of automorphic representations of $D^\times(\mathbb{A})$ satisfying (i)-(v) as well as (vi) $\omega_\pi(\mathbf{a}) = 1$, taken modulo $\mathbb{F}_q$-twists by a character of $\mathbb{Z}/n\deg(\mathbf{a})$. These twists are distinct. Hence $T(X_1, S_1^D, S_1^w, S_1^t, n; (c_{s_1}; s_1 \in S_1^w \sqcup S_1^t))$ is

$1/n\deg(\mathbf{a})$ times the number of automorphic representations of $D^\times(\mathbb{A})$ satisfying (i)-(vi). We may and will take $\mathbf{a}$ with degree 1.

4. Pseudo-coefficients

This section is used in the following sections but it is also of independent interest.

Let F be a local non-Archimedean field with ring $\mathcal{O}$ of integers and generator ϖ of the maximal ideal $\mathfrak{m}$. Put $q = |\mathcal{O}/\mathfrak{m}|$ for the residual cardinality.

Definition 4.1. The standard maximal compact subgroup $K = G(\mathcal{O})$ of $G = G(F)$, $G = \mathrm{GL}(n)$, $n \geq 2$, has an open subgroup K_c, the *c-conductor group*, consisting of the $g \in K$ with bottom row $(0, \dots, 0, 1) \bmod \varpi^c$, thus it is $(\varpi^c a_1, \dots, \varpi^c a_{n-1}, 1 + \varpi^c a_n)$, $a_j \in \mathcal{O}$, $1 \leq j \leq n$. Here c is a positive integer. We also put $K_0 = K$.

Lemma 4.1. *We have* $[K : K_c] = q^{(c-1)n}(q^n - 1)$*,* $0 < c \in \mathbb{Z}$*.*

Proof. Put $R_c = \mathcal{O}/\varpi^c\mathcal{O}$. The group $\mathrm{GL}(n, R_c)$ acts on the column space R_c^n by $g \cdot e = {}^t g^{-1} e$ (multiplication by transpose-inverse of g). The group K_c is the inverse image under $G(\mathcal{O}) \to G(R_c)$ of the ("mirabolic") subgroup of $\mathrm{GL}(n, R_c)$ fixing the last element e_n in the standard basis of the space R_c^n. The orbit of e_n consists of the e in R_c^n of the form $e \in R_c^n - (\varpi R_c)^n = \mathrm{Span}\{e_1, \dots, e_n\} - \mathrm{Span}\{\varpi e_1, \dots, \varpi e_n\}$. The index is $q^{cn} - q^{(c-1)n} = q^{(c-1)n}(q^n - 1)$. When $n = 1$ and $c \geq 1$ the cardinality of $(\mathcal{O}/(1+\varpi^c\mathcal{O}))^\times = R_c - \varpi R_c$ is $q^c - q^{c-1} = q^{c-1}(q-1)$. $\square$

Let (π, V_π) be an irreducible admissible representation of G over an algebraically closed field of characteristic zero, e.g. $\mathbb{C}$ (or $\overline{\mathbb{Q}}_\ell$). If C is an open subgroup of K put V_π^C for the space of C-fixed vectors in V_π.

To a generic π, and an additive $\overline{\mathbb{Q}}_\ell^\times$-valued character $\psi \neq 1$ of F, there are associated an L-factor $L(s, \pi) = P_\pi(q^s)^{-1}$, $P_\pi \in \mathbb{C}[x]$, $P_\pi(0) = 1$, and an ε-factor $\varepsilon(s, \pi, \psi) = \varepsilon_\pi(q^{-s}, \psi)$, $\varepsilon_\pi(x, \psi) = \gamma x^\mu$, $\gamma \in \mathbb{C}^\times$, $\mu \in \mathbb{Z}$. Suppose the exponent of ψ is 0, namely the maximal ideal on which ψ is 1 is $\mathcal{O}$. Then $\mu = \mu(\pi)$ depends only on π, and $\mu(\pi) \geq 0$.

Definition 4.2. Jacquet, Piatetski-Shapiro, Shalika [JPS81, Théorème, section 5], see also [J12], [M13], show that for each generic irreducible admissible π there is an integer $c \geq 0$ such that $V_\pi^{K_c} \neq 0$. The least such c is named the *conductor* $c(\pi)$ of π. It satisfies $\dim V_\pi^{K_{c(\pi)}} = 1$. It is equal to $\mu(\pi)$. A generic π is unramified precisely when $c(\pi) = 0$. A nonzero vector in $V_\pi^{K_{c(\pi)}}$ is called a *newform* of π.

Reeder [R91, Theorem 1] showed that $\dim V_\pi^{K_c}$ is $\binom{c-c(\pi)+n-1}{n-1}$ if $c \geq c(\pi)$, 0 if not.

Write $\pi^{c(\pi)}$ for a generic π if the conductor of π is $c(\pi)$. Fix a Haar measure on G. Let $f^{(c)}$ denote the characteristic function of K_c divided by the volume $|K_c|$ of K_c. Since $[K_0 : K_c] = q^{n(c-1)}(q^n - 1)$ if $c \geq 1$, we have $|K_c|^{-1} = q^{n(c-1)}(q^n - 1)|K_0|^{-1}$.

The convolution operator $\pi(f^{(c)})$ projects the space V_π of π to $V_\pi^{K_c}$. Hence $\operatorname{tr}\pi(f^{(c)})$ is $\binom{c-c(\pi)+n-1}{n-1}$ if $c \geq c(\pi)$, it is 0 if $c < c(\pi)$ (so it is $\binom{c-c(\pi)+n-1}{n-1}$ for all c, as $\binom{n}{k} = 0$ if $k > n$).

Theorem 4.2. *The function*

$$\widetilde{f}^{(c)} = f^{(c)} + \sum_{\max(0,c-n) \leq j < c} (-1)^{c-j} \binom{n}{c-j} f^{(j)} = f^{(c)} + \sum_{0 < k \leq \min(c,n)} (-1)^k \binom{n}{k} f^{(c-k)}$$

satisfies $\operatorname{tr}\pi^c(\widetilde{f}^{(c)}) = 1$ *and* $\operatorname{tr}\pi^j(\widetilde{f}^{(c)}) = 0$ *for* $j \neq c$.

Thus $\widetilde{f}^{(c)}$ is a *generic pseudo-coefficient* of the generic π with conductor c.

Proof. Define $\widetilde{f}^{(c)} = f^{(c)} + \sum_{0\le j<c} a_j f^{(j)} = f^{(c)} + \sum_{0<j\le c} a_{c-j} f^{(c-j)}$. Then by definition of the $f^{(j)}$ we have $\operatorname{tr}\pi^j(\widetilde{f}^{(c)}) = 0$ for $j > c$, and $\operatorname{tr}\pi^c(\widetilde{f}^{(c)}) = 1$. For k $(0 < k \le c)$ we then have $\operatorname{tr}\pi^{c-k}(\widetilde{f}^{(c)}) = \binom{k+n-1}{n-1} + \sum_{0<j\le k} a_{c-j}\binom{k-j+n-1}{n-1}$. We want to find a_j $(0 \le j < c)$ such that $\operatorname{tr}\pi^{c-k}(\widetilde{f}^{(c)}) = 0$ for all k $(0 < k \le c)$. These are given by $a_j = (-1)^{c-j}\binom{n}{c-j}$, $\max(0, c-n) \le j < c$ (otherwise $a_j = 0$), namely $a_{c-j} = (-1)^j\binom{n}{j}$ $(0 < j \le \min(c,n))$, otherwise $a_{c-j} = 0$. Here $\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j(j-1)\cdots 1}$ is 0 if $j > n$. For a fixed c we obtain c linear equations, for $k = 1,\dots,c$, in the c unknowns $a_0, a_1, \dots, a_{c-1}$, that can be solved inductively, in fact giving a unipotent matrix. For example, when $c = 3$ we have: [$k = 1$: $\binom{n}{n-1} + a_2 = 0$ so $a_2 = -n$], [$k = 2$: $\binom{n+1}{n-1} + a_2\binom{n}{n-1} + a_1 = 0$ so $a_1 = n(n-1)/2$], [$k = 3$: $\binom{n+2}{n-1} + a_2\binom{n+1}{n-1} + a_1\binom{n}{n-1} + a_0 = 0$ so $a_0 = -n(n-1)(n-2)/6$ ($= 0$ if $n = 2$)]. Namely the theorem follows from the well known identity:

Lemma. For all k $(0 < k \le c)$ we have

$$\sum_{0\le j\le\min(k,n)} (-1)^j \binom{n}{j}\binom{k-j+n-1}{n-1} = 0.$$

Proof. Note that the sum can be written as ranging over $j \ge 0$ or $0 \le j \le n$. Write t for $k+n-1$, note that $t \ge n$. The number of subsets of $\{1,2,\dots,t\}$ of size $n-1$ that contain all numbers $\{1,\dots,n\}$ is 0. Counting this set by the inclusion-exclusion principle we get: take all $\binom{t}{n-1}$ subsets of size $n-1$. Then for each $m \in \{1,2,\dots,n\}$ subtract the $\binom{t-1}{n-1}$ subsets that do not include m. Then add back the doubly subtracted subsets. Then subtract the triply subtracted subsets, etc.

Another proof is based on induction on $n \ge 1$ and $t \ge n$, using Pascal's $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. The induction hypothesis asserts the identity holds for $n = N-1$ and all $t \ge N-1$. It clearly implies that the identity holds for all $t \ge N$ when $n = N-1$. But we still need to show it holds for all $t \ge N$ when $n = N$. ☐

Corollary 4.3. *Let* χ *be a character of* $F^\times$. *Denote by the same letter the one dimensional representation* $\chi : g \mapsto \chi(\det g)$ *of* $G = G(F)$. *If* $\chi|\mathcal{O}^\times \neq 1$ *then the convolution operator* $\chi(f^{(j)})$ *is* 0. *If* $\chi|\mathcal{O}^\times = 1$ *then* $\operatorname{tr}\chi(f^{(c)}) = 1$, *and hence* $\operatorname{tr}\chi(\widetilde{f}^{(c)}) = \chi(\widetilde{f}^{(c)}) = \sum_{0\le k\le\min(c,n)}(-1)^k\binom{n}{k}$. *This is zero if* $c \ge n$; *it is* $\sum_{0\le k\le c}(-1)^k\binom{n}{k} = (-1)^c\binom{n-1}{c}$ *if* $c < n$.

The last equality follows from Pascal's identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and telescoping.

It extends to $c \ge n$ as $\binom{n}{k}$ signifies 0 when $k > n$.

Proposition 4.4. *The value of* $\widetilde{f}^{(c)}(e) = \sum_{0\le k\le\min(c,n)}(-1)^k\binom{n}{k}|K_{c-k}|^{-1}$ – *when* $c > n$ – *is*

$$\sum_{0\le k\le n} (-1)^k\binom{n}{k} q^{n(c-k-1)}(q^n - 1) = q^{n(c-n-1)}(q^n-1)^{n+1},$$

divided by $|K_0|$. *If* $c \le n$ *the value is* $\sum_{0\le k<c}(-1)^k\binom{n}{k}q^{n(c-k-1)}(q^n-1) + (-1)^c\binom{n}{c}$, *divided by* $|K_0|$.

The sum over c, $0 \leq c \leq n$, *of the last expression, is* $(q^n - 1)^n$, *divided by* $|K_0|$.

We often normalize the measure on G_v by $|K_v| = \mathrm{vol}(K_v) = 1$, to simplify notation.

Example 4.1. Compatibitity check: Suppose χ us an unramified character of $F^\times$, thus of $F^\times/\mathcal{O}^\times$, and $n = 2$. Then $\mathrm{tr}\,\mathrm{St}\otimes\chi(\widetilde{f}^{(1)}) = 1$, but $\mathrm{tr}\, I(\chi\nu^t, \chi\nu^{-t}; \widetilde{f}^{(1)}) = 0$ for all $t \in \mathbb{R}$ where $\nu(x) = |x|$, $x \in F^\times$, since $c(I(\chi\nu^t, \chi\nu^{-t})) = 2c(\chi) = 0 \neq 1$. When $t = \frac{1}{2}$ or $-\frac{1}{2}$ the induced $\mathrm{tr}\, I(\chi\nu^t, \chi\nu^{-t})$ has composition series consisting of the Steinberg St twisted by χ, and $\chi \circ \det$. Hence $\chi(\widetilde{f}^{(1)}) = -1$ as asserted by the corollary ($= (-1)^c\binom{n-1}{c}$ where $c = 1 < 2 = n$).

Suppose now F is a global field of positive characteristic, that is, the field of rational functions on a smooth projective irreducible curve X over a finite field $k = \mathbb{F}_q$ of q elements. Denote by $\mathbb{A}$ its ring of adèles, $\mathbb{A}^\times$ its group of idèles, and $\mathbf{a}$ a fixed idèle of degree 1. Then a character χ of $\mathbb{A}^\times/F^\times\mathbf{a}^\mathbb{Z}$ defines a one dimensional representation $g \mapsto \chi(\det g)$ of $G(\mathbb{A})$, also denoted by χ. When n is a prime then all discrete spectrum automorphic representations in $L^2(\mathbf{a}^\mathbb{Z} G(F)\backslash G(\mathbb{A}))$ that are not (cuspidal and) generic are one dimensional, of the form $\chi : g \mapsto \chi(\det g)$. In fact we are working with an inner form G of $\mathrm{GL}(n)$ anisotropic at the places of the set S^D, and force in the next section the components of π on $G(\mathbb{A})$ to be one dimensional at the places of S^D. This corresponds to Steinberg (or one-dimensional) components of the corresponding automorphic representation of $\mathrm{GL}(n, \mathbb{A})$. Hence there are no other discrete spectrum automorphic representations contributing to our formula other than cuspidal and one-dimensional, i.e., no automorphic representations with Speh components occur in our setting. As usual $\mathcal{O}^\times = \mathcal{O}^\times_\mathbb{A}$ is $\prod_{s\in|X|}\mathcal{O}^\times_s$, $\mathbb{A}^0 = \{a \in \mathbb{A}^\times;\ |a| = 1\}$.

Corollary 4.5. *Suppose* S *is a finite set of closed points on* X, *or places of* F. *Consider the test function* $\widetilde{f}^{(c_s; s\in S)} = \prod_{s\notin S} f_s^{(0)} \cdot \prod_{v\in S} \widetilde{f}_s^{(c_s)}$ *where* $f_s^{(0)}$ *is the characteristic function of the maximal compact* K_s *divided by the volume* $|K_s|$ *of* K_s. *Then the trace* $\mathrm{tr}\,\chi(\widetilde{f}^{(c_s;s\in S)})$ *is zero unless* $\chi|\mathcal{O}^\times_\mathbb{A}$ *is* 1 *and* $c_s < n$ *for all* $s \in S$, *in which case it is* $(-1)^{\sum_{s\in S} c_s}\prod_{s\in S}\binom{n-1}{c_s}$.

The number of χ on $\mathbb{A}^\times/\mathbf{a}^\mathbb{Z}F^\times\mathcal{O}^\times_\mathbb{A} \simeq \mathbb{A}^0/F^\times\mathcal{O}^\times_\mathbb{A} \simeq \mathrm{Pic}^0(X)(\mathbb{F}_q)$ is the Picard number h_1.

5. Trace formula

Theorem 1.1 is translated in section 3 to a statement purely about automorphic representations, which mentions no arithmetic-geometric terms such as $\overline{\mathbb{Q}}_\ell$-smooth sheaves. Let us first state it, then prove it using the trace formula.

Let X be a smooth projective absolutely irreducible curve of genus g over the finite field $\mathbb{F}_q$, $F = k(X)$ its function field, $\mathbb{A}$ its ring of adèles, $\mathbb{A}^\times$ its group of idèles, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_q$, S^D a set of places of F of cardinality $N^D \geq 2$, S^w a set of places of F of cardinality $N^w \geq 0$, S^t a set of places of F of cardinality $N^t \geq 0$, $n \geq 2$ an integer. Fix integers $c_v > n$ for each $v \in S^w$, $c_v \leq n$ for each $v \in S^t$. Assume S^D, S^w and S^t are disjoint and put $S = S^D \sqcup S^w \sqcup S^t$. Denote by D a division algebra of rank n over F such that $D_v = D \otimes_F F_v$ is a division algebra at each $v \in S^D$, and the matrix algebra $M(n \times n, F_v)$ for all places v of F outside S^D. Fix $\mathbf{a} \in \mathbb{A}^\times$ of degree 1. Put $q_v = q^{\deg(v)}$. Suppose $n \geq 2$, $N^D \geq 2$, N^D is even or $n > 2$ is odd. Then Theorem 1.1 is equivalent to:

Theorem 5.1. *Denote by* $T^{\mathrm{aut}} = T^{\mathrm{aut}}(X, S^D, S^w, S^t, n, (c_v;\ v \in S^w \sqcup S^t))$ *the cardinality of the set of equivalence classes under twisting by a character of* $\mathbb{Z}/n$ *composed with* $\deg \circ \det : D^\times(\mathbb{A}) \to \mathbb{Z}$, *of the automorphic representations* π *of* $D^\times(\mathbb{A})$ *satisfying (i)* π *is unramified outside* S; *(ii) for* v *in* S^D *the local component* π_v *is of the form* $\chi \circ \det$ *for* χ *an unramified character of* $F_v^\times$; *(iii) the component* π_v $(v \in S^w)$ *has the fixed conductor* c_v $(> n)$; *(iv) the component* π_v $(v \in S^t)$ *has the fixed conductor* c_v, $1 \le c_v \le n$; *(v)* $\dim \pi > 1$; *(vi) the value* $\omega_\pi(\mathbf{a})$ *at* $\mathbf{a}$ *of the central character* ω_π *of* π *is* 1. *When* $N^w \ge 1$, *fixed* $c_v > n$ *for all* $v \in S^w$, $1 \le c_v \le n$ *for all* $v \in S^t$, *the number* $T^{\mathrm{aut}}(X, S^D, S^w, S^t; n, (c_v;\ s \in S^w \sqcup S^t))$ *is* $\frac{h_1 I^D}{q^n - 1} \prod_{v \in S^w \sqcup S^t} I_v$, *where*

$$I^D = \prod_{1 \le j < n} \mathbf{h}_1(q^j) \frac{\prod_{v \in S^D}(q_v^j - 1)}{(q^j - 1)^2},$$

$I_v = I_v(c_v)$ *is* $\sum_{0 \le k < c_v} (-1)^k \binom{n}{k} q_v^{n(c_v - k - 1)}(q_v^n - 1) + (-1)^{c_v}\binom{n}{c_v}$ *if* $v \in S^t$, *where* $1 \le c_v \le n$;
$I_v = I_v(c_v)$ *is* $q_v^{n(c_v - n - 1)}(q_v^n - 1)^{n+1}$ *when* $v \in S^w$, *where* $c_v > n$.
Put

$$\delta(n) = \delta(n, S^w, S^t;\ \{c_v;\ v \in S^w \sqcup S^t\}) \quad \text{for} \quad (-1)^{\sum_{v \in S^t} c_v} \prod_{v \in S^t \sqcup S^w} \binom{n}{c_v}.$$

It is zero if $c_v > n$, *namely* $S^w \ne \emptyset$. *When* $N^w = 0$, *thus* S^w *is empty,* $N^t \ge 1$, *thus all* $c_v \ne 0$ *are* $\le n$, *the number* $T^{\mathrm{aut}}(X, S^D, S^w, S^t; n, (c_v;\ s \in S^w \sqcup S^t))$ *is*

$$\frac{h_1 I^D}{q^n - 1}\left(\prod_{v \in S^t} I_v - \delta(n)\right) + h_1(\delta(n) - \delta(n-1)) + \delta(n)(-1)^{N^D(n-1)} T(S^D).$$

The factor $(-1)^{N^D(n-1)}$ is 1 by our assumption in this paper that $N^D \ge 2$ is even or $n \ge 2$ is odd. The factor $T(S^D)$ is an integral multiple of h_1 ([DF13, Theorem 6.18]). For brevity, if n and the c_v are understood, we denote

$$T^{\mathrm{aut}}(X, S^D, S^w, S^t; n, (c_v;\ s \in S^w \sqcup S^t))$$

by $T(S^D, S^w, S^t)$. By $T(S^D)$ we mean the $T(X_1, S_1, n)$ of [DF13, Theorem 2.3] (with S_1 replaced by our S^D), or T of [F17, Theorem 1.1] with N equal our $N^D = |S^D|$. This is the case where our S^w and S^t are empty.

If $n = 1$ then S^D is empty, $I_v = q_v^{c_v - 2}(q_v - 1)^2$ if $c_v \ge 2$, $I_v = q_v - 2$ if $c_v = 1$, and $\sum_{v \in S^t} c_v = |S^t|$, as in Theorem 2.1.

The purpose of this section and the next is to prove the theorem. We use the trace formula for the compact quotient case of $D^\times(\mathbb{A})$, reminiscent of the "simple trace formula", and the pseudo-coefficients of section 4. Again, we expect the case where N^D is odd and ≥ 3 to reduce to the case ($n > 2$ is odd or N^D is even) as in [DF13, section 5], or rather using the trace formula as here, and a pseudo-coefficient of a Steinberg component at a fixed place of D, as in [F2].

In sections 5 and 6 we denote T^{aut} by T to simplify notation.

The trace formula is an expression for the sum $\sum_\pi \operatorname{tr} \pi(f\,dy)$ of traces of the convolution operators $\pi(f\,dy)$ over all irreducible constituents π in $L^2(\mathbf{a}^{\mathbb{Z}} G(F) \backslash G(\mathbb{A}))$, $G = D^\times$. Here we choose a Haar measure $dy = \otimes_v dy_v$ on $G(\mathbb{A})$, and let $f = \otimes_v f_v$ be a test function on $G(\mathbb{A})$. Thus $f_v \in C_c^\infty(G_v)$ for each $v \in |X|$ and f_v is f_v^0, the characteristic function of $K_v = G(\mathcal{O}_v)$ divided by $\mathrm{vol}(K_v)$, for almost all $v \in |X|$. So $f dy$ is a smooth compactly supported measure on $G(\mathbb{A})$. The convolution operator $(r(f\,dy)\phi)(x) = \int_{G(\mathbb{A})} f(y)\phi(xy)dy$ on the $G(\mathbb{A})$-module $L = L^2(\mathbf{a}^{\mathbb{Z}}G(F)\backslash G(\mathbb{A}))$ ($G(\mathbb{A})$ acts by right shifts: $(r(g)\phi)(h) = \phi(hg)$, $\phi \in L^2$, g, $h \in G(\mathbb{A})$) is of finite rank, hence has trace $\sum_\pi \mathrm{tr}\,\pi(f\,dy)$ as above, since L decomposes as the direct sum of the irreducibles π. The sum $\sum_\pi$ is finite for a given f.

We choose f_v to be f_v^0 for all $v \in |X| - S$. Then $\pi_v(f_v)$ (we suppress dy from the notation) is a projection on the space $\pi_v^{K_v}$ of K_v-fixed vectors in π_v, a space of dimension 1 if π_v is unramified (has a non-zero K_v-fixed vector), 0 otherwise. Thus $\mathrm{tr}\,\pi_v(f_v)$ is 1 if π_v is unramified, 0 if not. At $v \in S^D$ we similarly choose f_v to be f_v^0, the characteristic function of the multiplicative group $K_v = \mathcal{D}_v^\times$ of a maximal order $\mathcal{D}_v$ in the division algebra D_v, divided by its volume $|K_v|$. Then $\mathrm{tr}\,\pi_v(f_v)$ is 1 if π_v is one-dimensional, of the form $\chi_v \circ \det$ for χ_v a(n unramified) character of $F_v^\times/\mathcal{O}_v^\times$, or 0 if not. At the places $v \in S^{wt} = S^w \sqcup S^t$ we choose f_v to be the generic pseudo-coefficients $\widetilde{f_v^{(c_v)}}$, of conductor $c_v \geq 1$, as defined in Theorem 4.2. They have the property that $\mathrm{tr}\,\pi_v(\widetilde{f_v^{(c_v)}})$ is 1 if π_v is a *generic* admissible irreducible representation of $\mathrm{GL}(n, F_v)$ of conductor c_v, it is 0 if such π_v has conductor $c(\pi_v) \neq c_v$.

If π_v is one-dimensional $\chi_v \circ \det$, $\mathrm{tr}\,\pi_v(\widetilde{f_v^{(c_v)}})$ is 0 if $\chi_v|\mathcal{O}_v^\times \neq 1$, and if $\chi_v|\mathcal{O}_v^\times = 1$ it is $\sum_{0\leq k\leq \min(c_v,n)}(-1)^k\binom{n}{k}$, which is 0 if $c_v \geq n$; it is $\sum_{0\leq k\leq c_v}(-1)^k\binom{n}{k} = (-1)^{c_v}\binom{n-1}{c_v}$ if $c_v < n$, by Corollary 4.3.

The π that appear non-trivially in the sum $\sum_\pi \mathrm{tr}\,\pi(f)$ correspond to discrete series automorphic representations of $\mathrm{GL}(n, \mathbb{A})$, trivial on $\mathbf{a}^{\mathbb{Z}}$, that are cuspidal, or one-dimensional, since $S^D \neq \emptyset$. The components π_v ($v \in S^w \sqcup S^t$) of the π that correspond to cuspidal representations of $\mathrm{GL}(n, \mathbb{A})$ are generic for all $v \notin S^D$, and then $\mathrm{tr}\,\pi_v(\widetilde{f_v^{(c_v)}})$ is 1 if $c(\pi_v)$ is c_v, and 0 if not. The components π_v ($v \in S^w \sqcup S^t$) of the π that correspond to one-dimensional automorphic representations are one-dimensional of the form $\chi_v \circ \det$, hence $\mathrm{tr}\,\pi_v(\widetilde{f_v^{(c_v)}})$ is 0 if $\chi_v|\mathcal{O}_v^\times \neq 1$, and if $\chi_v|\mathcal{O}_v^\times = 1$ it is $\sum_{0\leq k\leq \min(c_v,n)}(-1)^k\binom{n}{k}$, so 0 if $c_v \geq n$, $(-1)^{c_v}\binom{n-1}{c_v}$ if $c_v < n$.

Here we used the assumption that $S^D \neq \emptyset$, that implies that the automorphic representations of $D^\times(\mathbb{A})$ that are not one-dimensional correspond to representations π' of $\mathrm{GL}(n, \mathbb{A})$ with Steinberg components at all $v \in S^D$, due to our choice of test functions at $v \in S^D$, hence the π' are cuspidal. In other words, discrete series automorphic representations that are non-tempered at all places outside S, namely constituents of the induced $I(\nu^{\frac{m-1}{2}}\rho, \nu^{\frac{m-3}{2}}\rho, \dots, \nu^{-\frac{m-1}{2}}\rho)$, $\nu(x) = |\det x|$, ρ a cuspidal representation of $\mathrm{GL}(n/m, \mathbb{A})$, $m < n$, with "Speh" components, contribute zero to our formula. Square integrable components such as "generalized Steinberg" may occur at the places of $S^t \sqcup S^w$ if they have the right conductor.

So $\sum_\pi \mathrm{tr}\,\pi(f)$ for our f is the number nT, $T = T(n, S^D, S^w, S^t)$, of irreducible representations of $G(\mathbb{A})$ in $L^2(\mathbf{a}^{\mathbb{Z}}G(F)\backslash G(\mathbb{A}))$, $G = D^\times$, corresponding to cuspidal irreducible representations of $\mathrm{GL}(n,\mathbb{A})$ unramified outside S; $\mathrm{St}\otimes\chi_v\circ\det$ at each $v\in S^D$ with unramified χ_v; of conductor $c_v>0$ at each $v\in S^w\sqcup S^t$; plus

$$nh_1\delta(n-1),\qquad \delta(n-1)=(-1)^{\sum_{v\in S^w\sqcup S^t}c_v}\prod_{v\in S^w\sqcup S^t}\binom{n-1}{c_v}.$$

Note that the product here is 0 unless $c_v<n$ for all $v\in S^w\sqcup S^t$, in particular if $|S^w|>0$.

Indeed, there are nh_1 one-dimensional automorphic representations of $D^\times(\mathbb{A})$ satisfying (i)-(iv), (vi) (but not (v) as here the dimension is 1). This is because the maps $\det: D^\times(\mathbb{A})\to\mathbb{A}^\times$, $D^\times\to F^\times$, $\mathcal{O}_{D,v}^\times\to\mathcal{O}_v^\times$ are onto. The representations under consideration may then be identified with the unramified characters χ of $\mathbb{A}^\times$ such that $\chi(\mathbf{a}^n)=1$. But the group $F^\times\backslash\mathbb{A}^\times/\prod_v\mathcal{O}_v^\times\cdot\mathbf{a}^{n\mathbb{Z}}$ is an extension of $\mathbb{Z}/n$ by $\mathrm{Pic}^0(X_1)(\mathbb{F}_q)$, of cardinality h_1, as required. In conclusion we have:

Summary. The trace formula has the form

$$T(n;\,S^D,S^w,S^t)+h_1\delta(n-1)=\frac{1}{n}\times\ \text{geometric side}.$$

$\delta(n-1)$, defined in the theorem, is 0 if S^w is non-empty or there is a v in S^t with $c_v=n$.

Since we work with the anisotropic group $G=G^D$ that ramifies at the places S^D and is anisotropic there, all elements γ in $G(F)$ are elliptic, namely central or not contained in any proper F-rational parabolic subgroup (that G does not have). Hence the only terms in the geometric side of the trace formula are the orbital integrals of the elliptic elements.

We show in Proposition 6.3 below that the orbital integrals of non-central $\gamma\in G(F)$ and central non-identity elements are zero for any test function with support in $G(\mathcal{O})$ and a component $f_v^{(j)}$, $j\geq 1$. This reduces the computation of the geometric side of the trace formula for our global function $\widetilde{f}^{(c)}=\widetilde{f}^{(c_v;v\in S^{wt})}=(\otimes_{v\in S^{wt}}\widetilde{f}_v^{(c_v)})\otimes f^{S^{wt},0}$, where $f^{S^{wt},0}=\otimes_{v\notin S^{wt}}f_v^0$, to the value at the identity, provided $c_v>n$ for some $v\in S^w$.

In the remaining case, where $1\leq c_v\leq n$ for all S^{wt} $(=S^t$ as S^w is empty), we note that

$$\widetilde{f}_v^{(c_v)}=f_v^{(c_v)}+\sum_{0<j<c_v}(-1)^{c_v-j}\binom{n}{c_v-j}f_v^{(j)}+(-1)^{c_v}\binom{n}{c_v}f_v^{(0)}.$$

So we *need to remove* $\delta(n)f^0$, $f^0=\otimes_{v\in|X|}f_v^0$, from our test function $\widetilde{f}^{(c)}$, in order to use Proposition 6.3, and compute the trace formula separately for f^0. The latter is done in [DF13, Theorem 2.3] and [F17, Theorem 1.1, 5.1].

Schematically, we are interested in the sum $\sum_{\pi\subset L_0}\operatorname{tr}\pi(\widetilde{f}^{(c)})$ over the set L_0 of infinite dimensional automorphic representations π of $G(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$. We add the sum over the set of one-dimensional π, that is equal to $\delta(n-1)nh_1$, to get the sum over all automorphic π. Then express the test function $\widetilde{f}^{(c)}$ as the sum of $\phi=\widetilde{f}^{(c)}-\delta(n)f^0$ and $\delta(n)f^0$. To $\sum_\pi\operatorname{tr}\pi(\phi)$ we apply the trace formula to get $|G(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}G(F)|\phi(e)$. The trace formula for f^0 is far more complicated. Fortunately it is worked out in [DF13], and with alternative exposition in [F17]. So we can just quote that result: $n(h_1+(-1)^{N^D(n-1)}T(S^D))$; nh_1 is the number of one-dimensional automorphic representations. Note again: $\delta(n)$ is 0 unless $c_v\leq n$ for all v.

Again, we are to compute (sums over $\pi \subset L_0$)

$$\frac{1}{n}\sum \operatorname{tr}\pi(\widetilde{f}^{(c)}) = \frac{1}{n}\sum \operatorname{tr}\pi(\widetilde{f}^{(c)} - \delta(n)f^0) + \frac{\delta(n)}{n}\sum \operatorname{tr}\pi(f^0).$$

The first sum on the right is $\frac{1}{n}$ times the geometric side of the trace formula, at $\widetilde{f}^{(c)} - \delta(n)f^0$, minus $\frac{1}{n}$ times $\sum_\chi \chi(\widetilde{f}^{(c)} - \delta(n)f^0)$, which is $nh_1(= |\mathbb{A}^\times/F^\times\alpha^{n\mathbb{Z}}\mathcal{O}_{\mathbb{A}}^\times|)$ times $\delta(n-1) - \delta(n)$, since $\chi_s(f_s^{(c_s)}) = (-1)^{c_s}\binom{n-1}{c_s}$ and $\delta(n-1)$ is the product of these factors over $s \in S^{tw}$. The sum $\frac{1}{n}\sum \operatorname{tr}\pi(f^0)$ is $(-1)^{N^D(n-1)}T(S^D)$ by [DF13, Theorem 2.3].

6. Geometric side

The convolution operator $(r(f\,dy)\phi)(x) = \int_{G(\mathbb{A})} f(y)\phi(xy)dy$ is of finite rank, for any test function f, smooth of compact support, hence has trace. The integral can be written as

$$\int_{\mathbf{a}^{\mathbb{Z}}\cdot G(F)\backslash G(\mathbb{A})} \left[\sum_{\gamma\in\mathbf{a}^{\mathbb{Z}}\cdot G(F)} f(x^{-1}\gamma y)\right]\phi(y)dy.$$

Hence the trace is equal to the integral of the kernel over the diagonal, thus to

$$\int_{G(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}\cdot G(F)} \left[\sum_{\gamma\in\mathbf{a}^{\mathbb{Z}}\cdot G(F)} f(x\gamma x^{-1})\right]dx.$$

As our $G = D^\times$ is anisotropic modulo center, the elements of $G(F)$ are either central, thus are $\gamma \in \mathbf{a}^{\mathbb{Z}} \cdot Z(F)$ where $Z(F) \simeq F^\times$ is the center of $G(F)$, or elliptic non-central, which generate a field extension $F(\gamma)$ of F of degree $d > 1$ dividing n. The centralizer $Z_\gamma(F)$ of γ in $G(F)$ is the multiplicative group of a central division algebra over $F(\gamma)$ of degree m with $md = n$. When γ is regular, thus $d = n$, this division algebra is $F(\gamma)$ itself, and the centralizer in $D^\times$ is an F-torus $T(F) \simeq F(\gamma)^\times$. We consider γ only up to conjugacy, denoted $\sim$, in $G(F)$. Then we have

Proposition 6.1. *The geometric side of the trace formula takes the form*

$$\operatorname{vol}(D^\times(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}\cdot D^\times)\cdot\sum_{\gamma\in F^\times\cdot\mathbf{a}^{\mathbb{Z}}} f(\gamma) + \sum_{\gamma\in\mathbf{a}^{\mathbb{Z}}(G(F)-Z(F))/\sim}\int_{D^\times(\mathbb{A})/Z_\gamma(F)\mathbf{a}^{\mathbb{Z}}} f(x\gamma x^{-1})dx.$$

We use this with the test function $f = \otimes_{v\in S^{wt}} f_v^{(c_v)} \otimes f^{S^{wt},0} - \delta(n)f^0$, that is supported in $G(\mathcal{O})$. It is a linear combination of $f' = f^{S^{wt},0} \otimes \otimes_{v\in S^{wt}} f_v^{(j_v)}$ with $j_v > 0$ at least for one v. The sum over γ in $F^\times \cdot \mathbf{a}^{\mathbb{Z}}$ is discussed in [F17, Proposition 5.1]. But note that in the displayed expression on p. 151, line 6, the exponent "-1" of "$\xi(D_v^\times)$" is missing (the expression in Proposition 5.1 is correct). If $f(\gamma) \neq 0$ then the scalar γ lies both in $\mathcal{O}_{\mathbb{A}}^\times$ and in $F^\times\mathbf{a}^{\mathbb{Z}}$, thus in $\mathbb{F}_q^\times$. In our case the components f'_v of f' at least at one $v \in S^w \sqcup S^t$ are supported on $K_{v,c_v} \subset K_v$, and this implies that $f(\gamma) \neq 0$ happens only when $\gamma = e$, the identity in $G(F)$. The value of $f(e)$ is then $1/\operatorname{vol}(\mathcal{D}_{\mathbb{A}}^\times)$, as in [F17, Proposition 5.1], times $\prod_{v\in S^w\sqcup S^t} I_v - \delta(n)f^0(e)$. Here $I_v = \widetilde{f}_v^{(c_v)}(e)$ is

$$I_v = I_v(c_v) = q_v^{n(c_v-n-1)}(q_v^n - 1)^{n+1}/|K_v|$$

if $v \in S^w$, thus $c_v > n$, and if $v \in S^t$, thus $1 \leq c_v \leq n$, it is

$$I_v = I_v(c_v) = \sum_{0 \leq k < c_v} (-1)^k \binom{n}{k} q_v^{n(c_v-k-1)}(q_v^n - 1) + (-1)^{c_v}\binom{n}{c_v},$$

divided by $|K_v|$. We often normalize the measure by $|K_v| = 1$, to simplify the notation. The value of $1/\operatorname{vol}(\mathcal{D}_{\mathbb{A}}^\times)$ is $\operatorname{vol}(\mathrm{GL}(n, \mathcal{O}_{\mathbb{A}}))^{-1} \prod_{v \in S^D} \xi(D_v^\times)$ as computed in [F17, Proposition 5.1]. The volume $\operatorname{vol}(D^\times(\mathbb{A})/\mathbf{a}^{\mathbb{Z}} D^\times)$ before the sum is $\operatorname{vol}(D_{\mathbb{A}}^0/D^0) \cdot \operatorname{vol}(\mathbb{A}^\times/F^\times \mathbf{a}^{n\mathbb{Z}})$, where $D_{\mathbb{A}}^0$ is the kernel of the reduced norm $D^\times(\mathbb{A}) \twoheadrightarrow \mathbb{A}^\times$. The second factor here is n times $\operatorname{vol}(\mathbb{A}^\times/F^\times \mathbf{a}^{\mathbb{Z}}) = h_1 \operatorname{vol}(\mathcal{O}_{\mathbb{A}}^\times)/(q-1)$ ([F17, p. 151, l. -11]). Also $\operatorname{vol}(\mathrm{GL}(n, \mathcal{O}_{\mathbb{A}}))/\operatorname{vol} \mathcal{O}_A^\times$ equals $\operatorname{vol} \mathrm{SL}(n, \mathcal{O}_{\mathbb{A}})$ for the choice of measures explained in [F17].

The Tamagawa volume $\operatorname{vol}_T \mathrm{SL}(n, \mathcal{O}_{\mathbb{A}})$ is $\prod_{1 \leq j < n} \zeta(X, q^j)^{-1}$ as computed in [F17, p. 152].

Altogether the terms of the scalars in the geometric side of the trace formula add up to

$$\frac{nh_1}{q-1} \prod_{1 \leq j < n} \zeta(X, q^j) \cdot \prod_{v \in S^D} \xi(D_v^\times) \cdot \left(\prod_{v \in S^w \sqcup S^t} \widetilde{f}_v^{(c_v)}(e) - \delta(n) \right),$$

where $\xi(D_v^\times) = \xi(d_v, m_v, q_v) = (\prod_{1 \leq j < n}(q_v^j - 1))/\prod_{1 \leq j < m_v}(q_v^{d_v j} - 1)$ if $D_v^\times = \mathrm{GL}(m_v, D_v')$ and D_v' is a division algebra central of rank $d_v = n/m_v$ over F_v, by [F17, Proposition 6.1]. In our case, as in [F17], $m_v = 1$, $d_v = n$, for all $v \in S^D$. Also $\zeta(X, t) = \mathbf{h}_1(t)/(1-t)(1-qt)$. In summary:

Proposition 6.2. *Divided by n, the identity contribution to the geometric side of the trace formula is h_1, times $I^d - \delta(n)$, where $I^d = \prod_{v \in S^w \sqcup S^t} I_v$, times*

$$\frac{1}{q-1} \prod_{1 \leq j < n} \frac{\mathbf{h}_1(q^j) \prod_{v \in S^D}(q_v^j - 1)}{(q^j - 1)(q^{j+1} - 1)} = \frac{I^D}{q^n - 1}, \quad I^D = \prod_{1 \leq j < n} \frac{\mathbf{h}_1(q^j) \prod_{v \in S^D}(q_v^j - 1)}{(q^j - 1)^2}.$$

This is $\frac{h_1 I^D}{q^n - 1}(I^d - \delta(n))$.

Our analysis of the non-scalar terms in the geometric part of the trace formula is based on [F17, section 7]. The non-scalar elements are elliptic since D is a division algebra. The contribution from these elements is the sum over the conjugacy classes in $D^\times$ of elements $\gamma \in \mathbf{a}^{\mathbb{Z}} \cdot (D^\times - F^\times)$, of $\int_{D^\times(\mathbb{A})/Z_\gamma(F)\mathbf{a}^{\mathbb{Z}}} f(x\gamma x^{-1})dx$. The centralizer $Z_\gamma(F)$ of a γ in $D^\times$ is the multiplicative group of a division algebra D_{d,F_m} central over $F_m = F(\gamma)$ of degree $d = n/m$, where $F(\gamma)$ is a field extension of F of degree m dividing n. Fix a subfield $\mathbb{F}_{q^n}$ in D.

Proposition 6.3. *If $\gamma \in \mathbf{a}^{\mathbb{Z}} \cdot (D^\times - F^\times)$ then $f(x\gamma x^{-1}) = 0$ for $f = \otimes_{v \in S^{tw}} f_v^{(c_v)} \otimes \otimes_{v \notin S^{tw}} f_v^0$.*

Proof. As shown in [F17, Lemma 7.1], if $f = \otimes_v f_v$, f_v supported on the maximal compact subgroup K_v for all $v \in |X|$, and $f(x\gamma x^{-1}) \neq 0$ for some $x \in G(\mathbb{A})$, then the characteristic polynomial of γ has coefficients in $\mathbb{F}_q$, so a conjugate of γ, denoted again by γ, lies in $\mathbb{F}_{q^n} - \mathbb{F}_q$ and $\mathbb{F}_q(\gamma) = \mathbb{F}_{q^m}$, $m|n$, and $L = F(\gamma)$ is $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$. Our $f = \otimes_v f_v$ is a product over v of $f_v = f_v^0$ for all $v \in |X| - S^t \sqcup S^w$, and $f_v = f_v^{(c_v)}$ for all $v \in S^t \sqcup S^w$, where $c_v \geq 1$ for all $v \in S^{tw}$. For such v the support of $f_v = f_v^{(c_v)}$ is contained in $K_{v,c_v} \subset K_v$, $c_v > 0$. Then $f_v^{(c_v)}(x\gamma x^{-1}) \neq 0$ implies that an eigenvalue of γ lies in $1 + \varpi_v^{c_v}\mathcal{O}_v$, $c_v > 0$, so it is pro-unipotent. On the other hand γ is of finite order, dividing q^m-1, $m|n$, as γ lies in $\mathbb{F}_{q^m}^{\times}$. Hence $\gamma=e$, the identity, contradicting its being elliptic non-scalar. □

This establishes theorem 5.1, hence also theorem 1.1.

7. Powers of Frobenius

When the formula of Theorem 1.1 is applied to $(X_m,\, S_m^D,\, S_m^{wt})$ we obtain the number $T(X_m,\, S_m^D,\, S_m^{wt},\, n,\, C,\, \iota\, :\, S_m^{wt} \twoheadrightarrow C)$ of fixed points of $\mathrm{Fr}^{*,m}$ acting on the set $E_n(X-S, S^D, S^{wt}, C, \iota)$. When $(X_1,\, S_1)$ over $\mathbb{F}_q$ is replaced by $(X_m,\, S_m)$ over $\mathbb{F}_{q^m}$, q is replaced by q^m, the cardinality N_1^* of S_1^* becomes N_m^*, the Fr^*-orbits $s\in S_1$ split to $\mathrm{Fr}^{*\,m}$-orbits, and the degrees $\deg(s)$ change too (to the number of geometric points in the $\mathrm{Fr}^{*\,m}$-orbits in $s\otimes_{\mathbb{F}_q}\mathbb{F}_{q^m}$). But we do not have the divisor (n/S_1) of n that occurs in [DF13], as in our case of a nonempty S^{wt} there is no contribution from the nonscalar elliptic elements to the trace formula that we consider, except of course in the case $c_v\le n$ for all v, and the term $T(S_1^D)$ of [DF13, Theorem 2.3], that is geometrized in [DF13, section 6].

Let us rewrite the formula of Theorem 1.1 to better follow these changes. As in [DF13, section 6.2], denote by (a) the multiset of the eignevalues of Frob acting on $H^1(X)$, counted with their multiplicities. It has $2g$ elements. The polynomial $\mathbf{h}_1(t)=\det[I-\mathrm{Frob}\cdot t;\, H^1(X)]\in\mathbb{Z}[t]$ is the product over $\alpha\in(a)$ of $(1-\alpha t)$, and $\mathbf{h}_m(q^m)=\prod_\alpha(1-\alpha^m q^m)$. Denote by (b_s) the set of $\deg(s)$ roots of unity. Let B^D denote the multiset sum $\sqcup_{s\in S_1^D}(b_s)$ of the (b_s), $s\in S_1^D$, and by B_2^D the multiset B^D minus twice $\{1\}$. Recall we assume S_1^D consists of $N_1^D\ge 2$ elements, and each (b_s) contains 1 once. Put $B_2^{D,(a)}$ for the sum of B_2^D and (a). Define B^c and $B_1^c=B^c-\{1\}$ using S_1^c instead of S_1^D. Then B^D is the multiset of the eigenvalues of the action of Frob on $\mathbb{Q}^{S^D}$, as S^D is the disjoint union of the fibers of the projection $S^D\twoheadrightarrow S_1^D$; the fiber at $s\in S_1^D$ has $\deg(s)$ elements, permuted cyclically by the Frobenius. $B_2^{D,(a)}$ consists of $2g+N^D-2$ elements.

The factor I^D is as appears in [DF13, p. 960, l. 6] with $m=1$: the product over $1\le j<n$ of $(-1)^{S_1^D}$ times

$$(1-q^j)^{-2}\cdot\prod_{\alpha\in(a)}(1-\alpha q^j)\cdot\prod_{s\in S_1^D}\prod_{\eta\in(b_s)}(1-\eta q^j)=\prod_{\beta\in B_2^D}(1-\beta q^j).$$

Considering q and the $\beta\in B_2^D$ as indeterminates Q and $X_1,\dots,X_{2g+N^D-2}$, we see we have a polynomial over $\mathbb{Z}$, and conclude as in [DF13, Corollary 6.8], Corollary 1.2: *When* $(X_1,\, S_1^D)/\mathbb{F}_q$ *is replaced by* $(X_m,\, S_m^D)/\mathbb{F}_{q^m}$, *the factor* I^D, *as a function of* $m\in\mathbb{Z}_{\ge 1}$, *has the form* $\sum_k n_k\gamma_k^m$ *where* $n_k\in\mathbb{Z}$ *and each* γ_k *is the product of a root of unity and a monomial in* q *and the eigenvalues of the action of* Frob *on* $H^1(X)$. Assuming the γ_k are distinct and the n_k nonzero, the expression is unique, and we may name the γ_k the eigenvalues occurring with nonzero multiplicity, and the n_k the multiplicity of γ_k. As explained in [DF13, Corollary 6.9] the same result holds for $h_1=\prod_{\alpha\in(a)}(1-\alpha)$, and hence for the product $h_1 I^D$.

The factor $(-1)^{|S_1^D|}$ is given by the trace of the Frobenius on the smooth sheaf $\varepsilon_\ell(\mathcal{S}^D)$ of [DF13, midpage 970].

Here is an alternative approach – of Deligne – to see that the tame factor I^D has the behavior described in Corollary 1.2, when $\mathbb{F}_q$ changes to $\mathbb{F}_{q^m}$. Consider the group $G = \mathrm{GSp}(2g, \mathbb{C}) \times \mathrm{Sym}_{|S|}$, where GSp is the group of symplectic similitudes and $\mathrm{Sym}_{|S|}$ is the symmetric group on $|S|$ letters, where we now put S for $S^D = S_1^D \otimes_{\mathbb{F}_q} \mathbb{F}$, and S_1 for S_1^D. If there is a representation R of G and a conjugacy class $\gamma \in G$ with $I^D = \mathrm{tr}(\gamma; R)$ then the required behavior holds for I^D, as passing from $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$ replaces γ by γ^m. We take the component of γ in $\mathrm{GSp}(2g, \overline{\mathbb{Q}})$ to be defined by the eigenvalues (a): $\alpha_1, \dots, \alpha_g, \alpha_{g+1}, \dots, \alpha_{2g}$, $\alpha_{g+i} = q/\overline{\alpha}_i$. Put $H : \mathrm{GSp}(2g) \hookrightarrow \mathrm{GL}(2g)$ for the natural embedding, $(1) : G \to \mathrm{GSp}(2g) \to \mathbb{G}_m$, $\gamma \mapsto q$, and $H(j)$ for the tensor product of H and $(j) = (1)^{\otimes j}$. Note that tensor products, sum, differences of virtual representations are virtual representations. Then

$$\mathbf{h}_1(q^j) = \mathrm{tr}[\gamma; \sum_{0 \le i \le 2g} (-1)^i \wedge^i H(j)] = \prod_{\alpha \in (a)} (1 - \alpha q^j).$$

Also we want to represent $\prod_{s \in S_1} (1 - q^{d_s}) \cdot (1-q)^{-2}$, $|S_1| \ge 2$, where twisting by (j) would replace q by q^j. Note that $s \in S_1$ defines a set of $d_s = \deg(s)$ embeddings $k(s) = \mathbb{F}_{q_s} = \mathbb{F}_{q^{\deg(s)}} \hookrightarrow \mathbb{F}$ over $\mathbb{F}_q \hookrightarrow \mathbb{F}$, that are permuted cyclically by the Frobenius. Thus S_1 defines an element $\sigma = \prod_{s \in S_1} (d_s)$ in the symmetric group Sym_N on $N = |S| = \sum_{s \in S_1} d_s$ (recall N denotes N^D here) letters, (d_s) denoting a cycle of length d_s. We assume $|S_1| \ge 2$, thus σ is not a cycle of order N. We are then looking for a virtual representation R of $\mathbb{G}_m \times \mathrm{Sym}_N$ with $\mathrm{tr}[(q, \sigma); R] = \prod_{s \in S_1} (1 - q^{d_s}) \cdot (1-q)^{-2}$ for all $\sigma = \prod_{s \in S_1} (d_s) \in \mathrm{Sym}_N$, σ not an N-cycle.

Denote by Perm the permutation representation of Sym_N on the N-dimensional $\mathbb{C}$-space spanned by the basis $e_1, \dots, e_N$, and by Perm(1) its tensor product with the natural one-dimensional representation $(1) : \mathbb{G}_m \to \mathrm{End}\, \mathbb{C}$. Then

$$\det[I - (q, \sigma);\ \mathrm{Perm}(1)] = \det[I - q\sigma;\ \mathrm{Perm}] = \prod_{s \in S_1} \det[I - q(d_s);\ \mathbb{C}^{d_s}]$$

$$= \prod_s (1 - q^{d_s}) = \prod_{s \in S_1} \prod_{\{\zeta;\, \zeta^{d_s} = 1\}} (1 - \zeta q).$$

Here $\sigma \in \mathrm{Sym}_N$ is not an N-cycle. Also replacing q by 1 we obtain 0.

Next write $\mathrm{Perm} = \mathbf{1} \oplus \overline{\mathrm{Perm}}$, $\mathbf{1}$ the trivial representation, on $e_1 + \cdots + e_N$. Then $\det[I - (q, \sigma);\ \mathbf{1}(-1)] = 1 - q$, and $\det[I - (q, \sigma);\ \overline{\mathrm{Perm}}(-1)] = \prod_{s \in S_1} (1 - q^{d_s}) \cdot (1-q)^{-1}$ for any $\sigma \in \mathrm{Sym}_N$ that is not an N-cycle. This is $\sum_{0 \le i \le N-1} (-1)^i \,\mathrm{tr}[(q, \sigma);\ \wedge^i (\overline{\mathrm{Perm}}(1))]$, a polynomial in q of degree $N - 1$.

The term in I^D that we wish to represent is $\prod_{s \in S_1} (1 - q^{d_s}) \cdot (1-q)^{-2}$, when $|S_1| \ge 2$, thus $\sigma = \prod_{s \in S_1} (d_s) \in \mathrm{Sym}_N$ is not an N-cycle. It is a polynomial in q of degree $N - 2$, equal to $\sum_{0 \le k \le N-2} a_k(\sigma) q^k$. The following implies Corollary 1.2.

Proposition 7.1. *For each k, $1 \le k \le N-2$, there exists a virtual character χ_k of* Sym_N *that agrees with a_k on* $\mathrm{Sym}_N - \{N\text{-cycles}\}$.

This does not determine χ_k as the values on N-cycles are missing.

Proof. We start with

$$(1-q)^{-1}\prod_{s\in S_1}(1-q^{d_s}) = \sum_{0\le i\le N-1}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]\cdot q^i$$

that holds for all $\sigma\in\mathrm{Sym}_N$. Now in the power series field $\mathbb{C}((t))$ we have $(1-q)^{-1}=\sum_{j\ge 0}q^j$, so

$$(1-q)^{-2}\prod_{s\in S_1}(1-q^{d_s}) = \sum_{j\ge 0}q^j\cdot\sum_{0\le i\le N-1}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]\cdot q^i$$

$$=\sum_{k\ge 0}q^k\sum_{0\le i\le k}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]$$

for all $\sigma\in\mathrm{Sym}_N$, not an N-cycle. The left side is a polynomial of degree $N-2$, so we conclude that $a_k(\sigma)=\sum_{0\le i\le k}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]$ for all k, $1\le k\le N-2$, as claimed. ☐

Moreover, for $k>N-2$, the virtual character $\chi_k:\sigma\mapsto\sum_{0\le i\le k}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]$ is 0 at all $\sigma\in\mathrm{Sym}_N-\{N\text{-cycles}\}$. As $\overline{\mathrm{Perm}}$ is of dimension $N-1$, $\chi_k=\chi_{N-1}$ for all $k\ge N-1$.

Consider $\chi=\chi_{N-1}$. Expanding in $\mathbb{C}((q))$:

$$(1-q)^{-1}\cdot(1-q^N)/(1-q)=\sum_{i\ge 0}q^i(1+q+\cdots+q^{N-1})$$

$$=\sum_{0\le k\le N-1}q^k(k+1)+\sum_{k\ge N}N\cdot q^k=\sum_{0\le k\le N-2}(k+1)q^k+\sum_{k>N-2}Nq^k,$$

we get the values of $\chi(\sigma)$ also at $\sigma=N$-cycle. In summary:
$\chi(\sigma)=\sum_{0\le i\le N-1}(-1)^i\,\mathrm{tr}[\sigma;\,\wedge^i(\overline{\mathrm{Perm}})]$ is N if $\sigma\in\mathrm{Sym}_N$ is an N-cycle, 0 if not.

There are $(N-1)\,!$ N-cycles, hence

$$(\chi,\chi)=\frac{1}{N\,!}\sum_{\sigma\in\mathrm{Sym}_N}\chi(\sigma)\chi(\sigma)=\frac{(N-1)\,!\,N^2}{N\,!}=N,$$

in agreement with: $\wedge^i(\overline{\mathrm{Perm}})$, $0\le i<N$, are irreducible representations of Sym_N.

For every irreducible character η of Sym_N,

$$(\eta,\chi)=\frac{1}{N\,!}\sum_{\sigma\in\mathrm{Sym}_N}\chi(\sigma)\eta(\sigma)=\frac{(N-1)\,!\,N}{N\,!}\eta(N\text{-cycle})=\eta(N\text{-cycle}).$$

Hence with $\eta=\wedge^i(\overline{\mathrm{Perm}})$ we have

$$\mathrm{tr}[N\text{-cycle};\,\wedge^i(\overline{\mathrm{Perm}})]=(\wedge^i\overline{\mathrm{Perm}},\chi=\sum_{0\le i\le N-1}(-1)^i\wedge^i(\overline{\mathrm{Perm}}))=(-1)^i,$$

thus the $\wedge^i(\overline{\mathrm{Perm}})$ are the only irreducible representations of Sym_N whose character does not vanish on the N-cycles: if there is an irreducible η with $\eta(N\text{-cycle})\ne 0$, then $(\eta,\chi)\ne 0$, so that $\eta=\wedge^i(\overline{\mathrm{Perm}})$ for some i, $0\le i\le N-1$.

Now consider the factor $-(1-q^n)^{-1}\prod_{s\in S_1^c} I_s$ for $c>n$. It is the product over $s\in S_1^c$ of

$$\prod_{(b_s)} q^{c-n-1}\prod_{\eta\in(b_s)}(1-\eta q^n)^n\prod_{\eta}(1-\eta q^n)$$

where the last product is over (b_s) for all s except one where it is over $(b_s)-\{1\}$, multiplied by $(-1)^{n+1}$ (or $(-1)^n$ if $(b_s)-\{1\}$). The product over $s\in S_1^c$ is then

$$=(-1)^{(n+1)|S_1^c|-1}q^{(c-n-1)|B^c|}\cdot\prod_{\beta\in B^c}(1-\beta q^n)^n\cdot\prod_{\beta\in B_1^c}(1-\beta q^n).$$

Then $\prod_{s\in S_1^c} I_s$ for $c>n$ is given by the same formula but with last product over B^c, not $B_1^c=B^c-\{1\}$, and the sign is $(-1)^{(n+1)|S_1^c|}$. Clearly we obtain a polynomial in q over $\mathbb{Z}$, depending only on $|B^c|$, and a power of $\pm\varepsilon_\ell(\mathcal{S})$.

The factors $I_s=I_s(c,q_s)$ when $c_s=c\le n$ are

$$-\sum_{0\le k<c}(-1)^k\binom{n}{k}\prod_{(b_s)}q^{n(c-k-1)}\cdot\prod_{(b_s)}(1-\eta q^n)+(-1)^c\binom{n}{c}.$$

Although (b_s) consists of $\deg(s)=|\overline{s}|$, $\overline{s}=s\otimes_{\mathbb{F}_q}\mathbb{F}$, elements, I do not see a way to express this, or the result when there is $c\le n$, in terms of $N^c=|B^c|$ and q alone, also when $n=1$.

References

[D15] P. Deligne, *Comptage de faisceaux ℓ-adiques.* Astérisque **369** (2015), 285-312, SMF.

[DF13] P. Deligne, Y. Flicker, *Counting local systems with principal unipotent local monodromy.* Ann. of Math. **178** (2013), 921-982.

[F14] Y. Flicker, *The trace formula for* GL(2) *over a function field.* Doc. Math. **19** (2014), 1-62.

[F15] Y. Flicker, *Counting Galois representations with at most one, unipotent, monodromy.* Amer. J. Math. **137** (2015), 1-25.

[F17] Y. Flicker, *Counting local systems via the trace formula.* J. reine angew. Math. **731** (2017), 139-157.

[F24] Y. Flicker, *Counting local systems with tame ramification.* Manuscripta Math. **173** (2024), 771-830.

[FÖ23] Y. Flicker, E. Özkan, *Local systems with tame, and a unipotent, local monodromy.* São Paulo J. Math. Sci. **17** (2023), 465-482.

[F1] Y. Flicker, *Counting tame local systems by anisotropic tools.* Preprint. `https://www.ariel.ac.il/wp/yuval-flicker/wp-content/uploads/sites/316/2022/10/anisotropicCountFull.pdf`

[F2] Y. Flicker, *Euler-Poincaré functions and local systems.* Preprint.

[GR10] B. H. Gross, M. Reeder, *Arithmetic invariants of discrete Langlands parameters.* Duke Math. J. **154** (2010), 431-508.

[J12] H. Jacquet, *A correction to Conducteur des representations du groupe lineaire.* Pacific J. Math. **260** (2012), 515-525.

[JPS81] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Conducteur des représentations du groupe linéaire.* Math. Ann. **256** (1981), 199-214.

[LR03] J. Lansky, A. Raghuram, *On the correspondence of representations between $GL(n)$ and division algebras.* Proc. Amer. Math. Soc. **131** (2003), 1641-1648.

[M13] N. Matringe, *Essential Whittaker functions for $GL(n)$.* Doc. Math. **18** (2013), 1191-1214.

[MP94] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups.* Invent. Math. **116** (1994), 393-408.

[MP96] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal K-types.* Comment. Math. Helvetici **71** (1996), 98-121.

[R91] M. Reeder, *Old Forms on GL_n*. Amer. J Math. **113** (1991), 911-930.
[Yu] Hongjie Yu, *ℓ-adic local systems and Higgs bundles: the generic case*. arXiv: 2304.06637

Ariel University, Ariel 40700, Israel; The Ohio State University, Columbus OH43210.
Email address: `yzflicker@gmail.com`